CHAPTER 2

ELEMENTARY HOMOTOPY THEORY

Homotopy theory, which is the main part of algebraic topology, studies topological objects up to homotopy equivalence. Homotopy equivalence is a weaker relation than topological equivalence, i.e., homotopy classes of spaces are larger than homeomorphism classes. Even though the ultimate goal of topology is to classify various classes of topological spaces up to a homeomorphism, in algebraic topology, homotopy equivalence plays a more important role than homeomorphism, essentially because the basic tools of algebraic topology (homology and homotopy groups) are invariant with respect to homotopy equivalence, and do not distinguish topologically nonequivalent, but homotopic objects.

The first examples of homotopy invariants will appear in this chapter: degree of circle maps in Section 2.4, the fundamental group in Section 2.8 and higher homotopy groups in Section 2.10, while homology groups will appear and will be studied later, in Chapter 8. In the present chapter, we will see how effectively homotopy invariants work in simple (mainly low-dimensional) situations.

2.1. Homotopy and homotopy equivalence

2.1.1. Homotopy of maps. It is interesting to point out that in order to define the homotopy equivalence, a relation between *spaces*, we first need to consider a certain relation between *maps*, although one might think that spaces are more basic objects than maps between spaces.

DEFINITION 2.1.1. Two continuous maps $f_0, f_1 : X \to Y$ between topological spaces are said to be *homotopic* if there exists a a continuous map $F : X \times [0,1] \to Y$ (the *homotopy*) that F *joins* f_0 *to* f_1, i.e., if we have $F(i,\cdot) = f_i$ for $i = 1, 2$.

A map $f : X \to Y$ is called *null-homotopic* if it is homotopic to a constant map $c : X \to \{y_0\} \subset Y$. If $f_0, f_1 : X \to Y$ are homeomorphisms, they are called *isotopic* if they can be joined by a homotopy F (the *isotopy*) which is a homeomorphism $F(t,\cdot)$ for every $t \in [0,1]$.

If two maps $f, g : X \to Y$ are homotopic, we write $f \simeq g$.

maybe add a picture (p. 37)

EXAMPLE 2.1.2. The identity map id: $\mathbb{D}^2 \to \mathbb{D}^2$ and the constant map $c_0 : \mathbb{D}^2 \to 0 \in \mathbb{D}^2$ of the disk $\mathbb{D}^2$ are homotopic. A homotopy between them may be defined by $F(t,(\rho,\varphi)) = ((1-t)\cdot\rho, \varphi)$, where (ρ,φ) are polar coordinates in $\mathbb{D}^2$. Thus the identity map of the disk is null homotopic.

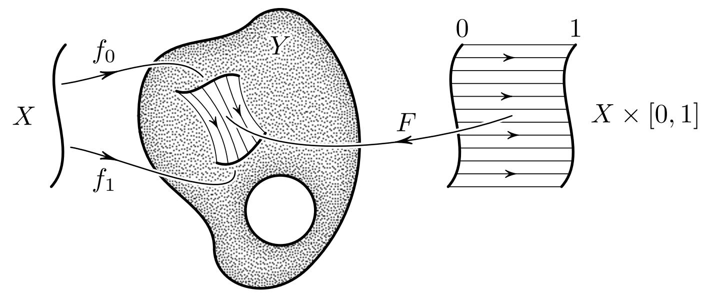

FIGURE 2.1.1. Homotopic maps

EXAMPLE 2.1.3. If the maps $f, g : X \to Y$ are both null-homotopic and Y is path connected, then they are homotopic to each other.

Indeed, suppose a homotopy F joins f with the constant map to the point $a \in Y$, and a homotopy G joins g with the constant map to the point $b \in Y$. Let $c : [0,1] \to Y$ be a path from a to b. Then the following homotopy

$$H(t,x) := \begin{cases} F(x,3t) & \text{when } 0 \le t \le \frac{1}{3}, \\ c(3t-1) & \text{when } \frac{1}{3} \le t \le \frac{2}{3} \\ G(x, 3-3t) & \text{when } \frac{2}{3} \le t \le 1. \end{cases}$$

joins the map f to g.

EXAMPLE 2.1.4. If A is the annulus $A = \{(x,y) | 1 \le x^2 + y^2 \le 2\}$, and the circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ is mapped homeomorphically to the outer and inner boundary circles of A according to the rules $f : e^{i\varphi} \mapsto (2, \varphi)$ and $g : e^{i\varphi} \mapsto (1, \varphi)$ (here we are using the polar coordinates (r, φ)) in the $(x,y) - plane$), then f and g are homotopic.

Indeed, $H(t, \varphi) := (t+1, \varphi)$ provides the required homotopy.

Further, it should be intuitively clear that neither of the two maps f or g is null homotopic, but at this point we do not possess the appropriate techniques for proving that fact.

2.1.2. Homotopy equivalence. To motivate the definition of homotopy equivalent spaces let us write the definition of homeomorphic spaces in the following form: topological spaces X and Y are homeomorphic if there exist maps $f : X \to Y$ and $g : Y \to X$ such that

$$f \circ g = \mathrm{Id}_X \quad \text{and} \quad g \circ f = \mathrm{Id}_Y .$$

If we now replace equality by homotopy we obtain the desired notion:

DEFINITION 2.1.5. Two topological spaces X, Y are called *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that

$$f \circ g : X \to X \quad \text{and} \quad g \circ f : Y \to Y$$

are homotopic to the corresponding identities Id_X and Id_Y.

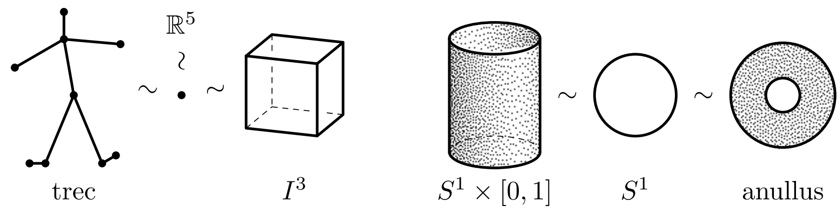

FIGURE 2.1.2. Homotopy equivalent spaces

EXAMPLE 2.1.6. The point, the disk, the Euclidean plane are all homotopy equivalent. To show that pt$\simeq$ $\mathbb{R}^2$, consider the maps $f\ :\ \mathrm{pt} \to 0 \in \mathbb{R}^2$ and $g : \mathbb{R}^2 \to$pt. Then $g \circ f$ is just the identity of the one point set pt, while the map $f \circ g : \mathbb{R}^2 \to \mathbb{R}^2$ is joined to the identity of $\mathbb{R}^2$ by the homotopy $H(t, (r, \varphi)) := ((1-t)r, \varphi)$.

EXAMPLE 2.1.7. The circle and the annulus are homotopy equivalent. Mapping the circle isometrically on the inner boundary of the annulus and projecting the entire annulus along its radii onto the inner boundary, we obtain two maps that comply with the definition of homotopy equivalence.

PROPOSITION 2.1.8. *The relation of being homotopic* (*maps*) *and being homotopy equivalent* (*spaces*) *are equivalence relations in the technical sense, i.e., are reflexive, symmetric, and transitive.*

PROOF. The proof is quite straightforward. First let us check transitivity for maps and reflexivity for spaces.

Suppose $f \simeq g \simeq h$. Let us prove that $f \simeq h$. Denote by F and G the homotopies joining f to g and g to h, respectively. Then the homotopy

$$H(t,x) := \begin{cases} F(2t) & \text{when } t \le \frac{1}{2}, \\ G(2t-1) & \text{when } t \ge \frac{1}{2} \end{cases}$$

joins f to h.

Now let us prove that for spaces the relation of homotopy equivalence is reflexive, i.e., show that for any topological space X we have $X \simeq X$. But the pair of maps $(\mathrm{id}_X, \mathrm{id}_X)$ and the homotopy given by $H(t,x) := x$ for any t shows that X is indeed homotopy equivalent to itself.

The proofs of the other properties are similar and are omitted. □

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PROPOSITION 2.1.9. *Homeomorphic spaces are homotopy equivalent.*

PROOF. If $h : X \to Y$ is a homeomorphism, then $h \circ h^{-1}$ and $h^{-1} \circ h$ are the identities of Y and X, respectively, so that the homotopy equivalence of X and Y is an immediate consequence of the reflexivity of that relation. □

In our study of topological spaces in the previous chapter, the main equivalence relation was homeomorphism. In homotopy theory, its role is played by homotopy equivalence. As we have seen, homeomorphic spaces are homotopy equivalent. The converse is not true, as simple examples show.

refer to the picture above

EXAMPLE 2.1.10. Euclidean space $\mathbb{R}^n$ and the point are homotopy equivalent but not homeomorphic since there is no bijection between them. Open and closed interval are homotopy equivalent since both are homotopy equivalent to a point but not homeomorphic since closed interval is compact and open is not.

EXAMPLE 2.1.11. The following five topological spaces are all homotopy equivalent but any two of them are not homeomorphic:

- the circle $\mathbb{S}^1$,
- the open cylinder $\mathbb{S}^1 \times \mathbb{R}$,
- the annulus $A = \{(x,y)|1 \leq x^2 + y^2 \leq 2\}$,
- the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$,
- the Möbius strip.

In all cases one can naturally embed the circle into the space and then project the space onto the embedded circle by gradually contracting remaining directions. Proposition 2.2.8 below also works for all cases but the last.

Absence of homeomorphisms is shown as follows: the circle becomes disconnected when two points are removed, while the other spaces are not; the annulus and the solid torus are compact, the open cylinder and the Möbius strip are not. The remaining two pairs are a bit more tricky since thy require making intuitively obvious statement rigorous: (i) the annulus has two boundary components and the solid torus one, and (ii) the cylinder becomes disconnected after removing any subset homeomorphic to the circle[1] while the Möbius strip remains connected after removing the middle circle.

As is the case with homeomorphisms in order to establish that two spaces are homotopy equivalent one needs just to produce corresponding maps while in order to establish the absence of homotopy equivalence an invariant is needed which can be calculated and shown to be different for spaces in question. Since homotopy equivalence is a more robust equivalence relation that homeomorphism there are fewer invariants and many simple homeomorphism invariants do not work, e.g. compactness and its derivative connectedness after removing one or more points and so on. In particular, we still lack means to show that the spaces from two previous examples are not homotopy equivalent. Those means will be provided in Section 2.4

2.2. Contractible spaces

Now we will study properties of contractible spaces, which are, in a natural sense, the trivial objects from the point of view of homotopy theory.

[1]This follows from the Jordan Curve Theorem Theorem 5.1.2

2.2.1. Definition and examples. As we will see from the definition and examples, contractible spaces are connected topological objects which have no "holes", "cycles", "apertures" and the like.

DEFINITION 2.2.1. A topological space X is called *contractible* if it is homotopically equivalent to a point. Equivalently, a space is contractible if its identity map is null-homotopic.

EXAMPLE 2.2.2. Euclidean and complex spaces $\mathbb{R}^n$, $\mathbb{C}^n$ are contractible for all n. So is the closed n-dimensional ball (disc) $\mathbb{D}^n$, any tree (graph without cycles; see Section 2.3), the wedge of two disks. This can be easily proven by constructing homotopy equivalence On the other hand, the sphere $\mathbb{S}^n$, $n \geq 0$, the torus $\mathbb{T}^n$, any graph with cycles or multiple edges are all not contractible. To prove this one needs to construct some*invariants*, i.e. quantities which are equal for homotopy equivalent spaces. An this point we do not have such invariants yet.

PROPOSITION 2.2.3. *Any convex subset of* $\mathbb{R}^n$ *is contractible.*

PROOF. Let C be a convex set in $\mathbb{R}^n$ ant let $x_0 \in C$, define

$$h(x,t) = x_0 + (1-t)(x - x_0).$$

By convexity for any $t \in [0,1]$ we obtain a map of C into itself. This is a homotopy between the identity and the constant map to x_0 □

REMARK 2.2.4. The same proof works for a broader class of sets than convex, namely star-shaped. A set $S \subset \mathbb{R}^n$ is called *star-shaped* if there exists a point x_0 such that the intersection of any half line with endpoint x_0 with S is an interval. hence any star-shaped set is contractible.

2.2.2. Properties. Contractible spaces have nice intrinsic properties and also behave well under maps.

PROPOSITION 2.2.5. *Any contractible space is path connected.*

PROOF. Let $x_1, x_2 \in X$, where X is contractible. Take a homotopy h between the identity and a constant map, to, say x_0. Let

$$f(t) := \begin{cases} h(x, 2t) & \text{when } t \leq \frac{1}{2}, \\ h(y, 2t-1) & \text{when } t \geq \frac{1}{2}. \end{cases}$$

Thus f is a continuous map of $[0,1]$ to X with $f(0) = x$ and $f(1) = y$. □

PROPOSITION 2.2.6. *If the space X is contractible, then any map of this space $f : X \to Y$ is null homotopic.*

PROOF. By composing the homotopy taking X to a point p and the map f, we obtain a homotopy of f and the constant map to $f(p)$. □

PROPOSITION 2.2.7. *If the space Y is contractible, then any map to this space $f : X \to Y$ is null homotopic.*

PROOF. By composing the map f with the homotopy taking Y to a point and, we obtain a homotopy of f and the constant map to that point. □

PROPOSITION 2.2.8. *If X is contractible, then for any topological space Y the product $X \times Y$ is homotopy equivalent to X.*

PROOF. If $h : Y \times [0,1] \to Y$ is a homotopy between the identity and a constant map of Y,that is, $h(y,0) = y$ and $h(y,1) = y_0$. Then for the map $H := \mathrm{Id}_X \times h$ one has $H(x,y,0) = (x,y)$ and $H(x,y,1) = (x,y_0)$. Thus the projection $\pi_1 : (x,y) \mapsto x$ and the embedding $i_{y_0} : x \mapsto (x,y_0)$ provide a homotopy equivalence. □

2.3. Graphs

In the previous section, we discussed contractible spaces, the simplest topological spaces from the homotopy point of view, i.e., those that are homotopy equivalent to a point. In this section, we consider the simplest type of space from the point of view of dimension and local structure: graphs, which may be described as one-dimensional topological spaces consisting of line segments with some endpoints identified.

We will give a homotopy classification of graphs, find out what graphs can be embedded in the plane, and discuss one of their homotopy invariants, the famous Euler characteristic.

2.3.1. Main definitions and examples.

Here we introduce (nonoriented) graphs as classes of topological spaces with an edge and vertex structure and define the basic related notions, but also look at abstract graphs as very general combinatorial objects. In that setting an extra orientation structure becomes natural.

DEFINITION 2.3.1. A (nonoriented) *graph* G is a topological space obtained by taking a finite set of line segments (called *edges* or *links*) and identifying some of their endpoints (called *vertices* or *nodes*).

Thus the graph G can be thought of as a finite sets of points (vertices) some of which are joined by line segments (edges); the sets of vertices and edges are denoted by $V(G)$ and $E(G)$, respectively. If a vertex belongs to an edge, we say that the vertex is *incident* to the edge or the edge is incident to the vertex. A *morphism* of graphs is a map of vertices and edges preserving incidence, an *isomorphism* is a bijective morphism.

It the two endpoints of an edge are identified, such an edge is called a *loop*. A *path* (or *chain*) is a ordered set of edges such that an endpoint of the first edge coincides with an endpoint of the second one, the other endpoint of the second edge coincides with an endpoint of the third edge, and so on, and finally an endpoint of the last edge coincides with an endpoint of the previous one. A closed path (i.e.,

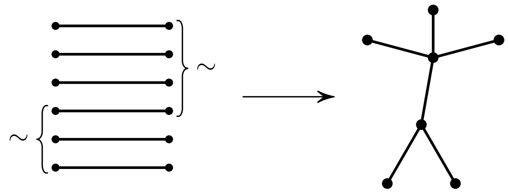

FIGURE 2.3.1. Constructing a graph by identifying endpoints of segments

a path whose first vertex coincides with its last one) is said to be a *cycle*; a loop is regarded as a particular case of a cycle.

A *tree* is a graph without cycles.

A graph is called *connected* if any two vertices can be joined by a path. This is equivalent to the graph being connected (or path-connected) as a topological space.

The number of edges with endpoints at a given vertex is called the *degree* of this vertex, the *degree of a graph* is the maximal degree of all its vertices.

A *complete graph* is a graph such that each pair of distinct vertices is joined by exactly one edge. check the terminology

EXERCISE 2.3.1. Prove that any graph can be embedded into $\mathbb{R}^3$, i.e. it is isomorphic to a graph which is a subset of the three-dimensional space $\mathbb{R}^3$.

A graph is called *planar* if it is isomorphic to a graph which is a subset of the plane $\mathbb{R}^2$.

EXAMPLE 2.3.2. The sets of vertices and edges of the n-simplex constitute a graph, which is connected and complete, and whose vertices are all of degree $n+1$. The sets of edges of an n-dimensional cube constitute a connected graph whose vertices are all of degree n, but which is not complete (if $n \geq 2$).

EXAMPLE 2.3.3. The figure shows two important graphs, $K_{3,3}$ and K_5, both of which are nonplanar. The first is the formalization of a famous (unsolvable) problem: to find paths joining each of three houses to each of three wells so that the paths never cross. In practice would have to build bridges or tunnels. The second is the complete graph on five vertices. The proof of their nonplanarity will be discussed on the next subsection.

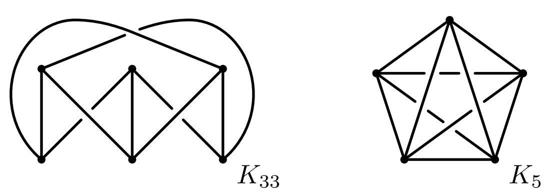

FIGURE 2.3.2. Two nonplanar graphs: $K_{3,3}$ and K_5

DEFINITION 2.3.4. An *oriented graph* is a graph with a chosen direction on each edge. Paths and cycles are defined as above, except that the edges must be

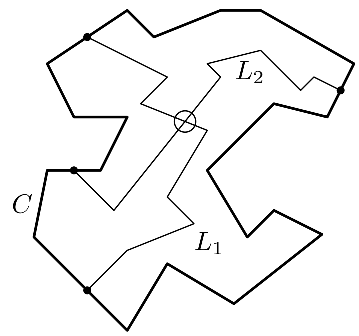

FIGURE 2.3.3. The polygonal lines L_1 and L_2 must intersect

coherently oriented. Vertices with only one edge are called *roots* if the edge is oriented away from the vertex, and *leaves* if it is oriented towards the vertex.

2.3.2. Planarity of graphs. The goal of this subsection is to prove that the graph $K_{3,3}$ is nonplanar, i.e., possesses no topological embedding into the plane $\mathbb{R}^2$. To do this, we first prove the polygonal version of the Jordan curve theorem and show that the graph $K_{3,3}$ has no polygonal embedding into the plane, and then show that it has no topological embedding in the plane.

PROPOSITION 2.3.5. [The Jordan curve theorem for broken lines] *Any broken line C in the plane without self-intersections splits the plane into two path connected components and is the boundary of each of them.*

PROOF. Let D be a small disk which C intersects along a line segment, and thus divides D into two (path) connected components. Let p be any point in $\mathbb{R}^2 \setminus C$. From p we can move along a polygonal line as close as we like to C and then, staying close to C, move inside D. We will then be in one of the two components of $D \setminus C$, which shows that $\mathbb{R}^2 \setminus C$ has *no more than two* components.

It remains to show that $\mathbb{R}^2 \setminus C$ is not path connected. Let ρ be a ray originating at the point $p \in \mathbb{R}^2 \setminus C$. The ray intersects C in a finite number of segments and isolated points. To each such point (or segment) assign the number 1 if C crosses ρ there and 0 if it stays on the same side. Consider the parity $\pi(p)$ of the sum S of all the assigned numbers: it changes continuously as ρ rotates and, being an integer, $\pi(p)$ is constant. Clearly, $\pi(p)$ does not change inside a connected component of $\mathbb{R}^2 \setminus C$. But if we take a segment intersecting C at a non-zero angle, then the parity π at its end points differs. This contradiction proves the proposition. $\square$

We will call a closed broken line without self-intersections a *simple polygonal line*.

COROLLARY 2.3.6. *If two broken lines L_1 and L_2 without self-intersections lie in the same component of $\mathbb{R}^2 \setminus C$, where C is a simple closed polygonal line, with their endpoints on C in alternating order, then L_1 and L_2 intersect.*

PROOF. The endpoints a and c of L_1 divide the polygonal curve C into two polygonal arcs C_1 and C_2. The curve C and the line L_1 divide the plane into three path connected domains: one bounded by C, the other two bounded by the closed curves $C_i \cup L$, $i = 1, 2$ (this follows from Proposition 2.3.5). Choose points b and d on L_2 close to its endpoints. Then b and d must lie in different domains bounded by L_1 and C and any path joining them and not intersecting C, in particular L_2, must intersect L_1. □

PROPOSITION 2.3.7. *The graph $K_{3,3}$ cannot be polygonally embedded in the plane.*

PROOF. Let us number the vertices $x_1, \dots, x_6$ of $K_{3,3}$ so that its edges constitute a closed curve $C := x_1x_2x_3x_4x_5x_6$, the other edges being

$$E_1 := x_1x_4, \quad E_2 := x_2x_5, \quad E_3 := x_3x_6.$$

Then, if $K_{3,3}$ lies in the plane, it follows from Proposition 2.3.5 that C divides the plane into two components. One of the two components must contain at least two of the edges E_1, E_2, E_3, which then have to intersect (by Corollary 2.3.6). This is a contradiction which proves the proposition. □

THEOREM 2.3.8. *The graph $K_{3,3}$ is nonplanar, i.e., there is no topological embedding $h : K_{3,3} \hookrightarrow \mathbb{R}^2$.*

The theorem is an immediate consequence of the nonexistence of a PL-embedding of $K_{3,3}$ (Proposition 2.3.7) and the following lemma.

LEMMA 2.3.9. *If a graph G is planar, then there exists a polygonal embedding of G into the plane.*

PROOF. Given a graph $G \subset \mathbb{R}^2$, we first modify it in small disk neighborhoods of the vertices so that the intersection of (the modified graph) G with each disk is the union of a finite number of radii of this disk. Then, for each edge, we cover its complement to the vertex disks by disks disjoint from the other edges, choose a finite subcovering (by compactness) and, using the chosen disks, replace the edge by a polygonal line. □

We conclude this subsection with a beautiful theorem, which gives a simple geometrical obstruction to the planarity of graphs. We do not present the proof (which is not easy), because this theorem, unlike the previous one, is not used in the sequel.

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THEOREM 2.3.10. [Kuratowski] *A graph is nonplanar if and only if it contains, as a topological subspace, the graph $K_{3,3}$ or the graph K_5.*

REMARK 2.3.11. The words “as a topological subspace” are essential in this theorem. They cannot be replaced by “as a subgraph”: if we subdivide an edge of K_5 by adding a vertex at its midpoint, then we obtain a nonplanar graph that does *not* contain either $K_{3,3}$ or K_5.

EXERCISE 2.3.2. Can the graph $K_{3,3}$ be embedded in (a) the Möbius strip, (b) the torus?

EXERCISE 2.3.3. Is there a graph that cannot be embedded into the torus?

EXERCISE 2.3.4. Is there a graph that cannot be embedded into the Möbius strip?

2.3.3. Euler characteristic of graphs and plane graphs. The *Euler characteristic* of a graph G is defined as

$$\chi(G) \ := \ V - E,$$

where V is the number of vertices and E is the number of edges.

The *Euler characteristic* of a graph G without loops *embedded in the plane* is defined as

$$\chi(G) \ := \ V - E + F,$$

where V is the number of vertices and E is the number of edges of G, while F is the number of connected components of $\mathbb{R}^2 \setminus G$ (including the unbounded component).

THEOREM 2.3.12. [Euler Theorem] *For any connected graph G without loops embedded in the plane,* $\chi(G) \ = \ 2$.

PROOF. At the moment we are only able to prove this theorem for polygonal graphs. For the general case we will need Jordan curve Theorem Theorem 5.1.2. The proof will be by induction on the number of edges. Without loss of generality, we can assume (by Lemma 2.3.9) that the graph is polygonal. For the graph with zero edges, we have $V = 1$, $E = 0$, $F = 1$, and the formula holds. Suppose it holds for all graphs with n edges; then it is valid for any connected subgraph H of G with n edges; take an edge e from G which is not in H but incident to H, and add it to H. Two cases are possible.

Case 1. Only one endpoint of e belongs to H. Then F is the same for G as for H and both V and E increase by one.

Case 2. Both endpoints of e belong to to H. Then e lies inside a face of H and divides it into two.[2] Thus by adding e we increase both E and F by one and leave V unchanged. Hence the Euler characteristic does not change. □

2.3.4. Homotopy classification of graphs. It turns out that, from the viewpoint of homotopy, graphs are classified by their Euler characteristic (which is therefore a *complete* homotopy invariant.)

EXERCISE 2.3.5. Prove that any tree is homotopy equivalent to a point.

THEOREM 2.3.13. *Any connected graph G is homotopy equivalent to the wedge of k circles, with* $k = \chi(G) - 1$.

[2]It is here that we need the conclusion of Jordan curve Theorem Theorem 5.1.2 in the case of general graphs. The rest of the argument remains the same as for polygonal graphs.

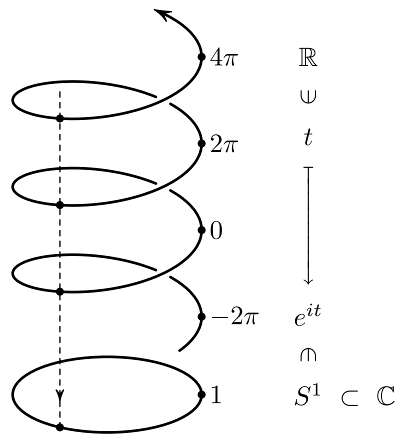

FIGURE 2.4.1. Exponential map

PROOF. Consider a maximal tree T which is a subgraph of G. The graph W obtained by identifying T into a single vertex p is homotopically equivalent to G. But any edge of W whose one endpoint is p must be a loop since otherwise T would not be a maximal tree in G. Since W is connected it has a single vertex p and hence is a wedge of several loops. $\square$

At this point we do not know yet that wedges of different numbers of circles are mutually not homotopically equivalent or, for that matter that they are not contractible. This will be shown with the use of the first non-trivial homotopy invariant which we will study in the next section. This will of course also imply that the Euler characteristic of a graph is invariant under homotopy equivalence.

2.4. Degree of circle maps

Now we will introduce a homotopy invariant for maps of the circle to itself. It turns out that this invariant can easily be calculated and have many impressive applications. Some of those applications are presented in three subsequent sections.

2.4.1. The exponential map. Recall the relation between the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and the line $\mathbb{R}$. There is a projection $\pi\colon \mathbb{R} \to \mathbb{S}^1$, $x \mapsto [x]$, where $[x]$ is the equivalence class of x in $\mathbb{R}/\mathbb{Z}$. Here the integer part of a number is written $\lfloor\cdot\rfloor$ and $\{\cdot\}$ stands for the fractional part.

PROPOSITION 2.4.1. *If $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ is continuous, then there exists a continuous map $F\colon \mathbb{R} \to \mathbb{R}$, called a* lift *of f to $\mathbb{R}$, such that*

$$f \circ \pi = \pi \circ F, \tag{2.4.1}$$

that is, $f([z]) = [F(z)]$. Such a lift is unique up to an additive integer constant and $\deg(f) := F(x+1) - F(x)$ *is an integer independent of $x \in \mathbb{R}$ and the lift F. It is called the* degree *of f. If f is a homeomorphism, then* $|\deg(f)| = \pm 1$.

PROOF. Existence: Pick a point $p \in \mathbb{S}^1$. Then we have $p = [x_0]$ for some $x_0 \in \mathbb{R}$ and $f(p) = [y_0]$ for some $y_0 \in \mathbb{R}$. From these choices of x_0 and y_0 define $F\colon \mathbb{R} \to \mathbb{R}$ by requiring that $F(x_0) = y_0$, that F be continuous, and that $f([z]) = [F(z)]$ for all $z \in \mathbb{R}$. One can construct such an F, roughly speaking, by varying the initial point p continuously, which causes $f(p)$ to vary continuously. Then there is no ambiguity of how to vary x and y continuously and thus $F(x) = y$ defines a continuous map.

To elaborate, take a $\delta > 0$ such that

$$d([x],[x']) \leq \delta \ \text{ implies } \ d(f([x]), f([x'])) < 1/2.$$

Then we can define F on $[x_0 - \delta, x_0 + \delta]$ as follows: If $|x - x_0| \leq \delta$ then $d(f([x]), q) < 1/2$ and there is a unique $y \in (y_0 - 1/2, y_0 + 1/2)$ such that $[y] = f([x])$. Define $F(x) = y$. Analogous steps extend the domain by another δ at a time, until F is defined on an interval of unit length. (One needs to check consistency, but it is straightforward.) Then $f([z]) = [F(z)]$ defines F on $\mathbb{R}$.

Uniqueness: Suppose $\tilde{F}$ is another lift. Then $[\tilde{F}(x)] = f([x]) = [F(x)]$ for all x, meaning $\tilde{F} - F$ is always an integer. But this function is continuous, so it must be constant.

Degree: $F(x + 1) - F(x)$ is an integer (now evidently independent of the choice of lift) because

$$[F(x+1)] = f([x+1]) = f([x]) = [F(x)].$$

By continuity $F(x+1) - F(x) =: \deg(f)$ must be a constant.

Invertibility: If $\deg(f) = 0$, then $F(x+1) = F(x)$ and thus F is not monotone. Then f is noninvertible because it cannot be monotone. If $|\deg(f)| > 1$, then $|F(x+1) - F(x)| > 1$ and by the Intermediate Value Theorem there exists a $y \in (x, x+1)$ with $|F(y) - F(x)| = 1$, hence $f([y]) = f([x])$, and $[y] \neq [x]$, so f is noninvertible. ☐

2.4.2. Homotopy invariance of the degree.

Here we show that the degree of circle maps is a homotopy invariant and obtain some immediate corollaries of this fact.

PROPOSITION 2.4.2. *Degree is a homotopy invariant.*

PROOF. The lift construction can be simultaneously applied to a continuous family of circle maps to produce a continuous family of lifts. Hence the degree must change continuously under homotopy. Since it is an integer, it is in fact constant. ☐

COROLLARY 2.4.3. *The circle is not contractible.*

PROOF. The degrees of any constant map is zero, whereas for the identity map it is equal to one. ☐

THEOREM 2.4.4. *Degree is a complete homotopy invariant of circle self–maps: for any $m \in \mathbb{Z}$ any map of degree m is homotopic to the map*

$$E_m := x \mapsto mx \pmod 1.$$

PROOF. Obviously, the map E_m lifts to the linear map $x \mapsto mx$ of R. On the other hand, every lift F of a degree m map f has the form $F(x) = mx + H(x)$, where H is a periodic function with period one. Thus the family of maps

$$F_t(x) := mx + (1 - t)H(x)$$

are lifts of a continuous family of maps of S^1 which provide a homotopy between f and E_m. $\square$

Since $E_m \circ E_n = E_{mn}$ we obtain

COROLLARY 2.4.5. *Degree of the composition of two maps is equal to the product of their degrees.*

EXERCISE 2.4.1. Show that any continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$ has at least $|\deg f - 1|$ fixed points.

EXERCISE 2.4.2. Prove Corollary 2.4.5 directly, not using Theorem 2.4.4.

EXERCISE 2.4.3. Given the maps $f : \mathbb{S}^1 \to \mathbb{D}^2$ and $g : \mathbb{D}^2 \to \mathbb{S}^1$, what can be said about the degree of their composition.

2.4.3. Degree and wedges of circles. In order to complete homotopy classification of graphs started in Section 2.3.4 we need to proof the following fact which will be deduced from the degree theory for circle maps.

PROPOSITION 2.4.6. *The wedges of k circles for $k = 0, 1, 2, \dots$ are pairwise not homotopy equivalent.*

PROOF. We first show that the wedge of any number of circles is not contractible. For one circle this has been proved already (2.4.3). Let W be the wedge of $k > 1$ circles and $p \in W$ be the common point of the circles. If W is contractible then the identity map Id_W of W is homotopic to the constant map c_P of W to p. Let S be one of the circles comprising W and let U be the union of remaining circles. Then one can identify U into a single point (naturally identified with p and project the homotopy to the identification space which is naturally identified with the circle S and thus provides a homotopy between the identity and a constant map on the circle, a contradiction. More specifically we apply the following process which looks like cutting the graph of a continuous function at a constant level when the function exceeds this level. As long as the images of a point $x \in S$ stay in S we change nothing. When it reaches p and leaves S we replace the images by the constant p.

Now assume that the wedge W of m circles is homotopically equivalent to the wedge of $n < m$ circles which can be naturally identified with a subset U of W consisting of n circles. This implies that there exists a homotopy between Id_W and the map $c_U : W \to W$ which is equal to the identity on U and maps $m - n$ circles comprising $W \setminus U$ into the common point p of all circles in W. As before, we identify U into a point and project the homotopy into the identification space which is naturally identified with the wedge of $m - n$ circles. Thus we obtain a homotopy between a homotopy between the identity map and the constant map c_p which is impossible by the previous argument. □

this does not follow from homotopy equivalence directly. Need to be argued or replaces by another argument

Now we can state the homotopy classification of graphs as follows.

COROLLARY 2.4.7. *Two graphs are homotopy equivalent if and only if they have the same Euler characteristic. Any graph with Euler characteristic E is homotopy equivalent to the wedge of $E + 1$ circles.*

2.4.4. Local definition of degree. One of the central ideas in algebraic topology is extension of the notion of degree of a self-map from circles to spheres of arbitrary dimension and then to a broad class of compact manifold. Definition which follows from Proposition 2.4.1 stands no chance of generalization since the exponential map is a phenomenon specific for the circle and, for example in has no counterparts for spheres of higher dimensions. Now we give another definition which is equivalent to the previous one for the circle but can be generalized to other manifolds.

We begin with piecewise strictly monotone maps of the circle into itself. For such a map every point $x \in \mathbb{S}^1$ has finitely many pre-images and for if we exclude finitely many values at the endpoints of the interval of monotonicity each pre-image $y \in f^{-1}(x)$ lies on a certain interval of monotonicity where the function f either "increases", i.e. preserves orientation on the circle or "decreases", i.e. reverses orientation. In the first case we assign number 1 to the point y and call it a positive pre-image and in the second the number -1 and call it a negative pre-image of x. Adding those numbers for all $y \in f^{-1}(x)$ we obtain an integer which we denote $d(x)$.

provide picture(s)

THEOREM 2.4.8. *The number $d(x)$ is independent of x and is equal to the degree of f.*

REMARK 2.4.9. Since any continuous map of the circle can be arbitrary well approximated by a piecewise monotone map (in fact, even by a piecewise linear one) and by the above theorem the number thus defined for piecewise monotone maps (call it the local degree) is the same for any two sufficiently close maps we can *define* degree of an arbitrary continuous map $f : \mathbb{S}^1 \to \mathbb{S}^1$ as the the local degree of any piecewise monotone map g sufficiently close to f. This is a "baby version" of the procedure which will be developed for other manifolds in **??**.

PROOF. Call a value $x \in \mathbb{S}^1$ *critical* if $x = f(y)$ where x is an endpoint of an interval of monotonicity for f which we will call critical points. Obviously $d(x)$

does not change in a neighborhood of any non-critical value. It does not change at a critical value either since each critical value is the image of several critical points and near each such point either there is one positive and one negative pre-image for nearby values on one side and none on the other or vise versa. Thus $d(x)$ is constant which depends only on the map f and can thus be denoted by $d(f)$.

For any piecewise monotone map f let as call its *piecewise linear approximation* f_{PL} the map which has the same intervals of monotonicity and is linear on any of them. Obviously $d(f_{PL}) = d(f)$; this follows from a simple application of the intermediate value theorem from calculus. Consider the straight-line deformation of the map f_{PL} to the linear map $E_{\deg f}$. Notice that since f_{PL} is homotopic to f (by the straight line on each monotonicity interval) $\deg f_{PL} = \deg f$. This homotopy passes through piecewise linear maps which we denote by g_t and hence the local degree is defined. A small point is that for some values of t the map g_t may be constant on certain intervals of monotonicity of of f but local degree is defined for such maps as well. It remains to notice that the local degree does not change during this deformation. But this is obvious since any non-critical value of g_t remains non-critical with a small change of t and for each t all but finitely many values are non-critical. Since local degree can be calculated at any non-critical value this shows that

$$d(f) = d(f_{PL}) = d(E_{\deg f}) = \deg f.$$

□

2.5. Brouwer fixed point theorem in dimension two

In the general case, the Brouwer theorem says that any (continuous) self-map of the disk $\mathbb{D}^n$ (a closed ball in $\mathbb{R}^n$) has a fixed point, i.e., there exists a $p \in \mathbb{D}^n$ such that $f(p) = p$.

The simplest instance of this theorem (for $n = 1$) is an immediate corollary of the intermediate value theorem from calculus since a continuous map f of a closed interval $[a, b]$ into itself can be considered as a real-values function such that $f(a) \geq a$ and $f(b) \geq b$. Hence by the intermediate value theorem the function $f(x) - x$ has a zero on $[a, b]$.

The proof in dimension two is based on properties of the degree.

THEOREM 2.5.1. [Brouwer fixed-point theorem in dimension two.] *Any continuous map of a closed disk into itself* (*and hence of any space homeomorphic to the disk*) *has a fixed point.*

PROOF. We consider the standard closed disc

$$\mathbb{D}^2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Suppose $f : \mathbb{D}^2 \to \mathbb{D}^2$ is a continuous map without fixed points. For $p \in \mathbb{D}^2$ consider the open halfline (ray) beginning at $F(p)$ and passing through the point p. This halfline intersects the unit circle $\mathbb{S}^1$, which is the boundary of the disc $\mathbb{D}^2$, at a single point which we will denote by $h(p)$. Notice that for $p \in \partial\mathbb{D}^2,\ h(p) = p$

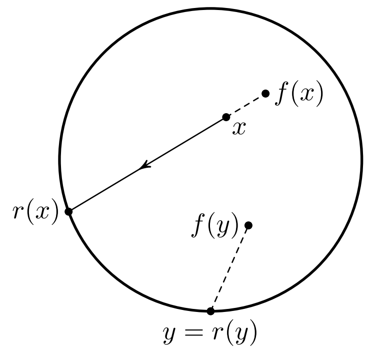

FIGURE 2.5.1. Retraction of the disk onto the circle

The map $h : \mathbb{D}^2 \to \partial\mathbb{D}^2$ thus defined is continuous by construction (exactly because f has no fixed points) and is homotopic to the identity map $\mathrm{Id}_{\mathbb{D}^2}$ via the straight-line homotopy $H(p,t) = (1-t)p + th(p)$. Now identify $\partial\mathbb{D}^2$ with the unit circle $\mathbb{S}^1$. Taking the composition of h with this identification, we obtain a map $\mathbb{D}^2 \to \mathbb{S}^1$, which we will denote by g. Let $i : \mathbb{S}^1 \to \mathbb{D}^2$ be the standard embedding. We have

$$g \circ i : \mathbb{S}^1 \to \mathbb{S}^1 = \mathrm{Id}_{\mathbb{S}^1}, \;\; i \circ g = h \;\; \text{is homotopic to } \; \mathrm{Id}_{\mathbb{D}^2} .$$

Thus the pair (i, g) gives a homotopy equivalence between $\mathbb{S}^1$ and $\mathbb{D}^2$.

But this is impossible, since the disc is contractible and the circle is not (Corollary 2.4.3). Hence such a map h cannot be constucted; this implies that F has a fixed point at which the halfline in question cannot be uniquely defined. □

EXERCISE 2.5.1. Deduce the general form of the Brouwer fixed–point theorem: *Any continuous map of a closed n-disc into itself has a fixed point*, from the fact that the identity map on the sphere of any dimension is not null homotopic. The latter fact will be proved later (**??**).

2.6. Index of a point w.r.t. a curve

In this section we study curves and points lying in the plane $\mathbb{R}^2$ and introduce an important invariant: the index $\mathrm{ind}(p,\gamma)$ of a point p with respect to a curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$. This invariant has many applications, in particular it will help us prove the so-called "Fundamental Theorem of Algebra" in the next section.

2.6.1. Main definition and examples. By a *curve* we mean the image $C = f(\mathbb{S}^1)$ of a continuous map $f : \mathbb{S}^1 \to \mathbb{R}^2$, not necessarily injective. Recall that C is compact by Proposition 1.5.11 Let p be a point in the open complement $\mathbb{R}^2 - C$ of the curve. The complement is nonempty since C is compact but R^2 is not. Notice however that C may have an interior if f is a so-called *Peano curve* **??** or somehting similar. Denote by φ the angular parameter on $\mathbb{S}^1$ and by V_φ the vector joining the points p and $f(\varphi)$. As φ varies from 0 to 2π, the endpoint of the unit vector $V_\varphi/|V_\varphi|$ moves along the unit circle S_0 centered at p, defining a map $\gamma_f : S_0 \to S_0$.

DEFINITION 2.6.1. The *index of the point p with respect to the curve f* is defined as the degree of the map γ, i.e.,

$$\operatorname{ind}(p, f) := \deg(\gamma_f)$$

Clearly, $\operatorname{ind}(p, f)$ does not change when p varies inside a connected component of $\mathbb{R}^2 \backslash C)$: indeed, the function ind is continuous in $p \notin C$ and takes integer values, so it has to be a constant when p varies in a connected component of $\mathbb{R}^2 \backslash C)$.

If the point p is "far from" $f(S^1)$ (i.e., in the connected component of $\mathbb{R}^2 \backslash f(\mathbb{S}^1)$ with noncompact closure), then $\deg(p, f) = 0$; indeed, if p is sufficiently far from C (which is compact), then C is contained in an acute angle with vertex at p, so that the vector $f(\varphi)$ remains within that angle as φ varies from 0 to 2π and γ must have degree 0.

A concrete example of a curve in $\mathbb{R}^2$ is shown on Figure ??, (a); on it, the integers indicate the values of the index in each connected component of its complement. is this 2.6.1?; wheree is (b)?

2.6.2. Computing the index for immersed curves. When the curve is nice enough, there is a convenient method for computing the index of any point with respect to the curve. To formalize what we mean by "nice" we introduce the following definition.

DEFINITION 2.6.2. A curve $f : \mathbb{S}^1 \to \mathbb{R}^2$ is said to be an *immersion* if f is differentiable, has a nonzero tangent vector, and has a finite number of self-intersections, all of them transversal, i.e. with all tangent vectors making non-zero angles with each other.

In order to compute the index of p with respect to an immersed curve f, let us join p by a (nonclosed) smooth curve α transversal to f to a far away point a and move from a to p along that curve. At the start, we put $i(a) = 0$, and, moving along α, we add one to i when we cross $f(\mathbb{S}^1)$ in the positive direction (i.e., so that the tangent vector to f looks to the right of α) and subtract one when we cross it in the negative direction. When we reach the connected component of the complement to the curve containing p, we will obtain a certain integer $i(p)$.

EXERCISE 2.6.1. Prove that the integer $i(p)$ obtained in this way is actually the index of p w.r.t. f (and so $i(p)$ does not depend on the choice of the curve α).

Turn this into a proposition and provide a proof.

EXERCISE 2.6.2. Compute the indices of the connected components of the complements to the curve shown on Figure ??(b) by using the algorithm described above.

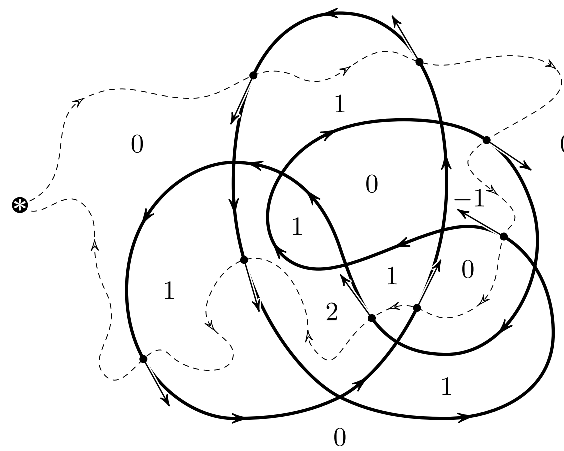

FIGURE 2.6.1. Index of points w.r.t. a curve

2.7. The fundamental theorem of algebra

2.7.1. Statement and commentary. In our times the term "fundamental theorem of algebra" reflects historical preoccupation of mathematicians with solving algebraic equations, i.e. finding roots of polynomials. Its equivalent statement is that the field of complex numbers is *algebraically complete* i.e. that no need to extend it in order to perform algebraic operations. This in particular explain difficulties with constructing "hyper-complex" numbers; in order to do that in a meaningful way, one needs to relax some of the axioms of the field (e.g. commutativity for the four-dimensional *quaternions*).[3] Thus, in a sense, the theorem is fundamental but not so much for algebra where the field of complex numbers is only one of many objects of study, and not the most natural one at that, but for analysis, analytic number theory and classical algebraic geometry.

THEOREM 2.7.1. *Any polynomial*

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \quad a_n \neq 0, \quad n > 0,$$

with complex coefficients has a least one complex root.[4]

REMARK 2.7.2. This theorem has many different proofs, but no "purely algebraic" ones. In all existing (correct!) proofs, the crucial point is topological. In the proof given below, it ultimately comes down to the fact that a degree n self map of the circle is not homotopic to the identity provided $n \geq 2$.

[3]This by no means implies that one cannot include complex number to a larger field. General algebraic constructions such as fields of rational functions provide for that.

[4]The fact that the polynomial $x^2 + 1$ has no *real* roots is the most basic motivation for introducing complex numbers.

2.7.2. Proof of the theorem. By dividing all coefficients by a_n which does not change the roots we may assume that $a_n = 1$. Furthermore, if $a_0 = 0$ than $p(0) = 0$. Thus we can also assume that $a_0 \neq 0$.

Consider the curve $f_n : S^1 \to \mathbb{R}^2$ given by the formula $e^{i\varphi} \mapsto R_0^n e^{in\varphi}$, where R_0 is a (large) positive number that will be fixed later. Further, consider the family of curves $f_{p,R} : S^1 \to \mathbb{R}^2$ given by the formula $e^{i\varphi} \mapsto p(Re^{i\varphi})$, where $R \leq R_0$. We can assume that the origin O does not belong to $f_{p,R_0}(\mathbb{S}^1)$ (otherwise the theorem is proved).

LEMMA 2.7.3. *If R_0 is sufficiently large, then*

$$\operatorname{ind}(O, f_{p,R_0}) = \operatorname{ind}(O, f_n) = n.$$

Before proving the lemma, let us show that it implies the theorem.

By the lemma, $\operatorname{ind}(O, f_{p,R_0}) = n$. Let us continuously decrease R from R_0 to 0. If for some value of R the curve $f_{p,R}(\mathbb{S}^1)$ passes through the origin, the theorem is proved. So we can assume that $\operatorname{ind}(O, f_{p,R})$ changes continuously as $R \to 0$; but since the index is an integer, it remains constant and equal to n. However, if R is small enough, the curve $f_{p,R}(\mathbb{S}^1)$ lies in a small neighborhood of a_0; but for such an R we have $\operatorname{ind}(O, f_{p,R}) = 0$. This is a contradiction, because $n \geq 1$.

It remains to prove the lemma. The equality $\operatorname{ind}(O, f_n) = n$ is obvious. To prove the other equality, it suffices to show that for any φ the difference Δ between the vectors $V_p(\varphi)$ and $V_n(\varphi)$ that join the origin O with the points $f_p(R_0e^{i\varphi})$ and $f_n(R_0e^{i\varphi})$, respectively, is small in absolute value (as compared to $R_0^n = |V_p(\varphi)|$) if R_0 is large enough. Indeed, by the definition of degree, if the mobile vector is replaced by another mobile vector whose direction always differs from the direction of the first one by less than $\pi/2$, the degree will be the same for the two vectors.

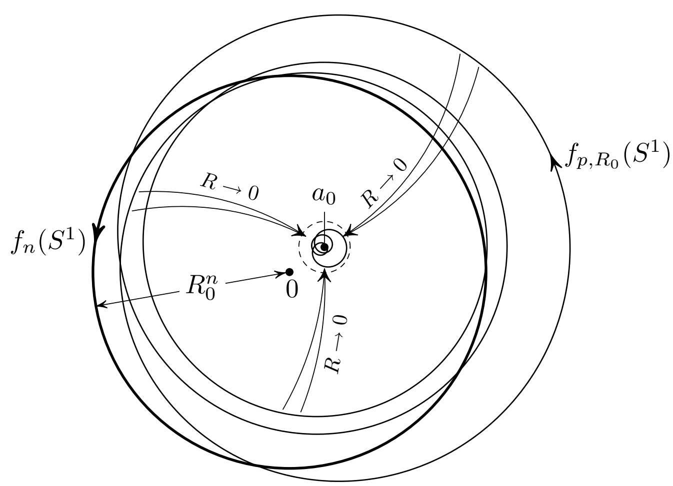

FIGURE 2.7.1. Proof of the fundamental theorem of algebra

Clearly, $|\Delta| = |a_{n-1}z^{n-1}+\cdots+a_1z+a_0|$. Let us estimate this number, putting $z = R_0e^{\varphi}$ and $A = \max\{a_{n-1}, a_{n-2}, \ldots, a_0\}$ (here without loss of generality we assume that $R_0 > 1$). We then have

$$|\Delta| = |a_{n-1}z^{n-1} + \cdots + a_1z + a_0| \leq |A(R_0^{n-1} + R_0^{n-2} \cdots + 1)| \leq A \cdot n \cdot R_0^{n-1}.$$

Now if we put $R_0 := K \cdot A$, where K is a large positive number, we will obtain $|\Delta| \leq nA(KA)^{n-1} = nK^{n-1}A^n$. Let us compare this quantity to R_0^n; the latter equals $R_0^n = K^nA^n$, so for K large enough the ratio $|\Delta|/R_0^n$ is as small as we wish. This proves the lemma and concludes the proof of the theorem. □

2.8. The fundamental group; definition and elementary properties

calculations for Mobius and Klein bottle(the latter need to be defined in Chapter 1

The fundamental group is one of the most important invariants of homotopy theory. It also has numerous applications outside of topology, especially in complex analysis, algebra, theoretical mechanics, and mathematical physics. In our course, it will be the first example of a "functor", assigning a group to each path-connected topological space and a group homomorphism to each continuous map of such spaces, thus reducing topological problems about spaces to problems about groups, which can often be effectively solved. In a more down-to-earth language this will be the first sufficiently universal non-trivial invariant of homotopy equivalence, defined for all path connected spaces and calculable in many natural situations.

2.8.1. Main definitions. Let M be a topological space with a marked point $p \in M$.

DEFINITION 2.8.1. A curve $c\colon [0,1] \to M$ such that $c(0) = c(1) = p$ will be called a *loop* with *basepoint* p. Two loops c_0, c_1 with basepoint p are called *homotopic rel* p if there is a homotopy $F : [0,1]\times[0,1] \to M$ joining c_0 to c_1 such that $F(t, x) = p$ for all $t \in [0,1]$.

If c_1 and c_2 are two loops with basepoint p, then the loop $c_1 \cdot c_2$ given by

$$c_1 \cdot c_2(t) := \begin{cases} c_1(2t) & \text{if } t \leq \frac{1}{2}, \\ c_2(2t-1) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

is called the *product* of c_1 and c_2.

PROPOSITION 2.8.2. *Classes of loops homotopic rel p form a group with respect to the product operation induced by* $\cdot$.

PROOF. First notice that the operation is indeed well defined on the homotopy classes. For, if the paths c_i are homotopic to $\tilde{c}_i$, $i = 1, 2$ via the maps h_1 : $[0,1] \times [0,1] \to M$, then the map h, defined by

$$h(t,s) := \begin{cases} h_1(2t,s) & \text{if } \ t \leq \frac{1}{2}, \\ h_2(2t-1,s) & \text{if } t \geq \frac{1}{2} \end{cases}$$

is a homotopy rel p joining c_1 to c_2.

Obviously, the role of the unit is played by the homotopy class of the constant map $c_0(t) = p$. Then the inverse to c will be the homotopy class of the map $c'(t) := c(1-t)$. What remains is to check the associative law: $(c_1 \cdot c_2) \cdot c_3$ is homotopic rel p to $c_1 \cdot (c_2) \cdot c_3)$ and to show that $c \cdot c'$ is homotopic to c_0. In both cases the homotpy is done by a reparametrization in the preimage, i.e., on the square $[0,1] \times [0,1]$.

For associativity, consider the following continuous map ("reparametrization") of the square into itself

$$R(t,s) = \begin{cases} (t(1+s), s) & \text{if } 0 \le t \le \frac{1}{4}, \\ (t + \frac{s}{4}, s) & \text{if } \frac{1}{4} \le t \le \frac{1}{2}, \\ (1 - \frac{1}{1+s} + \frac{t}{1+s}, s) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then the map $c_1 \cdot (c_2 \cdot c_3) \circ R : [0,1] \times [0,1] \to M$ provides a homotopy rel endpoints joining the loops $c_1 \cdot (c_2 \cdot c_3)$ and $(c_1 \cdot c_2) \cdot c_3$.

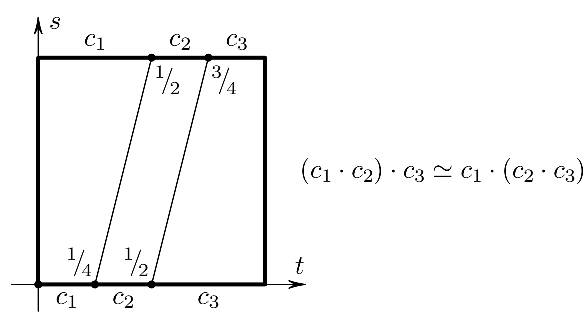

FIGURE 2.8.1. Associativity of multiplication

Similarly, a homotopy joining $c \cdot c'$ to c_0 is given by $c \cdot c' \circ I$, where the reparametrization $I : [0,1] \times [0,1] \to [0,1] \times [0,1]$ is defined as

$$I(t,s) = \begin{cases} (t, s) & \text{if } 0 \le t \le \frac{1-s}{2}, \text{ or } \frac{1+s}{2} \le t \le 1, \\ (\frac{1-s}{2}, s) & \text{if } \frac{1-s}{2} \le t \le \frac{1+s}{2}, \end{cases}$$

Notice that while the reparametrization I is discontinuous along the wedge $t = (1 \pm s)/2$, the map $(c \cdot c') \circ I$ is continuous by the definition of c'. $\square$

DEFINITION 2.8.3. The group described in Proposition 2.8.2 is called the *fundamental group* of M at p and is denoted by $\pi_1(M, p)$.

It is natural to ask to what extent $\pi_1(M, p)$ depends on the choice of the point $p \in M$. The answer is given by the following proposition.

PROPOSITION 2.8.4. *If p and q belong to the same path connected component of M, then the groups $\pi_1(M, p)$ and $\pi_1(M, q)$ are isomorphic.*

PROOF. Let $\rho : [0,1] \to M$ be a path connecting points p and q. It is natural to denote the path $\rho \circ S$ where $S(t) = 1 - t$ by ρ^{-1}. It is also natural to extend the " $\cdot$ " operation to paths with different endpoints if they match properly. With these conventions established, let us associate to a path $c\colon [0,1] \to M$ with $c(0) = c(1) = p$ the path $c' := \rho^{-1} \cdot c \cdot \rho$ with $c'(0) = c'(1) = q$. In order to finish the proof, we must show that this correspondence takes paths homotopic rel p to paths homotopic rel q, respects the group operation and is bijective up to homotopy. These staments are proved using appropriate rather natural reparametrizations, as in the proof of Proposition 2.8.2. $\square$

REMARK 2.8.5. By mapping the interval $[0,1]$ to the circle with a marked point e first and noticing that, if the endpoints are mapped to the e, than the homotopy can also be interpreted as a map of the closed cylinder $\mathbb{S}^1 \times [0,1]$ to the space with a based point which maps $e \times [0,1]$ to the base point p, we can interpret the construction of the fundamental group as the group of homotopy classes of maps $(\mathbb{S}^1, e)$ into $M, p)$. Sometimes this language is more convenient and we will use both versions interchangeably.

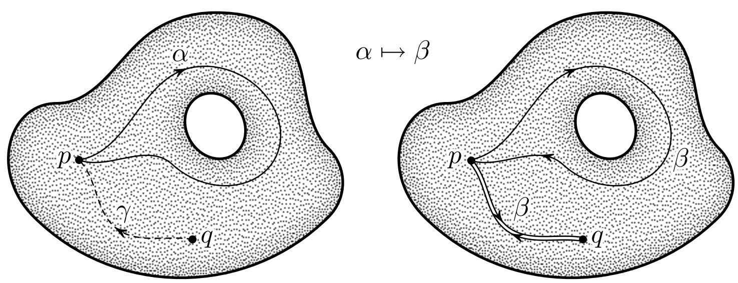

FIGURE 2.8.2. Change of basepoint isomorphism

REMARK 2.8.6. It follows from the construction that different choices of the connecting path ρ will produce isomorphisms between $\pi_1(M,p)$ and $\pi_1(M,q)$ which differ by an inner automorphism of either group.

If the space M is path connected then the fundamental groups at all of its points are isomorphic and one simply talks about the *fundamental group of* M and often omits the basepoint from its notation: $\pi_1(M)$.

DEFINITION 2.8.7. A path connected space with trivial fundamental group is said to be *simply connected* (or sometimes 1-*connected*).

REMARK 2.8.8. Since the fundamental group is defined modulo homotopy, it is the same for homotopically equivalent spaces, i.e., it is a homotopy invariant.

The *free* homotopy classes of curves (i.e., with no fixed base point) correspond exactly to the conjugacy classes of curves modulo changing base point, so there is a natural bijection between the classes of freely homotopic closed curves and conjugacy classes in the fundamental group.

That this object has no natural group structure may sound rather unfortunate to many a beginner topologist since the main tool of algebraic topology, namely, translating difficult geometric problems into tractable algebraic ones, have to be applied here with fair amount of care and caution here.

2.8.2. Functoriality. Now suppose that X and Y are path connected, $f : X \to Y$ is a continuous maps with and $f(p) = q$. Let $[c]$ be an element of $\pi_1(X, p)$, i.e., the homotopy class rel endpoints of some loop $c : [0, 1] \to X$. Denote by $f_\#(c)$ the loop in (Y, q) defined by $f_\#(t) := f(c(t))$ for all $t \in [0, 1]$. $\|f_*^n(v)\| \geq Ce^{-\mu|n|}\|v\|, \quad$ for all $\quad n > 0 \quad$ and $\quad v \in E^c(p)$. The following simple but fundamental fact is proven by a straightforward checking that homotopic rel based points loop define homotopic images.

PROPOSITION 2.8.9. *The assignment* $c \mapsto f_\#(c)$ *is well defined on classes of loops and determines a homomorphism* (*still denoted by* $f_\#$) *of fundamental groups:*

$$f_\# : \pi_1(X, p) \to \pi_1(Y, q)$$

(*refered to as the homomorphism induced by* f), *which possesses the following properties* (*called functorial*)*:*

- $(f \circ g)_\# = f_\# \circ g_\#$ (*covariance*)*;*
- $(id_X)_\# = id_{\pi_1(X,p)}$ (*identity maps induce identity homomorphisms*).

The fact that the construction of an invariant (here the fundamental group) is functorial is very convenient for applications. For example, let us give another proof of the Brouwer fixed point theorem for the disk by using the isomorphism $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ (see Proposition 2.8.12 below) and $\pi_1(\mathbb{D}^2) = 0$ (since $\mathbb{D}^2$ is contractible) and the functoriality of $\pi_1(\cdot)$.

We will prove (by contradiction) that there is no *retraction* of $\mathbb{D}^2$ on its boundary $\mathbb{S}^1 = \partial\mathbb{D}^2$ i.e. a map $\mathbb{D}^2 \to \mathbb{S}^1$ which is identity on $\mathbb{S}^1$, Let $r : \mathbb{D}^2 \to \mathbb{S}^1$ be such a retraction, let $i : \mathbb{S}^1 \to \mathbb{D}^2$ be the inclusion; choose a basepoint $x_0 \in \mathbb{S}^1 \subset \mathbb{D}^2$. Note that for this choice of basepoint we have $i(x_0 = r(x_0) = x_0)$. Consider the sequence of induced maps:

$$\pi_1(\mathbb{S}^1, x_0) \xrightarrow{i_*} \pi_1(\mathbb{D}^2, x_0) \xrightarrow{r_*} \pi_1(\mathbb{S}^1, x_0).$$

In view of the isomorphisms noted above, this sequence is actually

$$\mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{r_*} \mathbb{Z}.$$

But such a sequence is impossible, because by functoriality we have

$$r_* \circ i_* = (r \circ i)_* = \mathrm{Id}_* = \mathrm{Id}_{\mathbb{Z}}\,. \quad \square$$

In addition to functoriality the fundamental group behaves nicely with respect to the product.

PROPOSITION 2.8.10. *If X and Y are path connected spaces, then*

$$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y).$$

PROOF. Let us construct an isomorphism of $\pi_1(X) \times \pi_1(Y)$ onto $\pi_1(X \times Y)$. Let x_0, y_0 be the basepoints in X and Y, respectively. For the basepoint in $X \times Y$, let us take the point (x_0, y_0). Now to the pair of loops α and β in X and Y let us assign the loop $\alpha \times \beta$ given by $\alpha \times \beta(t) := (\alpha(t), \beta(t))$. The verification of the fact that this assignment determines a well-defined isomorphism of the appropriate fundamental groups is quite straightforward. For example, to prove surjectivity, for a given loop γ in $X \times Y$ with basepoint (x_0, y_0), we consider the two loops $\alpha(t) := (\mathrm{pr}_X \circ \gamma)(t)$ and $\beta(t) := (\mathrm{pr}_Y \circ \gamma)(t)$, where pr_X and pr_Y are the projections on the two factors of $X \times Y$. $\square$

COROLLARY 2.8.11. *If C is contractible, then $\pi_1(X \times C) = \pi_1(X)$*

EXERCISE 2.8.1. Prove that for any path connected topological space X we have $\pi_1(\text{Cone(X)}) = 0$.

2.8.3. Examples and applications. The first non-trivial example is an easy corollary of degree theory.

PROPOSITION 2.8.12. *The fundamental group of the circle $\mathbb{R}/\mathbb{Z}$ is $\mathbb{Z}$ and in additive notation for $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with 0 being the base point the element $n \in \mathbb{Z}$ is represented by the map E_m.*

PROOF. This is essentially a re-statement of Theorem 2.4.4. Since this is a very fundamental fact of homotopy theory we give a detailed argument.

Let $\gamma : (\mathbb{S}^1, 0) \to (\mathbb{S}^1, 0)$ be a loop. LIft it in a unique fashion to a map $\Gamma : (\mathbb{R}, 0) \to \mathbb{R}, 0)$. A homotopy rel 0 between any two maps $\gamma, \gamma' : (\mathbb{S}^1, 0) \to (\mathbb{S}^1, 0)$ lifts uniquely between a homotopy between lifts. Hence $\deg \gamma$ is a homotopy invariant of γ. On the other hand the "straight-line homotopy" between Γ and the linear map $x \deg \gamma x$ projects to a homotopy rel 0 between γ and $E_{\deg \gamma}$. $\square$

Proposition 2.8.12 and Proposition 2.8.10 immediately imply

COROLLARY 2.8.13. $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$.

Notice that $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and $\pi_1(\mathbb{T}^n)$ is isomorphic to the subgroup $\mathbb{Z}^n$ by which $\mathbb{R}^n$ is factorized. This is not accidental but the first instance of *universal covering* phenomenon, see Section 6.2.2.

On the other hand here is an example of a space, which later will be shown to be non-contractible, with trivial fundamental group.

PROPOSITION 2.8.14. *For any $n \geq 2$,* $\pi_1(\mathbb{S}^n) = 0$.

PROOF. The main idea of the proof is to make use of the fact that $\mathbb{S}^n$ is the one-point compactification of the contractible space $\mathbb{R}^n$ and that for $n \geq 2$ any loop is homotopic to one which avoids this single point. For such a loop the contraction (deformation) of $\mathbb{S}^n$ with one point removed to the base point of the loop also produces a homotopy of the loop to the trivial one. However exotic a loop whose image covers the whole sphere may look such loops exist (see Peano curves, **??**). Still any loop is homotopic to a loop which consist of a finite number of arcs of great circles and hence does not cover the whole sphere. The method we use here is interesting since we will make use of a *geometric* structure (spherical geometry on this occasion) to prove a purely topological statement, so we will describe it in detail.

For any two points p and q on the standard unit n-sphere in $\mathbb{R}^{n+1}$, which are not diametrically opposite, there is a unique shortest curve connecting these points, namely the shorter of the two arcs of the *great circle* which can be described as the intersection of the two-dimensional plane passing through p, q and the origin. Such curves give the next simplest example after straight lines in the Euclidean space of *geodesics* which play a central role in Riemannian geometry, the core part of differential geometry. We will mention that subject somewhat more extensively in **??** and will describe the basics of a systematic theory in **??**. An important thing to remember is that any geodesic is provided with the natural *length parameter* and that they depend continuously on the endpoints as long those are not too far away (e.g. are not diametrically opposite in the case of the standard round sphere).

Now come back to our general continuous loop γ in $\mathbb{S}^n$. By compactness one can find finitely many points $0 = t_0 < t_1 < \cdots < t_{m-1}, t_m = 1$ such that for $k = 0, 1 \ldots, m-1$ the set $\Gamma_k := \gamma[t_k, t_{k+1}]$ lies is a sufficiently small ball. In fact for our purpose it would be sufficient if this set lies within an open half-sphere. Now for any open half-sphere $H \subset \mathbb{S}^n$ and any $p, q \in H$ there is a canonical homotopy of H into the arc of the great circle C in H connecting p and q keeping these two points fixed. Namely, first for any $x \in H$ consider the unique arc A_x of the great circle perpendicular to the great circle C and connecting x with C and lying in H. Our homotopy moves x along A_x according to the length parameter normalized to 1/2. The result is a homotopy of H to $C \cap H$ keeping every point on $C \cap H$ fixed. After that one contracts $C \cap H$ to the arc between p and q by keeping all points on that arc fixed and uniformly contracting the length parameter normalized to 1/2 on the remaining two arcs. This procedure restricted to $\gamma[t_k, t_{k+1}]$ on each interval $[t_k, t_{k+1}]$, $k = 0, \ldots, m-1$ produces a homotopy of γ to a path whose image is a finite union of arcs of great circles and hence does not cover the whole sphere. $\square$

it would be nice to have illustrations here

Now we can make an advance toward a solution of a natural problem which concerned us since we first introduced manifolds: invariance of dimension. We proved that one-dimensional manifolds and higher dimensional ones are not homeomorphic by an elementary observation that removing a single point make the former disconnected locally while the latter remains connected. Now we can make a step forward from one to two. This will be the first instance when we prove absence of homeomorphism by appealing to homotopy equivalence.

PROPOSITION 2.8.15. *Any two-dimensional manifold and any n-dimensional manifold for $n \geq 3$ are not homeomorphic.*

PROOF. First let us show that $\mathbb{R}^2$ and $\mathbb{R}^n$ for $n \geq 3$ are not homeomorphic. By removing one point we obtain in the first case the space homotopically equivalent to the circle which hence has fundamental group $\mathbb{Z}$ by Proposition 2.8.12 and in the second the space homotopically equivalent to $\mathbb{S}^{n-1}$ which is simply connected by Proposition 2.8.14.

Now assume that $h : M^n \to M^2$ is a homeomorphism from an n-dimensional manifold to a two-dimensional manifold. Let $h(p) = q$. Point p has a base of neighborhoods homeomorphic to $\mathbb{R}^n$. Hence any loop in such a neighborhood which does not touch p can be contracted to a point within the neighborhood without the homotopy touching p. On the other hand, q has a base of neighborhoods homeomorphic to $\mathbb{R}^2$ which do not possess this property. Let $N \ni q$ be such a neighborhood and let $N' \ni p$ be a neighborhood of p homeomorphic to $\mathbb{R}^n$ such that $h(N') \supset N$. Let $\gamma : [0,1] \to N' \setminus \{p\}$ be a loop which is hence contractible in $N' \setminus \{p\}$. Then $h \circ \gamma : [0,1] \to h(N') \setminus \{q\}$ is a loop which is contractible in $h(N') \setminus \{q\}$ and hence in $N \setminus \{q\}$, a contradiction. □

REMARKS 2.8.16.
(1) In order to distinguish between the manifolds of dimension higher than two the arguments based on the fundamental group are not sufficient. One needs either higher homotopy group s introduced below in Section 2.10 or degree theory for maps of spheres of higher dimension **??**.
(2) Our argument above by no means shows that manifolds of different dimension are not homotopically equivalent; obviously all $\mathbb{R}^n$s are since they are all contractible. More interestingly even, the circle and Móbius strip are homotopically equivalent as we already know. However a proper even more general version of degree theory (which is a basic part of homology theory for manifolds) will allow as to show that dimension is an invariant of homotopy equivalence for *compact* manifolds.

2.8.4. The Seifert–van Kampen theorem. In this subsection we state a classical theorem which relates the fundamental group of the union of two spaces with the fundamental groups of the summands and of their intersection. The result turns out to give an efficient method for computing the fundamental group of a “complicated” space by putting it together from “simpler” pieces.

In order to state the theorem, we need a purely algebraic notion from group theory.

DEFINITION 2.8.17. Let G_i, $i = 1, 2$, be groups, and let $\varphi_i : K \to G_i$, $i = 1, 2$ be monomorphisms. Then the *free product with amalgamation* of G_1 and G_2 with respect to φ_1 and φ_2, denoted by $G_1 *_K G_2$ is the quotient group of the free product $G_1 * G_2$ by the normal subgroup generated by all elements of the form $\varphi_1(k)(\varphi_2(k))^{-1}$, $k \in K$.

THEOREM 2.8.18 (Van Kampen's Theorem). *Let the path connected space X be the union of two path connected spaces A and B with path connected intersection containing the basepoint $x_0 \in X$. Let the inclusion homomorphisms*

$$\varphi_A : \pi_1(A \cap B) \to \pi_1(A), \quad \varphi_B : \pi_1(A \cap B) \to \pi_1(B)$$

be injective. Then $\pi_1(X, x_0)$ is the amalgamated product

$$\pi_1(X, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0).$$

For a proof see G, Bredon, Geometry and Topology, Theorem 9.4.

2.9. The first glance at covering spaces

A covering space is a mapping of spaces (usually manifolds) which, locally, is a homeomorphism, but globally may be quite complicated. The simplest nontrivial example is the exponential map $\mathbb{R} \to \mathbb{S}^1$ discussed in Section 2.4.1.

2.9.1. Definition and examples.

DEFINITION 2.9.1. If M, M' are topological manifolds and $\pi \colon M' \to M$ is a continuous map such that $\operatorname{card} \pi^{-1}(y)$ is independent of $y \in M$ and every $x \in \pi^{-1}(y)$ has a neighborhood on which π is a homeomorphism to a neighborhood of $y \in M$ then π is called a *covering map* and M' (or (M', π)) is called a *covering (space)* or *cover* of M. If $n = \operatorname{card} \pi^{-1}(y)$ is finite, then (M', π) is said to be an *n-fold covering*.

If $f \colon N \to M$ is continuous and $F \colon N \to M'$ is such that $f = \pi \circ F$, then F is said to be a *lift* of f. If $f \colon M \to M$ is continuous and $F \colon M' \to M'$ is continuous such that $f \circ \pi = \pi \circ F$ then F is said to be a *lift* of f as well.

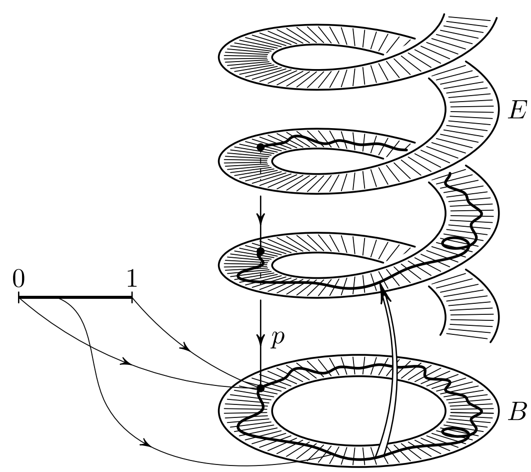

FIGURE 2.9.1. Lift of a closed curve

DEFINITION 2.9.2. A simply connected covering is called the *universal cover*. A homeomorphism of a covering M' of M is called a *deck transformation* if it is a lift of the identity on M.

EXAMPLE 2.9.3. $(\mathbb{R}, \exp(2\pi i(\cdot)))$ is a covering of the unit circle. Geometrically one can view this as the helix $(e^{2\pi ix}, x)$ covering the unit circle under projection. The map defined by taking the fractional part likewise defines a covering of the circle $\mathbb{R}/\mathbb{Z}$ by $\mathbb{R}$.

PROPOSITION 2.9.4. *If $\pi\colon M' \to M$ and $\rho\colon N' \to N$ are covering maps, then $\pi \times \rho\colon M' \times N' \to M \times N$ is a covering map.*

EXAMPLE 2.9.5. The torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is covered by the cylinder $\mathbb{S}^1 \times \mathbb{R}$ which is in turn covered by $\mathbb{R}^2$. Notice that the fundamental group $\mathbb{Z}$ of the cylinder is a subgroup of that of the torus ($\mathbb{Z}^2$) and $\mathbb{R}^2$ is a simply connected cover of both.

EXAMPLE 2.9.6. The maps $E_m,\ |m| \geq 2$ of the circle define coverings of the circle by itself.

EXAMPLE 2.9.7. The natural projection $\mathbb{S}^n \to \mathbb{R}P(n)$ which send points x and $-x$ into their equivalence class is a two-fold covering. On the other, hand, the identification map $\mathbb{S}^{2n-1} \to \mathbb{C}P(n)$ is not a covering since the pre-image on any point is a continuous curve.

EXERCISE 2.9.1. Describe two-fold coverings of
(1) the (open) Möbius strip by the open cylinder $\mathbb{S}^1 \times \mathbb{R}$;
(2) the Klein bottle by the torus $\mathbb{T}^2$.

2.9.2. Role of the fundamental group. One of the remarkable aspects of any covering space $p : X \to B$ is that it is, in a sense, entirely governed by the fundamental groups of the spaces B and X, or more precisely, by the induced homomorphism $p_\# : \pi_1(X) \to \pi_1(B)$ of their fundamental groups. We shall observe this in the two examples given below, postponing the exposition of the general theory to Chapter 6.

EXAMPLE 2.9.8. Let B be the plane annulus given by the inequalities $1 \leq r \leq 2$ in the polar coordinates (r, φ) on the plane $\mathbb{R}^2$, and let X be another copy of this annulus. Consider the map $p : X \to B$ given by $(r, \varphi) \mapsto (r, 3\varphi)$. It is obviously a covering space. Geometrically, it can be viewed as in the figure, i.e., as the vertical projection of the strip $aba'b'$ (with the segments ab and $a'b'$ identified) onto the horizontal annulus.

FIGURE ?? A triple covering of the annulus

The fundamental group of B (and of X) is isomorphic to $\mathbb{Z}$, and the induced homomorphism $p_\# : \pi_1(X) \to \pi_1(B)$ is the monomorphism of $\mathbb{Z}$ into $\mathbb{Z}$ with image $3\mathbb{Z} \subset \mathbb{Z}$. The deck transformations constitute a group isomorphic to $\mathbb{Z}_3 \cong \mathbb{Z}\big/3\mathbb{Z}$.

This is a fairly general situation. The homomorphism $p_\#$ is always injective (for any covering space p) and, provided $\mathrm{Im}(p_\#)$ is a normal subgroup of $\pi_1(B)$, [5] the deck transformations form a group isomorphic to the quotient $\pi_1(B)\big/\mathrm{Im}(p_\#)$.

More remarkable is that the covering map p is entirely determined (up equivalence, defined in a natural way) by the choice of a subgroup of $\pi_1(B)$, in our case, of the infinite cyclic subgroup of $\pi_1(B)$ generated by the element $3e$, where e is the generator of $\pi_1(B) \cong \mathbb{Z}$. There is in fact a geometric procedure for constructing the covering space X, which in our case will yield the annulus.

Another way of defining the geometric structure of a covering space in algebraic terms is via the action of a discrete group in some space X. Then the covering is obtained as the quotient map of X onto the orbit space of the group action. In our case the space X is the annulus, the discrete group is $\mathbb{Z}_3$ and it acts on X by rotations by the angles 0, $2\pi/3$, $4\pi/3$, the orbit space is B (another annulus), and the quotient map is p.

EXAMPLE 2.9.9. Let B be the torus $\mathbb{S}^1 \times \mathbb{S}^1$ with coordinates (φ, ψ) and X be the cylinder $r = 1$ in 3-space endowed with the cylindrical coordinates (r, θ, h). Consider the map $p : X \to B$ given by

$$(1, \varphi, h) \mapsto (2\varphi,\ h \mod 2\pi).$$

It is obviously a covering space map. Geometrically, it can be described as wrapping the cylinder an infinite number of times along the parallels of the torus and simultaneously covering it twice along the meridians.

The fundamental group of B is isomorphic to $\mathbb{Z}$, that of X is $\mathbb{Z} \oplus \mathbb{Z}$ and the induced homomorphism $p_\# : \pi_1(X) \to \pi_1(B)$ is the monomorphism of $\mathbb{Z}$ into $\mathbb{Z} \oplus \mathbb{Z}$ with image $2\mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z}$. The deck transformations constitute a group isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z}$.

Here also the covering p can be obtained by an appropriate choice of a discrete group acting on the cylinder X; then p will be the quotient map of X onto the orbit space of this action.

For an arbitrary "sufficiently nice" space B, say a manifold, there is natural bijection between conjugacy classes of subgroups of $\pi_1(M)$ and classes of covering spaces modulo homeomorphisms commuting with deck transformations. This bijection will be described in detail in Chapter 6, where it will be used, in particular, to prove the uniqueness of the universal cover.

[5] This is an important condition which prevents pathologies which may appear for other coverings

2.10. Definition of higher homotopy groups

The fundamental group has natural generalizations (with $\mathbb{S}^1$ replaced by $\mathbb{S}^n$, $n \geq 2$) to higher dimensions, called (higher)*homotopy groups* (and denoted by $\pi_n(\cdot)$). The higher homotopy groups are just as easy (in a sense easier) to define than the fundamental group, and, unlike the latter, they are commutative.

Let X be a topological space with a marked point $p \in X$. On the sphere $\mathbb{S}^n$, fix a marked point $q \in \mathbb{S}^n$, and consider a continuous map

$$f : \mathbb{S}^n \to X \quad \text{such that} \quad f(q) = p.$$

Such a map is called a *spheroid*. Two spheroids are considered *equivalent* if they are homotopic rel basepoints, i.e., if there exists a homotopy $h_t : \mathbb{S}^n \to X$, $t \in [0,1]$, joining the two spheroids and satisfying $h_t(q) = p$ for all $t \in [0,1]$. By an abuse of language, we will also refer to the corresponding equivalence classes as spheroids. It is sometimes more convenient to regard spheroids as homotopy classes of maps

$$f : (\mathbb{D}^n, \partial\mathbb{D}^n) \to (X, p),$$

where the homotopy h_t must take $\partial\mathbb{D}^n$, the $n-1$-dimensional sphere $\mathbb{S}^{n-1}$, to p for all $t \in [0,1]$.

Let us denote by $\pi_n(X,p)$ the set of all (equivalence classes of) spheroids and introduce a binary operation in that set as follows. Suppose $f, g : (\mathbb{S}^n, q) \to (X,p)$ are two spheroids; then their product is the spheroid $fg : (\mathbb{S}^n, q) \to (X,p)$ obtained by pulling the equator of $\mathbb{S}^n$ containing p to a point and then defining fg by using f on one of the two spheres in the obtained wedge and g on the other (see the figure).

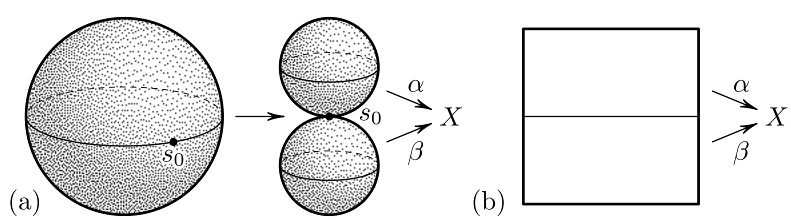

FIGURE 2.10.1. The product of two spheroids

Note that for $n = 1$ this definition coincides with the product of loops for the fundamental group $\pi_1(X,p)$. We will also sometimes consider the set $\pi_0(X,p)$, which by definition consists of the path connected components of X and has no natural product operationdefined on it.

PROPOSITION 2.10.1. *For $n \geq 2$ and all path connected spaces X, the set $\pi_n(X,p)$ under the above definition of product becomes an Abelian group, known as the n-th* homopoty group of X with basepoint p.

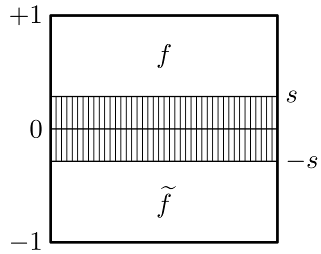

FIGURE 2.10.2. Inverse element in $\pi_n(X, p)$

PROOF. The verification of the fact that $\pi_n(X, p))$ is a group is straightforward; we will only show how inverse elements are constructed. This construction is shown on the figure.

On the figure $f : (\mathbb{D}^n, \partial\mathbb{D}^n) \to (X, p)$ is a spheroid. Denote by $\widetilde{f} : I^{n-1} \times [-1, 1] \to X$ the spheroid given by $\widetilde{f}(x, s) := f(x, -s)$. Then the map (spheroid) $f\widetilde{f}$ satisfies $f\widetilde{f}(x, s) = \widetilde{f}f(x, -s)$ (look at the figure again). Therefore we can consider the family of maps

$$h_t(x, s) = \begin{cases} f\widetilde{f}(x, s), & \text{for } |s| \geq t, \\ \widetilde{f}f(x, -s), & \text{for } |s| \leq t. \end{cases}$$

For this family of maps we have $h_0 = f\widetilde{f}$, while h_1 is the constant map. For the map $h(\cdot, s)$ the shaded area is mapped to p. This shows that every map has an inverse.

To see that the group $\pi_n(X, p))$, $n \geq 2$, is abelian, the reader is invited to look at the next figure, which shows a homotopy between fg and gf, where g and f are arbitrary spheroids. □

PROPOSITION 2.10.2. *For $n \geq 2$ and all path connected spaces X, the groups $\pi_n(X, p)$ and $\pi_n(X, q)$, where $p, q \in X$, are isomorphic, but the isomorphism is not canonical, it depends on the homotopy class (rel endpoints) of the path joining p to q.*

PROOF. The proof is similar to that of an analogous fact about the fundamental group. □ elaborate a bit

PROPOSITION 2.10.3. *The homotopy groups are homotopy invariants of path connected spaces.*

PROOF. The proof is a straightforward verification similar to that of an analogous statement about the fundamental group. □ elaborate

EXERCISE 2.10.1. Prove that all the homotopy groups of a contractible space are trivial.

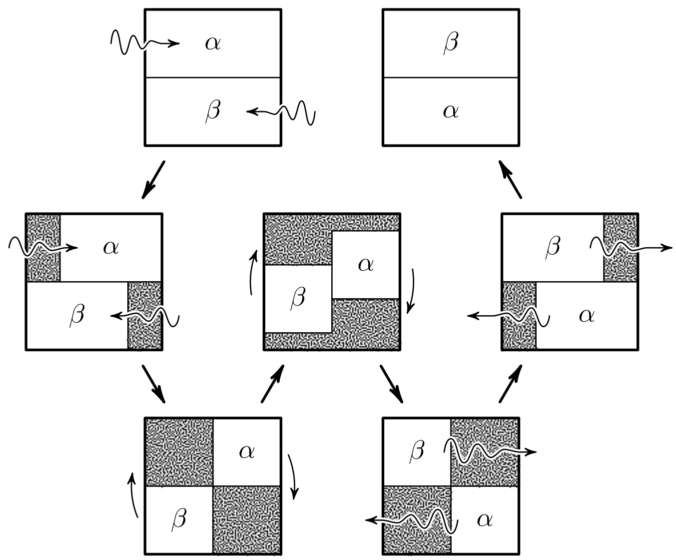

FIGURE 2.10.3. Multiplication of spheroids is commutative

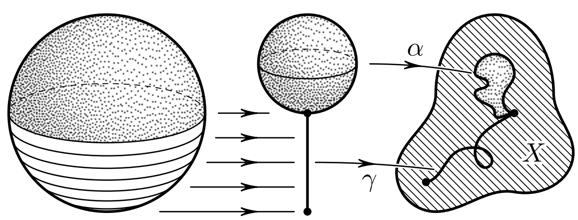

FIGURE 2.10.4. Change of basepoint isomorphism for spheroids

2.11. Hopf fibration

Unlike the fundamental group and homology groups (see Chapter 8), for which there exist general methods and algorithm for computation , the higher homotopy groups are extremely difficult to compute. The are certain "easy" cases: for example $\pi_k(\mathbb{S}^n) = 0$ for $k < n$ which we will be able to show by a proper extension of the method used in Proposition 2.8.14. Furthermore, $\pi_n(\mathbb{S}^n) = \mathbb{Z}$. this will be shown in **??** after developing a proper extension of degree theory.

However already computation of $\pi_k(\mathbb{S}^n) = 0$ for $k > n$ present very difficult problem which has not been completely solved. The first nontrivial example is the computation of π_3 for the sphere $\mathbb{S}^2$, based on one of the most beautiful constructions in topology – the Hopf fibration which we will describe now. Computation of $\pi_3(\mathbb{S}^2)$ is presented later in Chapter ??. The Hopf fibration appears in a number of problems in topology, geometry and differential equations.

make more specific

Consider the unit sphere in $\mathbb{C}^2$:

$$\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$$

and the action H of the circle on it by scalar multiplication: for $\lambda \in \mathbb{S}^1$ put

$$H_\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2).$$

PROPOSITION 2.11.1. *The identification space of this action is homeomorphic to* $\mathbb{S}^2$.

PROOF. This identification space is the same as the identification space of $\mathbb{C}^2$ where all proportional vectors are identified; it is simply the restriction of this equivalence relation to the unit sphere. The identification space is $\mathbb{C}P(1)$ which is homeomorphic to $\mathbb{S}^2$. $\square$

The Hopf fibration is defined by a very simple formula. To help visualize we think of the sphere $\mathbb{S}^3$ as the one-point compactification of $\mathbb{R}^3$, so that we can actually draw the preimages of the Hopf map $h : \mathbb{S}^3 \to \mathbb{S}^2$ (which are circles) in the way shown on the figure.

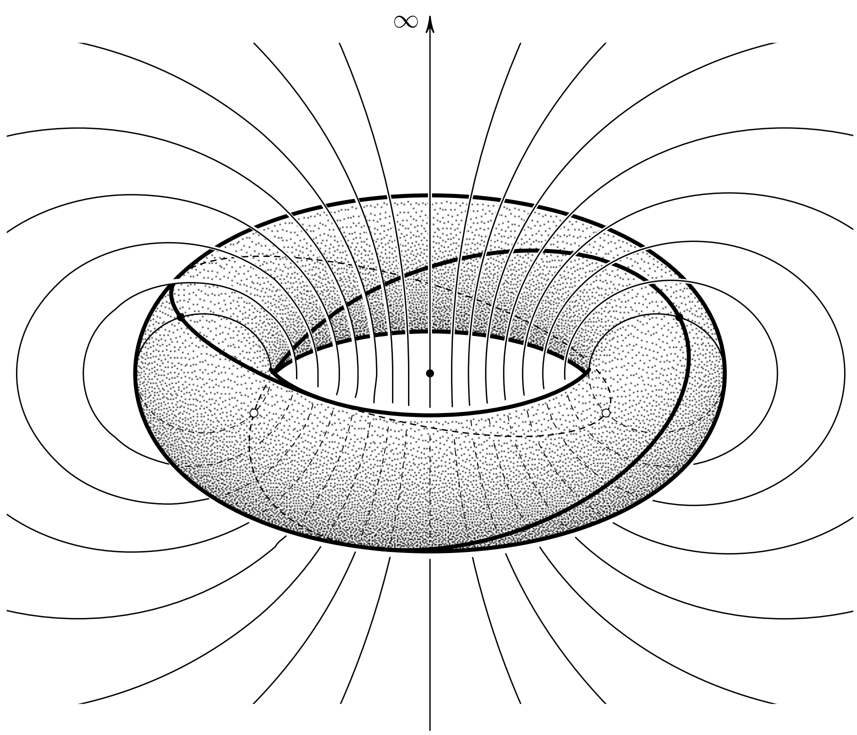

FIGURE 2.11.1. The Hopf fibration

EXERCISE 2.11.1. Let $z(t) = (z_1(t), z_2(t)) \in \mathbb{C}^2$. Consider the system of differential equations $\dot{z} = iz$ and restrict it to the 3-sphere $\{z \in \mathbb{C}^2 \mid |z| = 1\}$. Show that the trajectories of this system are circles constituting the Hopf fibration.

2.12. Problems

EXERCISE 2.12.1. Prove that in $\mathbb{S}^3$, represented as $\mathbb{R}^3 \cup \{\infty\}$, the complement of the unit circle in the xy-plane centered at the origin is homotopy equivalent to the circle.

EXERCISE 2.12.2. Prove that the 2-sphere with three points removed is homotopy equivalent to the figure eight (the wedge of two circles).

EXERCISE 2.12.3. The torus with three points removed is homotopy equivalent to the wedge of four circles.

EXERCISE 2.12.4. Let $f : \mathbb{S}^1 \to \mathbb{D}^2$ and $g : \mathbb{D}^2 \to \mathbb{S}^1$ be any continuous maps. Prove that their composition $g \circ f$ is homotopic to the constant map.

EXERCISE 2.12.5. For any finite cyclic group C there exists a compact connected three-dimensional manifold whose fundamental group is isomorphic to C.
Hint: Use the Hopf fibration.

EXERCISE 2.12.6. Show that the complex projective plane $\mathbb{C}P(2)$ (which is a four–dimensional manifold) is simply connected, i.e. its fundamental group is trivial.

EXERCISE 2.12.7. Consider the following map f of the torus $\mathbb{T}^2$ into itself:

$$f(x, y) = (x + \sin 2\pi y, 2y + x + 2\cos 2\pi x) \pmod 1.$$

Describe the induced homomorphism f_* of the fundamental group.
Hint: You may use the description of the fundamental group of the direct product $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

EXERCISE 2.12.8. Let $X = \mathbb{R}^2 \setminus \mathbb{Q}^2$. Prove that $\pi_1(X)$ is uncountable.

EXERCISE 2.12.9. The real projective space $\mathbb{R}P(n)$ is not simply connected.
Note: Use the fact that $\mathbb{R}P(n)$ is the sphere $\mathbb{S}^n$ with diametrically opposed points identified.

EXERCISE 2.12.10. For any abelian finitely generated group A there exists a compact manifold whose fundamental group is isomorphic to A.

EXERCISE 2.12.11. The fundamental group of any compact connected manifold is no more than countable and is finitely generated.

EXERCISE 2.12.12. Let X be the quotient space of the disjoint union of $\mathbb{S}^1$ and $\mathbb{S}^2$ with a pair of points $x \in S^1$ and $y \in S^2$ identified. Calculate $\pi_1(X)$.