CHAPTER 5

TOPOLOGY AND GEOMETRY OF SURFACES

Compact (and some noncompact) surfaces are a favorite showcase for various branches of topology and geometry. They are two-dimensional topological manifolds, which can be supplied with a variety of naturally defined differentiable and Riemannian structures. Their complete topological classification, which coincides with their smooth (differentiable) classification, is obtained via certain simple invariants. These invariants allow a variety of interpretations: combinatorial, analytical and geometrical.

Surfaces are also one-dimensional complex manifolds; but, surprisingly, the complex stuctures are not all equivalent (except for the case of the sphere), although they can be classified. This classification if the first result in a rather deep area at the junction of analysis, geometry, and algebraic geometry known as *Teichmüller theory*, which recently has led to spectacular applications in theoretical physics.

In this chapter we study the classification of compact surfaces (two-dimensional manifolds) from various points of view. We start with a fundamental preparatory result, which we will prove by using a beautiful argument based on combinatorial considerations.

5.1. Two big separation theorems: Jordan and Schoenflies

The goal of this section is to prove the famous Jordan Curve Theorem, which we will need in the next section, and which is constantly used in many areas of analysis and topology. Note that although the statement of the theorem seems absolutely obvious, it does not have a simple proof.

5.1.1. Statement of the theorem and strategy of proof. Here we state the theorem and outline the main steps of the proof.

DEFINITION 5.1.1. A simple closed curve on a manifold M (in particular on the plane $\mathbb{R}^2$) is the homeomorphic image of the circle $\mathbb{S}^1$ in M, or equivalently the image of $\mathbb{S}^1$ under a topological embedding $\mathbb{S}^1 \to M$.

THEOREM 5.1.2 (Jordan Curve Theorem). *A simple closed curve C on the plane $\mathbb{R}^2$ separates the plane into two connected components.*

COROLLARY 5.1.3. *A simple closed curve C on the sphere $\mathbb{S}^2$ separates the sphere into two connected components.*

PROOF. The proof is carried out by a simple but clever reduction of the Jordan Curve Theorem to the nonplanarity of the graph $K_{3,3}$, established in ??

Suppose that C is an arbitrary (not necessarily polygonal) simple closed curve in the plane $\mathbb{R}^2$. Suppose l and m are parallel support lines of C and p is a line perpendicular to them and not intersecting the curve. Let A_1 and A_2 be points of the intersections of C with l and m, respectively. Further, let B_3 be the intersection point of l and p. The points A_1 and A_2 divide the curve C into two arcs, the "upper" one and the "lower" one. Take a line q in between l and m parallel to them. By compactness, there is a lowest intersection point B_1 of q with the upper arc and a highest intersection point B_2 of q with the lower arc. Let A_3 be an inner point of the segment $[B_1, B_2]$ (see the figure).

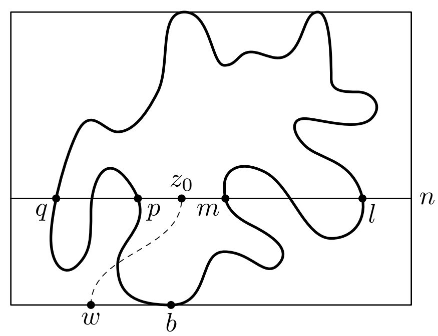

FIGURE 5.1.1. Proof of the Jordan Curve Theorem

We claim that $\mathbb{R}^2 \setminus C$ is not path connected, in fact there is no path joining A_3 and B_3. Indeed, if such a path existed, by Lemma **??** there would be an arc joining these two points. Then we would have nine pairwise nonintersecting arcs joining each of the points A_1, A_2, A_3 with all three of the points B_1, B_2, B_3. This means that we have obtained an embedding of the graph $K_{3,3}$ in the plane, which is impossible by Theorem 5.2.4. $\square$

5.1.2. Schoenflies Theorem. The Schoenflies Theorem is an addition to the Jordan curve theorem asserting that the curve actually bounds a disk. We state this theorem here without proof.

THEOREM 5.1.4 (Schoenflies Theorem). *A simple closed curve C on the plane $\mathbb{R}^2$ separates the plane into two connected components; the component with bounded closure is homeomorphic to the disk, that is,*

$$\mathbb{R}^2 \setminus C = \mathcal{D}_1 \cup \mathcal{D}_2, \text{ where } \mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing \text{ and } \overline{\mathcal{D}_1} \approx \mathbb{D}^2.$$

COROLLARY 5.1.5. *A simple closed curve C on the sphere $\mathbb{S}^2$ separates the sphere into two connected components, each of which has closure homeomorphic to the disk, that is,*

$$\mathbb{S}^2 \setminus C = \mathcal{D}_1 \cup \mathcal{D}_2, \quad \text{where } \mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing \text{ and } \overline{\mathcal{D}_i} \approx \mathbb{D}^2,\ i = 1, 2.$$

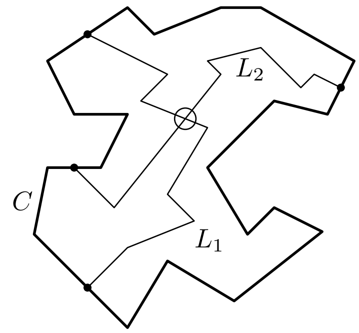

FIGURE 5.2.1. The polygonal lines L_1 and L_2 must intersect

5.2. Planar and non-planar graphs

5.2.1. Non-planarity of $K_{3,3}$. We first show that the graph $K_{3,3}$ has no polygonal embedding into the plane, and then show that it has no topological embedding in the plane.

PROPOSITION 5.2.1. [The Jordan curve theorem for broken lines] *Any broken line C in the plane without self-intersections splits the plane into two path connected components and is the boundary of each of them.*

PROOF. Let D be a small disk which C intersects along a line segment, and thus divides D into two (path) connected components. Let p be any point in $\mathbb{R}^2 \setminus C$. From p we can move along a polygonal line as close as we like to C and then, staying close to C, move inside D. We will then be in one of the two components of $D \setminus C$, which shows that $\mathbb{R}^2 \setminus C$ has *no more than two* components.

It remains to show that $\mathbb{R}^2 \setminus C$ is not path connected. Let ρ be a ray originating at the point $p \in \mathbb{R}^2 \setminus C$. The ray intersects C in a finite number of segments and isolated points. To each such point (or segment) assign the number 1 if C crosses ρ there and 0 if it stays on the same side. Consider the parity $\pi(p)$ of the sum S of all the assigned numbers: it changes continuously as ρ rotates and, being an integer, $\pi(p)$ is constant. Clearly, $\pi(p)$ does not change inside a connected component of $\mathbb{R}^2 \setminus C$. But if we take a segment intersecting C at a non-zero angle, then the parity π at its end points differs. This contradiction proves the proposition. □

We will call a closed broken line without self-intersections a *simple polygonal line*.

COROLLARY 5.2.2. *If two broken lines L_1 and L_2 without self-intersections lie in the same component of $\mathbb{R}^2 \setminus C$, where C is a simple closed polygonal line, with their endpoints on C in alternating order, then L_1 and L_2 intersect.*

PROOF. The endpoints a and c of L_1 divide the polygonal curve C into two polygonal arcs C_1 and C_2. The curve C and the line L_1 divide the plane into three path connected domains: one bounded by C, the other two bounded by the closed curves $C_i \cup L$, $i = 1, 2$ (this follows from Proposition 5.2.1). Choose points b and d on L_2 close to its endpoints. Then b and d must lie in different domains bounded by L_1 and C and any path joining them and not intersecting C, in particular L_2, must intersect L_1. $\square$

PROPOSITION 5.2.3. *The graph $K_{3,3}$ cannot be polygonally embedded in the plane.*

PROOF. Let us number the vertices $x_1, \dots, x_6$ of $K_{3,3}$ so that its edges constitute a closed curve $C := x_1x_2x_3x_4x_5x_6$, the other edges being

$$E_1 := x_1x_4, \quad E_2 := x_2x_5, \quad E_3 := x_3x_6.$$

Then, if $K_{3,3}$ lies in the plane, it follows from Proposition 5.2.1 that C divides the plane into two components. One of the two components must contain at least two of the edges E_1, E_2, E_3, which then have to intersect (by Corollary 5.2.2). This is a contradiction which proves the proposition. $\square$

THEOREM 5.2.4. *The graph $K_{3,3}$ is nonplanar, i.e., there is no topological embedding $h : K_{3,3} \hookrightarrow \mathbb{R}^2$.*

The theorem is an immediate consequence of the nonexistence of a PL-embedding of $K_{3,3}$ (Proposition 5.2.3) and the following lemma.

LEMMA 5.2.5. *If a graph G is planar, then there exists a polygonal embedding of G into the plane.*

PROOF. Given a graph $G \subset \mathbb{R}^2$, we first modify it in small disk neighborhoods of the vertices so that the intersection of (the modified graph) G with each disk is the union of a finite number of radii of this disk. Then, for each edge, we cover its complement to the vertex disks by disks disjoint from the other edges, choose a finite subcovering (by compactness) and, using the chosen disks, replace the edge by a polygonal line. $\square$

5.2.2. Euler characteristic and Euler theorem. The *Euler characteristic* of a graph G without loops *embedded in the plane* is defined as

$$\chi(G) \ := \ V - E + F,$$

where V is the number of vertices and E is the number of edges of G, while F is the number of connected components of $\mathbb{R}^2 \setminus G$ (including the unbounded component).

THEOREM 5.2.6. [Euler Theorem] *For any connected graph G without loops embedded in the plane, $\chi(G) \ = \ 2$.*

PROOF. At the moment we are only able to prove this theorem for polygonal graphs. For the general case we will need Jordan curve Theorem Theorem 5.1.2. The proof will be by induction on the number of edges. For the graph with zero edges, we have $V = 1$, $E = 0$, $F = 1$, and the formula holds. Suppose it holds for all graphs with n edges; then it is valid for any connected subgraph H of G with n edges; take an edge e from G which is not in H but incident to H, and add it to H. Two cases are possible.

Case 1. Only one endpoint of e belongs to H. Then F is the same for G as for H and both V and E increase by one.

Case 2. Both endpoints of e belong to to H. Then e lies inside a face of H and divides it into two.[1] Thus by adding e we increase both E and F by one and leave V unchanged. Hence the Euler characteristic does not change. □

5.2.3. Kuratowski Theorem.

We conclude this subsection with a beautiful theorem, which gives a simple geometrical obstruction to the planarity of graphs. We do not present the proof (which is not easy), because this theorem, unlike the previous one, is not used in the sequel.

small print for parts outside of the main line: no proofs or too difficult

THEOREM 5.2.7. [Kuratowski] *A graph is nonplanar if and only if it contains, as a topological subspace, the graph $K_{3,3}$ or the graph K_5.*

REMARK 5.2.8. The words "as a topological subspace" are essential in this theorem. They cannot be replaced by "as a subgraph": if we subdivide an edge of K_5 by adding a vertex at its midpoint, then we obtain a nonplanar graph that does *not* contain either $K_{3,3}$ or K_5.

EXERCISE 5.2.1. Can the graph $K_{3,3}$ be embedded in (a) the Möbius strip, (b) the torus?

EXERCISE 5.2.2. Is there a graph that cannot be embedded into the torus?

EXERCISE 5.2.3. Is there a graph that cannot be embedded into the Möbius strip?

5.3. Surfaces and their triangulations

In this section, we define (two-dimensional) surfaces, which are topological spaces that locally look like $\mathbb{R}^2$ (and so are supplied with local systems of coordinates). It can be shown that surfaces can always be triangulated (supplied with a PL-structure) and smoothed (supplied with a smooth manifold structure). We will not prove these two assertions here and limit ourselves to the study of triangulated surfaces (also known as two-dimensional PL-manifolds). The main result is a neat classification theorem, proved by means of some simple piecewise linear techniques and with the help of the Euler characteristic.

proof will be added here or later

an easy consequence of PL

[1]It is here that we need the conclusion of Jordan curve Theorem Theorem 5.1.2 in the case of general graphs. The rest of the argument remains the same as for polygonal graphs.

5.3.1. Definitions and examples.

DEFINITION 5.3.1. A *closed surface* is a compact connected 2-manifold (without boundary), i.e., a compact connected space each point of which has a neighborhood homeomorphic to the open 2-disk $\operatorname{Int}\mathbb{D}^2$. In the above definition, connectedness can be replaced by path connectedness without loss of generality (see ??)

A *surface with boundary* is a compact space each point of which has a neighborhood homeomorphic to the open 2-disk $\operatorname{Int}\mathbb{D}^2$ or to the open half disk

$$\operatorname{Int}\mathbb{D}^2_{1/2} = \{(x,y)\in\mathbb{R}^2 | x \geqslant 0, x^2+y^2<1\}.$$

EXAMPLE 5.3.2. Familiar surfaces are the 2-sphere $\mathbb{S}^2$, the projective plane $\mathbb{R}P^2$, and the torus $\mathbb{T}^2 = S^1 \times S^1$, while the disk $\mathbb{D}^2$, the annulus, and the Möbius band are examples of surfaces with boundary.

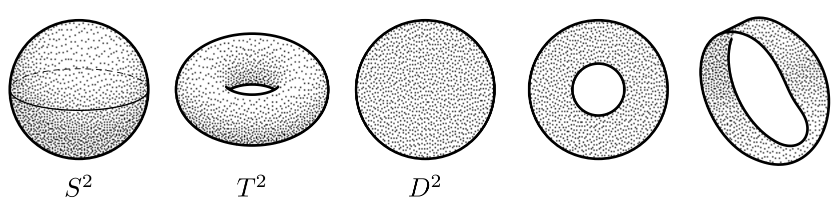

FIGURE 5.3.1. Examples of surfaces

DEFINITION 5.3.3. The *connected sum* $M_1\#M_2$ of two surfaces M_1 and M_2 is obtained by making two small holes (i.e., removing small open disks) in the surfaces and gluing them along the boundaries of the holes

EXAMPLE 5.3.4. The connected sum of two projective planes $\mathbb{R}P^2\#\mathbb{R}P^2$ is the famous *Klein bottle*, which can also be obtained by gluing two Möbius bands along their boundaries (see Fig.??). The connected sum of three tori $\mathbb{T}^2\#\mathbb{T}^2\#\mathbb{T}^2$ is (topologically) the surface of a pretzel (see Fig.??).

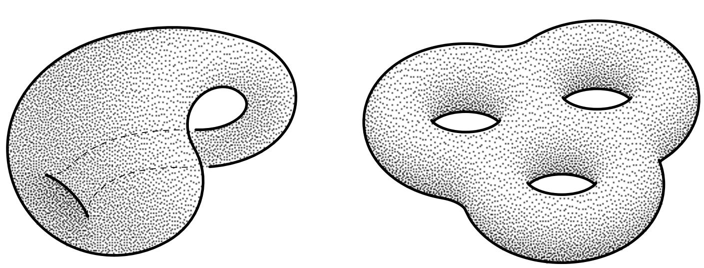

FIGURE 5.3.2. Klein bottle and pretzel

5.3.2. Polyhedra and triangulations. Our present goal is to introduce a combinatorial structure (called PL-structure) on surfaces. First we we give the corresponding definitions related to PL-structures.

A (finite) 2-*polyhedron* is a topological space represented as the (finite) union of triangles (its *faces* or *2-simplices*) so that the intersection of two triangles is either empty, or a common side, or a common vertex. The sides of the triangles are called *edges* or 1-*simplices*, the vertices of the triangles are called *vertices* or 0-*simplices* of the 2-polyhedron.

Let P be a 2-polyhedron and $v \in P$ be a vertex. The (closed) *star* of v in P (notation $\mathrm{Star}(v, P)$) is the set of all triangles with vertex v. The *link* of v in P (notation $\mathrm{Link}(v, P)$) is the set of sides opposite to v in the triangles containing v.

A finite 2-polyhedron is said to be a *closed PL-surface* (or a *closed triangulated surface*) if the star of any vertex v is homeomorphic to the closed 2-disk with v at the center (or, which is the same, if the links of all its vertices are homeomorphic to the circle).

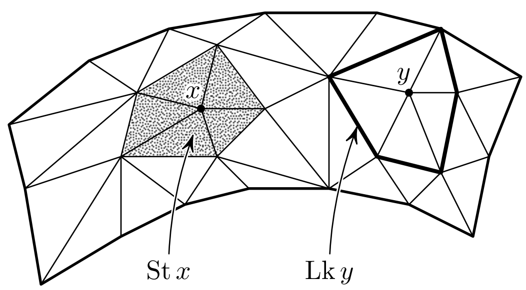

FIGURE 5.3.3. Star and link of a point on a surface

A finite 2-polyhedron is said to be a *PL-surface with boundary* if the star of any vertex v is homeomorphic either to the closed 2-disk with v at the center or to the closed disk with v on the boundary (or, which is the same, if the links of all its vertices are homeomorphic either to the circle or to the line segment). It is easy to see that in a PL-surface with boundary the points whose links are segments (they are called *boundary points*) constitute a finite number of circles (called *boundary circles*). It is also easy to see that each edge of a closed PL-surface (and each nonboundary edge of a surface with boundary) is contained in exactly two faces.

A PL-surface (closed or with boundary) is called *connected* if any two vertices can be joined by a sequence of edges (each edge has a common vertex with the previous one). Further, unless otherwise stated, we consider only connected PL-surfaces.

A PL-surface (closed or with boundary) is called *orientable* if its faces can be coherently oriented; this means that each face can be oriented (i.e., a cyclic order of its vertices chosen) so that each edge inherits opposite orientations from the orientations of the two faces containing this edge. An *orientation* of an orientable surface is a choice of a coherent orientation of its faces; it is easy to see that that any orientable (connected!) surface has exactly two orientations.

A *face subdivision* is the replacement of a face (triangle) by three new faces obtained by joining the baricenter of the triangle with its vertices. An *edge subdivision* is the replacement of the two faces (triangles) containing an edge by four new faces obtained by joining the midpoint of the edge with the two opposite vertices of the two triangles. A *baricentric subdivision* of a face is the replacement of a face (triangle) by six new faces obtained by constructing the three medians of the triangles. A *baricentric subdivision of a surface* is the result of the baricentric subdivision of all its faces. Clearly, any baricentric subdivision can be obtained by means of a finite number of edge and face subdivisions. A *subdivision* of a PL-surface is the result of a finite number of edge and face subdivisions.

Two PL-surfaces M_1 and M_2 are called *isomorphic* if there exists a homeomorphism $h : M_1 \to M_2$ such that each face of M_1 is mapped onto a face of M_2. Two PL-surfaces M_1 and M_2 are called *PL-homeomorphic* if they have isomorphic subdivisions.

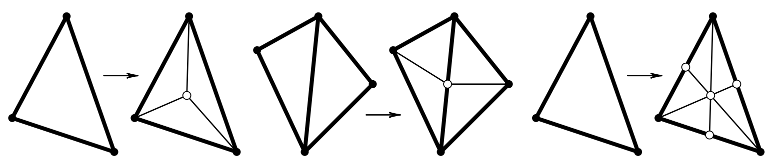

FIGURE 5.3.4. Face, edge, and baricentric subdivisions

EXAMPLE 5.3.5. Consider any convex polyhedron P; subdivide each of its faces into triangles by diagonals and project this radially to a sphere centered in any interior point of P. The result is a triangulation of the sphere.

If P is a tetrahedron the triangulation has four vertices. This is the minimal number of vertices in a triangulation of any surface. In fact, any triangulation of a surface with four vertices is equivalent of the triangulation obtained from a tetrahedron and thus for any surface other than the sphere the minimal number of vertices in a triangulation is greater then four.

EXERCISE 5.3.1. Prove that there exists a triangulation of the projective plane with any given number $N > 4$ of vertices.

EXERCISE 5.3.2. Prove that minimal number of vertices in a triangulation of the torus is six.

5.4. Euler characteristic and genus

In this section we introduce, in an elementary combinatorial way, one of the simplest and most important homological invariants of a surface M – its Euler characteristic $\chi(M)$. The Euler characteristic is an integer (actually defined for a much wider class of objects than surfaces) which is topologically invariant (and, in fact, also homotopy invariant). Therefore, if we find that two surfaces have different Euler characteristics, we can conclude that they are not homeomorphic.

5.4.1. Euler characteristic of polyhedra.

DEFINITION 5.4.1. The *Euler characteristic* $\chi(M)$ of a two-dimensional polyhedron, in particular of a PL-surface, is defined by

$$\boxed{\chi(M) := V - E + F},$$

where V, E, and F are the numbers of vertices, edges, and faces of M, respectively.

PROPOSITION 5.4.2. *The Euler characteristic of a surface does not depend on its triangulation. PL-homeomorphic PL-surfaces have the same Euler characteristic.*

PROOF. It follows from the definitions that we must only prove that the Euler characteristic does not change under subdivision, i.e., under face and edge subdivision. But these two facts are proved by a straightforward verification. □

EXERCISE 5.4.1. Compute the Euler characteristic of the 2-sphere, the 2-disk, the projective plane and the 2-torus.

EXERCISE 5.4.2. Prove that $\chi(M\#N) = \chi(M) + \chi(N) - 2$ for any PL-surfaces M and N. Use this fact to show that adding one handle to an oriented surface decreases its Euler characteristic by 2.

5.4.2. The genus of a surface. Now we will relate the Euler characteristic with a a very visual property of surfaces – their genus (or number of handles). The genus of an oriented surface is defined in the next section (see ??), where it will be proved that the genus g of such a surface determines the surface up to homeomorphism. The model of a surface of genus g is the *sphere with g handles*; for $g = 3$ it is shown on the figure.

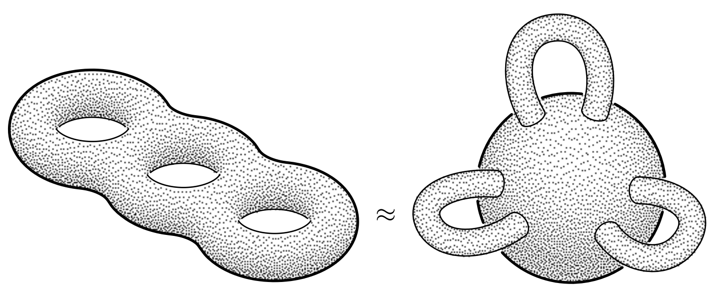

FIGURE 5.4.1. The sphere with three handles

PROPOSITION 5.4.3. *For any closed surface M, the genus $g(M)$ and the Euler characteristic $\chi(M)$ are related by the formula*

$$\boxed{\chi(M) \;=\; 2 - 2g(M)}.$$

PROOF. Let us prove the proposition by induction on g. For $g = 0$ (the sphere), we have $\chi(\mathbb{S}^2) = 2$ by Exercise **??**. It remains to show that adding one handle decreases the Euler characteristic by 2. But this follows from Exercise **??** □

REMARK 5.4.4. In fact $\chi = \beta_2 - \beta_1 + \beta_0$, where the β_i are the Betti numbers (defined in **??**). For the surface of genus g, we have $\beta_0 = \beta_2 = 1$ and $\beta_1 = 2g$, so we do get $\chi = 2 - 2g$.

5.5. Classification of surfaces

In this section, we present the topological classification (which coincides with the combinatorial and smooth ones) of surfaces: closed orientable, closed nonorientable, and surfaces with boundary.

5.5.1. Orientable surfaces. The main result of this subsection is the following theorem.

THEOREM 5.5.1 (Classification of orientable surfaces). *Any closed orientable surface is homeomorphic to one of the surfaces in the following list*

$$\mathbb{S}^2,\ \mathbb{S}^1 \times \mathbb{S}^1\ (\textit{torus}),\ (\mathbb{S}^1 \times \mathbb{S}^1)\#(\mathbb{S}^1 \times \mathbb{S}^1)\ (\textit{sphere with 2 handles}), \dots$$

$$\dots, (\mathbb{S}^1 \times \mathbb{S}^1)\#(\mathbb{S}^1 \times \mathbb{S}^1)\# \dots \#(\mathbb{S}^1 \times \mathbb{S}^1)\ (\textit{sphere with k handles}), \dots$$

Any two surfaces in the list are not homeomorphic.

PROOF. By **??** we may assume that M is triangulated and take the double baricentric subdivision M'' of M. In this triangulation, the star of a vertex of M'' is called a *cap*, the union of all faces of M'' intersecting an edge of M but not contained in the caps is called a *strip*, and the connected components of the union of the remaining faces of M'' are called *patches*.

Consider the union of all the edges of M; this union is a graph (denoted G). Let G_0 be a maximal tree of G. Denote by M_0 the union of all caps and strips surrounding G_0. Clearly M_0 is homeomorphic to the 2-disk (why?). If we successively add the strips and patches from $M - M_0$ to M_0, obtaining an increasing sequence

$$M_0 \subset M_1 \subset M_2 \subset \dots \subset M_p = M,$$

we shall recover M.

Let us see what happens when we go from M_0 to M_1.

If there are no strips left, then there must be a patch (topologically, a disk), which is attached along its boundary to the boundary circle Σ_0 of M_0; the result is a 2-sphere and the theorem is proved.

Suppose there are strips left. At least one of them, say S, is attached along one end to Σ_0 (because M is connected) and its other end is also attached to Σ_0 (otherwise S would have been part of M_0). Denote by K_0 the closed collar neighborhood of Σ_0 in M_0. The collar K_0 is homoeomorphic to the annulus (and not to the Möbius strip) because M is orientable. Attaching S to M_0 is the same as

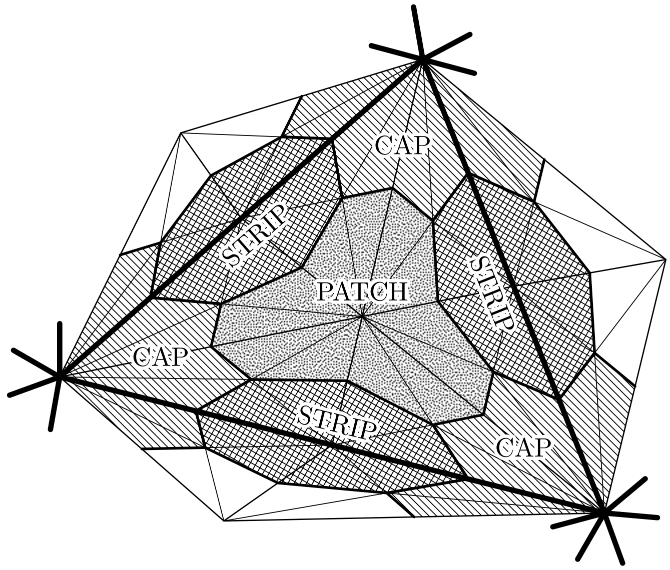

FIGURE 5.5.1. Caps, strips, and patches

attaching another copy of $K \cup S$ to M_0 (because the copy of K can be homeomorphically pushed into the collar K). But $K \cup S$ is homeomorphic to the disk with two holes (what we have called "pants"), because S has to be attached in the orientable way in view of the orientability of M (for that reason the twisting of the strip shown on the figure cannot occur). Thus M_1 is obtained from M_0 by attaching the pants $K \cup S$ by the waist, and M_1 has two boundary circles.

FIGURE ??? This cannot happen

Now let us see what happens when we pass from M_1 to M_2.

If there are no strips left, there are two patches that must be attached to the two boundary circles of M_1, and we get the 2-sphere again.

Suppose there are patches left. Pick one, say S, which is attached at one end to one of the boundary circles, say Σ_1 of M_1. Two cases are possible: either

(i) the second end of S is attached to Σ_2, or

(ii) the second end of S is attached to Σ_1.

Consider the first case. Take collar neighborhoods K_1 and K_2 of Σ_1 and Σ_2; both are homoeomorphic to the annulus (because M is orientable). Attaching S to M_1 is the same as attaching another copy of $K_1 \cup K_2 \cup S$ to M_1 (because the copy of $K_1 \cup K_2$ can be homeomorphically pushed into the collars K_1 and K_2).

FIGURE ??? Adding pants along the legs

But $K - 1 \cup K_2 \cup S$ is obviously homeomorphic to the disk with two holes. Thus, in the case considered, M_2 is obtained from M_1 by attaching pants to M_1 along the legs, thus decreasing the number of boundary circles by one,

The second case is quite similar to adding a strip to M_0 (see above), and results in attaching pants to M_1 along the waist, increasing the number of boundary circles by one.

What happens when we add a strip at the ith step? As we have seen above, two cases are possible: either the number of boundary circles of M_{i-1} increases by one or it decreases by one. We have seen that in the first case "inverted pants" are attached to M_{i-1} and in the second case "upright pants" are added to M_{i-1}.

FIGURE ??? Adding pants along the waist

After we have added all the strips, what will happen when we add the patches? The addition of each patch will "close" a pair of pants either at the "legs" or at the "waist". As the result, we obtain a sphere with k handles, $k \geqslant 0$. This proves the first part of the theorem.

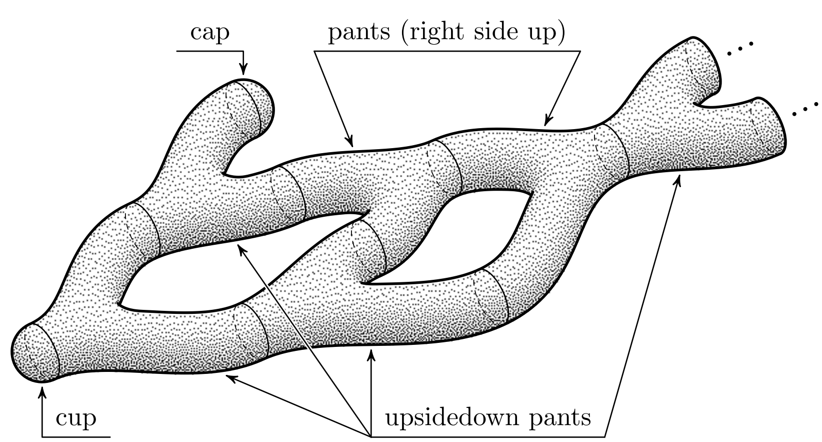

FIGURE 5.5.2. Constructing an orientable surface

To prove the second part, it suffices to compute the Euler characteristic (for some specific triangulation) of each entry in the list of surfaces (obtaining $2, 0, -2, -4, \dots$, respectively). $\square$

5.5.2. Nonorientable surfaces and surfaces with boundary. Nonorientable surfaces are classified in a similar way. It is useful to begin with the best-known example, the *Möbius strip*, which is the nonorientable surface with boundary obtained by identifying two opposite sides of the unit square $[0,1]\times[0,1]$ via $(0,t)\sim(1,1-t)$. Its boundary is a circle.

Any compact nonorientable surface is obtained from the sphere by attaching several *Möbius caps*, that is, deleting a disk and identifying the resulting boundary circle with the boundary of a Möbius strip. Attaching m Möbius caps yields a surface of genus $2-m$. Alternatively one can replace any pair of Möbius caps by a handle, so long as at least one Möbius cap remains, that is, one may start from a sphere and attach one or two Möbius caps and then any number of handles.

All compact surfaces with boundary are obtained by deleting several disks from a closed surface. In general then a sphere with h handles, m Möbius strips, and d deleted disks has Euler characteristic

$$\chi = 2 - 2h - m - d.$$

In particular, here is the finite list of surfaces with nonnegative Euler characteristic:

Surface	h	m	d	χ	Orientable?
Sphere	0	0	0	2	yes
Projective plane	0	1	0	1	no
Disk	0	0	1	1	yes
Torus	1	0	0	0	yes
Klein bottle	0	2	0	0	no
Möbius strip	0	1	1	0	no
Cylinder	0	0	2	0	yes

5.6. The fundamental group of compact surfaces

Using the Seifert–van Kampen theorem (see ???), here we compute the fundamental groups of closed surfaces.

5.6.1. π_1 for orientable surfaces.

THEOREM 5.6.1. *The fundamental group of the orientable surface of genus g can be presented by $2g$ generators $p_1, m_1, \dots, p_n, m_n$ satisfying the following defining relation:*

$$p_1 m_1 p_1^{-1} m_1^{-1} \dots p_n m_n p_n^{-1} m_n^{-1} = 1.$$

PROOF. ++++++++++++++++++++++++++++++++++++ □

5.6.2. π_1 for nonorientable surfaces.

THEOREM 5.6.2. *The fundamental group of the nonorientable surface of genus g can be presented by the generators $c_1, \dots c_n$, where $n := 2g + 1$, satisfying the following defining relation:*

$$c_1^2 \dots c_n^2 = 1.$$

PROOF. ++++++++++++++++++++++++++++++++++++ □

5.7. Vector fields on the plane

The notion of vector field comes from mechanics and physics. Examples: the velocity field of the particles of a moving liquid in hydrodynamics, or the field of gravitational forces in Newtonian mechanics, or the field of electromagnetic induction in electrodynamics. In all these cases, a vector is given at each point of some domain in space, and this vector changes continuously as we movefrom point to point.

In this section we will study, using the notion of degree (see??) a simpler model situation: vector fields on the plane (rather than in space).

5.7.1. Trajectories and singular points. A *vector field* V in the plane $\mathbb{R}^2$ is a rule that assigns to each point $p \in \mathbb{R}^2$ a vector $V(p)$ issuing from p. Such an assignment may be expressed in the coordinates x, y of $\mathbb{R}^2$ as

$$X = \alpha(x, y) \quad Y = \beta(x, y),$$

where $\alpha : \mathbb{R}^2 \to \mathbb{R}$ and $\beta : \mathbb{R}^2 \to \mathbb{R}$ are real-valued functions on the plane, (x, y) are the coordinates of the point p, and (X, Y) are the coordinates of the vector $V(p)$. If the functions α and β are continuous (respectively differentiable), then the vector field V is called *continuous* (resp. *smooth*).

A *trajectory* through the point $p \in \mathbb{R}^2$ is a curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ passing through p and tangent at all its points to the vector field (i.e., the vector $V(q)$ is tangent to the curve $C := \gamma(\mathbb{R})$ at each point $q \in C$). A *singular point* p of a vector field V is a point where V vanishes: $V(p) = 0$; when V is a velocity field, such a point is often called a *rest point*, when V is a field of forces, it is called an *equilibrium point*.

5.7.2. Generic singular points of plane vector fields. We will now describe some of the simplestf singular points of plane vector fields. To define these points, we will not write explicit formulas for the vectors of the field, but instead describe the topological picture of its trajectories near the singular point and give physical examples of such singularities.

The *node* is a singular point contained in all the nearby trajectories; if all the trajectories move towards the point, the node is called *stable* and *unstable* if all the trajectories move away from the point. As an example, we can consider the gravitational force field of water droplets flowing down the surface $z = x^2 + y^2$ near the point $(0, 0, 0)$ (stable node) or down the surface $z = -x^2 - y^2$ near the same point (unstable node).

The *saddle* is a singular point contained in two transversal trajectories, called *separatrices*, one of which is ingoing, the other outgoing, the other trajectories behaving like a family of hyperbolas whose asymptotes are the separatrices. As an example, we can consider the gravitational force field of water droplets flowing down the surface $z = x^2 - y^2$ near the point $(0, 0, 0)$; here the separatrices are the coordinates axes.

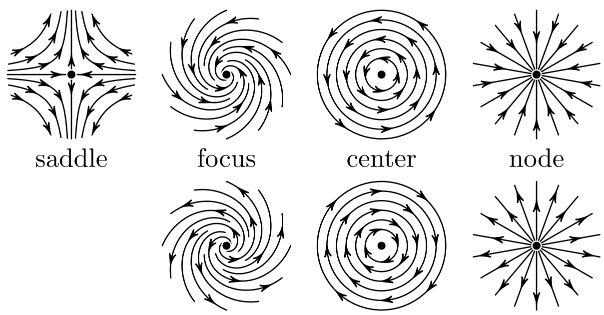

FIGURE 5.7.1. Simplest singular points of vector fields

The *focus* is a singular point that ressembles the node, except that the trajectories, instead of behaving like the set of straight lines passing through the point, behave as a family of logarithmic spirals converging to it (*stable focus*) or diverging from it (*unstable focus*).

topology of the node is right but geometry is wrong: in general "parabolas" tangent to the horizontal line plus horizontal and vertical lines

The *center* is a singular point near which the trajectories behave like the family of concentric circles centered at that point; a center is called *positive* if the trajectories rotate counterclockwise and *negative* if they rotate clockwise. As an example, we can consider the velocity field obtained by rotating the plane about the origin with constant angular velocity.

REMARK 5.7.1. From the topological point of view, there is no difference between a node and a focus: we can unfurl a focus into a node by a homeomorphism which is the identity outside a small neighborhood of the singular point. However, we can't do this by means of a diffeomorphism, so that the node differs from the focus in the smooth category.

A singular point is called *generic* if it is of one of the first three types described above (node, saddle, focus). A vector field is called *generic* if it has a finite number of singular points all of which are generic. In what follows we will mostly consider generic vector fields.

REMARK 5.7.2. Let us explain informally why the term generic is used here. Generic fields are, in fact, the "most general" ones in the sense that, first, they occur "most often" (i.e., as close as we like to any vector field there is a generic one) and, second, they are "stable" (any vector field close enough to a generic one is also generic, has the same number of singular points, and those points are of the same types). Note that the center is *not* generic: a small perturbation transforms it into a focus. These statements are not needed in this course, so we will not make them more precise nor prove them.

REMARK 5.7.3. It can be proved that the saddle and the center are *not* topologically equivalent to each other and not equivalent to the node or to the focus; however, the focus and the node *are* topologically equivalent, as we noted above.

5.7.3. The index of plane vector fields. Suppose a vector field V in the plane is given. Let $\gamma : S^1 \to \mathbb{R}^2$ be a closed curve in the plane not passing through any singular points of V; denote $C := \gamma(S^1)$. To each vector $V(c)$, $c \in C$, let us assign the unit vector of the same direction as $V(c)$ issuing from the origin of coordinates $O \in \mathbb{R}^2$; we then obtain a map $g : C \to S^1_1$ (where $S^1_1 \subset \mathbb{R}^2$ denotes the unit circle centered at O), called the *Gauss map* corresponding to the vector field V and to the curve γ. Now we define the *index of the vector field* V *along the curve* γ as the degree of the Gauss map $g : S^1 \to S^1$ (for the definition of the degree of circle maps, see section 5, §3): $\mathrm{Ind}(\gamma, V) := \deg(g)$. Intuitively, the index is the total number of revolutions in the positive (counterclockwise) direction that the vector field performs when we go around the curve once.

REMARK 5.7.4. A simple way of computing $\mathrm{Ind}(\gamma)$ is to fix a ray issuing from O (say the half-axis Ox) and count the number of times p the endpoint of $V(c)$ passes through the ray in the positive direction and the number of times q in the negative one; then $\mathrm{Ind}(\gamma) = p - q$.

THEOREM 5.7.5. *Suppose that a simple closed curve* γ *does not pass through any singular points of a vector field* V *and bounds a domain that also does not contain any singular points of* V. *Then*

$$\boxed{\mathrm{Ind}(\gamma, V) = 0\ .}$$

PROOF. By the Schoenflies theorem, we can assume that there exists a homeomorphism of $\mathbb{R}^2$ that takes the domain bounded by $C := \gamma(\mathbb{S}^1)$ to the unit disk centered at the origin O. This homeomorphism maps the vector field V to a vector field that we denote by V'. Obviously,

$$\mathrm{Ind}(\gamma, V) = \mathrm{Ind}(S^1_0, V),$$

where S^1_O denotes the unit circle centered at O. Consider the family of all circles S^1_r of radius $r < 1$ centered at O. The vector $V'(O)$ is nonzero, hence for a small enough r_0 all the vectors $V'(s)$, $s \in S^1_{r_0}$, differ little in direction from $V'(O)$, so that $\mathrm{Ind}(S^1_r, V) = 0$. But then by continuity $\mathrm{Ind}(S^1_r, V) = 0$ for all $r \leqslant 1$. Now the theorem follows from (1). □

Now suppose that V is a generic plane vector field and p is a singular point of V. Let C be a circle centered at p such that no other singular points are contained in the disk bounded by C. Then the *index of* V *at the singular point* p is defined as $\mathrm{Ind}(p, V) := \mathrm{Ind}(C, V)$. This index is well defined, i.e., it does not depend on the radius of the circle C (provided that the disk bounded by C does not contain any other singular points); this follows from the next theorem.