

MATH 502: REAL AND COMPLEX ANALYSIS

SPRING 2003

A.Katok

SHORT SYLLABUS

1. Geometry of Banach and Hilbert spaces including the three principles of linear analysis.
2. Complex analysis.
3. Elements of spectral theory of linear operators in Banach and Hilbert spaces.

Text: *W. Rudin, Real and Complex Analysis*

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PROBLEM SET # 1: January 17

GEOMETRY OF BANACH AND HILBERT SPACES

Due on Friday 1-31-03

1. Prove that a linear space of countable dimension (i.e. a space where each vector is a finite linear combination of vectors from a countable system, such as the space of all polynomials) is never complete with respect to a norm.

Hint: You may use Baire Category Theorem.

2. Consider the space B of functions of bounded variation on the interval $[0, 1]$. Prove that $\text{Var}f + |f(0)|$ defines a norm on B and makes it into a Banach space.

3. Introduce the norm as above, i.e. $\|f\| = \text{Var}f + |f(0)|$ in the space of polynomials on $[0, 1]$. Describe the completion and identify it with a subspace of the space B from the previous problem.

4. Prove that the space l^1 is not isomorphic to any l^p for $p > 1$, i.e. there is no continuous linear bijection between the spaces with continuous inverse.

Hint: Find an invariant of isomorphism which distinguishes the spaces.

5. Consider the space S of finite linear combinations of characteristic functions of intervals in $[0, 1]$ with the supremum norm. Identify the closure with a certain space of functions on $[0, 1]$.

Hint: Consider how discontinuities of functions in the completion may look like.

6. Prove that the unit sphere in an infinite-dimensional Hilbert space is not compact.

7. Let $a = (a_1, \dots) \in l^2$. Consider the following set $C_a \subset l^2$:

$$\{x = (x_1 \dots) : |x_n| \leq |a_n|, n = 1, \dots\}.$$

Prove that C_a is compact and is homeomorphic to the product of the countable number of copies of the unit interval with the product topology.

8. Prove that in x_n is a sequence of elements of a Hilbert space which converges weakly to an element x and $\lim_{n \rightarrow \infty} \|x_n\| = \|x\|$ then x_n converges to x , i.e. $\|x_n - x\| \rightarrow 0$.

9. Prove that a convex closed subset of a Hilbert space is closed in weak topology.

10. Let H be a separable Hilbert space, $L \subset H$ a subspace such that both L and its orthogonal complement L^\perp are infinite dimensional. Prove that there exists an invertible isometry between H and l^2 which maps L to the subspace of vectors whose even coordinates vanish and L^\perp into the space of vectors whose odd coordinates vanish.

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PROBLEM SET # 2: February 2, 2003

HILBERT SPACES AND BANACH SPACES, cont.

Due on Friday 2-14-03

11. Consider the closure of the space of trigonometric polynomials with respect to the following scalar product: for $f = \sum_{|m| \leq N} f_m \exp 2\pi i m x$ and $g = \sum_{|m| \leq N} g_m \exp 2\pi i m x$,

$$\langle f, g \rangle = \sum f_m \bar{g}_m (1 + |m|^{2k+1}).$$

Prove that every element of the completion may be naturally identified with a function which has k derivatives.

12. Give an example of a trigonometric series which converges in L_2 (and hence is a Fourier series of an L_2 function) but which diverges in a dense set of points.

13. Let X be a Banach space with the norm $\|\cdot\|_X$ and Y be its dense linear subspace provided with a complete norm $\|\cdot\|_Y$, such that $\|x\|_Y \geq \|x\|_X$. Obviously any continuous linear functional on $(X, \|\cdot\|_X)$ determines a continuous linear functional on $(Y, \|\cdot\|_Y)$. Prove that there exists a continuous linear functional on $(Y, \|\cdot\|_Y)$ which does not extend to a continuous linear functional on $(X, \|\cdot\|_X)$.

14. Consider the space $C^1([0, 1])$ of all continuously differentiable functions on the interval $[0, 1]$ with the norm $\|f\| = \sup |f| + \sup |f'|$. Find a general form of a continuous linear functional on the space $C^1([0, 1])$. Give an example of a functional satisfying the assertion of the previous problem.

15. Consider the space $CL([0, 1])$ of all bounded functions on the unit interval which have finite limits from the left and from the right at every point and are continuous from the left, provided with the sup norm. Find a general form of a continuous linear functional on $CL([0, 1])$.

16. Prove that there is no norm (not necessarily complete) on the space $C([0, 1])$ of continuous functions on the unit interval such that the norm convergence is equivalent to point-wise convergence.

17. Rudin, p. 114, N16 (third edition)

18. Rudin, p. 114, N 18

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PROBLEM SET # 3: February 18, 2003

Due on Monday 3-3-03

19. Is the unit ball in the space $C([0, 1])$ compact in the weak topology?
20. Is the set of all continuous functions with norm and variation ≤ 1 compact in $C([0, 1])$?
21. Prove that a compact linear operator in an infinite-dimensional Banach space is never surjective.
22. Prove that a compact operator in a Banach space can only have finitely many eigenvalues of absolute value greater than any given positive number r and the total multiplicity of such eigenvalues is bounded.
23. Give an example of a compact linear operator in a separable Hilbert space which has no eigenvectors.
24. Let B be a Banach space and $S \subset B$ a closed subspace. A vector $v \in B$ is called a *perpendicular* to S if $\|v\| = \inf_{w \in S} \|v - w\|$. Give an example of a Banach space and a closed subspace which has no perpendicular vectors.
Hint: Use may use information about the linear functionals in $C([0, 1])$.
25. Prove that every closed subspace S of a Banach space B has an *almost perpendicular* vector: for every $\epsilon > 0$ there exists a vector v such that $\|v\| = 1$ and $\inf_{w \in S} \|v - w\| > 1 - \epsilon$
26. A linear operator A in a Hilbert space is called *normal* if it commutes with its conjugate A^* : $AA^* = A^*A$. Prove that the orthogonal complement to a subspace invariant under a normal operator is invariant
27. Prove that any compact normal operator in a complex Hilbert space has an eigenvector
Hint: Try to reduce this to a statement for self-adjoint operators.
28. Find a necessary and sufficient condition for an integral operator A in $L_2(X, \mu)$ $(Af)(x) = \int_X K(x, y)f(y)d\mu(y)$ to be normal.

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FIRST MIDTERM EXAMINATION

Thursday March 6, 2003

For a perfect score you should give complete solutions of two problems from each of the two sections.

SECTION 1

1.1. Let B be a real Banach space, $v_1, \dots, v_n \in B$ be n linearly independent vectors and x_1, \dots, x_n be arbitrary vectors in \mathbb{R}^n . Prove that there exists a bounded linear operator $L: B \rightarrow \mathbb{R}^n$ such that $Lv_i = x_i$, $i = 1, \dots, n$.

1.2. Let \mathcal{L}_n be the class of all functions f on $[0, 1]$ such that

(1) $|f| \leq 1$:

(2) f is linear on any interval $[k/n, (k+1)/n]$, $k = 0, \dots, n-1$;

(3) for $k = 0, \dots, n-1$, $f((k+1)/n) - f(k/n) = \epsilon/n$ where $\epsilon = 0, +1$, or -1 .

Is the set $\bigcup_n \mathcal{L}_n$ precompact in $C([0, 1])$?

You will get extra bonus points if you correctly describe the closure of $\bigcup_n \mathcal{L}_n$ in $C([0, 1])$ (even without proof).

1.3. Consider the space $L_\infty([0, 1])$ of equivalence classes up to a set of measure zero of bounded measurable functions on $[0, 1]$. Prove that there is no norm in $L_\infty[0, 1]$ (not necessarily complete) such that the norm convergence is equivalent to the convergence in probability (also called convergence in measure).

1.4. Consider the space \mathcal{R} of *Riemann integrable* functions on $[0, 1]$ (recall that those are exactly bounded functions for which the set of points of discontinuity have *Lebesgue* measure zero). This space is closed in the sup norm. Prove that extension of the Riemann integral from $C([0, 1])$ to linear functional of norm one with respect to the sup norm on \mathcal{R} is not unique.

SECTION 2

2.1. Suppose a continuous linear operator A in a complex Hilbert space has an orthonormal basis of eigenvectors. Prove that A is normal.

2.2. Let A be a linear operator in a Hilbert space such that both A and A^* are isometries, i.e. preserve the scalar product. Prove that A is invertible.

2.3. Suppose the norm in a real Banach space satisfies the *parallelogram identity*: for any two vectors x, y :

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Prove that the norm is determined by a scalar product.

2.4. Let H be an infinite-dimensional Hilbert space. Given a number t , $0 \leq t \leq 1$ construct an example of a sequence of vectors $x_n \in H$, $n = 1, 2, \dots$, all of norm one, which converges weakly to a vector x of norm t .

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PROBLEM SET # 4: March 7, 2003

Due on Monday 3-24-03

29. Rudin, p. 227, N2, modified. If for a holomorphic function at least one coefficient in the power series expansion at every point of an open set is equal to zero, the function is a polynomial.

30. Rudin, p. 227, N3.

31. Rudin, p. 227, N4.

32. Let f be a function on the unit circle $C \subset \mathbb{C}$ whose Fourier series converges absolutely. Prove that there exist two functions f_1 , defined and holomorphic inside the unit circle, and f_2 , defined and holomorphic outside the unit circle, such that for any $z_0 \in C$, $\lim_{z \rightarrow z_0} f_1(z)$ and $\lim_{z \rightarrow z_0} f_2(z)$ exist, and the sum of these limits is equal to $f(z_0)$.

33. Prove that any function f defined on \mathbb{C} with finitely many poles and no other singularities, and such that either $|f|$ is bounded outside a large circle, or $\lim_{z \rightarrow \infty} f(z) = \infty$, is rational.

Interpret this statement in terms of maps of the Riemann sphere

34. Prove that any non-constant holomorphic map of the Riemann sphere into itself is surjective.

35. If an entire function takes any value only at a finite (but not necessarily bounded) number of points, it is a polynomial.

Hint: Use the Open Mapping Theorem

36. Suppose f is a holomorphic function defined in a certain domain D and the radius of convergence of its power series at $z_0 \in D$ is equal to r . Then there exists z , $|z - z_0| = r$, such that f is not holomorphic in any neighborhood of z .

37. For any positive integer n give an example of a function and f and z_0 such that there are exactly n points z satisfying the assertion of the previous problem.

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PROBLEM SET # 5, Part 1: April 2, 2003

Due on Monday 4-14-03

38. Prove that there is no one-to-one conformal map from \mathbb{C} with 0 removed onto an annulus.

39. Construct all one-to-one conformal maps from the intersection of two discs (we assume it is open) onto the upper half-plane.

40. Rudin p. 293, N3

41. Rudin p. 294, N8.

42. Rudin p. 294, N9(a).

43. Rudin p. 295, N22.

44. Describe the Riemann surface of the function $z^{1/2} + (z - 1)^{1/2}$.

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SECOND MIDTERM EXAMINATION

Friday April 18, 2003

For a perfect score you should give complete solutions of two problems from each of the two sections.

SECTION 1

1.1. Let f be a holomorphic function in the unit disc. Suppose that for any real $x \in (-1, 1)$ a certain derivative $f^{(n)}(x)$ vanishes. Prove that f is a polynomial.

1.2. Let n be a natural number. Suppose that an entire function f takes any value exactly n times. Prove that f is linear.

1.3. Let f_n , $n = 1, 2, \dots$ be a sequence of functions defined and continuous in the closed unit disc \bar{U} , holomorphic in the open unit disc U , uniformly bounded on the unit circle and converging to zero at every point $1/m$, $m = 2, 3, \dots$. Prove that f_n converges to zero in U .

1.4. Let f be a meromorphic function in the unit disc which takes finite real values on the circle $|z| = 1/2$. Prove that f is rational.

Hint: Analytic continuation works for functions with poles.

SECTION 2

2.1. Given a simply connected domain D , $z_0 \in D$ and any nonzero complex number w prove existence of a bijective conformal map f from D to the upper half-plane P such that $f'(z_0) = w$.

2.2. Prove that there is no bijective conformal mapping f of the positive quadrant $\text{Re } z > 0, \text{Im } z > 0$ to the unit disc such that $f(1+i) = 0$ and $f(2+2i) = i/10$.

2.3. Prove that there is no bijective conformal mapping f of the positive quadrant $\text{Re } z > 0, \text{Im } z > 0$ to the unit disc such that $f(1+i) = 0$ and $f'(1+i) = 10$.

2.4. Construct (or prove existence of) a holomorphic function f in the unit disc U such that the image $f(U)$ is the annulus $1 < |z| < 2$.