CHAPTER 3

METRIC SPACES AND UNIFORM STRUCTURES

The general notion of topology does not allow to compare neighborhoods of different points. Such a comparison is quite natural in various geometric contexts. The general setting for such a comparison is that of a *uniform structure*. The most common and natural way for a uniform structure to appear is via a metric, which was already mentioned on several occasions in Chapter 1, so we will postpone discussing the general notion of union structure to Section 3.11 until after detailed exposition of metric spaces. Another important example of uniform structures is that of topological groups, see Section 3.12 below in this chapter. Also, as in turns out, a Hausdorff compact space carries a natural uniform structure, which in the separable case can be recovered from any metric generating the topology. Metric spaces and topological groups are the notions central for foundations of analysis.

3.1. Definition and basic constuctions

3.1.1. Axioms of metric spaces. We begin with listing the standard axioms of metric spaces, probably familiar to the reader from elementary real analysis courses, and mentioned in passing in Section 1.1, and then present some related definitions and derive some basic properties.

DEFINITION 3.1.1. If X is a set, then a function $d\colon X \times X \to \mathbb{R}$ is called a *metric* if

(1) $d(x, y) = d(y, x)$ *(symmetry)*,
(2) $d(x, y) \geq 0$; $d(x, y) = 0 \Leftrightarrow x = y$ *(positivity)*,
(3) $d(x, y) + d(y, z) \geq d(x, z)$ *(the triangle inequality)*.

If d is a metric, then (X, d) is called a *metric space*.

The set

$$B(x, r) := \{y \in X \mid d(x, y) < r\}$$

is called the *(open) r-ball* centered at x. The set

$$B_c(x, r) = \{y \in X \mid d(x, y) \leq r\}$$

is called the *closed r-ball* at (or around) x.

The *diameter* of a set in a metric space is the supremum of distances between its points; it is often denoted by diam A. The set A is called *bounded* if it has finite diameter.

A map $f : X \to Y$ between metric spaces with metrics d_X and d_Y is called as *isometric embedding* if for any pair of points $x, x' \in X$ $d_X(x, x') = d_Y(f(x), f(x'))$. If an isometric embedding is a bijection it is called an *isometry*. If there is an isometry between two metric spaces they are called *isometric*. This is an obvious equivalence relation in the category of metric spaces similar to homeomorphism for topological spaces or isomorphism for groups.

3.1.2. Metric topology. $O \subset X$ is called *open* if for every $x \in O$ there exists $r > 0$ such that $B(x,r) \subset O$. It follows immediately from the definition that open sets satisfy Definition 1.1.1. Topology thus defined is sometimes called the *metric topology* or *topology, generated by the metric* d. Naturally, different metrics may define the same topology.

Metric topology automatically has some good properties with respect to bases and separation.

Notice that the closed ball $B_c(x,r)$ contains the closure of the open ball $B(x,r)$ but may not coincide with it (Just consider the integers with the the standard metric: $d(m,n) = |m-n|$.)

Open balls as well as balls or rational radius or balls of radius $r_n,\ n = 1, 2, \dots,$ where r_n converges to zero, form a base of the metric topology.

PROPOSITION 3.1.2. *Every metric space is first countable. Every separable metric space has countable base.*

PROOF. Balls of rational radius around a point form a base of neighborhoods of that point.

By the triangle inequality, every open ball contains an open ball around a point of a dense set. Thus for a separable spaces balls of rational radius around points of a countable dense set form a base of the metric topology. □

Thus, for metric spaces the converse to Proposition 1.1.12 is also true.

Thus the closure of $A \subset X$ has the form

$$\bar{A} = \{x \in X \mid \forall r > 0, \quad B(x,r) \cap A \neq \varnothing\}.$$

For any closed set A and any point $x \in X$ the *distance from* x *to* A,

$$d(x,A) := \inf_{y \in A} d(x,y)$$

is defined. It is positive if and only if $x \in X \setminus A$.

THEOREM 3.1.3. *Any metric space is normal as a topological space.*

PROOF. For two disjoint closed sets $A, B \in X$, let

$$\mathcal{O}_A := \{x \in X \mid d(x,A) < d(x,B), \ \mathcal{O}_B := \{x \in X \mid d(x,B) < d(x,A).$$

These sets are open, disjoint, and contain A and B respectively. □

Let $\varphi : [0,\infty] \to \mathbb{R}$ be a nondecreasing, continuous, concave function such that $\varphi^{-1}(\{0\}) = \{0\}$. If (X,d) is a metric space, then $\phi \circ d$ is another metric on d which generates the same topology.

It is interesting to notice what happens if a function d as in Definition 3.1.1 does not satisfy symmetry or positivity. In the former case it can be symmetrized producing a metric $d_S(x,y) := \max(d(x,y), d(y,x))$. In the latter by the symmetry and triangle inequality the condition $d(x, y) = 0$ defines an equivalence relation and a genuine metric is defined in the space of equivalence classes. Note that some of the most impotrant notions in analysis such as spaces L^p of functions on a measure space are actually not spaces of actual functions but are such quotient spaces: their elements are equivalence classes of functions which coincide outside of a set of measure zero.

3.1.3. Constructions.

1. *Inducing*. Any subset A of a metric space X is a metric space with an *induced metric* d_A, the restriction of d to $A \times A$.

2. *Finite products*. For the product of finitely many metric spaces, there are various natural ways to introduce a metric. Let $\varphi : ([0, \infty])^n \to \mathbb{R}$ be a continuous concave function such that $\varphi^{-1}(\{0\}) = \{(0, \ldots, 0)\}$ and which is nondecreasing in each variable.

Given metric spaces $(X_i, d_i),\ i = 1, \ldots, n$, let

$$d^\varphi := \varphi(d_1, \ldots, d_n) : (X_1 \times \ldots X_n) \times (X_1 \times \ldots X_n) \to \mathbb{R}.$$

EXERCISE 3.1.1. Prove that d^φ defines a metric on $X_1 \times \ldots X_n$ which generates the product topology.

Here are examples which appear most often:

• the *maximum metric* corresponds to

$$\varphi(t_1, \ldots, t_n) = \max(t_1, \ldots, t_n);$$

• the l^p *metric* for $1 \leq p < \infty$ corresponds to

$$\varphi(t_1, \ldots, t_n) = (t_1^p + \cdots + t_n^p)^{1/p}.$$

Two particularly important cases of the latter are $t = 1$ and $t = 2$; the latter produces the Euclidean metric in $\mathbb{R}^n$ from the standard (absolute value) metrics on n copies of $\mathbb{R}$.

3. *Countable products*. For a countable product of metric spaces, various metrics generating the product topology can also be introduced. One class of such metrics can be produced as follows. Let $\varphi : [0, \infty] \to \mathbb{R}$ be as above and let $a_1, a_2, \ldots$ be a suquence of positive numbers such that the series $\sum_{n=1}^{\infty} a_n$ converges. Given metric spaces $(X_1, d_1), (X_2, d_2) \ldots$, consider the metric d on the infinite product of the spaces $\{X_i\}$ defined as

$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) := \sum_{n=1}^{\infty} a_n \varphi(d_n(x_n, y_n)).$$

EXERCISE 3.1.2. Prove that d is really a metric and that the corresponding metric topology coincides with the product topology.

4. *Factors*. On the other hand, projecting a metric even to a very good factor space is problematic. Let us begin with an example which exhibits some of the characteristic difficulties.

EXAMPLE 3.1.4. Consider the partition of the plane $\mathbb{R}^2$ into the level sets of the function xy, i.e. the hyperboli $xy = const \neq 0$ and the union of coordinate axes. The factor topology is nice and normal. It is easy to see in fact that the function xy on the factor space establishes a homeomorphism between this space and the real line. On the other hand, there is no natural way to define a metric in the factor space based on the Euclidean metric in the plane. Any two elements of the factor contain points arbitrary close to each other and arbitrary far away from each other so manipulating with infimums and supremums of of distances between the points in equivalence classes does not look hopeful.

We will see later that when the ambient space is compact and the factor-topology is Hausdorff there is a reasonable way to define a metric as the *Hausdorff metric* (see Definition 3.10.1) between equivalence classes considered as closed subsets of the space.

Here is a very simple but beautiful illustration how this may work.

EXAMPLE 3.1.5. Consider the real projective space $\mathbb{R}P(n)$ as the factor space of the sphere $\mathbb{S}^n$ with opposite points identified. Define the distance between the pairs $(x, -x)$ and $(y, -y)$ as the minimum of distances between members of the pairs. Notice that this minimum is achieved simultaneously on a pair and the pair of opposite points. This last fact allows to check the triangle inequality (positivity and symmetry are obvious) which in general would not be satisfied for the minimal distance of elements of equivalence classes even if those classes are finite.

EXERCISE 3.1.3. Prove the triangle inequality for this example. Prove that the natural projection from $\mathbb{S}^n$ to $\mathbb{R}P(n)$ is an isometric embedding in a neighborhood of each point. Calculate the maximal size of such a neighborhood.

Our next example is meant to demonstrate that the chief reason for the success of the previous example is not compactness but the fact that the factor space is the orbit space of an action by isometries (and of course is Hausdorff at the same time):

EXAMPLE 3.1.6. Consider the natural projection $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$. Define the distance $d(a\mathbb{Z}^n, b\mathbb{Z}^n)$ on the torus as the minimum of Euclidean distances between points in $\mathbb{R}^n$ in the equivalence classes representing corresponding points on the torus. Notice that since translations are isometries the minimum is always achieved and if it is achieved on a pair (x, y) it is also achieved on any integer translation of (x, y).

EXERCISE 3.1.4. Prove the triangle inequality for this example. Prove that the natural projection from $\mathbb{R}^n$ to $\mathbb{T}^n$ is an isometric embedding in any open ball of radius 1/2 and is not an isometric embedding in any open ball of any greater radius.

3.2. Cauchy sequences and completeness

3.2.1. Definition and basic properties. The notion of Cauchy sequence in Euclidean spaces and the role of its convergence should be familiar from elementary real analysis courses. Here we will review this notion in the most general setting, leading up to general theorems on completion, which play a crucial role in functional analysis.

DEFINITION 3.2.1. A sequence $\{x_i\}_{i\in\mathbb{N}}$ is called a *Cauchy sequence* if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_i, x_j) < \varepsilon$ whenever $i, j \geq \mathbb{N}$; X is said to be *complete* if every Cauchy sequence converges.

PROPOSITION 3.2.2. *A subset A of a complete metric space X is a complete metric space with respect to the induced metric if and only if it is closed.*

PROOF. For a closed $A \in X$ the limit of any Cauchy sequence in A belongs to A. If A is not closed take a sequence in A converging to a point in $\bar{A} \setminus A$. It is Cauchy but does not converge in A. □

The following basic property of complete spaces is used in the next two theorems.

PROPOSITION 3.2.3. *Let $A_1 \supset A_2 \supset \dots$ be a nested sequence of closed sets in a complete metric space, such that* $\operatorname{diam} A_n \to 0$ *as $n \to \infty$. Then $\bigcap_{n=1}^{\infty} A_n$ is a single point.*

PROOF. Since $\operatorname{diam} A_n \to 0$ the intersection cannot contain more than one point. Take a sequence $x_n \in A_n$. It is Cauchy since $\operatorname{diam} A_n \to 0$. Its limit x belongs to $\bar{A}_n$ for any n. Since the sets A_i are closed, it follows that $x \in A_n$ for any n. □

3.2.2. The Baire category theorem.

THEOREM 3.2.4 (Baire Category Theorem). *In a complete metric space, a countable intersection of open dense sets is dense. The same holds for a locally compact Hausdorff space.*

PROOF. If $\{O_i\}_{i\in\mathbb{N}}$ are open and dense in X and $\varnothing \neq B_0 \subset X$ is open then inductively choose a ball B_{i+1} of radius at most ε/i for which we have $\bar{B}_{i+1} \subset O_{i+1} \cap B_i$. The centers converge by completeness, so

$$\varnothing \neq \bigcap_i \bar{B}_i \subset B_0 \cap \bigcap_i O_i.$$

For locally compact Hausdorff spaces take B_i open with compact closure and use the finite intersection property. □

The Baire Theorem motivates the following definition. If we want to mesure massivenes of sets in a topological or in particular metric space, we may assume that nowhere dense sets are small and their complements are massive. The next natural step is to introduce the following concept.

DEFINITION 3.2.5. Countable unions of nowhere dense sets are called *sets of first (Baire) category*.

The complement to a set of first baire category is called a *residual* set.

The Baire category theorem asserts that, at least for complete metric spaces, sets of first category can still be viewed as small, since they cannot fill any open set.

The Baire category theorem is a simple but powerful tool for proving *existence* of various objects when it is often difficult or impossible to produce those constructively.

3.2.3. Minimality of the Cantor set. Armed with the tools developed in the previous subsections, we can now return to the Cantor set and prove a universality theorem about this remarkable object.

THEOREM 3.2.6. *(cf. Exercise 1.10.14)*
Any uncountable separable complete metric space X contains a closed subset homeomorphic to the Cantor set.

PROOF. First consider the following subset

$$X_0\colon \{x \in X | \text{any neigbourhood of } x \text{ contains uncountably many points}\}$$

Notice that the set X_0 is perfect, i.e., it is closed and contains no isolated points.

LEMMA 3.2.7. *The set $X \setminus X_0$ is countable.*

PROOF. To prove the lemma, for each point $x \in X \setminus X_0$ find a neighborhood from a countable base which contains at most countably many points (Proposition 3.1.2). Thus $X \setminus X_0$ is covered by at most countably many sets each containing at most countably many points. □

Thus the theorem is a consequence of the following fact.

PROPOSITION 3.2.8. *Any perfect complete metric space X contains a closed subset homeomorphic to the Cantor set.*

PROOF. To prove the the proposition, pick two points $x_0 \neq x_1$ in X and let $d_0 := d(x_0, x_1)$. Let

$$X_i := \overline{B(x_i, (1/4)d_0)}, \quad i = 0, 1$$

and $C_1 := X_0 \cup X_1$.

Then pick two different points $x_{i,0}, x_{i,1} \in \operatorname{Int} X_i,\ i = 0, 1$. Such choices are possible because any open set in X contains infinitely many points. Notice that $d(x_{i,0}, x_{i,1}) \le (1/2)d_0$. Let

$$Y_{i_1,i_2} := \overline{B(x_{i_1,i_2}, (1/4)d(x_{i_1,0}, x_{i_1,1}))},\ i_1, i_2 = 0, 1,$$

$$X_{i_1,i_2} := Y_{i_1,i_2} \cap C_1 \quad \text{and } C_2 = X_{0,0} \cup X_{0,1} \cup X_{1,0} \cup X_{1,1}.$$

Notice that $\operatorname{diam}(X_{i_1,i_2}) \le d_0/2$.

Proceed by induction. Having constructed

$$C_n = \bigcup_{i_1,\dots,i_n \in \{0,1\}} X_{i_1,\dots,i_n}$$

with $\operatorname{diam} X_{i_1,\dots,i_n} \le d_0/2^n$, pick two different points $x_{i_1,\dots,i_n,0}$ and $x_{i_1,\dots,i_n,1}$ in $\operatorname{Int} X_{i_1,\dots,i_n}$ and let us successively define

$$Y_{i_1,\dots,i_n,i_{n+1}} := \bar{B}(x_{i_1,\dots,i_n,i_{n+1}}, d(x_{i_1,\dots,i_n,0}, x_{i_1,\dots,i_n,1})/4),$$

$$X_{i_1,\dots,i_n,i_{n+1}} := Y_{i_1,\dots,i_n,i_{n+1}} \cap C_n,$$

$$C_{n+1} := \bigcup_{i_1,\dots,i_n,i_{n+1} \in \{0,1\}} X_{i_1,\dots,i_n,i_{n+1}}.$$

Since $\operatorname{diam} X_{i_1,\dots,i_n} \le d_0/2^n$, each infinite intersection

$$\bigcap_{i_1,\dots,i_n,\dots \in \{0,1\}} X_{i_1,\dots,i_n,\dots}$$

is a single point by Heine–Borel (Proposition 3.2.3). The set $C := \bigcap_{n=1}^{\infty} C_n$ is homeomorphic to the countable product of the two point sets $\{0, 1\}$ via the map

$$\bigcap_{i_1,\dots,i_n,\dots \in \{0,1\}} X_{i_1,\dots,i_n,\dots} \mapsto (i_1, \dots, i_n \dots).$$

By Proposition 1.7.3, C is homeomorphic to the Cantor set. □

The theorem is thus proved. □

3.2.4. Completion. Completeness allows to perform limit operations which arise frequently in various constructions. Notice that it is not possible to define the notion of Cauchy sequence in an arbitrary topological space, since one lacks the possibility of comparing neighborhoods at different points. Here the uniform structure (see Section 3.11) provides the most general natural setting.

A metric space can be made complete in the following way:

DEFINITION 3.2.9. If X is a metric space and there is an isometry from X onto a dense subset of a complete metric space $\hat{X}$ then $\hat{X}$ is called the *completion* of X.

THEOREM 3.2.10. *For any metric space X there exists a completion unique up to isometry which commutes with the embeddings of X into a completion.*

PROOF. The process mimics the construction of the real numbers as the completion of rationals, well–known from basic real analysis. Namely, the elements of the completion are equivalence classes of Cauchy sequences by identifying two sequences if the distance between the corresponding elements converges to zero. The distance between two (equivalence classes of) sequences is defined as the limit of the distances between the corresponding elements. An isometric embedding of X into the completion is given by identifying element of X with constant sequences. Uniqueness is obvious by definition, since by uniform continuity the isometric embedding of X to any completion extends to an isometric bijection of the standard completion. □

3.3. The p-adic completion of integers and rationals

This is an example which rivals the construction of real numbers in its importance for various areas of mathematics, especially to number theory and algebraic geometry. Unlike the construction of the reals, it gives infinitely many differnt nonisometric completions of the rationals.

3.3.1. The p-adic norm. Let p be a positive prime number. Any rational number r can be represented as $p^m \frac{k}{l}$ where m is an integer and k and l are integers realtively prime with p. Define the *p-adic norm* $\|r\|_p := p^{-m}$ and the distance $d_p(r_1, r_2) := \|r_1 - r_2\|_p$.

EXERCISE 3.3.1. Show that the p-adic norm is *multiplicative*, i.e., we have $\|r_1 \cdot r_2\|_p = \|r_1\|_p \|r_2\|_p$.

PROPOSITION 3.3.1. *The inequality*

$$d_p(r_1, r_3) \leq \max(d_p(r_1, r_2), d_p(r_2, r_3))$$

holds for all $r_1, r_2, r_3 \in \mathbb{Q}$.

REMARK 3.3.2. A metric satisfying this property (which is stronger than the triangle inequality) is called an *ultrametric*.

PROOF. Since $\|r\|_p = \|-r\|_p$ the statement follows from the property of p-norms:

$$\|r_1 + r_2\|_p \leq \|r_1\|_p + \|r_2\|_p.$$

To see this, write $r_i = p_i^m \frac{k_i}{l_i}$, $i = 1, 2$ with k_i and l_i relatively prime with p and assume without loss of generality that $m_2 \geq m_1$. We have

$$r_1 + r_2 = p_1^m \frac{k_1 l_2 + p^{m_2 - m_1} k_2 l_1}{l_1 l_2}.$$

The numerator $k_1 l_2 + p^{m_2 - m_1} k_2 l_1$ is an integer and if $m_2 > m_1$ it is relatively prime with p. In any event we have $\|r_1 + r_2\|_p \leq p^{-m_1} = \|r_1\|_p = \max(\|r_1\|_p, \|r_2\|_p)$. □

3.3.2. The p-adic numbers and the Cantor set. Proposition 3.3.1 and the multiplicativity prorerty of the p-adic norm allow to extend addition and multiplication from $\mathbb{Q}$ to the completion. This is done in exacly the same way as in the real analysis for real numbers. The existence of the opposite and inverse (the latter for a nonzero element) follow easily.

Thus the completion becomes a field, which is called the *field of p-adic numbers* and is usually denoted by $\mathbb{Q}_p$. Restricting the procedure to the integers which always have norm ≤ 1 one obtains the subring of $\mathbb{Q}_p$, which is called the *ring of p-adic integers* and is usually denoted by $\mathbb{Z}_p$.

The topology of p–adic numbers once again indicates the importance of the Cantor set.

PROPOSITION 3.3.3. *The space $\mathbb{Z}_p$ is homeomorphic to the Cantor set; $\mathbb{Z}_p$ is the unit ball (both closed and open) in $\mathbb{Q}_p$.*

The space $\mathbb{Q}_p$ is homeomorphic to the disjoint countable union of Cantor sets.

PROOF. We begin with the integers. For any sequence

$$a = \{a_n\} \in \prod_{n=1}^{\infty} \{0, 1 \ldots, p-1\}$$

the sequence of integers

$$k_n(a) := \sum_{i=1}^{n} a_n p^i$$

is Cauchy; for different $\{a_n\}$ these sequences are non equivalent and any Cauchy sequence is equivalent to one of these. Thus the correspondence

$$\prod_{n=1}^{\infty} \{0, 1 \ldots, p-1\} \to \mathbb{Z}_p, \quad \{a_n\} \mapsto \text{the equivalence class of } k_n(a)$$

is a homeomorphism. The space $\prod_{n=1}^{\infty} \{0, 1 \ldots, p-1\}$ can be mapped homeomorphically to a nowhere dense perfect subset of the interval by the map

$$\{a_n\}_{n=1}^{\infty} \mapsto \sum_{n=1}^{\infty} a_n (2p-1)^{-i}$$

. Thus the statement about $\mathbb{Z}_p$ follows from Proposition 1.7.5.

Since $\mathbb{Z}$ is the unit ball (open and closed) around 0 in the matric d_p and any other point is at a distance at least 1 from it, the same holds for the completions.

Finally, any rational number can be uniquely represented as

$$k + \sum_{i=1}^{n} a_i p^{-i}, \quad k \in \mathbb{Z}, \quad a_i \in \{0, \ldots, p-1\},\ i = 1, \ldots, n.$$

If the corresponging finite sequences a_i have different length or do not coincide, then the p-adic distance between the rationals is at least 1. Passing to the completion we see that any $x \in \mathbb{Q}_p$ is uniquely represented as $k + \sum_{i=1}^{n} a_i p^{-i}$ with $k \in \mathbb{Z}_p$. with pairwise distances for different a_i's at least one. □

EXERCISE 3.3.2. Where in the construction is it important that p is a prime number?

3.4. Maps between metric spaces

3.4.1. Stronger continuity properties.

DEFINITION 3.4.1. A map $f : X \to Y$ between the metric spaces (X, d), (Y, dist) is said to be *uniformly continuous* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ we have $\mathrm{dist}(f(x), f(y)) < \varepsilon$. A uniformly continuous bijection with uniformly continuous inverse is called a *uniform homeomorphism*.

PROPOSITION 3.4.2. *A uniformly continuous map from a subset of a metric space to a complete space uniquely extends to its closure.*

PROOF. Let $A \subset X$, $x \in \bar{A}$, $f \colon A \to Y$ uniformly continuous. Fix an $\epsilon > 0$ and find the corresponding δ from the definition of uniform continuity. Take the closed $\delta/4$ ball around x. Its image and hence the closure of the image has diameter $\leq \epsilon$. Repeating this procedure for a sequence $\epsilon_n \to 0$ we obtain a nested sequence of closed sets whose diameters converge to zero. By Proposition 3.2.3 their intersection is a single point. If we denote this point by $f(x)$ the resulting map will be continuous at x and this extension is unique by uniqueness of the limit since by construction for any sequence $x_n \in A,\ x_n \to x$ one has $f(x_n) \to f(x)$. □

DEFINITION 3.4.3. A family $\mathcal{F}$ of maps $X \to Y$ is said to be *equicontinuous* if for every $x \in X$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies

$$\mathrm{dist}(f(x), f(y)) < \varepsilon \quad \text{for all } y \in X \text{ and } f \in \mathcal{F}.$$

DEFINITION 3.4.4. A map $f \colon X \to Y$ is said to be *Hölder continuous* with exponent α, or α*-Hölder*, if there exist $C, \varepsilon > 0$ such that $d(x, y) < \varepsilon$ implies

$$d(f(x), f(y)) \leq C(d(x, y))^{\alpha},$$

Lipschitz continuous if it is 1-Hölder, and biLipschitz if it is Lipschitz and has a Lipschitz inverse.

It is useful to introduce local versions of the above notions. A map $f \colon X \to Y$ is said to be Hölder continuous with exponent α, at the point $x \in X$ or α*-Hölder*, if there exist $C, \varepsilon > 0$ such that $d(x, y) < \varepsilon$ implies

$$d(f(x), f(y)) \leq C(d(x, y))^{\alpha},$$

Lipschitz continuous at x if it is 1-Hölder at x.

3.4.2. Various equivalences of metric spaces. Besides the natural relation of isometry, the category of metric spaces is endowed with several other equivalence relations.

DEFINITION 3.4.5. Two metric spaces are *uniformly equivalent* if there exists a homeomorphism between the spaces which is uniformly continuous together with its inverse.

PROPOSITION 3.4.6. *Any metric space uniformly equivalent to a complete space is complete.*

PROOF. A uniformly continuous map obviously takes Cauchy sequences to Cauchy sequences. □

EXAMPLE 3.4.7. The open interval and the real line are homeomorphic but not uniformly equivalent because one is bounded and the other is not.

EXERCISE 3.4.1. Prove that an open half–line is not not uniformly equivalent to either whole line or an open interval.

DEFINITION 3.4.8. Metric spaces are *Hölder equivalent* if there there exists a homeomorphism between the spaces which is Hölder together with its inverse.

Metric spaces are *Lipschitz equivalent* if there exists a biLipschitz homeomorphism between the spaces.

EXAMPLE 3.4.9. Consider the standard middle–third Cantor set C and the subset C_1 of $[0, 1]$ obtained by a similar procedure but with taking away at every step the open interval in the middle of one half of the length. These two sets are Hólder equivalent but not Lipschitz equivalent.

EXERCISE 3.4.2. Find a Hölder homeomorphism with Hölder inverse in the previous example.

As usual, it is easier to prove existence of an equivalence that absence of one. For the latter one needs to produce an invariant of Lipschitz equivalence calculate it for two sets and show that the values (which do not have to be numbers but may be mathematical objects of another kind) are different. On this occasion one can use asymptotics of the minimal number of ϵ-balls needed to cover the set as $\epsilon \to 0$. Such notions are called *capacities* and are related to the important notion of *Hausdorff dimension* which, unlike the topological dimension, is not invariant under homeomorphisms. See **??**.

EXERCISE 3.4.3. Prove that the identity map of the product space is biLIpschitz homeomorphism between the space provided with the maximal metric and with any l^p metric.

EXAMPLE 3.4.10. The unit square (open or closed) is Lipschitz equivalent to the unit disc (respectively open or closed), but not isometric to it.

EXERCISE 3.4.4. Consider the unit circle with the metric induced from the $\mathbb{R}^2$ and the unit circle with the angular metric. Prove that these two metric spaces are Lipschitz equivalent but not isometric.

3.5. Role of metrics in geometry and topology

3.5.1. Elementary geometry. The study of metric spaces with a given metric belongs to the realm of geometry. The natural equivalence relation here is the strongest one, mentioned above, the isometry. Recall that the classical (or "elementary") Euclidean geometry deals with properties of simple objects in the plane or in the three-dimensional space invariant under isometries, or, according to some interpretations, under a larger class of similarity transformations since the absolute unit of length is not fixed in the Euclidean geometry (unlike the prototype non-Euclidean geometry, the hyperbolic one!).

Isometries tend to be rather rigid: recall that in the Euclidean plane an isometry is uniquely determined by images of three points (not on a line) , and in the Euclidean space by the images of four (not in a plane), and those images cannot be arbitrary.

EXERCISE 3.5.1. Prove that an isometry of $\mathbb{R}^n$ with the standard Euclidean metric is uniquely determined by images of any points $x_1, \dots, x_{n+1}$ such that the vectors $x_k - x_1,\ k = 2, \dots, n+1$ are linearly independent.

3.5.2. Riemannian geometry. The most important and most central for mathematics and physics generalization of Euclidean geometry is *Riemannian geometry*. Its objects are manifolds (in fact, differentiable or smooth manifolds which are defined and discussed in Chapter 4) with an extra structure of a *Riemannian metric* which defines Euclidean geometry (distances and angles) *infinitesimally* at each point, and the length of curves is obtained by integration. A smooth manifolds with a fixed Riemannian metric is called a *Riemannian manifold*. While we will wait till Section 13.2 for a systematic introduction to Riemannian geometry, instances of it have already appeared, e.g. the metric on the standard embedded sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ where the distance is measured along the great circles, (and is *not* induced from $\mathbb{R}^{n+1}$), its projection to $\mathbb{R}P(n)$, and projection of Euclidean metric in $\mathbb{R}^n$ to the torus $\mathbb{T}^n$. More general and more interesting classes of Riemannian manifolds will continue to pop up along the way, e.g. in **??** and **??**.

EXERCISE 3.5.2. Prove that in the spherical geometry the sum of angels of a triangle whose sides are arcs of great circles is always greater than π

3.5.3. More general metric geometries. Riemannian geometry is the richest and the most important but by no means only and not the most general way metric spaces appear in geometry. While Riemannian geometry, at least classically, has been inspired mostly by analytic methods of classical geometries (Euclidean, spherical and suchlike) there are other more contemporary directions which to a large extent are developing the synthetic methods of classical geometric reasoning; an outstanding example is the geometry of *Aleksandrov spaces*.

EXERCISE 3.5.3. Let $a > 0$ and denote by C_a the surface of the cone in $\mathbb{R}^3$ given by the conditions $a^2z^2 = x^2 + y^2$, $z \geq 0$. Call a curve in C_a a line segment if it is the shortest curve between its endpoints. Find all line segments in C_a.

3.5.4. Metric as a background and a base for Lipschitz structure. The most classical extensions of Euclidean geometry dealt (with the exception of spherical geometry) not with other metrics spaces but with geometric structures more general that Euclidean metric, such as affine and projective structures. To this one should add conformal structure which if of central importance for complex analysis. In all these geometries metrics appear in an auxiliary role such as the metric from Example 3.1.5 on real projective spaces.

EXERCISE 3.5.4. Prove that there is no metric on the projective line $\mathbb{R}P(1)$ generating the standard topology which is invariant under projective transformations.

EXERCISE 3.5.5. Prove that there is no metric in $\mathbb{R}^2$ generating the standard topology and invariant under all area preserving affine transformations, i.e transformations of the form $x \mapsto Ax + b$ where A is a matrix with determinanat ± 1 and b is a vector.

The role played by metrics in the principal branches of topology, algebraic and differential topology, is somewhat similar. Most spaces studied in those disciplines are metrizable; especially in the case of differential topology which studies smooth manifolds and various derivative objects, fixing a Riemannian metric on the manifold is very useful. It allows to bring precise measurements into the picture and provides various function spaces associated with the manifold such as spaces of smooth functions or differential forms, with the structure of a Banach space. But the choice of metric is usually arbitrary and only in the special cases, when the objects of study possess many symmetries, a particular choice of metric sheds much light on the core topological questions.

One should also point out that in the study of non-compact topological spaces and group actions on such spaces often a natural class of biLipschitz equivalent metrics appear. The study of such structures has gained importance over last two decades.

3.6. Separation properties and metrizability

As we have seen any metric topology is first countable (Proposition 3.1.2) and normal (Theorem 3.1.3). Conversely, it is natural to ask under what conditions a topological space has a metric space structure compatible with its topology.

A topological space is said to be *metrizable* if there exists a metric on it that induces the given topology. The following theorem gives necessary and sufficient conditions for metrizability for second countable topological spaces. Theorem 9.10 from Bredon.

THEOREM 3.6.1. *[Urysohn Metrization Theorem]*
A normal space with a countable base for the topology is metrizable.

PROOF. +++++++++++++++++++++++++ □

Theorem 3.6.1 and Proposition 1.5.4 imply

COROLLARY 3.6.2. *Any compact Hausdorff space with a countable base is metrizable.*

Example: normal first countable not metrizable?

3.7. Compact metric spaces

3.7.1. Sequential compactness.

PROPOSITION 3.7.1. *Any compact metric space is complete.*

PROOF. Suppose the opposite, that is, X is a compact metric space and a Cauchy sequence $x_n,\ n = 1, 2, \dots$ does not converge. By taking a subsebuence if necessary we may assume that all points x_n are different. The union of the elements of the sequence is closed since the sequence does not converge. Let

$$\mathcal{O}_n := X \setminus \bigcup_{i=n}^{\infty} \{x_n\}.$$

These sets form an open cover of X but since they are increasing there is no finite subcover. □

DEFINITION 3.7.2. Given $r > 0$ a subset A of a metric space X is called an *r-net* if for any $x \in X$ there is $a \in A$ such that the distance $d(x, a)$. Equivalently r-balls around the points of A cover X.

A set $A \subset X$ is called *r-separated* if the distance between any two different points in A is greater than r.

The following observation is very useful in the especially for quantifying the notion of compactness.

PROPOSITION 3.7.3. *Any maximal r-separated set is an r-net.*

PROOF. If A is r-separated and is not an r-net then there is a point $x \in X$ at a distance $\geq r$ from every point of A Hence the set $A \cup \{x\}$ is r-separated □

PROPOSITION 3.7.4. *The following properties of a metric space X are equivalent*

(1) *X is compact;*
(2) *for any $\epsilon > 0$ X contains a finite ϵ-net, or, equivalently, any r-separated set for any $r > 0$ is finite;*
(3) *every sequence contains a congerving subsequence.*

PROOF. (1) → (2). If X is compact than the cover of X by all balls of radius ϵ contains a finite subcover; centers of those balls form a finite ϵ-net.

(2) → (3) By Proposition 3.7.1 it is sufficient to show that every sequence has a Cauchy subsequence. Take a sequence $x_n,\ n = 1, 2, \dots$ and consider a finite 1-net. There is a ball of radius 1 which contains infinitely many elements of the sequence. Consider only these elements as a subsequence. Take a finite 1/2-net and find a subsequence which lies in a single ball of radius 1/2. Continuing by induction we find nested subsequences of the original sequence which lie in balls of radius $1/2^n$. Using the standard diagonal process we construct a Cauchy subsequence.

(3) → (1). Let us first show that the space must be separable. This implies that any cover contains a countable subcover since the space has countable base. If the space is not separable than there exists an $\epsilon > 0$ such that for any countable (and hence finite) collection of points there is a point at the distance greater than ϵ from all of them. This allows to construct by induction an infinite sequence of points which are pairwise more than ϵ apart. Such a sequence obviously does not contain a converging subsequence.

Now assume there is an open countable cover $\{\mathcal{O}_1, \mathcal{O}_2, \dots\}$ without a finite subcover. Take the union of the first n elements of the cover and a point x_n outside of the union. The sequence $x_n,\ n = 1, 2, \dots$ thus defined has a converging subsequence $x_{n_k} \to x$. But x belong to a certain element of the cover, say $\mathcal{O}_N$. Then for a sufficinetly large $k,\ n_k > N$ hence $x_{n_k} \notin \mathcal{O}_N$, a contradiction to convergence. □

An immediate corollary of the proof is the following.

PROPOSITION 3.7.5. *Any compact metric space is separable.*

Aside from establishing equivalence of compactness and sequential compactness for metric spaces Proposition 3.7.4 contains a very useful criterion of compactness in the form of property (2). Right away it gives a necessary and sufficient condition for a (in general incomplete) metric space to have compact completion. As we see it later in Section 3.7.5 it is also a starting point for developing qualitative notions related to the "size" of a metric space.

DEFINITION 3.7.6. A metric space (X, d) is *totally bounded* if it contains a finite ϵ-net for any $\epsilon > 0$, or, equivalently if any r-separates subset of X for any $r > 0$ is finite.

Since both completion and any subset of a totally bounded space are totally bounded Proposition 3.7.4 immediately implies

COROLLARY 3.7.7. *Completion of a metric space is compact if and only if the space is totally bounded.*

EXERCISE 3.7.1. Prove that an isometric embedding of a compact metric space into itself is an isometry.

3.7.2. Lebesgue number.

PROPOSITION 3.7.8. *For an open cover of a compact metric space there exists a number δ such that every δ-ball is contained in an element of the cover.*

PROOF. Suppose the opposite. Then there exists a cover and a sequence of points x_n such that the ball $B(x_n, 1/2^n)$ does not belong to any element of the cover. Take a converging subsequence $x_{n_k} \to x$. Since the point x is covered by an open set, a ball of radius $r > 0$ around x belongs to that element. But for k large enough $d(x, x_{n_k}) < r/2$ and hence by the triangle inequality the ball $B(x_{n_k}, r/2)$ lies in the same element of the cover. □

The largest such number is called the *Lebesgue number* of the cover.

3.7.3. Characterization of Cantor sets.

THEOREM 3.7.9. *Any perfect compact totally disconnected metric space X is homeomorphic to the Cantor set.*

PROOF. Any point $x \in X$ is contained in a set of arbitrally small diameter which is both closed and open. For x is the intersection of all sets which are open and closed and contain x. Take a cover of $X \setminus X$ by sets which are closed and open and do not contain x Adding the ball $B(x, \epsilon)$ one obtains a cover of X which has a finite subcover. Union of elements of this subcover other than $B(x, \epsilon)$ is a set which is still open and closed and whose complement is contained in $B(x, \epsilon)$.

Now consider a cover of the space by sets of diameter ≤ 1 which are closed and open. Take a finite subcover. Since any finite intersection of such sets is still both closed an open by taking all possible intersection we obtain a *partition* of the space into finitely many closed and open sets of diameter ≤ 1. Since the space is perfect no element of this partition is a point so a further division is possible. Repeating this procedure for each set in the cover by covering it by sets of diameter $\leq 1/2$ we obtain a finer partition into closed and open sets of of diameter $\leq 1/2$. Proceeding by induction we obtain a nested sequence of finite partitions into closed and open sets of positive diameter $\leq 1/2^n$, $n = 0, 1, 2, \ldots$. Proceeding as in the proof of Proposition 1.7.5, that is, mapping elements of each partition inside a nested sequence of contracting intervals, we constuct a homeomorphism of the space onto a nowhere dense perfect subset of $[0, 1]$ and hence by Proposition 1.7.5 our space is homeomorphic to the Cantor set. □

3.7.4. Universality of the Hilbert cube. Theorem 3.2.6 means that Cantor set is in some sense a minimal nontrivial compact metrizable space. Now we will find a maximal one.

THEOREM 3.7.10. *Any compact separable metric space X is homeomorphic to a closed subset of the Hilbert cube H.*

PROOF. First by multiplying the metric by a constant if nesessary we may assume that the diameter of X is less that 1. Pick a dense sequence of points $x_1, x_2 \dots$ in X. Let $F : X \to H$ be defined by

$$F(x) = (d(x, x_1), d_(x, x_2), \dots).$$

This map is injective since for any two distict points x and x' one can find n such that $d(x, x_n) < (1/2)d(x', x_n)$ so that by the triangle inequality $d(x, x_n) < d(x', x_n)$ and hence $F(x) \neq F(x')$. By Proposition 1.5.11 $F(X) \subset H$ is compact and by Proposition 1.5.13 F is a homeomorphism between X and $F(X)$. □

EXERCISE 3.7.2. Prove that the infinite-dimensioanl torus $\mathbb{T}^\infty$, the product of the countably many copies of the unit circle, has the same universality property as the Hilbert cube, that is, any compact separable metric space X is homeomorphic to a closed subset of $\mathbb{T}^\infty$.

3.7.5. Capacity and box dimension. For a compact metric space there is a notion of the "size" or capacity inspired by the notion of volume. Suppose X is a compact space with metric d. Then a set $E \subset X$ is said to be *r-dense* if $X \subset \bigcup_{x \in E} B_d(x, r)$, where $B_d(x, r)$ is the r-ball with respect to d around x (see **??**). Define the *r-capacity* of (X, d) to be the minimal cardinality $S_d(r)$ of an r-dense set.

For example, if $X = [0, 1]$ with the usual metric, then $S_d(r)$ is approximately $1/2r$ because it takes over $1/2r$ balls (that is, intervals) to cover a unit length, and the $\lfloor 2 + 1/2r \rfloor$-balls centered at $ir(2 - r)$, $0 \leq i \leq \lfloor 1 + 1/2r \rfloor$ suffice. As another example, if $X = [0, 1]^2$ is the unit square, then $S_d(r)$ is roughly r^{-2} because it takes at least $1/\pi r^2$ r-balls to cover a unit area, and, on the other hand, the $(1 + 1/r)^2$-balls centered at points (ir, jr) provide a cover. Likewise, for the unit cube $(1 + 1/r)^3$, r-balls suffice.

In the case of the ternary Cantor set with the usual metric we have $S_d(3^{-i}) = 2^i$ if we cheat a little and use closed balls for simplicity; otherwise, we could use $S_d((3 - 1/i)^{-i}) = 2^i$ with honest open balls.

One interesting aspect of capacity is the relation between its dependence on r [that is, with which power of r the capacity $S_d(r)$ increases] and dimension.

If $X = [0, 1]$, then

$$\lim_{r \to 0} -\frac{\log S_d(r)}{\log r} \geq \lim_{r \to 0} -\frac{\log(1/2r)}{\log r} = \lim_{r \to 0} \frac{\log 2 + \log r}{\log r} = 1$$

and

$$\lim_{r \to 0} -\frac{\log S_d(r)}{\log r} \leq \lim_{r \to 0} -\frac{\log \lfloor 2 + 1/2r \rfloor}{\log r} \leq \lim -\frac{\log(1/r)}{\log r} = 1,$$

so $\lim_{r \to 0} - \log S_d(r)/\log r = 1 = \dim X$. If $X = [0, 1]^2$, then

$$\lim_{r \to 0} - \log S_d(r)/\log r = 2 = \dim X,$$

and if $X = [0,1]^3$, then

$$\lim_{r\to 0} -\log S_d(r)/\log r = 3 = \dim X.$$

This suggests that $\lim_{r\to 0} -\log S_d(r)/\log r$ defines a notion of dimension.

DEFINITION 3.7.11. If X is a totally bounded metric space (Definition 3.7.6), then

$$\operatorname{bdim}(X) := \lim_{r\to 0} -\frac{\log S_d(r)}{\log r}$$

is called the *box dimension* of X.

Let us test this notion on a less straightforward example. If C is the ternary Cantor set, then

$$\operatorname{bdim}(C) = \lim_{r\to 0} -\frac{\log S_d(r)}{\log r} = \lim_{n\to\infty} -\frac{\log 2^i}{\log 3^{-i}} = \frac{\log 2}{\log 3}.$$

If C_α is constructed by deleting a middle interval of relative length $1-(2/\alpha)$ at each stage, then $\operatorname{bdim}(C_\alpha) = \log 2/\log\alpha$. This increases to 1 as $\alpha \to 2$ (deleting ever smaller intervals), and it decreases to 0 as $\alpha \to \infty$ (deleting ever larger intervals). Thus we get a small box dimension if in the Cantor construction the size of the remaining intervals decreases rapidly with each iteration.

This illustrates, by the way, that the box dimension of a set may change under a homeomorphism, because these Cantor sets are pairwise homeomorphic. Box dimension and an associated but more subtle notion of *Hausdorff dimension* are the prime exhibits in the panoply of "fractal dimensions", the notion surrounded by a certain mystery (or mystique) at least for laymen. In the next section we will present simple calculations which shed light on this notion.

3.8. Metric spaces with symmetries and self-similarities

3.8.1. Euclidean space as an ideal geometric object and some of its close relatives. An outstanding, one may even say, the central, feature of Euclidean geometry, is an abundance of isometries in the Euclidean space. Not only there is isometry which maps any given point to any other point (e.g. the parallel translation by the vector connecting those points) but there are also isometries which interchange any given pair of points, e.g the central symmetry with respect to the midpoint of the interval connecting those points, or the reflection in the (hyper)plane perpendicular to that interval at the midpoint. The latter property distinguishes a very important class of Riemannian manifolds, called *symmetric spaces*. The next obvious examples of symmetric space after the Euclidean spaces are spheres $\mathbb{S}^n$ with the standard metric where the distance is measure along the shorter arcs of great circles. Notice that the metric induced from the embedding of $\mathbb{S}^n$ as the unit sphere into $\mathbb{R}^{n+1}$ also possesses all there isometries but the metric is not a Riemanninan metric, i.e. the distance cannot be calculated as the minimum of lengths of curves connecting two points, and thus this metric is much less interesting.

EXERCISE 3.8.1. How many isometries are there that interchange two points $x, y \in \mathbb{R}^n$ for different values of n?

EXERCISE 3.8.2. How many isometries are there that interchange two points $x, y \in \mathbb{S}^n$ for different values of n and for different configurations of points?

EXERCISE 3.8.3. Prove that the real projective space $\mathbb{R}P(n)$ with the metric inherited from the sphere (**??**) is a symmetric space.

EXERCISE 3.8.4. Prove that the torus $\mathbb{T}^n$ is with the metric inherited from $\mathbb{R}^n$ a symmetric space.

There is yet another remarkable property of Euclidean spaces which is not shared by other symmetric spaces: existence of *similarities*, i.e. transformations which preserve angles and changes all distances with the same coefficient of proportionality. It is interesting to point out that in the long quest to "prove" Euclid's fifth postulate, i.e. to deduce it from other axioms of Euclidean geometry, one among many equivalent formulations of the famous postulate is existence of a single pair of similar but not equal (not isometric) triangles. In the non-Euclidean hyperbolic geometry which results from adding the negation of the fifth postulates there no similar triangles and instead there is absolute unit of length! Incidentally the hyperbolic plane (as well as its higher-dimensional counterparts) is also a symmetric space. Existence of required symmetries can be deduced synthetically form the axioms common to Euclidean and non-Euclidean geometry, i.e. it belong s to so-called *absolute geometry*, the body of statement which can be proven in Euclidean geometry without the use of fifth postulate.

Metric spaces for which there exists a self-map which changes all distance with the same coefficient of proportionality different from one are called *self-similar*.

Obviously in a compact globally self-similar space which contain more one point the coefficient of proportionality for any similarity transformation must be less than one and such a transformation cannot be bijective; for non-compact spaces this is possible however.

3.8.2. Metrics on the Cantor set with symmetries and self-similarities. There is an interesting example of a similarity on the middle-third Cantor set, namely, $f_0\colon [0,1] \to [0,1]$, $f_0(x) = x/3$. Since f_0 is a contraction, it is also a contraction on every invariant subset, and in particular on the Cantor set. The unique fixed point is obviously 0. There is another contraction with the same contraction coefficient 1/3 preserving the Cantor set, namely $f_1(x) = \frac{x+2}{3}$ with fixed point 1. Images of these two contractions are disjoint and together they cover the whole Cantor set

EXERCISE 3.8.5. Prove that any similarity of the middle third Cantor set belongs to the semigroup generated by f_0 and f_1.

EXERCISE 3.8.6. Find infinitely many different self-similar Cantor sets on $[0,1]$ which contain both endpoints 0 and 1.

FIGURE 3.8.1. Sierpinski carpet and Sierpinski gasket.

FIGURE 3.8.2. The Koch snowflake.

3.8.3. Other Self-Similar Sets. Let us describe some other interesting self-similar metric spaces that are of a different form. The *Sierpinski carpet* (see **??**) is obtained from the unit square by removing the "middle-ninth" square $(1/3, 2/3) \times (1/3, 2/3)$, then removing from each square $(i/3, i + 1/3) \times (j/3, j + 1/3)$ its "middle ninth," and so on. This construction can easily be described in terms of ternary expansion in a way that immediately suggests higher-dimensional analogs.

Another very symmetric construction begins with an equilateral triangle with the bottom side horizontal, say, and divide it into four congruent equilateral triangles of which the central one has a horizontal top side. Then one deletes this central triangle and continues this construction on the remaining three triangles. he resulting set is sometimes called *Sierpinski gasket*.

The *von Koch snowflake* is obtained from an equilateral triangle by erecting on each side an equilateral triangle whose base is the middle third of that side and continuing this process iteratively with the sides of the resulting polygon It is attributed to Helge von Koch (1904).

A three-dimensional variant of the Sierpinski carpet S is the Sierpinski sponge or Menger curve defined by $\{(x, y, z) \in [0, 1]^3 \mid (x, y) \in S,\ (x, z) \in S\ (y, z) \in S\}$. It is obtained from the solid unit cube by punching a 1/3-square hole through the center from each direction, then punching, in each coordinate direction, eight 1/9-square holes through in the right places, and so on. Both Sierpinski carper and Menger curve have important universality properties which we do not discuss in this book.

Let as calculate the box dimension of these new examples. For the square Sierpinski carpet we can cheat as in the capacity calculation for the ternary Cantor set and use closed balls (sharing their center with one of the small remaining cubes at a certain stage) for covers. Then $S_d(3^{-i}/\sqrt{2}) = 8^i$ and

$$\operatorname{bdim}(S) = \lim_{n\to\infty} -\frac{\log 8^i}{\log 3^{-i}/\sqrt{2}} = \frac{\log 8}{\log 3} = \frac{3\log 2}{\log 3},$$

which is three times that of the ternary Cantor set (but still less than 2, of course). For the triangular Sierpinski gasket we similarly get box dimension $\log 3/\log 2$.

The Koch snowflake K has $S_d(3^{-i}) = 4^i$ by covering it with (closed) balls centered at the edges of the ith polygon. Thus

$$\operatorname{bdim}(K) = \lim_{n\to\infty} -\frac{\log 4^i}{\log 3^{-i}} = \frac{\log 4}{\log 3} = \frac{2\log 2}{\log 3},$$

which is less than that of the Sierpinski carpet, corresponding to the fact that the iterates look much "thinner". Notice that this dimension exceeds 1, however, so it is larger than the dimension of a curve. All of these examples have (box) dimension that is not an integer, that is, fractional or "fractal". This has motivated calling such sets *fractals*.

Notice a transparent connection between the box dimension and coefficients of self-similarity on all self-similar examples.

3.9. Spaces of continuous maps

If X is a compact metrizable topological space (for example, a compact manifold), then the space $C(X,X)$ of continuous maps of X into itself possesses the C^0 or *uniform* topology. It arises by fixing a metric ρ in X and defining the distance d between $f, g \in C(X,X)$ by

$$d(f,g) := \max_{x\in X} \rho(f(x), g(x)).$$

The subset $\mathrm{Hom}(X)$ of $C(X,X)$ of homeomorphisms of X is neither open nor closed in the C^0 topology. It possesses, however, a natural topology as a complete metric space induced by the metric

$$d_H(f,g) := \max(d(f,g), d(f^{-1}, g^{-1})).$$

If X is σ-compact we introduce the compact–open topologies for maps and homeomorphisms, that is, the topologies of uniform convergence on compact sets.

We sometimes use the fact that equicontinuity gives some compactness of a family of continuous functions in the uniform topology.

THEOREM 3.9.1 (Arzelá–Ascoli Theorem). *Let X, Y be metric spaces, X separable, and $\mathcal{F}$ an equicontinuous family of maps. If $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{F}$ such that $\{f_i(x)\}_{i\in\mathbb{N}}$ has compact closure for every $x \in X$ then there is a subsequence converging uniformly on compact sets to a function f.*

Thus in particular a closed bounded equicontinuous family of maps on a compact space is compact in the uniform topology (induced by the maximum norm).

Let us sketch the proof. First use the fact that $\{f_i(x)\}_{i\in\mathbb{N}}$ has compact closure for every point x of a countable dense subset S of X. A diagonal argument shows that there is a subsequence f_{i_k} which converges at every point of S. Now equicontinuity can be used to show that for every point $x \in X$ the sequence $f_{i_k}(x)$ is Cauchy, hence convergent (since $\{f_i(x)\}_{i\in\mathbb{N}}$ has compact, hence complete, closure). Using equicontinuity again yields continuity of the pointwise limit. Finally a pointwise convergent equicontinuous sequence converges uniformly on compact sets. elaborate

EXERCISE 3.9.1. Prove that the set of Lipschitz real-valued functions on a compact metric space X with a fixed Lipschitz constant and bounded in absolute value by another constant is compact in $C(x, \mathbb{R})$.

EXERCISE 3.9.2. Is the closure in $C([0,1], \mathbb{R})$ (which is usually denoted simply by $C([0,1])$) of the set of all differentiable functions which derivative bounded by 1 in absolute value and taking value 0 at 1/2 compact?

3.10. Spaces of closed subsets of a compact metric space

3.10.1. Hausdorff distance: definition and compactness. An interesting construction in the theory of compact metric spaces is that of the Hausdorff metric:

DEFINITION 3.10.1. If (X, d) is a compact metric space and $K(X)$ denotes the collection of closed subsets of X, then the *Hausdorff metric* d_H on $K(X)$ is defined by

$$d_H(A, B) := \sup_{a \in A} d(a, B) + \sup_{b \in B} d(b, A),$$

where $d(x, Y) := \inf_{y \in Y} d(x, y)$ for $Y \subset X$.

Notice that d_H is symmetric by construction and is zero if and only if the two sets coincide (here we use that these sets are closed, and hence compact, so the "sup" are actually "max"). Checking the triangle inequality requires a little extra work. To show that $d_H(A, B) \leq d_H(A, C) + d_H(C, B)$, note that $d(a, b) \leq d(a, c) + d(c, b)$ for $a \in A$, $b \in B$, $c \in C$, so taking the infimum over b we get $d(a, B) \leq d(a, c) + d(c, B)$ for $a \in A$, $c \in C$. Therefore, $d(a, B) \leq d(a, C) + \sup_{c \in C} d(c, B)$ and $\sup_{a \in A} d(a, B) \leq \sup_{a \in A} d(a, C) + \sup_{c \in C} d(c, B)$. Likewise, one gets $\sup_{b \in B} d(b, A) \leq \sup_{b \in B} d(b, C) + \sup_{c \in C} d(c, A)$. Adding the last two inequalities gives the triangle inequality.

PROPOSITION 3.10.2. *The Hausdorff metric on the closed subsets of a compact metric space defines a compact topology.*

PROOF. We need to verify total boundedness and completeness. Pick a finite $\epsilon/2$-net N. Any closed set $A \subset X$ is covered by a union of ϵ-balls centered at points of N, and the closure of the union of these has Hausdorff distance at most ϵ from A. Since there are only finitely many such sets, we have shown that this metric is totally bounded. To show that it is complete, consider a Cauchy sequence (with respect to the Hausdorff metric) of closed sets $A_n \subset X$. If we let $A := \bigcap_{k \in \mathbb{N}} \overline{\bigcup_{n \geq k} A_n}$, then one can easily check that $d(A_n, A) \to 0$. □

EXERCISE 3.10.1. Prove that for the Cantor set C the space $K(C)$ is homeomorphic to C.

EXERCISE 3.10.2. Prove that $K([0, 1])$ contains a subset homeomorphic to the Hilbert cube.

3.10.2. Existence of a minimal set for a continuous map. Any homeomorphism of a compact metric space X induces a natural homeomorphism of the collection of closed subsets of X with the Hausdorff metric, so we have the following:

PROPOSITION 3.10.3. *The set of closed invariant sets of a homeomorphism f of a compact metric space is a closed set with respect to the Hausdorff metric.*

PROOF. This is just the set of fixed points of the induced homeomorphism; hence it is closed. □

We will now give a nice application of the Hausdorff metric. Brouwer fixed point Theorem (Theorem 2.5.1 and Theorem 9.3.7) does not extend to continuous maps of even very nice spaces other than the disc. The simplest example of a continuous map (in fact a self–homeomorphism) which does not have have fixed points is a rotation of the circle; if the angle of rotation is a rational multiple of π all points are periodic with the same period; otherwise there are no periodic points. However, there is a nice generalization which works for any compact Hausdorff spaces. An obvious property of a fixed or periodic point for a continuous map is its minimality: it is an invariant closed set which has no invariant subsets.

DEFINITION 3.10.4. An invariant closed subset A of a continuous map $f\colon X \to X$ is *minimal* if there are no nonempty closed f-invariant subsets of A.

THEOREM 3.10.5. *Any continuous map f of a compact Hausdorff space X with a countable base into itself has an invariant minimal set.*

PROOF. By Corollary 3.6.2 the space X is metrizable. Fix a metric d on X and consider the Hausdorff metric on the space $K(X)$ of all closed subsets of X. Since any closed subset A of X is compact (Proposition 1.5.2) $f(A)$ is also compact (Proposition 1.5.11) and hence closed (Corollary 3.6.2). Thus f naturally induces a map $f_*\colon K(X) \to K(X)$ by setting $f_*(A) = f(A)$. A direct calculation shows that the map f_* is continuous in the topology induced by the Hausdorff metric. Closed f-invariant subsets of X are fixed points of f_*. The set of all such sets is closed, hence compact subset $I(f)$ of $K(X)$. Consider for each $B \in I(f)$ all $A \in I(f)$ such that $A \subset B$. Such A form a closed, hence compact, subset $I_B(f)$. Hence the function on $I_B(f)$ defined by $d_H(A, B)$ reaches its maximum, which we denote by $m(B)$, on a certain f-invariant set $M \subset B$.

Notice that the function $m(B)$ is also continuous in the topology of Hausdorff metric. Hence it reaches its minimum m_0 on a certain set N. If $m_0 = 0$, the set N is a minimal set. Now assume that $m_0 > 0$.

Take the set $M \subset B$ such that $d_H(M, B) = m(B) \geq m_0$. Inside M one can find an invariant subset M_1 such that $d_H(M_1, M) \geq m_0$. Notice that since $M_1 \subset M,\ \ d_H(M_1, B) \geq d_H(M, B) = m(B) \geq m_0$.

Continuing by induction we obtain an infinite sequence of nested closed invariant sets $B \supset M \supset M_1 \supset M_2 \supset \cdots \supset M_n \supset \dots$ such that the Hausdorff distance between any two of those sets is at least m_0. This contradicts compactness of $K(X)$ in the topology generated by the Hausdorff metric. □

EXERCISE 3.10.3. Give detailed proofs of the claims used in the proof of Theorem 3.10.5:

- the map $f_*\colon K(X) \to K(X)$ is continuous;
- the function $m(\cdot)$ is continuous;
- $d_H(M_i, M_j) \geq m_0$ for $i, j = 1, 2, \dots; i \neq j$.

EXERCISE 3.10.4. For every natural number n give an example of a homeomorphism of a compact path connected topological space which has no fixed points and has exactly n minimal sets.

3.11. Uniform structures

3.11.1. Definitions and basic properties. The main difference between a metric topology and an even otherwise very good topology defined abstractly is the possibility to choose "small" neighborhoods for all points in the space simultaneously; we mean of course fixing an (arbitrary small) positive number r and taking balls $B(x, r)$ for all x. The notion of uniform structure is a formalization of such a possibility without metric (which is not always possible under the axioms below)

3.11.2. Uniform structure associated with compact topology.

3.12. Topological groups

In this section we introduce groups which carry a topology invariant under the group operations. A *topological group* is a group endowed with a topology with respect to which all *left translations* $L_{g_0}: g \mapsto g_0 g$ and *right translations* $R_{g_0}: g \mapsto g g_0$ as well as $g \mapsto g^{-1}$ are homeomorphisms. Familiar examples are $\mathbb{R}^n$ with the additive structure as well as the circle or, more generally, the n-torus, where translations are clearly diffeomorphisms, as is $x \mapsto -x$.

3.13. Problems

EXERCISE 3.13.1. Prove that every metric space is homeomorphic to a bounded space.

EXERCISE 3.13.2. Prove that in a compact set A in metric space X there exists a pair or points $x, y \in A$ such that $d(x, y) = \operatorname{diam} A$.

EXERCISE 3.13.3. Suppose a function $d: X \times X \to \mathbb{R}$ satisfies conditions (2) and (3) of Definition 3.1.1 but not (1). Find a natural way to modify this function so that the modified function becomes a metric.

EXERCISE 3.13.4. Let S be a smooth surface in $\mathbb{R}^3$, i.e. it may be a non-critical level of a smooth real-valued function, or a closed subset locally given as a graph when one coordinate is a smooth function of two others. S carries two metrics: (i) induced from $\mathbb{R}^3$ as a subset of a metric space, and (ii) the natural internal distance given by the minimal length of curves in S connecting two points.

Prove that if these two metrics coincide then S is a plane.

EXERCISE 3.13.5. Introduce a metric d on the Cantor set C (generating the Cantor set topology) such that (C, d) cannot be isometrically embedded to $\mathbb{R}^n$ for any n.

EXERCISE 3.13.6. Introduce a metric d on the Cantor set C such that (C, d) is not Lipschitz equivalent to a subset of $\mathbb{R}^n$ for any n.

EXERCISE 3.13.7. Prove that the set of functions which are not Hölder continuous at any point is a residual subset of $C([0,1])$.

EXERCISE 3.13.8. Let $f\colon [0,1]\mathbb{R}^2$ be α-Höder with $\alpha > 1/2$. Prove that $f([0,1)]$ is nowhere dense.

EXERCISE 3.13.9. Find a generalization of the previous statement for the maps of the m-dimensional cube I^m to $\mathbb{R}^n$ with $m < n$.

EXERCISE 3.13.10. Prove existence of 1/2-Hölder surjective map $f\colon [0,1] \to I^2$. (Such a map is usually called a *Peano curve*).

EXERCISE 3.13.11. Prove that any connected topological manifold is metrizable. check!

EXERCISE 3.13.12. Find a Riemannian metric on the complex projective space $\mathbb{C}P(n)$ which makes it a symmetric space.

EXERCISE 3.13.13. Prove that $\mathbb{S}^n$ is not self-similar.