CHAPTER 4

REAL AND COMPLEX SMOOTH MANIFOLDS

The notion of smooth or differentiable manifold is one of the central concepts of modern mathematics and its applications, and is also of fundamental importance in theoretical mechanics and physics. Roughly speaking, a smooth manifold is a topological space which may have a complicated global structure, but locally is like Euclidean space, i.e. it is a topological manifold as in Section 1.8 (it possesses "local coordinates"), with the transition from one system of local coordinates to a neighboring one being ensured by smooth functions. The fact that the transition functions are smooth allows the use of the whole machinery of the multivariable differential and integral calculus, which interacts very efficiently with geometric and topological tools in that setting.

This chapter is only a first introduction to real (and complex) smooth manifolds. We will return to this topic in Chapter 10, where, after having further developed some of these tools, in particular homology theory, we will have a glance at deep connections between algebraic and differential topology.

4.1. Differentiable manifolds, smooth maps and diffeomorphisms

4.1.1. Definitions.

DEFINITION 4.1.1. A Hausdorff topological space M with countable base is said to be an *n-dimensional differentiable* (or *smooth*) *manifold* if it is covered by a family $\mathcal{A} = \{(U_\alpha, h_\alpha)\}_{\alpha\in A}$ of open sets U_α called *charts* and supplied with homeomorphisms into $\mathbb{R}^n$,

$$\bigcup_\alpha U_\alpha = M, \quad h_\alpha : U_\alpha \to \mathbb{R}^n$$

(the index set A may be finite, countable or uncountable) that satisfy the *compatibility* condition: for any two charts (U_1, h_1) and (U_2, h_2) in $\mathcal{A}$ with $h_i\colon U_i \to B_i \subset \mathbb{R}^n$ the *coordinate change* $h_2 \circ h_1^{-1}$ (also sometimes called *transition function*) is differentiable on $h_1(U_1 \cap U_2) \subset B_1$.

Here "differentiable" can be taken to mean C^r for any $r \in \mathbb{N} \cup \infty$, or analytic. A collection of such charts covering M is called an *atlas* of M. Any atlas defines a unique *maximal atlas* obtained by taking all charts compatible with the present ones. A maximal atlas is called a *differentiable* (or *smooth*) *structure*.

DEFINITION 4.1.2. A *smooth* or *differentiable map* of one smooth manifold to another is a map $f : M \to N$ which is expressed by differentiable functions in the local coordinates of any chart. More precisely, for any charts (U, h), $U \ni x$

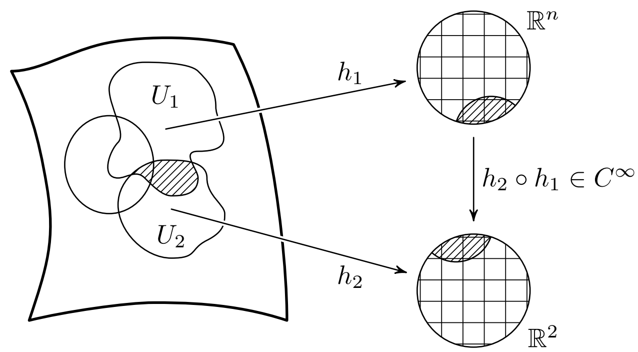

FIGURE 4.1.1. Definition of a smooth manifold

and (V, k), $V \ni f(x)$, the map $k \circ f \circ h^{-1}$ is a differentiable map of one domain of Euclidean space into another.

In view of the compatibility condition, in order to check the smoothness of a map $f : M \to N$ it suffices to check that it is smooth on any cover of M by charts and not on all charts of the maximal atlas.

DEFINITION 4.1.3. A *diffeomorphism* between smooth manifolds is a bijective smooth map with smooth inverse.

Obviously, any diffeomorphism is a homeomorphism,

EXERCISE 4.1.1. Give an example of a homeomorphism which is not a diffeomorphism.

The notion of diffeomorphism provides the natural concept of isomorphism of the smooth structures of manifolds: diffeomorphic manifolds are undistinguishable as differentiable manifolds. [1]

REMARK 4.1.4. In the definition above the local model for a differentiable manifold is $\mathbb{R}^n$ with its differentiable structure. It follows form the definition that any manifold diffeomorphic to $\mathbb{R}^n$ may serve as an alternative model. Two useful special cases are an open ball in $\mathbb{R}^n$ and the open unit disc $(0, 1)^n$. This follows from the fact that all those models are diffeomorphic smooth manifolds, see Exercise 4.1.2 and Exercise 4.1.3.

The Inverse Function Theorem from multi-variable calculus provides the following criterion which can be checked on most occasions.

PROPOSITION 4.1.5. *A map $f\colon M \to N$ between differentiable manifolds is a diffeomorphism if and only if (i) it is bijective and (ii) there exists atlases $\mathcal{A}$ and $\mathcal{B}$ for the differentiable structures in M and N correspondingly such that for any $x \in M$ there exist $(U, h) \in \mathcal{A},\ x \in U$ with $h(x) = p \in \mathbb{R}^n$ and $(V, k) \in \mathcal{B},\ f(x) \in V$ such that* $\det A \neq 0$ *where A the matrix of partial derivatives of $h^{-1} \circ f \circ k$ at p.*

[1] However, the same manifold may have different representations which, for example, may carry different geometric structures.

PROOF. Necessity follows directly from the definition.

To prove sufficiency first notice that by the chain rule for the coordinate changes in $\mathbb{R}^n$ condition (ii) is independent from a choice of (U, h) and (V, k) from the maximal atlases providing $x \in U$ and $f(x) \in V$.

The Inverse Function Theorem guarantees that $h^{-1} \circ f \circ k$ is a diffeomorphism between a sufficiently small ball around p and its image. Taking such balls and their images for covers of M and N correspondingly we see that both f and f^{-1} are smooth. $\square$

PROPOSITION 4.1.6. *Let M be a differentiable manifold, $A \subset M$ an open set. Then A has a natural structure of differentiable manifold compatible with that for M.*

PROOF. Let $x \in A$ ant let (U, h) be an element of the atlas for M such that $x \in U$. Then since A is open so is $h(U \cap A \subset \mathbb{R}^n$. Hence there is an open ball $B \subset U \cap A$ centered at $h(x)$. let $V := h^{-1}(B)$ and h' be the restriction of h to V. By Remark 4.1.4 pairs (V, h') obtained this way from various points $x \in A$ form an atlas compatible with the differentiable structure on M. $\square$

Smooth manifolds constitute a category, whose morphisms are appropriately called smooth (or differentiable) maps. An important class of smooth maps of a fixed manifold is the class of its maps to $\mathbb{R}$, or smooth functions. We will see that smooth functions form an $\mathbb{R}$-algebra from which the manifold can be entirely reconstructed.

A real-valued function $f : M \to \mathbb{R}$ on a smooth manifold M is called *smooth* (or *differentaible*) if on each chart (U, h) the composition $f \circ h^{-1}$ is a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}$. Using the compatibility condition, it is easy to verify that it suffices to check differentiability for any set of charts covering M (rather than for all charts of its maximal atlas).

maybe we should do this at least once?

The set of all smooth functions on M will be denoted by $C^\infty(M)$ (or $C^n(M)$, $n \in \mathbb{N}$, depending on the differentiability class under consideration).

One of the remarkable mathematical discoveries of the mid-twentieth century was the realization that a topological manifold can have more than one differentiable structure: even the sphere (e.g. in dimension 7) can have several different smooth structures. Further, certain topological manifolds have no smooth structure compatible with their topology. These delicate questions will not be discussed in this course.

4.1.2. First examples.

EXAMPLE 4.1.7. $\mathbb{R}^n$ is a smooth manifold with an atlas consisting of a single chart: the identity of $\mathbb{R}^n$.

Any open subset of $\mathbb{R}^n$ is also an n-dimensional differentiable manifold by Proposition 4.1.6. However, it may not be diffeomorphic to $\mathbb{R}^n$ and hence in general would not possess an atlas with single chart.

EXAMPLE 4.1.8. An interesting specific example of this kind is obtained by viewing the linear space of $n \times n$ matrices as $\mathbb{R}^{n^2}$. The condition $\det A \neq 0$ then defines an open set, hence a manifold (of dimension n^2), which is familiar as the *general linear group* $GL(n, \mathbb{R})$ of invertible $n \times n$ matrices.

EXERCISE 4.1.2. Construct an explicit diffeomorphism between $\mathbb{R}^n$ and the open unit ball B^n.

EXERCISE 4.1.3. Prove that any convex open set in $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

this exercise is better turned into a proposition; at least it requires an extensive hint

EXAMPLE 4.1.9. The standard sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ is a differentiable manifold. As charts modeled on the open ball one can take six appropriately chosen parallel projections of hemispheres to the coordinate planes. More economically, one gets a cover by two charts modeled on $\mathbb{R}^2$ by the two stereographic projections of the sphere from its north and south poles. As a forward reference we notice that if $\mathbb{R}^2$ is identified with $\mathbb{C}$ the latter method also provides $\mathbb{S}^2$ with the structure of one-demansional complex manifold (see Section 4.9.1).

EXAMPLE 4.1.10. The embedded torus

$$\mathbb{T}^2 = \left\{(x, y, z) \in \mathbb{R}^3 : \left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2 = 1\right\}$$

can be covered by overlapping pieces of parametrized surfaces

$$W \ni (u, v) \mapsto \big(x(u, v), y(u.v), z(u, v)\big) \in U \subset \mathbb{T}^2$$

whose inverses $U \to W$ (see the figure) constitute an atlas of $\mathbb{T}^2$, so it has the structure of a two-dimensional smooth manifold.

FIGURE ??? Chart on the embedded torus

EXERCISE 4.1.4. Using the square with identified opposite sides as the model of the torus, construct a smooth atlas of the torus with four charts homeomorphic to the open disk.

EXERCISE 4.1.5. Construct a smooth atlas of the projective space $\mathbb{R}P(3)$ with as few charts as possible.

EXAMPLE 4.1.11. The surface of a regular tetrahedron can be endowed with the structure of a two-dimensional smooth manifold by embedding it into 3-space, projecting it from its center of gravity G onto a 2-sphere of large radius centered at G, and pulling back the charts of the sphere to the surface.

Intuitively, there is something unnatural about this smooth structure, because the embedded tetrahedron has "corners", which are not "smooth" in the everyday sense. We will see below that a rigorous definition corresponds to this intuitive feeling: the embedded tetrahedron is not a "submanifold" of $\mathbb{R}^3$ (see Definition 4.2.1).

4.1.3. Manifolds defined by equations. Joint level sets of smooth functions into $\mathbb{R}$ or $\mathbb{R}^m$ corresponding to regular values are an interesting general class of manifolds. This is the most classical source of examples of manifolds.

Charts are provided by the implicit function theorem. Due to importance of this method we will give a detailed exposition here.

THEOREM 4.1.12 (Implicit Function Theorem). *Let $O \subset \mathbb{R}^n \times \mathbb{R}^m$ be open and $f\colon O \to \mathbb{R}^n$ a C^r map. If there is a point $(a, b) \in O$ such that $f(a, b) = 0$ and $D_1 f(a, b)$ is invertible then there are open neighborhoods $U \subset O$ of (a, b), $V \subset \mathbb{R}^m$ of b such that for every $y \in V$ there exists a unique $x =: g(y) \in \mathbb{R}^n$ with $(x, y) \in U$ and $f(x, y) = 0$. Furthermore g is C^r and $Dg(b) = -(D_1 f(a, b))^{-1} D_2 f(a, b)$.*

Proof of this theorem can be found in ??.

Examples are the sphere in $\mathbb{R}^n$ (which is the level set of one function, e.g. $F(x, y, z) = x^2 + y^2 + z^2$, for which 1 is a regular value) and the *special linear group* $SL(n, \mathbb{R})$ of $n \times n$ matrices with unit determinant. Viewing the space of $n \times n$ matrices as $\mathbb{R}^{n^2}$, we obtain $SL(n, \mathbb{R})$ as the manifold defined by the equation $\det A = 1$. One can check that 1 is a regular value for the determinant. Thus this is a manifold defined by one equation.

4.2. Principal constructions

Now we will look at how the notion of smooth manifold behaves with respect to the basic constructions. This will provide as with two principal methods of constructing smooth manifolds other than directly describing an atlas: embeddings as submanifolds, and projections into factor-spaces.

4.2.1. Submanifolds. In the case of a topological or a metric space, any subset automatically acquires the corresponding structure (induced topology or metric). For smooth manifolds, the situation is more delicate: arbitrary subsets of a smooth manifold do not necessarily inherit a differentiable structure from the ambient manifold. The following definition provides a natural generalization of the notion described in the previous subsection.

DEFINITION 4.2.1. A *submanifold* V of M (of dimension $k \le n$) is a differentiable manifold that is a subset of M such that the maximal atlas for M contains charts $\{(U, h)\}$ for which the restrictions $h_{\restriction U \cap V}$ map $U \cap V$ to $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ define charts for V compatible with the differentiable structure of V.

EXAMPLE 4.2.2. An open subset of a differentiable manifold M with the induced atlas as described in Proposition 4.1.6 is a submanifold of dimension n.

EXAMPLE 4.2.3. Let C be simple closed polygonal curve in $\mathbb{R}^2$ and let $h : C \to \mathbb{S}^1$ be a homeomorphism; then C acquires a smooth structure (via the atlas $\mathcal{A}_h$, the pullback by h of the standard atlas of $\mathbb{S}^1$). The curve C with this smooth structure is *not* a submanifold of the smooth manifold $\mathbb{R}^2$ (because of the "corners"). The same can be said of the tetrahedron embedded in 3-space, see Exercise 4.1.11.

EXERCISE 4.2.1. Prove that the n-dimensional torus in $\mathbb{R}^{2n}$:

$$x_{2k-1}^2 + x_{2k}^2 = \frac{1}{n}, \quad k = 1, \dots, n$$

is a smooth submanifold of the $(2n-1)$-dimensional sphere

$$\sum_{i=1}^{2n} x_i^2 = 1.$$

EXERCISE 4.2.2. Prove that the upper half of the cone

$$x^2 + y^2 = z^2, \;\; z \geq 0$$

is not a submanifold of $\mathbb{R}^3$, while the punctured one

$$x^2 + y^2 = z^2, \;\; z > 0$$

is a submanifold of $\mathbb{R}^3$.

Conversely, every smooth n-manifold can be viewed as a submanifold of R^N for a large enough N (see Theorem 4.5.1 and **??**).

4.2.2. Direct products. The Cartesian product of two smooth manifolds M and N of dimensions m and n automatically acquires the structure of an $(n+m)$–dimensional manifold in the following (natural) way. In the topological space $M \times N$, consider the atlas consisting of the products $U_i \times V_j$ of all pairs of charts of M and N with the natural local coordinates

$$l_{ij} := h_i \times k_j : U_i \times V_j \to \mathbb{R}^{m+n}.$$

It is easy to see that these charts are compatible and constitute an atlas of $M \times N$.

EXERCISE 4.2.3. Show that the smooth structure obtained on the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ in the above way coincides with that induced from the standard embedding of the torus in 3-space.

4.2.3. Quotient spaces. Identification spaces can also be smooth manifolds, for example, the unit circle viewed as $\mathbb{R}/\mathbb{Z}$, the torus as $\mathbb{R}^n/\mathbb{Z}^n$, or compact factors of the hyperbolic plane **??**.

properly discontinuous actions by diffeomorphisms (discrete); examples with continuous fibers (implicit function)

Note that, conversely, given a covering map of a smooth manifold, its smooth structure always lifts to a smooth structure of the covering space.

EXERCISE 4.2.4. Prove that the following three smooth structures on the torus $\mathbb{T}^2$ are equivalent, i.e. the torus provided with any of these structure is diffeomorphic to the one provided with another:

- $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ with the product structure;
- $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the factor-structure;
- The embedded torus of revolution in $\mathbb{R}^3$
$$\mathbb{T}^2 = \left\{(x, y, z) \in \mathbb{R}^3 : \left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2 = 1\right\}$$
with the submanifold structure.

4.3. Orientability and degree

4.3.1. Orientation and orientability.

4.3.2. Easy part of Sard theorem.

4.3.3. Degree for maps of compact orientable manifolds.

4.3.4. Calculation of $\pi_n(\mathbb{S}^n)$.

4.4. Paracompactness and partition of unity

An important result for analysis on manifolds is the fact that (using our assumption of second countability, that is, that there is a countable base for the topology) every smooth manifold admits a partition of unity (used below, in particular, to define the volume element of a manifold), which is defined as follows.

DEFINITION 4.4.1. A *partition of unity* subordinate to a cover $\{U_i\}$ of a smooth manifold M is a collection of continuous real-valued functions $\varphi_i : M \to [0, 1]$ such that

• the collection of functions φ_i is *locally finite*, i.e., any point $x \in M$ has a neighborhood V which intersects only a finite number of sets $\operatorname{supp}(\varphi_i)$ (recall that the support of a function is the closure of the set of points at which it takes nonzero values);
• $\sum_i \varphi_i(x) = 1$ for any $x \in M$;
• $\operatorname{supp}(\varphi) \subset U_i$ for all i.

PROPOSITION 4.4.2. *For any locally finite cover of a smooth manifold M, there exists a partition of unity subordinate to this cover.*

PROOF. Define the functions $g_i : M \to [0, 2^{-i}]$ by setting

if we need a SMOOTH partition of unity, I can give another proof

$$g_i := \min\{d(x, M \setminus U_i,\, 2^{-i}\},$$

where $d(\cdot\,,\,\cdot)$ denotes the distance between a point and a set and $\{U_i\}$ is the given cover of M. Then we have $g_i(x) > 0$ for $x \in U_i$ and $g_i(x) = 0$ for $x \notin U_i$. Further define

$$G(x) := \lim_{N\to\infty} G_N(x) = \lim_{N\to\infty} \sum_{i=1}^{N} f_i(x).$$

Since $\{U_i\}$ is a cover, it follows that $G(x) > 0$ for all $x \in M$.

Now put

$$f_i(x) := \max\Big\{g_i(x) - \frac{1}{3}G(x), 0\Big\}.$$

It is then easy to see that $\operatorname{supp}(f_i) \subset U_i$, and, since the cover $\{U_i\}$ is locally finite, so is the system of functions $\{f_i\}$.

Now let us show that

$$F(x) := \sum_{i=1}^{\infty} f_i(x) > 0 \quad \text{for all} x \in M,$$

i.e., for any $x \in M$ there is an i for which $f_i(x) > 0$. We do know that $g_j(x > 0)$ for some j and $g_n(x) < 2^{-n}$, hence $\sup_{j\in\mathbb{N}} g_j(x) = g_{i_0}(x)$ for a certain i_0 such that $g_{i_0} > 0$. The definition of the function $G(x)$ implies

$$G(x) = \sum_{j=0}^{\infty} 2^{-j} g_j(x) \le \sum_{j=0}^{\infty} 2^{-j} g_{i_0}(x) = 2g_{i_0}.$$

Therefore

$$f_{i_0}(x) \ge g_{i_0}(x) - \frac{2g_{i_0}(x)}{3} = \frac{g_{i_0}(x)}{3} > 0.$$

Now we can define the required partition of unity by setting

$$\varphi(x) := f_i(x)/F(x).$$

The proof of the facts that the φ_i are continuous, form a locally finite family, and add up to 1 at any point $x \in M$ is a straightforward verification that we leave to the reader. $\square$

A topological space X is called *paracompact* if a locally finite open cover can be inscribed in in any open cover of X.

PROPOSITION 4.4.3. *Any smooth manifold M is paracompact.*

PROOF. Let $\{U_i\}$ be an open cover of M, which we assume countable without loss of generality. Then the interiors of the supports of the functions φ_i obtained by the construction (which does not use the local finiteness of the covering $\{U_i\}$) in the proof of the previous proposition will form a locally finite open cover of M subordinated to $\{U_i\}$. $\square$

COROLLARY 4.4.4. *Any smooth manifold possesses a locally finite cover with a partition of unity subordinate to it.*

4.5. Embedding into Euclidean space

In this section we will prove that any compact differentiable manifold is diffeomorphic to a submanifold of a Euclidean space of a sufficiently high dimension.

THEOREM 4.5.1. *Any smooth compact manifold M^n can be smoothly embedded in Euclidean space $\mathbb{R}^N$ for sufficiently large N .*

Why only compact here? This is true for any manifold

PROOF. Since the manifold M^n is compact, it possesses a finite family of charts $f_i : U_i \to \mathbb{R}^n$, $i = 1, \dots, k$, such that

(1) the sets $f_i(U_i)$ are open balls of radius 2 centered at the origin of $\mathbb{R}^n$;

(2) the inverse images (denoted V_i) by f_i of the unit balls centered at the origin of $\mathbb{R}^n$ cover M^n.

We will now construct a smooth "cut off" function $\lambda : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lambda(x) = \begin{cases} 1 & \text{for} \quad \|y\| \le 1, \\ 0 & \text{for} \quad \|y\| \ge 2, \end{cases}$$

and $0 < \lambda(y) < 1$ for $1 < \|y\| < 2$. To do this, we first consider the function

$$\alpha(x) := \begin{cases} 0 & \text{for} \quad \|x\| \le 0, \\ e^{-1/x} & \text{for} \quad \|x\| > 0, \end{cases}$$

and then put $\beta(t) := \alpha(x-1)\alpha(2-x)$; the function β is positive on the open interval $(1, 2)$. Finally, we define

$$\gamma(\tau) := \Big(\int_\tau^2 \beta(t)dt\Big) \Big/ \Big(\int_1^2 \beta(t)dt\Big)$$

and put $\lambda(y) := \gamma(\|y\|)$. This function obviously satisfies the conditions listed above.

We set $\lambda_i(x) := \lambda(f_i(x))$ (see the figure).

FIGURE ??? The cut off function λ_i

Now let us consider the map $h : M^n \to \mathbb{R}^{(n+1)k}$ given by the formula

$$x \mapsto \big(\lambda_1(x),\, \lambda_1(x)f_1(x), \dots, \lambda_k(x), \lambda_k(x)f_k(x)\big).$$

The map h is one-to-one. Indeed, let $x_1, x_2 \in M^n$. Then x_1 belongs to V_i for some i, and two cases are possible: $x_2 \in V_i$ and $x_2 \notin V_i$. In the first case, we have $\lambda_i(x_1) = \lambda_i(x_2) = 1$, and therefore the relation

$$\lambda_i(x_1) f_i(x_1) = \lambda_i(x_2) f_i(x_2)$$

is equivalent to $f_i(x_1) = f_i(x_2)$ and so $x_1 = x_2$. In the second case (when $x_2 \notin V_i$), we have $\lambda_i(x_1) = 1$ while $\lambda_i(x_2) < 1$, and so $h(x_1) \neq h(x_2)$.

Now the restriction of the map $x \mapsto \lambda_i(x) f_i(x)$ to U_i is an immersion (i.e., at any point its Jacobian is of rank n), because the inclusion $x \in U_i$ implies $\lambda_i(x) = 1$), while the map $x \mapsto f_i(x)$ is a local diffeomorphism. Hence the map h is also an immersion.

But we know (see ??) that any one-to-one map of a compact space into a Hausdorff space (in our case $h : M^n \mathbb{R}^{(n+1)k}$) is a homeomorphism onto its image. Thus h is a smooth embedding into $\mathbb{R}^{(n+1)k}$. $\square$

4.6. Derivatives and the tangent bundle

4.6.1. Derivations as classes of curves. Recall that the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}$ in the direction of a vector $v = (v_1, \dots, v_n)$ is defined in calculus courses as

$$D_v(f) := v_1 \frac{\partial f}{\partial x_1} + \cdots + v_n \frac{\partial f}{\partial x_n}.$$

Derivations form a linear space of dimension n whose canonical basis is constituted by the partial derivatives

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}.$$

In order to give a similar definition of the derivative of a function on a smooth manifold, we must, first of all, define what we mean by the direction along which we differentiate. We will do this by defining tangent vectors as equivalence classes of curves. The underlying intuitive consideration is that curves passing through a point are viewed as trajectories, two curves being regarded as equivalent if the "velocity of motion" at the chosen point is the same.

DEFINITION 4.6.1. Let M be a C^∞ manifold and $p \in M$. Consider curves $c\colon (a, b) \to M$, where $a < 0 < b$, $c(0) = p$ such that $h \circ c$ is differentiable at 0 for one (hence any) chart (U, h) with $p \in U$. Each such curve c passing through the point p assigns to each function $f \in C^\infty(M)$ the real number

$$D_{c,p}(f) := \frac{d}{dt}\big(f(c(t))\big|_{t=0},$$

the *derivative of f at p along c*. Two curves c' and c'' are called *equivalent* if in some chart (U, h) (and hence, by compatibility, in all charts) containing p, we have

$$\frac{d}{dt}\big(h(c'(t))\big|_{t=0} = \frac{d}{dt}\big(h(c''(t))\big|_{t=0}.$$

An equivalence class of curves at the point p is called a *tangent vector* to M at p and denoted by $v = v(c)$, where c is any curve in the equivalence class. The *derivative of f in the direction of the vector v* can now be (correctly!) defined by the formula

$$D_{v,p}(f) := \frac{d}{dt}\big(f(c(t))\big|_{t=0}, \quad \text{for any } c \in v.$$

The space of all the derivations at p (i.e., equivalence classes of curves at p) obtained in this way, has a linear space structure (since each derivation is a real-valued function) which turns out to have dimension n. It is called the *tangent space* at p of M and denoted T_pM.

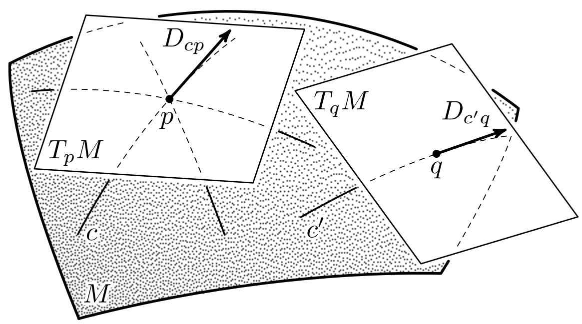

FIGURE 4.6.1. Tangent spaces to a manifold

Given a specific chart (U, h), we define the standard basis

$$\frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p$$

of T_pM by taking the canonical basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ and setting

$$\frac{\partial}{\partial x_i}\Big|_p(f) := \frac{d}{dt}\big(f(c_i(t))\big)\Big|_{t=0}, \quad \text{where } c_i(t) = h^{-1}(h(p) + te_i)$$

for all $i = 1, \dots, n$.

4.6.2. Derivations as linear operators. Another intrinsic way of defining derivatives, more algebraic than the geometric approach described in the previous subsection, is to define them by means of linear operators satisfying the Leibnitz rule.

DEFINITION 4.6.2. Let p be a point of a smooth manifold M. A *derivation* of $C^\infty(M)$ at the point p is a linear functional $D : C^\infty(M) \to \mathbb{R}$ satisfying the Leibnitz rule, i.e.,

$$D(f \cdot g) = Df \cdot g(p) + f(p) \cdot Dg.$$

The derivations at p (in this sense) obviously constitute a linear space. If we choose a fixed chart (U, h) with coordinates $(x_1, \dots, x_n)$ containing p, then we can determine a basis $(\partial_1, \dots, \partial_n)$ of this space by setting

$$D_1(f) := \frac{\partial}{\partial x_1}(h \circ f)\Big|_{f(p)}, \dots, D_n(f) := \frac{\partial}{\partial x_2}(h \circ f)\Big|_{f(p)};$$

here $\partial/\partial x_i$ denotes the usual partial derivative in the target space $\mathbb{R}^n$ of our chart h.

EXERCISE 4.6.1. Prove that the linear space of derivations can be identified with the tangent space $T_p(M)$ defined in the previous subsection, so that the derivations defined above are nothing but tangent vectors and the basis $\{D_i\}$ can be identified with the basis $\{(\partial/\partial x_i)|_p\}$.

REMARK 4.6.3. Note that the definition of derivation given in this subsection yields a purely algebraic approach to the differential calculus on smooth manifolds: none of the classical tools of analysis (e.g. limits, continuity via the $\varepsilon-\delta$ language, infinite series, etc.) are involved.

4.6.3. The tangent bundle. We define the *tangent bundle* of M to be the disjoint union

$$TM := \bigcup_{p \in m} T_p M$$

of the tangent spaces with the *canonical projection* $\pi \colon TM \to M$ given by $\pi(T_p M) = \{p\}$. Any chart (U, h) of M then induces a chart

$$\Big(U \times \bigcup_{p \in U} T_p U, H\Big), \quad \text{where} \quad H(p, v) := (h(p), (v^1, \dots, v^n)) \in \mathbb{R}^n \times \mathbb{R}^n;$$

here the v^i are the coefficients of $v \in T_p M$ with respect to the basis

$$\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}$$

of $T_p M$. In this way TM is a differentiable manifold (of dimension 2) itself.

A *vector field* is a map $X \colon M \to TM$ such that $\pi \circ X = \mathrm{Id}_M$, that is, X assigns to each p a tangent vector at p. We denote by $\Gamma(M)$ the space of *smooth* vector fields on M, i.e., vector fields defined by a smooth map of the manifold M to the manifold TM. Thus smooth vector fields determine operators (that we will sometimes denote by D_X) on $C^\infty(M)$ by acting on functions via derivations, i.e., $D_X(f) := X(p)(f)$.

We shall see later that $\pounds_v w := [v, w] := vw - wv$ also acts on functions by derivations, that is, as a vector field, and we call $[v, w]$ the *Lie bracket* of v and w and $\pounds_v$ the *Lie derivative* .

4.7. Smooth maps and the tangent bundle

As we already noted, smooth manifolds, like any other self-respecting mathematical objects, form a category: we defined their morphisms (called smooth maps) and their "isomorphisms" (called diffeomorphisms) at the beginning of the present chapter. We now return to these notions and look at them from the perspective of tangent bundles.

4.7.1. Main definitions. We now define the morphisms of the differentiable structure.

DEFINITION 4.7.1. Let M and N be differentiable manifolds. Recall that a map $f\colon M \to N$ is said to be *smooth* if for any charts (U,h) of M and (V,g) of N the map $g \circ f \circ h^{-1}$ is differentiable on $h(U \cap f^{-1}(V))$.

A smooth map f acts on derivations by sending curves $c\colon (a,b) \to M$ to $f \circ c\colon (a,b) \to N$. Differentiability means that curves inducing the same derivation have images inducing the same derivation. Thus we define the differential of f to be the map

$$Df\colon TM = \bigcup_{p\in M} T_pM \to TN = \bigcup_{q\in N} T_qN$$

that takes each vector $v \in T_pM$ determined by a curve c to the vector $w \in T_{f(p)}$ given by the curve $f \circ c$. It is easy to deduce from the definition of equivalence of curves (see ??) that the definition of w does not depend on the choice of curve $c \in v$. The restriction of Df to T_pM (which takes T_pM to $T_{f(p)}N$) is denoted by $Df\big|_p$.

A *diffeomorphism* is a differentiable map with differentiable inverse. Two manifolds M, N are said to be *diffeomorphic* or *diffeomorphically equivalent* if and only if there is a diffeomorphism $M \to N$. An *embedding*of a manifold M in a manifold N is a diffeomorphism $f\colon M \to V$ of M onto a submanifold V of N. We often abuse terminology and refer to an embedding of an open subset of M into N as a (*local*) *diffeomorphism* as well. An *immersion* of a manifold M into a manifold N is a differentiable map $f\colon M \to V$ onto a subset of N whose differential is injective everywhere.

4.7.2. Examples. Smooth maps must be compatible, in a sense, with the differentiable structure of the source and target manifolds. As we shall see, not all naturally defined maps (e.g. some projections) have this property.

EXAMPLE 4.7.2. The orthogonal projection on the (x,y)-plane of the standard unit sphere $x^2 + y^2 + z^2 = 1$ is not a smooth map.

Further, even injectively immersed manifolds may fail to be smooth submanifolds.

EXAMPLE 4.7.3. Choose a point on the standard embedding of the torus $\mathbb{T}^2$ and consider a curve passing through that point and winding around $\mathbb{T}^2$ with irrational slope (forming the same irrational angle at all its intersections with the parallels of the torus). In that way, we obtain a (dense) embedding of $\mathbb{R}$ into $\mathbb{T}^2$, which is a smooth map *locally*, but is *not* a smooth map of $\mathbb{R}$ to $\mathbb{T}^2$.

Clearly, diffeomorphic manifolds are homeomorphic. The converse is, however, not true. As we mentioned above, there are "exotic" spheres and other manifolds whose smooth structure is not diffeomorphic to the usual smooth structure.

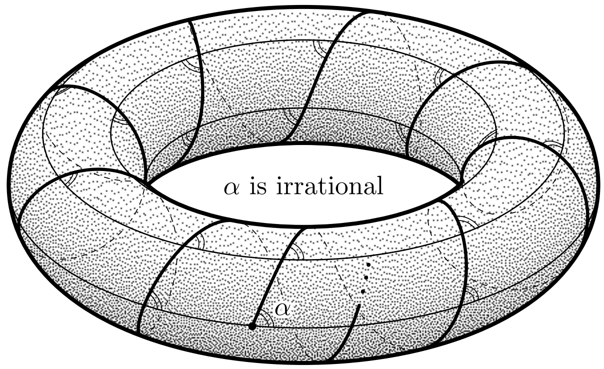

FIGURE 4.7.1. Dense embedded trajectory on the torus

EXAMPLE 4.7.4. In the space $\mathbb{R}^9$ with coordinates $(x_1, \dots, x_9)$, consider the cone C given by

$$x_1^7 + 3x_7^4 x_2^3 + x_5^6 x_6 = 0$$

and take the intersection of C with the standard unit 8-sphere $\mathbb{S}^8 \subset \mathbb{R}^9$. The intersection $\Sigma := C \cap \mathbb{S}^8$ is clearly homeomorphic to the 7-sphere. It turns out that Σ with the smooth structure induced on from $\mathbb{R}^9$ is not diffeomorphic to $\mathbb{S}^7$ with the standard smooth structure. (The proof of this fact lies beyond the scope of the present book.)

4.8. Manifolds with boundary

The notion of real smooth manifold with boundary is a generalization of the notion of real smooth manifold obtained by adding the half-space

$$\mathbb{R}^n_+ := \{(x_1, \dots, x_n \in \mathbb{R}^n \mid x_n \geq 0\}$$

to $\mathbb{R}^n$ as the possible target space of the charts (U_i, h_i); we must also appropriately modify the compatibility condition: we now require that, whenever $U_r \cap U_s \neq \varnothing$, there must exist two mutually inverse diffeomorphisms $\varphi_{r,s}$ and $\varphi_{s,r}$ of *open* sets in $\mathbb{R}^n$ whose restrictions are $h_r \circ h_s^{-1}$ and $h_s \circ h_r^{-1}$. (The necessity of such a version of the compatibility condition is in that smooth maps are defined only on open subsets of $\mathbb{R}^n$, whereas an open set in $\mathbb{R}^n_+$, e.g. $h_i(U_i)$, may be non open in $\mathbb{R}^n$.)

If M is a smooth manifold with boundary, then it has two types of points: the *interior* points (those contained in only in those charts (U_i, h_i) for which $U_i \subset M$ is open) and the *boundary* points (those not contained in any such charts). It seems obvious that the *boundary* ∂M of a manifold with boundary (i.e., the set of its boundary points) coincides with the set

$$\bigcup_j h_j^{-1}((x_1, \dots, x_{n-1}, 0)),$$

where the intersection is taken over only those h_j whose target space is $\mathbb{R}^n_+$. However, this fact is rather nontrivial, and we state it as a lemma.

LEMMA 4.8.1. *The two definitions of boundary point of a manifold with boundary coincide.*

PROOF. We need to prove that any point contained in an *open* chart U_i cannot be mapped by h_i to a boundary point $(x_1, \dots, x_{n-1}, 0) \in \mathbb{R}^n_+$. This can be done by using the inverse function theorem. We omit the details. $\square$

Sometimes, in order to stress that some M is an ordinary manifold (not a manifold with boundary), we will say that M is a "manifold without boundary". It may happen that the set of boundary points of a manifold with boundary M is empty. In that case, all the charts of its maximal atlas targeted to $\mathbb{R}^n_+$ are in fact redundant; deleting them, we obtain a smooth manifold without boundary.

PROPOSITION 4.8.2. *The set of boundary points ∂M of a manifold with boundary has the natural structure of a smooth $(n-1)$-dimensional manifold (without boundary).*

PROOF. An atlas for ∂M is obtained by taking the restrictions of the charts h_i to the sets $h_i^{-1}(h_i(U_i) \cap \mathbb{R}^n_+)$. $\square$

4.9. Complex manifolds

4.9.1. Main definitions and examples. Complex manifolds are defined quite similarly to real smooth manifolds by considering charts with values in $\mathbb{C}^n$ instead of $\mathbb{R}^n$ and requiring the coordinate changes between charts to be holomorphic. Since holomorphic maps are much more rigid that differentiable maps, the resulting theory differs from the one above in several aspects. For example the one–dimensional complex manifolds (Riemann surfaces) is a much richer subject than one- and even two-dimensional differentiable manifolds.

Complex manifolds form a category, the natural notion of morphism $\varphi : M \to N$ being defined similarly to that of smooth map for their real counterparts, except that the maps $k \circ \varphi h^{-1}$ (where h and k are charts in M and N) must now be holomorphic rather than differentiable.

In this course, we do not go deeply into the theory of complex manifolds, limiting our study to some illustrative examples.

EXAMPLE 4.9.1. The *Riemann sphere*, $\mathbb{C} \cup \{\infty\}$, which is homeomorphic to S^2, becomes a one–dimensional complex manifold by considering an atlas of two charts $(\mathbb{C}, \mathrm{Id})$ and $(\mathbb{C} \cup \{\infty\} \setminus \{0\}, I)$, where

$$I(z) = \begin{cases} 1/z & \text{if } z \in \mathbb{C} \\ 0 & \text{if } z = \infty \end{cases}.$$

EXERCISE 4.9.1. Identify $\mathbb{R}^2$ with $\mathbb{C}$ and define the torus $\mathbb{T}^2$ as the quotient space $\mathbb{C}/\mathbb{Z}^2$.

EXERCISE 4.9.2. Describe a complex atlas for the complex projective space $\mathbb{C}P^n$.

EXERCISE 4.9.3. Describe a complex atlas for the group $U(n)$ of unitary matrices

4.9.2. Riemann surfaces. An attractive showcase of examples of complex manifolds comes from *complex algebraic curves* (or *Riemann surfaces*, as they are also called), which are defined as zero sets of complex polynomials of two variables in the space $\mathbb{C}^2$.

More precisely, consider the algebraic equation

$$p(z,w) := a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0, \quad a_0(z) \neq 0, \tag{4.9.1}$$

where the $a_i(z)$ are polynomials in the complex variable $z \in \mathbb{C}$ with complex coefficients and $w = w(z)$ is an unknown complex-valued function.

Already in the simplest cases (e.g. for $w^2 - z = 0$), this equation does not have a univalent analytic solution $w : \mathbb{C} \to \mathbb{C}$ defined for all $z \in \mathbb{C}$. However, as Riemann noticed, such a solution exists provided we replace the domain of definition of the solution by an appropriately chosen surface that we will now define.

To do this, it will be convenient to replace $\mathbb{C}$ by its natural compactification $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, the *Riemann sphere*, which is of course homeomorphic to the ordinary sphere $\mathbb{S}^2$). We now regard equation (4.5.1.) as given on $\bar{\mathbb{C}} \times \bar{\mathbb{C}}$ and define the corresponding *Riemann surface* S_p as the set of zeros of this equation, i.e., as

$$S_p := \big\{(z,w) \in \bar{\mathbb{C}} \times \bar{\mathbb{C}} \,|\, p(z,w) = 0\big\}.$$

Now the projection (given by the assignment $(z,w) \mapsto w$) of S_p on the second factor of the product $\bar{\mathbb{C}} \times \bar{\mathbb{C}}$ is by definition univalent, so that on the Riemann surface S_p equation (4.5.1.) defines a single-valued function $w = w(z)$.

It is of course difficult to visualize Riemann surfaces, which are two-dimensional objects embedded in a four-dimensional manifold homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$, but we will see that there is an effective geometric construction that, given $p(z,w)$, specifies the topological structure of S_p.

We will now consider several examples of this construction.

EXAMPLE 4.9.2. Consider the equation

$$p(z) := w^2 - z = 0.$$

Obviously, there are two values of w that satisfy this equation for a fixed (nonzero) value of $z = re^{\varphi}$, namely $w_1 = +\sqrt{r}\, e^{i\varphi/2}$ and $w_2 = -\sqrt{r}\, e^{i\varphi/2}$. These determine the two "sheets" of the solution; when we go around the origin of the z-plane, we "jump" from one sheet to the other. Let us cut the z-plane along the real axis, or more precisely cut the Riemann sphere $\bar{\mathbb{C}}$ along the arc arc of the great circle joining the points 0 and ∞. Take another copy of $\bar{\mathbb{C}}$ (which will be the second sheet of our Riemann surface), make the same cut joining 0 and ∞, and identify the "shores" of the cuts (see the figure below).

Thus we see that the Riemann surface of the equation under consideration is the sphere.

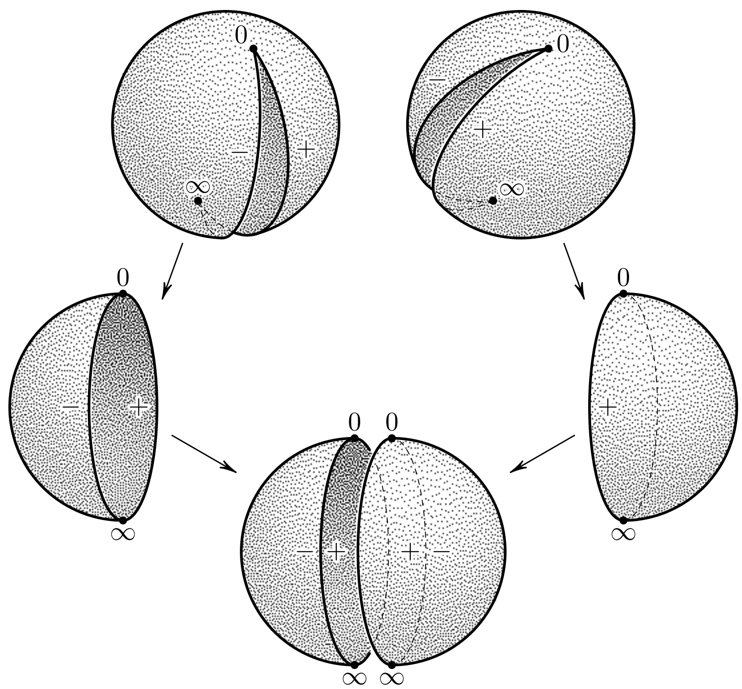

FIGURE 4.9.1. The Riemann surface of a polynomial linear in z

EXERCISE 4.9.4. Show that the Riemann surface of the quadratic equation

$$w^2 - (z - a_1)(z - a_2) = 0,$$

where a_1 and a_2 are distinct complex numbers, is the sphere $\S^2_b$.

EXAMPLE 4.9.3. Consider the cubic equation

$$q(z, w) := w^2 - (z - a_1)(z - a_2)(z - a_3) = 0,$$

where a_1, a_2, a_3 are distinct complex numbers. This function also has two sheets, but the passage from one sheet to the other is more complicated than in the previous example: if we circle around one of the points a_1, a_2, a_3, or ∞, we pass from one sheet to the other, if we circle around any two of them, we stay on the same sheet, if we circle around three, we switch sheets again. To obstruct these switches, we perform cuts along the arcs a_1a_2 and $a_3\infty$ on two copies of the Riemann sphere and glue the two copies along the shores of the cuts. The construction is shown on the figure.

The result will clearly be homeomorphic to the torus.

EXERCISE 4.9.5. Find a polynomial whose zero set is a complex curve homeomorphic to the sphere with two handles.

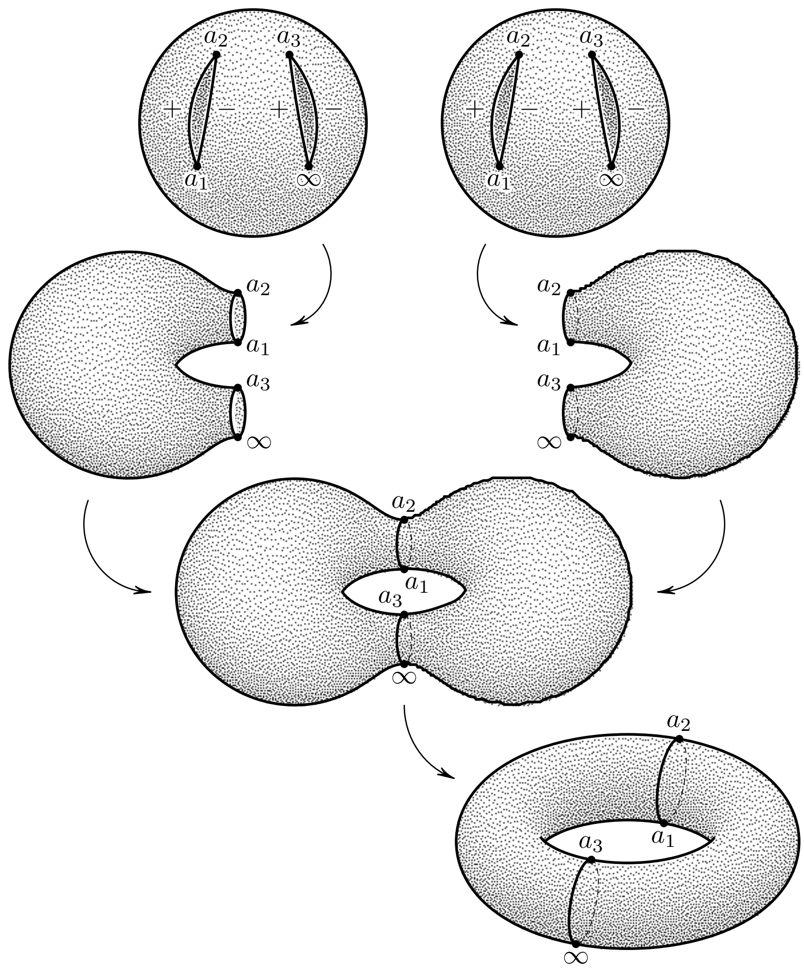

FIGURE 4.9.2. The Riemann surface of a polynomial cubic in z

4.10. Lie groups: first examples

DEFINITION 4.10.1. An *n-dimensional Lie group* is an n-dimensional smooth manifold G with a group operation such that the product map $G \times G \to G\colon (x, y) \mapsto xy$ and the inverse map $G \to G\colon x \mapsto x^{-1}$ are differentiable.

Lie groups G and H are *isomorphic* is there exists a group isomorphism $i\colon G \to H$ which is at the same time a diffeomorphism between smooth manifolds.

A *Lie subgroup* of a Lie group G is a smooth submanifold H of G which is also a subgroup. [2]

Lie groups form one of the most important and interesting classes of smooth manifolds. Here we discuss few examples of the classical Lie groups and mention

[2]In fact, any closed subgroup of a Lie group is a Lie subgroup. This is one of the fundamental results of the Lie group theory which is used quite often. Its proof is far from elementary.

some of their properties. More systematic study of Lie groups in their connection to geometry and topology will be presented in Chapter 11.

Notice that any groups with discrete topology is a zero-dimensional Lie group. Direct product of Lie groups also has natural Lie group structure. Thus in the structural theory of Lie groups interest in concentrated primarily on connected Lie groups. However *discrete subgroups* of connected Lie groups are of great interest.

Abelian Lie groups have rather simple structure. First, $\mathbb{R}^n$ with addition as the group operation is a Lie group. All its closed subgroups and factor-groups by closed subgroups are also Lie groups. Proofs of those facts will be given in Chapter 11. Now we consider natural examples.

EXAMPLE 4.10.2. Any linear subspace of $\mathbb{R}^n$ is a Lie subgroup isomorphic to $\mathbb{R}^k$ for some $k < n$.

The integer lattice $\mathbb{Z}^k \subset \mathbb{R}^k \subset R^n$ is a discrete subgroup and the factor group $\mathbb{R}^n/\mathbb{Z}^k$ is isomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$ and is a Lie group. In particular the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a compact connected abelian Lie group.

EXERCISE 4.10.1. Prove that the group $\mathbb{C}^*$ of non-zero complex numbers with multiplication as group operation is isomorphic to $\mathbb{R} \times \mathbb{S}^1$.

The group $GL(n, \mathbb{R})$ is the group of all invertible $n \times n$ matrices with differentiable structure inherited from its representation as the open subset of $\mathbb{R}^{n^2}$ determined by the condition $\det A \neq 0$ as in Example 4.1.8. Those groups play in the theory of Lie groups role somewhat similar to that played by the Euclidean spaces in the theory of differentiable manifolds. Many manifolds naturally appear as submanifolds of $\mathbb{R}^n$ and many more are diffeomorphic to submanifolds of $\mathbb{R}^n$ (see Theorem 4.5.1). The situation with Lie groups is similar. Most Lie groups naturally appear as Lie subgroups of $GL(n, \mathbb{R})$; such groups are called *linear groups*.

EXAMPLE 4.10.3. The *orthogonal group* $O(n)$ consists of all matrices A satisfying $AA^t = \mathrm{Id}$. Here the superscript t indicates transposition. It consists of two connected components according to the value of the determinant: +1 or -1. The former is also a group which is usually called the *special orthogonal group* and is denoted by $SO(n)$.

EXERCISE 4.10.2. Prove that $SO(2)$ is isomorphic to $\mathbb{S}^1$.

EXERCISE 4.10.3. Prove that $O(n)$ consist of matrices which represent all isometries of the Euclidean space $\mathbb{R}^n$ fixing the origin, or, equivalently, all isometries of the the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

Many geometric structures naturally give rise to Lie groups, namely groups of transformations preserving the structure. In the example above the structure was the standard symmetric Riemannian metric on the sphere with $O(n)$ as the group of isometries. An even more basic example is given by $GL(n, \mathbb{R})$, the group of automorphisms of $\mathbb{R}^n$, the structure being that of linear space.

However, one needs to be cautious: this happens if for the group of transformations preserving the structure is *finite–dimensional*. For example, if one considers

$\mathbb{R}^n$ as a smooth manifold its automorphism group, the group of all diffeomorphisms, is not a Lie group.

EXERCISE 4.10.4. Identify isometries of the Euclidean plane with certain 3×3 matrices and prove that they form a linear group. Calculate its dimension.

For representation of groups of Euclidean isometries and affine transformations as linear groups see Exercise 4.11.14 and Exercise 4.11.15.

Notice that projective structure does not give as much new it terms of its group of automorphisms: projective transformations of $\mathbb{R}P(n)$ are simply linear transformations of $\mathbb{R}^{n+1}$. However, scalar matrices act identically on $\mathbb{R}P(n)$ so the group of projective transformations is not simply $GL(n+1,\mathbb{R})$ but its factor group.

If n is even and hence $n+1$ is odd one can find unique transformation with determinant one in each equivalence class, simply my multiplying all elements of a given matrix by the $(n+1)$ root of its determinant. hence in this case the group of projective transformations is isomorphic to $SL(n+1,\mathbb{R})$.

If n is odd the above procedure only works for matrices with positive determinant but it still leaves one non-identity matrix acting as identity, namely $-\operatorname{Id}$ which has determinant one in this case. On the other hand, matrices with negative determinant can be reduced to those with determinant -1, again with a similar identification. Thus the group of projective transformations in this case has a factor goup of index two which is isomorphic to $PSL(n+1,\mathbb{R}){:=}SL(n+1,\mathbb{R})/\{\pm\operatorname{Id}\}$.

EXAMPLE 4.10.4. The group $GL(n,\mathbb{C})$ of invertible $n\times n$ matrices with complex entries is a Lie group since it is an open subset $\det A\neq 0$ in the space of all $n\times n$ complex matrices which is isomorphic to $\mathbb{R}^{2n^2}$.

It is also a linear group since every complex number $a+bi$ can be identified with 2×2 real matrix $\begin{pmatrix} a & b\\ -b & a\end{pmatrix}$ and any $n\times n$ complex matrix can be associated with an $2n\times 2n$ real matrix by substituting each matrix element with the corresponding 2×2 matrix. This correspondence preserves addition and multiplication.

Its Lie subgroup $SL(n,\mathbb{C})$ consists of matrices with determinant one.

The group $GL(n,\mathbb{C})$ can be interpreted as the group of linear automorphisms of $\mathbb{R}^{2n}$ preserving and extra structure which in complex form corresponds to the multiplication of all coordinates of a vector by i.

EXAMPLE 4.10.5. The group $U(n)$ appears as groups of transformations of the space $\mathbb{C}^n$ preserving the *Hermitian product* $\sum_{i=1}^n z_i\bar{w}_i$ for $z=(z_1,\dots,z_n),\ w=(w_1,\dots,w_n)$. It is embedded into $GL(n,\mathbb{C})$ as the Lie subgroup of matrices A such that $AA^*\operatorname{Id}$. Here A^* is the matrix conjugate to A: its (i,j) matrix element is equal to the complex conjugate to the (j,i) element of A.

EXAMPLE 4.10.6. The symplectic group of $2n\times 2n$ consists of matrices A satisfying

$$AJA^t=J,\ \text{ where }\ J=\begin{pmatrix} 0 & \operatorname{Id}\\ -\operatorname{Id} & 0\end{pmatrix},$$

4.11. Problems

The next exercises are examples of smooth manifolds. Many examples of manifolds are given by configuration and phase spaces of mechanical systems. One can think of the configuration space of a mechanical system as a topological space whose points are different "positions" of the system, and neighborhoods are "nearby" positions (i.e., positions that can be obtained from the given one by motions of "length" smaller than a fixed number). The phase space of a mechanical system moving in time is obtained from its configuration space by supplying it with all possible velocity vectors.

EXERCISE 4.11.1. Describe the configuration space of the mechanical system consisting of a rod rotating in space about a fixed hinge at its extremity. What configuration space is obtained if the hinge is fixed at the midpoint of the rod?

EXERCISE 4.11.2. The *double pendulum* consists of two rods AB and CD moving in a vertical plane, connected by a hinge joining the extremities B and C, while the extremity A is fixed by a hinge in that plane. Find the configuration space of this mechanical system.

EXERCISE 4.11.3. On a round billiard table, a pointlike ball moves with uniform velocity, bouncing off the edge of the table according to the law saying that the angle of incidence is equal to the angle of reflection. Find the phase space of this system.

EXERCISE 4.11.4. Show that the configuration space of an asymetric solid rotating about a fixed hinge in 3-space is $\mathbb{R}P^3$.

EXERCISE 4.11.5. In $\mathbb{R}^9$ consider the set of points satisfying the following system of algebraic equations:

$$\begin{array}{ll} x_1^2+x_2^2+x_3^2=1; & x_1x_4+x_2x_5+x_3x_6=0; \\ x_4^2+x_5^2+x_6^2=1; & x_1x_7+x_2x_8+x_3x_9=0; \\ x_1^2+x_8^2+x_9^2=1; & x_4x_7+x_5x_8+x_6x_9=0. \end{array}$$

Show that this set is a smooth 3-dimensional submanifold of $\mathbb{R}^9$ and describe it. (Solution sets of systems of algebraic equations are not necessarily smooth manifolds: they may have *singularities*.)

EXERCISE 4.11.6. Show that the topological spaces obtained by identifying diametrically opposed points of the 3-sphere $\mathbb{S}^3$ and by identifying diametrically opposed boundary points of the 3-disk

$$\mathbb{D}^3 := \left\{(x_1,x_2,x_3)\in\mathbb{R}^3 \,|\, x_1^2+x_2^2+x_3^2\le 1\right\}$$

have a natural smooth manifold structure and are homeomorphic to each other.

EXERCISE 4.11.7. Seven rods of length 1 in the plane are joined end to end by hinges, and the two "free" ends are fixed to the plane by hinges at the distance 6.5 from each other. Find the configuration space of this mechanical system.

EXERCISE 4.11.8. Five rods of length 1 in the plane are joined end to end by hinges, and the two "free" ends are fixed to the plane by hinges at the distance 1 from each other. Find the configuration space of this mechanical system.

EXERCISE 4.11.9. Prove that the group $O(2)$ of orthogonal transformations of the plane is not isomorphic to $\mathbb{S}^1 \times C_2$.

define standard notation for cyclic groups

EXERCISE 4.11.10. Prove that the Lie group $SO(3)$ is diffeomorphic to the real projective space $\mathbb{R}P(3)$.

EXERCISE 4.11.11. Prove that the Lie group $SU(2)$ is diffeomorphic to the sphere $\mathbb{S}^3$.

EXERCISE 4.11.12. Represent the torus $\mathbb{T}^n$ as a linear group.

EXERCISE 4.11.13. What is the minimal value of m such that $\mathbb{T}^n$ is isomorphic to a Lie subgroup of $GL(m, \mathbb{R})$?

EXERCISE 4.11.14. Prove that the group of Euclidean isometries of of $\mathbb{R}^n$ is isomorphic to a Lie subgroup of $GL(n+1, \mathbb{R})$. Calculate its dimension.

EXERCISE 4.11.15. Prove that the group of affine transformations of $\mathbb{R}^n$ is isomorphic to a Lie subgroup of $GL(n+1, \mathbb{R})$. Calculate its dimension.