CHAPTER 6

# COVERING SPACES AND DISCRETE GROUPS

We have already met covering spaces in Chapter 2, where our discussion was geometric and basically limited to definitions and examples. In this chapter, we return to this topic from a more algebraic point of view, which will allow us to produce numerous examples coming from group actions and to classify all covering spaces with given base (provided the latter is "nice" enough).

The main tools will be groups:

• discrete groups acting on manifolds, as the source of numerous examples of covering spaces;

• the fundamental groups of the spaces involved (and the homomorphisms induced by their maps), which will play the key role in the classification theorems of covering spaces.

## 6.1. Coverings associated with discrete group actions

We already mentioned (see ???) that one classical method for obtaining covering spaces is to consider discrete group actions on nice spaces (usually manifolds) and taking the quotient map to the orbit space. In this section, we dwell on this approach, providing numerous classical examples of covering spaces and conclude with a discussion of the generality of this method.

**6.1.1. Discrete group actions.** Here we present some basic definitions and facts related to group actions.

DEFINITION 6.1.1. The *action* of a group $G$ on a set $X$ is a map

$$G \times X \to X, \quad (g, x) \mapsto g(x),$$

such that

(1) $(gh)(x) = g(h(x))$;

(2) $e(x) = x$ if $e$ is the unit element of $G$.

We will be interested in the case in which the group $G$ is discrete (i.e., possesses, besides its group structure, the structure of a discrete topological space) and $X$ is a Hausdorff topological space (more often than not a manifold). We then require the action to be a continuous map of topological spaces.

DEFINITION 6.1.2. The *orbit* of a point $x \in X$ is the set $\{g(x) \mid g \in G\}$. It immediately follows from definitions that the orbits constitute a partition of $X$. The *orbit space* of the action of $G$ on $X$ is the quotient space of $X$ under the equivalence relation identifying all points lying in the same orbit; it is standardly denoted by $X/G$, and we have the *quotient map* $X \to X/G$ *of the action*.

We say that $G$ *acts by homeomorphisms* if there is an inclusion (a monomorphism) $\mu : G \to \text{Homeo}(X)$ of the group $G$ into the group Homeo($X$) of homeomorphisms of $X$ if $\mu$ satisfies $g(x) = (\mu(g))(x)$ for all $x \in X$ and all $g \in G$. This means that the assignment $(x, g) \mapsto g(x)$ is a homeomorphism of $X$. Similarily, for a metric space $X$, we say that a group $G$ *acts by isometries* on $X$ if there is a monomorphism $\mu : G \to \text{Isom}(X)$ of the group $G$ into the group Isom($X$) of isometries of $X$ such that $g(x) = (\mu(g))(x)$ for all $x \in X$ and all $g \in G$. A similar meaning is assigned to the expressions *act by rotations*, *act by translations*, *act by homotheties*, etc.

Suppose $G$ acts by homeomorphisms on a Hausdorff space $X$; we then say that the action of $G$ is *normal* if any $x \in X$ has a neighborhood $U$ such that all the images $g(U)$ for different $g \in G$ are disjoint, i.e.,

$$g_1(U) \cap g_2(U) \neq \varnothing \implies g_1 = g_2.$$

REMARK 6.1.3. There is no standard terminology for actions that we have called "normal"; sometimes the expressions "properly discontinuous action" or "covering space action" are used. We strongly favor "normal" – the reason for using it will become apparent in the next subsection.

EXAMPLE 6.1.4. Let $X$ be the standard unit square centered at the origin of $\mathbb{R}^2$ and let $G$ be its isometry group, acting on $X$ in the natural way. Then the orbits of this action consist of 8, or 4, or 1 points (see the figure), the one-point orbit being the orbit of the origin. The orbit space can be visualized as a right isoceles triangle with the hypothenuse removed. Obviously, this action is *not* normal.

FIGURE ??? Isometry group acting on the square

EXAMPLE 6.1.5. Let the two-element group $\mathbb{Z}_2$ act on the 2-sphere $\mathbb{S}^2$ by symmetries with respect to its center. Then all the orbits consist of two points, the action is normal, and the orbit space is the projective plane $\mathbb{R}P^2$.

EXAMPLE 6.1.6. Let the permutation group $S_3$ act on the regular tetrahedron $X$ by isometries. Then the orbits consist of 6, 4, or 1 point, and the action is not normal.

EXERCISE 6.1.1. Suppose $X$ is the union of the boundary of the equilateral triangle and its circumscribed circle (i.e., the graph on 3 vertices of valency 3 and 6 edges). Find a normal group action for which the orbit space is the figure eight (i.e., the one-vertex graph with two edges (loops)).

EXERCISE 6.1.2. Find a normal action of the cyclic group $\mathbb{Z}_5$ on the annulus so that the orbit space is also the annulus.

**6.1.2. Coverings as quotient maps to orbit spaces.** The main contents of this subsection are its examples and exercises, which are all based on the following statement.

PROPOSITION 6.1.7. *The quotient map $X \to X/G$ of a Hausdorff topological space $X$ to its orbit space $X/G$ under a normal action* (*in the sense of Definition ???*) *of a discrete group $G$ is a covering space .*

PROOF. By the definition of normal action, it follows that each orbit is in bijective correspondence with $G$ (which plays the role of the fiber $F$ in the definition of covering space). If $x$ is any point of $X$ and $U$ is the neighborhood specified by the normality condition, each neighborhood of the family $\{g(U) \,|\, g \in G\}$ is projected homeomorphically onto another copy of $U$ in the quotient space, which means that $p :\to X/G$ is a covering space. $\square$

REMARK 6.1.8. We will see later that in the situation of the proposition, the covering space will be normal (or regular, in another terminology), which means that the subgroup $p_{\#}(\pi_1(X)) \subset \pi_1(X/G)$ is normal. This explains our preference for the term "normal" for such group actions and such covering spaces.

EXAMPLE 6.1.9. Let the lattice $\mathbb{Z}^2$ act on the plane $\mathbb{R}^2$ in the natural way (i.e., by parallel translations along integer vectors). Then the orbit space of this action is the torus $\mathbb{T}^2$, and the corresponding quotient map $\tau : \mathbb{R}^2 \to \mathbb{T}^2$ is a covering. The covering map $\tau$ is actually a universal covering (i.e., it covers any other covering, see Definition ???), but we cannot prove this yet.

EXAMPLE 6.1.10. In the hyperbolic plane $\mathbb{H}^2$, choose a regular polygon $P$ of $4g$ sides, $g \geq 2$, with inner angle $\pi/2g$. Consider the natural action of the subgroup $G$ of isometries of $\mathbb{H}^2$ generated by parallel translations identifying opposite sides of the polygon. Then the entire hyperbolic plane will be covered by nonoverlapping copies of $P$.

Then the orbit space of this action is $M_g^2$, the sphere with $g$ handles and the corresponding quotient map $\mu : \mathbb{H}^2 \to M_g^2$ is a covering. Since $\mathbb{H}^2$ is contractible, the map $\mu$ is the universal covering of $M_g^2$.

FIGURE ??? Universal covering of the sphere with two handles

EXAMPLE 6.1.11. Let $X$ be the real line with identical little 2-spheres attached at its integer points. Let the group $\mathbb{Z}$ act on $X$ by integer translations. The covering space corresponding to this action is represented in the figure.

FIGURE ??? Universal covering of the wedge of $\mathbb{S}^1$ and $\mathbb{S}^2$

EXERCISE 6.1.3. Construct a space $X$ and an action of the cyclic group $\mathbb{Z}_3$ whose orbit space (i.e., the base of the corresponding covering space) is the wedge of the circle and the 2-sphere.

EXERCISE 6.1.4. Construct a 5-fold covering of the sphere with 11 handles over the sphere with 3 handles. Generalize to $n$-fold coverings (find $k$ and $l$ such that $M_k$ is the $n$-fold cover of $M_l$)

**6.1.3. Group actions and deck transformations.** Recall that a deck transformation (see Definition ???) of a covering space $p : X \to B$ is an isomorphism of $p$ to itself, i.e., a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \Big\downarrow p & & \Big\downarrow p \\ B & \xrightarrow{\mathrm{Id}} & B \end{array}$$

PROPOSITION 6.1.12. *The group of deck transformations of the covering space obtained as the quotient map of a Hausdorff space $X$ to its orbit space $X/G$ under a normal action* (*see Definition ???*) *of a discrete group $G$ is isomorphic to the group $G$.*

PROOF. This immediately follows from the definition of normal action: the fiber of the covering space under consideration is an orbit of the action of $G$, and property (1) of the definition of an action (see ???) implies that the deck transformations can be identified with $G$. $\square$

**6.1.4. Subgroup actions and associated morphisms.** If $G$ is a group acting by homeomorphisms on a Hausdorff space $X$, and $H$ is a subgroup of $G$, then $H$ acts by homeomorphisms on $X$ in the obvious way.

PROPOSITION 6.1.13. *A subgroup $H$ of a discrete group $G$ possessing a normal action on a Hausdorff space $X$ induces an injective morphism of the covering space $p_H$ corresponding to $H$ into the covering space $p_G$ corresponding to $G$.*

In this case the image of $p_H$ in $p_G$ is called a *subcovering* of $p_G$.

PROOF. The statement of the proposition is an immediate consequence of definitions. □

EXAMPLE 6.1.14. Let $w_n : \mathbb{S}^1 \to \mathbb{S}^1$ be the $n$-fold covering of the cicle by another copy of the circle, where $n = pq$ with $p$ and $q$ coprime. We can then consider two more coverings of the circle by itself, namely the $p$- and $q$-fold coverings $w_p$, $w_q : \mathbb{S}^1 \to \mathbb{S}^1$. Both of them are subcoverings of $w_n$, and their composition is (isomorphic to) $w_n$.

EXERCISE 6.1.5. Prove that the $p$-fold covering of the circle by itself has no subcoverings (other than the identity map) if $p$ is prime.

EXERCISE 6.1.6. Describe (up to isomorphism) all the subcoverings of the universal covering of the torus.

## 6.2. The hierarchy of coverings, universal coverings

In this section we are interested in the "social life" of covering spaces, i.e., in how they interact with each other. In this study, a number of natural questions arise, for example concerning the "hierarchy" of coverings $p : X \to B$ over a fixed space $B$ and the so-called universal coverings. These questions will mainly be answered in the next sections, but here the reader will find many useful constructions and examples.

**6.2.1. Definitions and examples.** Recall that the definition of covering space $p : X \to B$ implies that $p$ is a local homeomorphism, and the number of points of $p^{-1}(b)$ (which can be finite or countable) does not depend on the choice of $b \in B$. Recall further that covering spaces form a category, morphisms being pairs of maps $f : B \to B'$, $F : X \to X'$ for which $f \circ p = p' \circ F$, i.e., the square diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & X' \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

is commutative. If the maps $f$ and $F$ are homeomorphisms, then the two covering spaces are isomorphic. We do not distinguish isomorphic covering spaces: the classification of covering spaces will always be performed up to isomorphism.

Consider a fixed Hausdorff topological space $B$ and the set (which is actually a category) of covering spaces with base $B$. Our aim is to define an order in this set.

DEFINITION 6.2.1. We say that the covering space $p' : X' \to B$ *supersedes* or *covers* the covering space $p : X \to B$ (and write $p' \gg p$) if there is a covering space $q : X' \to X$ for which the following diagram

$$\begin{array}{ccc} X & \xrightarrow{q} & X' \\ \Big\downarrow p & & \Big\downarrow p \\ B & \xrightarrow{\mathrm{Id}} & B' \end{array}$$

is commutative.

The relation $\gg$ is obviously reflexive and transitive, so it is a partial order relation. Thus we obtain a hierarchy of covering spaces over each fixed base.

EXAMPLE 6.2.2. Let $w_n : \mathbb{S}^1 \to \mathbb{S}_1$ denote the $n$-fold covering of the circle by itself. Then $w_m$ supersedes $w_n$ iff $n$ divides $m$. Further, the least (in the sense of the order $\gg$) covering $w_d$ that supersedes both $w_r$ and $w_s$ is $w_m$, where $m$ is the least common multiple of $r$ and $s$, while the greatest covering superseded by both $w_r$ and $w_s$ is $w_d$, where $d$ is the greatest commom divisor of $r$ and $s$.

EXERCISE 6.2.1. Prove that no covering of the projective plane $\mathbb{R}P^2$ supersedes its double covering by the sphere $\mathbb{S}^2$.

EXERCISE 6.2.2. Prove that no covering of the circle $\mathbb{S}^1$ supersedes its covering by $\mathbb{R}$ (via the exponential map).

**6.2.2. Universal coverings.** Recall that the universal covering space $s : E \to B$ of a given space $B$ was defined as a covering satisfying the condition $\pi_1(E) = 0$. It turns out that such a covering $s : E \to B$ exists, is unique (up to isomorphism), and coincides with the maximal covering of $B$ with respect to the relation $\gg$, provided $B$ is nice enough (if it has no local pathology, e.g. is a manifold, a simplicial space, or a CW-space). We could prove this directly now, but instead we will prove (later in this chapter) a more general result from which the above statements follow.

In this subsection, using only the definition of universal covering (the condition $\pi_1(E) = 0$), we will accumulate some more examples of universal coverings. Recall that we already know several: $\mathbb{R}$ over $\mathbb{S}^1$, $\mathbb{S}^2$ over $\mathbb{R}P^2$, $\mathbb{R}^2$ over $\mathbb{T}^2$.

EXAMPLE 6.2.3. The universal covering of the sphere $\mathbb{S}^n$, $n \geq 2$, and more generally of any simply connected space, is the identity map.

EXAMPLE 6.2.4. The universal covering of the wedge sum $\mathbb{S}^1 \vee \mathbb{S}^1$ of two circles is the infinite 4-valent graph $\Gamma$ (shown on the figure) mapped onto $\mathbb{S}^1 \vee \mathbb{S}^1$ in the following way.

FIGURE ??? Universal covering of the wedge of $\mathbb{S}^1$ and $\mathbb{S}^1$

Each vertex of $\Gamma$ is sent to the point of tangency of the wedge and each edge is wrapped once around one of the circles (you can think of this as an infinite process, beginning at the center of the graph and moving outwards in a uniform way). It follows that we have obtained the universal covering of $\mathbb{S}^1 \vee \mathbb{S}^1$, since $\Gamma$ is (obviously) simply connected and the map described above is indeed a covering with fiber $\mathbb{Z}$ (this can be checked directly in the two different types of points of $\mathbb{S}^1 \vee \mathbb{S}^1$ – the tangency point, and all the others).

EXERCISE 6.2.3. Find an appropriate group $G$ and define a normal action of $G$ on the graph $\Gamma$ from the previous example so as to obtain the universal covering of the wedge sum of two circles.

EXERCISE 6.2.4. Describe the universal covering of the wedge sum of three circles.

EXERCISE 6.2.5. Describe the universal covering of the union of three circles, two of which are tangent (at different points) to the third.

## 6.3. Path lifting and covering homotopy properties

In this section, we prove two important technical assertions which allow, given a covering space $p : X \to B$, to lift "upstairs" (i.e., to $X$) continuous processes taking place "downstairs" (i.e., in $B$). The underlying idea has already been exploited when we defined the degree of circle maps by using the exponential map (see ???), and we will now be generalizing the setting from the exponential map to arbitrary covering spaces.

**6.3.1. Path lifting.** Let $p : X \to B$ be a covering space. Recall that the *lift* of a map $f : A \to B$ was defined (see ???) as any map $\widetilde{f} : A \to X$ such that $p \circ \widetilde{f} = f$.

LEMMA 6.3.1 (Path lifting lemma). *Any path in the base of a covering space can be lifted to the covering, and the lift is unique if its initial point in the covering is specified. More precisely, if* $p : X \to B$ *is a covering space,* $\alpha : [0,1] \to B$ *is any path, and* $x_0 \in p^{-1}(\alpha(0))$*, then there exists a unique map* $\widetilde{\alpha} : [0,1] \to X$ *such that* $p \circ \widetilde{\alpha} = \alpha$ *and* $\widetilde{\alpha}(0) = x_0$*.*

PROOF. By the definition of covering space, for each point $b \in \alpha([0,1]))$ there is a neighborhood $U_b$ whose inverse image under $p$ falls apart into disjoint neighborhoods each of which is projected homeomorphically by $p$ onto $U_b$. The set of all such $U_b$ covers $\alpha([0,1])$ and, since $\alpha([0,1])$ is compact, it possesses a finite subcover that we denote by $U_0, U_1, \dots U_k$.

Without loss of generality, we assume that $U_0$ contains $b_0 := \alpha(0)$ and denote by $\widetilde{U}_0$ the component of $p^{-1}(U_0)$ that contains the point $x_0$. Then we can lift a part of the path $\alpha$ contained in $U_0$ to $\widetilde{U}_0$ (uniquely!) by means of the inverse to the homeomorphism between $\widetilde{U}_0$ and $U_0$.

Now, again without loss of generality, we assume that $U_1$ intersects $U_0$ and contains points of $\alpha[0,1]$ not lying in $U_0$. Let $b_1 \in \alpha([0,1])$ be a point contained both in $U_0$ and $U_1$ and denote by $\widetilde{b}_1$ the image of $b_1$ under $p^{-1}\big|_{U_0}$. Let $\widetilde{U}_1$ be the component of the inverse image of $U_1$ containing $\widetilde{b}_1$. We now extend the lift of our path to its part contained in $U_1$ by using the inverse of the homeomorphism between $\widetilde{U}_1$ and $U_1$. Note that the lift obtained is the only possible one. Our construction is schematically shown on the figure.

FIGURE ??? Path lifting construction

Continuing in this way, after a finite number of steps we will have lifted the entire path $\alpha([0,1])$ to $X$, and the lift obtained will be the only one obeying the conditions of the lemma.

To complete the proof, it remains to show that the lift that we have constructed is unique and continuous. We postpone the details of this argument (which uses the fact that $X$ and $B$ are "locally nice", e.g. CW-spaces) to Subsection ??? . $\square$

REMARK 6.3.2. Note that the lift of a closed path is *not* necessarily a closed path, as we have already seen in our discussion of the degree of circle maps.

Note that if all paths (i.e., maps of $A = [0,1]$) can be lifted, it is not true that all maps of *any* space $A$ can be lifted.

EXAMPLE 6.3.3. Let $\tau : \mathbb{R}^2 \to \mathbb{T}^2$ be the standard covering of the torus by the plane. Then the map $\alpha : \mathbb{S}^1 \to \mathbb{T}^2$ taking the circle to some meridian of the torus cannot be lifted to the plane. Indeed, if such a lift existed, it would be a continuous map of a compact set ($\mathbb{S}^1$) with a noncompact image.

EXERCISE 6.3.1. Give an example of a map $\alpha : A \to \mathbb{R}P^2$ which cannot be lifted to the standard covering space $p : \mathbb{S}^2 \to \mathbb{R}P^2$.

**6.3.2. Homotopy lifting.** Now we generalize the path lifting lemma to homotopies, having in mind that a path is actually a homotopy, namely a homotopy of the one-point space. This trivial observation is not only the starting point of the formulation of the covering homotopy theorem, but also the key argument in its proof.

THEOREM 6.3.4 (Covering homotopy theorem). *Any homotopy in the base of a covering space can be lifted to the covering, and the homotopy is unique if its initial map in the covering is specified as a lift of the initial map of the given homotopy. More precisely, if $p : X \to B$ is a covering, $F : A \times [0,1] \to B$ is any homotopy whose initial map $f_0(\cdot) := F(\cdot, 0)$ possesses a lift $\widetilde{f_0}$, then there exists a unique homotopy $\widetilde{F} : A \times [0,1] \to X$ such that $p \circ \widetilde{F} = F$ and $\widetilde{F}(\cdot, 0) = \widetilde{f_0}(\cdot)$.*

PROOF. The theorem will be proved by means of a beautiful trick, magically reducing the theorem to the path lifting lemma from the previous subsection. Fix some point $a \in A$. Define $\alpha_a(t) := F(a,t)$ and denote by $x_a$ the point $\widetilde{f_0}(a)$. Then $\alpha_a$ is a path, and by the path lifting lemma, there exists a unique lift $\widetilde{\alpha}_a$ of this path such that $\widetilde{\alpha}(0) = x_a$. Now consider the homotopy defined by

$$\widetilde{F}(a,t) := \widetilde{\alpha}_a(t), \quad \text{for all} \quad a \in A, \quad t \in [0,1].$$

Then, we claim that $\widetilde{F}$ satisfies all the conditions of the theorem.

To complete the proof, one must verify that $\widetilde{F}$ is continuous and unique. We leave this verification to the reader. □

REMARK 6.3.5. Although the statement of the theorem is rather technical, the underlying idea is of fundamental importance. The *covering homotopy property* that it asserts holds not only for covering spaces, but more generally for arbitrary fiber bundles. Still more generally, this property holds for a very important class of fibrations, known as Serre fibrations (see ???), which are defined as precisely those which enjoy the covering homotopy property.

EXAMPLE 6.3.6. Let $X$ be the union of the lateral surface of the cone and the half-line issuing from its vertex $v$, and let $p : X \to B$ be the natural projection of $X$ on the line $B = \mathbb{R}$ (see the figure).

FIGURE ??? A map without the covering homotopy property

Then the covering homotopy property does not hold for $p$. Indeed, the path lifting property already fails for paths issuing from $p(v)$ and moving to the left of $p(v)$ (i.e., under the cone). Of course, lifts of such paths exist, but they are not unique, since they can wind around the cone in different ways.

## 6.4. Classification of coverings with given base via $\pi_1$

As we know, a covering space $p : X \to B$ induces a homomorphism $p_\# : \pi_1(X) \to \pi_1(B)$ (see ???). We will see that when the spaces $X$ and $B$ are "locally nice", $p_\#$ entirely determines (up to isomorphism) the covering space $p$ over a given $B$.

More precisely, in this section we will show that, provided that the "local nicety" condition holds, $p_\#$ is a monomorphism and that, given a subgroup $G$ of $\pi_1(B)$, we can effectively construct a unique space $X$ and a unique (up to isomorphism) covering map $p : X \to B$ for which $G$ is the image of $\pi_1(X)$ under $p_\#$. Moreover, we will prove that there is a bijection between conjugacy classes of subgroups of $\pi_1(B)$ and isomorphism classes of coverings, thus achieving the classification of all coverings over a given base $B$ in terms of $\pi_1(B)$.

Note that here we are *not* assuming that $G$ is a normal subgroup of $\pi_1(B)$, and so the covering space is not necessarily normal and $G$ does not necessarily coincide with the group of deck transformations.

**6.4.1. Injectivity of the induced homomorphism.** The goal of this subsection is to prove the following theorem.

THEOREM 6.4.1. *The homomorphism $p_\# : \pi_1(X) \to \pi_1(B)$ induced by any covering space $p : X \to B$ is a monomorphism.*

PROOF. The theorem is an immediate consequence of the homotopy lifting property proved in the previous section. Indeed, it suffices to prove that a nonzero element $[\alpha]$ of $\pi_1(X)$ cannot be taken to zero by $p_\#$. Assume that $p_\#([\alpha]) = 0$. This means that the loop $p \circ \alpha$, where $\alpha \in [\alpha]$, is homotopic to a point in $B$. By the homotopy lifting theorem, we can lift this homotopy to $X$, which means that $[\alpha] = 0$. $\square$

**6.4.2. Constructing the covering space.** Here we describe the main construction of this chapter: given a space and a subgroup of its fundamental group, we construct the associated covering. This construction works provided the space considered is "locally nice" in a sense that will be specified in the next subsection, and we will postpone the conclusion of the proof of the theorem until then.

THEOREM 6.4.2. *For any "locally nice" space $B$ and any subgroup $G \subset \pi_1(B, b_0)$ there exists a unique covering space $p : X \to B$ such that $p_\#(X) = G$.*

PROOF. The theorem is proved by means of another magical trick. Let us consider the set $P(B, b_0)$ of all paths in $B$ issuing from $b_0$. Two such paths $\alpha_i : [0,1] \to B$, $i = 1, 2$ will be identified (notation $\alpha_1 \sim \alpha_2$) if they have a common endpoint and the loop $\lambda$ given by

$$\lambda(t) = \begin{cases} \alpha_1(2t) & \text{if } \ 0 \le t \le 1/2, \\ \alpha_2(2-2t) & \text{if } \ 1/2 \le t \le 1. \end{cases}$$

determines an element of $\pi_1(B)$ that belongs to $G$. (The loop $\lambda$ can be described as first going along $\alpha_1$ (at double speed) and then along $\alpha_2$ from its endpoint back to $b_0$, also at double speed.)

Denote by $X := P(B, b_0)/_{\sim}$ the identification space of $P(B, b_0)$ by the equivalence relation just defined. Endow $X$ with the natural topology (the detailed definition appears in the next subsection) and define the map $p : X \to B$ by stipulating that it takes each equivalence class of paths in $P(B, b_0)$ to the endpoint of one of them (there is no ambiguity in this definition because equivalent paths have the same endpoint).

Then $p : X \to B$ is the required covering space. It remains to prove that

(i) $p$ is continuous;

(ii) $p$ is a local homeomorphism;

(iii) $p_{\#}(\pi_1(X))$ coincides with $G$;

(iv) $p$ is unique.

This will be done in the next subsection. □

To better understand the proof, we suggest that the reader do the following exercise.

EXERCISE 6.4.1. Prove the theorem in the particular case $G = 0$, i.e., construct the universal covering of $B$,

**6.4.3. Proof of continuity and uniqueness.** In this subsection, we fill in the missing details of the previous exposition: we specify what is meant by "locally nice" and use that notion to prove the continuity of different key maps constructed above and give rigorous proofs of their uniqueness properties.

DEFINITION 6.4.3. A topological space $X$ is called *locally path connected* if for any point $x \in X$ and any neighborhood $U$ of $x$ there exists a smaller neighborhood $V \subset U$ of $x$ which is path connected. A topological space $X$ is called *locally simply connected* if for any point $x \in X$ and any neighborhood $U$ of $x$ there exists a smaller neighborhood $V \subset U$ of $x$ which is simply connected.

EXAMPLE 6.4.4. Let $X \subset \mathbb{R}^2$ be the union of the segments

$$\{(x, y) \,|\, y = 1/n,\ 0 \leq x \leq 1\} \quad n = 1, 2, 3, \ldots$$

and the two unit segments $[0, 1]$ of the $x$-axis and $y$-axis (see the figure). Then $X$ is path connected but not locally path connected (at all points of the interval $(0, 1]$ of the $x$-axis).

FIGURE ??? Not locally connected and not simply connected spaces

EXAMPLE 6.4.5. Let $X \subset \mathbb{R}^2$ be the union of the circles

$$\{(x,y) \,|\, x^2 + (y - 1/n)^2 = 1/n^2\} \quad n = 1, 2, 3, \dots;$$

the circles are all tangent to the $x$-axis and to each other at the point $(0,0)$ (see the figure). Then $X$ is path connected but not locally simply connected (at the point$(0,0)$).

We will now conclude, step by step, the proof of the main theorem of the previous subsection under the assumption that $B$ is locally path connected and locally simply connected.

(o) *Definition of the topology in* $X = P(B, b_0)/_\sim$. In order to define the topology, we will specify a base of open sets of rather special form, which will be very convenient for our further considerations. Let $U$ be an open set in $B$ and $x \in X$ be a point such that $p(x) \in U$. Let $\alpha$ be one of the paths in $x$ with initial point $x_0$ and endpoint $x_1$. Denote by $(U, x)$ the set of equivalence classes (with respect to $\sim$) of extensions of the path $\alpha$ whose segments beyond $x_1$ lie entirely inside $U$. Clearly, $(U, x)$ does not depend on the choice of $\alpha \in x$.

We claim that $(U, x)$ actually does not depend on the choice of the point $x$ in the following sense: if $x_2 \in (U, x_1)$, then $(U, x_1) = (U, x_2)$. To prove this, consider the points $b_1 := p(x_1)$ and $b_2 := p(x_2)$. Join the points $b_1$ and $(b_2)$ by a path (denoted $\beta$) contained in $U$ (see the figure).

FIGURE ??? Defining the topology in the covering space

Let $\alpha\alpha_1$ denote an extension of $\alpha$, with the added path segment $\alpha_1$ contained in $U$. Now consider the path $\alpha\beta\beta^{-1}\alpha_1$, which is obviously homotopic to $\alpha\alpha_1$. On the other hand, it may be regarded as the extension (beyond $x_2$) of the path $\alpha\beta$ by the path $\beta^{-1}\alpha_{-1}$. Therefore, the assignment $\alpha\alpha_1 \mapsto \alpha\beta\beta^{-1}\alpha_1$ determines a bijection between $(U, x_1)$ and $(U, x_2)$, which proves our claim.

Now we can define the topology in $X$ by taking for a base of the topology the family of all sets of the form $(U, x)$. To prove that this defines a topology, we must check that a nonempty intersection of two elements of the base contains an element of the base. Let the point $x$ belong to the intersection of the sets $(U_1, x_1)$ and $(U_2, x_2)$. Denote $V := U_1 \cap U_2$ and consider the set $(V, x)$; this set is contained in the intersection of the sets $(U_1, x_1)$ and $(U_2, x_2)$ (in fact, coincides with it) and contains $x$, so that $\{(U, x)\}$ is indeed a base of a topology on $X$.

(i) *The map $p$ is continuous.* Take $x \in X$. Let $U$ be any path connected and simply connected neighborhood of $p(x)$ (it exists by the condition imposed on

$B$). The inverse image of $U$ under $p$ is consists of basis open sets of the topology of $X$ (see item (o)) and is therefore open, which establishes the continuity at the (arbitrary) point $x \in X$.

(ii) *The map $p$ is a local homeomorphism.* Take any point $x \in X$ and denote by $p|_U : (U, x) \to U$ the restriction of $p$ to any basis neighborhood $(U, x)$ of $x$, so that $U$ will be an open path connected and simply connected set in $B$. The path connectedness of $U$ implies the surjectivity of $p|_U$ and its simple connectedness, the injectivity of $p|_U$.

(iii) *The subgroup $p_\#(\pi_1(X))$ coincides with $G$.* Let $\alpha$ be a loop in $B$ with basepoint $b_0$ and $\widetilde{\alpha}$ be the lift of $\alpha$ initiating at $x_0$ ($\widetilde{\alpha}$ is not necessarily a closed path). The subgroup $p_\#(\pi_1)(X)$ consists of homotopy classes of the loops $\alpha$ whose lifts $\widetilde{\alpha}$ are closed paths. By construction, the path $\widetilde{\alpha}$ is closed iff the equivalence class of the loop $\alpha$ corresponds to the point $x_0$, i.e., if the homotopy class of $\alpha$ is an element of $G$.

(iv) *The map $p$ is unique.* To prove this we will need the following lemma.

LEMMA 6.4.6 (Map Lifting Lemma). *Suppose $p : X \to B$ is a covering space, $f : A \to B$ is a (continuous) map of a path connected and locally path connected spaces $A$ and $B$, and $f_\#$ is a monomorphism of $\pi_1(A, a_0)$ into $p_\#(\pi_1(X, x_0))$. Then there exists a unique lift $\widetilde{f}$ of the map $f$, i.e., a unique map $\widetilde{f} : A \to X$ satisfying $p \circ \widetilde{f} = f$ and $\widetilde{f}(a_0) = x_0$.*

PROOF. Consider an arbitrary path $\alpha$ in $A$ joining $a_0$ to some point $a$. The map $f$ takes it to to the path $f \circ \alpha$. By the Path Lifting Lemma (see ???), we can lift $f \circ \alpha$ to a (unique) path $\widetilde{a}$ in $X$ issuing from the point $x_0$. Let us define $\widetilde{f} : A \to X$ by setting $\widetilde{f}(a) = x$, where $x$ is the endpoint of the path $\widetilde{\alpha}$.

First let us prove that $\widetilde{f}$ is well defined, i.e., does not depend on the choice of the path $\alpha$. Let $\alpha_i$, $i = 1, 2$, be two paths joining $a_0$ to $a$. Denote by $\lambda$ the loop at $a_0$ defined by

$$\lambda(t) = \begin{cases} \alpha_1(2t) & \text{if } \ 0 \le t \le 1/2, \\ \alpha_2(2 - 2t) & \text{if } \ 1/2 \le t \le 1. \end{cases}$$

Then the lift of the loop $f \circ \lambda$ issuing from $x_0$ should be a closed path in $X_0$, i.e., the class of the loop $f \circ \alpha$ should lie in $p_\# \pi_1(X, x_0)$, i.e., we should have $f_\# \pi_1(A, a_0) \subset p_\# \pi_1(X, x_0)$. But this holds by assumption. Thus $\widetilde{f}$ is well defined.

It remains to prove that $\widetilde{f}$ is continuous. Let $a \in A$ and $x := \widetilde{f}(a)$. For the point $p(x)$, let us choose a path connected neighborhood $U$ from the definition of covering space. Let $\widetilde{U}$ be the path connected component of $p^{-1}(U)$ containing the point $x_0$. Since $f$ is continuous, $f^{-1}(U)$ contains a certain neighborhood of the point $a$. Since $A$ is locally path connected, we can assume that $V$ is path connected. Now we claim that $\widetilde{f}(V) \subset \widetilde{U}$ (which means that $\widetilde{f}$ is continuous). Indeed, any point $a_1 \in V$ can be joined to $\alpha_0$ by a path $\alpha$ entirely contained in $V$. Its image

$\alpha \circ f$ lies in $U$, therefore $\alpha \circ f$ can be lifted to a path entirely contained in $\widetilde{U}$. But this means that $\widetilde{f}(a) \in \widetilde{U}$. $\square$

Now to prove the uniqueness of $p$ (the covering space corresponding to the given subgroup of $\pi_1(B)$, where $B$ is path connected and locally path connected), suppose that we have two coverings $p_i : X_i \to B,\ \ i = 1, 2$ such that

$$(p_1)_\#\big(\pi_1(X-1, x_1)\big) \subset (p_2)_\#\big(\pi_1(X_2, x_2)\big).$$

something is wrong here

By the Map Lifting Lemma, we can lift $p_1$ to a (unique) map $h : X_1 \to X_2$ such that $h(x_1) = x_2$ and lift $p_2$ to a (unique) map $k : X_2 \to X_1$ such that $k(x_2) = x_1$. The maps $k$ is the inverse of $h$, so that $h$ is a homeomorphism, which proves that $p_1$ and $p_2$ are isomorphic.

EXAMPLE 6.4.7. This example, due to Zeeman, shows that for a covering space $p : X \to B$ with non locally path connected space $X$, the lift $\widetilde{f}$ of a map $f : A \to B$ (which always exists and is unique) may be discontinuous.

The spaces $A$, $B$, and $X$ consist of a central circle, one (or two) half circles, one (or two) infinite sequences of segments with common end points as shown on the figure. Obviously all three of these spaces fail to be locally path connected at the points $a, b, c, d, e$.

FIGURE ??? Zeeman's example

The covering space $p : X \to B$ is obtained by wrapping the central circle of $X$ around the central circle of $B$ twice, and mapping the segments and arcs of $X$ homeomorphically onto the corresponding segments and the arc of $B$. The map $f : A \to B$ (which also happens to be a covering) is defined exactly in the same way. Then we have

$$f_\#\big(\pi_1(A)\big) = p_\#\big(\pi_1(X)\big) \cong 2\mathbb{Z} \subset \mathbb{Z} \cong \pi_1(B),$$

but the map $f$ has no continuous lift $\widetilde{f}$, because the condition $f \circ \widetilde{f} = p$ implies the uniqueness of $\widetilde{f}$, but this map is necessarily discontinuous at the points $a$ and $b$.

EXERCISE 6.4.2. Can the identity map of $\mathbb{S}^1$ to $\mathbb{S}^1$ be lifted to the exponential covering $\mathbb{R} \to \mathbb{S}^1$?

EXERCISE 6.4.3. Prove that a map of $\mathbb{S}^1$ to $\mathbb{S}^1$ can be lifted to the exponential covering $\mathbb{R} \to \mathbb{S}^1$ if and only if its degree is zero.

EXERCISE 6.4.4. Describe the maps of $\mathbb{S}^1$ to $\mathbb{S}^1$ that can be lifted to the $n$-sheeted covering $w_n : \mathbb{S}^1 \to \mathbb{S}^1$.

EXERCISE 6.4.5. What maps of the circle to the torus $\mathbb{T}^2$ can be lifted to the universal covering $\mathbb{R}^2 \to \mathbb{T}^2$?

## 6.5. Coverings of surfaces and the Euler characteristic

In this section we compute the fundamental group of compact surfaces, use it to investigate the covering spaces of surfaces, and investigate the behavior of the Euler characteristic under the corresponding covering maps.

**6.5.1. Behavior of the Euler characteristic.** Here we will see that for a covering of a surface by another surface, the Euler characteristics of the surfaces are directly related to the number of sheets of the covering. In fact, we have the following theorem.

THEOREM 6.5.1. *For any $n$-sheeted covering $p : M \to N$ of the compact surface $N$ by the compact surface $M$, the Euler characteristics of the surfaces are related by the formula*

$$\boxed{\chi(N) \ = \ n \cdot \chi(M)}$$

PROOF. We will prove the theorem for triangulated surfaces, which does not restrict generality by Theorem ??? Let $U_1, \dots, U_k$ be a covering of $N$ by neighborhoods whose inverse images consist of $n$ disjoint sets each of which is mapped homeomorphically onto the corresponding $U_i$. Take a triangulation of $N$ and subdivide it barycentrically until each 2-simplex is contained in one of the $U_i$. Using the projection $p$, pull back this triangulation to $M$. Then the inverse image of each 2-simplex of $N$ will consist of $n$ 2-simplices of $M$ and we will have

$$\begin{aligned} \chi(M) &= V_M - E_M + F_M = nV_M - nE_M + nF_M \\ &= n(V_M - E_M + F_M) = n\chi(N), \end{aligned}$$

where $V$, $E$, and $F$ (with subscripts) stand for the number of vertices, edges, and faces of the corresponding surface. $\square$

EXAMPLE 6.5.2. There is no covering of the surface $M_{(4)}$ of genus 4 by the surface $M_{(6)}$ of genus 6. Indeed, the Euler characteristic of $M_{(4)}$ is $-6$, while the Euler characteristic of $M_{(6)}$ is $-10$, and $-6$ does not divide $-10$.

EXERCISE 6.5.1. Can the surface of genus 7 cover that of genus 5?

EXERCISE 6.5.2. State and prove a theorem similar to the previous one for coverings of graphs.

**6.5.2. Covering surfaces by surfaces.** After looking at some examples, we will use the previous theory to find the genus of surfaces that can cover a surface of given genus.

EXAMPLE 6.5.3. The surface $M_{(9)}$ of genus 9 is the 4-fold covering of the surface of genus $M_{(3)}$. To see this, note that $M_{(9)}$ has a $\mathbb{Z}_4$ rotational symmetry (this is clear from the figure) which can be regarded as a normal action of $\mathbb{Z}_4$ on $M_{(9)}$. Obviously, the corresponding quotient space is $M_{(3)}$.

FIGURE ??? A four-sheeted covering of surfaces

EXERCISE 6.5.3. Generalizing the previous example, construct a $d$-fold covering $p : X \to B$ of of the orientable surface of genus $k$ by the orientable surface of genus $d(k-1)+1$.

THEOREM 6.5.4. *A compact orientable surface $M$ covers a compact orientable surface $N$ if and only if the Euler characteristic of $N$ divides that of $M$.*

PROOF. If $\chi(N)$ does not divide $\chi(M)$, then $M$ cannot cover $N$ by the theorem in the previous subsection, which proves the "only if" part of the theorem. To prove the "if" part, assume that $\chi(M) = d \cdot \chi(N)$. Since the Euler characteristic of a surface can be expressed via its genus as $\chi = 2 - 2g$, we have $d(2-g_N) = 2-2g_M$, whence $g_M = dg_N - d - 1$. Now the construction from the previous exercise (which is a straightforward generalization of the one described in the previous example) completes the proof of the theorem. □

EXERCISE 6.5.4. Describe all possible coverings of nonorientable surfaces by nonorientable surfaces.

EXERCISE 6.5.5. Describe all possible coverings of orientable surfaces by nonorientable ones.

## 6.6. Branched coverings of surfaces

In this and the next section, we study the theory of branched coverings (also called ramified coverings) of two-dimensional manifolds (surfaces). This is a beautiful theory, originally coming from complex analysis, but which has drifted into topology and, recently, into mathematical physics, where it is used to study such fashionable topics as moduli spaces and Gromov–Witten theory.

**6.6.1. Main definitions.** Suppose that $M^2$ and $N^2$ are two-dimensional manifolds. Recall that a continuous map $p : M^2 \to N^2$ is said to be a *covering* (with *fiber* $\Gamma$, where $\Gamma$ is a fixed discrete space) if for every point $x \in N^2$ there exists a neighborhood $U$ and a homeomorphism $\varphi : p^{-1}(U) \to U \times \Gamma$ such that the restriction of $p$ to $p^{-1}(U)$ coincides with $\pi \circ \varphi$, where the map $\pi : U \times \Gamma \to U$ is the projection on the first factor. Then $M^2$ is called the *covering manifold* or *covering space*, while $N^2$ is the *base manifold* or *base*. If the fiber $\Gamma$ consists of $n$ points, then the covering $p$ is said to be $n$*-fold*.

A continuous map $p : M^2 \to N^2$ is said to be a *branched* (or *ramified*) *covering* if there exists a finite set of points $x_1, \dots, x_n \in N^2$ such that the set $p^{-1}(\{x_1, \dots, x_n\})$ is discrete and the restriction of the map $p$ to the set $M^2 - p^{-1}(\{x_1, \dots, x_n\})$ is a covering. In other words, after we delete a finite set of points, we get a covering. The points $x_1, \dots, x_n \in N^2$ that must be deleted are called the *branch points* of $p$. The following obvious statement not only provides an example of a branched covering, but shows how branched coverings behave near branch points.

PROPOSITION 6.6.1. *Let $D^2 = \{z \in \mathbb{C} : |z| \leqslant 1\}$ and let $p : D^2 \to D^2$ be the map given by the formula $p(z) = z^m$. Then $p$ is an $m$-fold branched covering with unique branch point $z = 0$.* $\square$

The example in the proposition for different $m$ describes the structure of an arbitrary branched covering near its branch points. Indeed, it turns out that if $p$ is an $n$-fold branched covering and $U$ is a sufficiently small disk neighborhood of a branch point, then $p^{-1}(U)$ consists of one or several disks on which $p$ has the same structure as the map in the proposition (in general, with different values of $m$). We shall not prove this fact in the general case, but in all the examples considered it will be easy to verify that this is indeed the case. If in a small neighborhood of a point $x$ of the covering manifold the covering map is equivalent to the map $z \mapsto z^m$, we shall say that $x$ has *branching index* $m$. The following proposition is obvious.

PROPOSITION 6.6.2. *For any $n$-fold branched covering, the sum of branching indices of all the preimages of any branch point is equal to $n$.*

Here is another, less trivial, example of a branched covering, which will allow us to construct a branched covering of the open disk (actually, the open set with boundary an ellipse) by the open annulus.

PROPOSITION 6.6.3. *Consider the map $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ given by the formula $f(z) = 2(z + 1/z)$. This map is a 2-fold branched covering with branch points*

$\pm 4$. *The preimages of these points are the points* $\pm 1$, *and the branching index of each is* 2.

PROOF. The equation $2(z + 1/z) = c$ is quadratic. Its discriminant $c^2/4 - 4$ vanishes iff $c = \pm 4$. This value is assumed by the function $f$ when $z = \pm 1$. $\square$

PROPOSITION 6.6.4. *Let* $p$ *be the restriction of the map* $f$ *from the previous proposition to the annulus* $C = \{z \in \mathbb{C} : 1/2 < |z| < 2\}$. *If* $z = \rho e^{i\varphi}$, *then we have*

$$p(z) = 2((\rho + 1/\rho)\cos\varphi + i(\rho - 1/\rho)\sin\varphi),$$

*so that the image of the annulus* $C$ *is the set of points located inside the ellipse* (*see Fig. ??*): $\{\rho = 5\cos\varphi + 3i\sin\varphi,\ 0 \leqslant \varphi < 2\pi\}$. $\square$

FIGURE ?? Branched covering of the ellipse by the annulus

A more geometric description of $p$ is the following. Imagine the open annulus $C$ as a sphere with two holes (closed disks) and an axis of symmetry $l$ (Fig.??). Let us identify points of the set $C$ symmetric with respect to $l$. It is easy to see that the resulting space is homeomorphic to the open disk $D^2$. The quotient map $p : C \to D^2$ thus constructed is a 2-fold branched covering with two branch points (the intersection points of $l$ with the sphere).

FIGURE ?? Sphere with two holes and symmetry axis

EXERCISE 6.6.1. Prove that if the base manifold $M^2$ of a branched covering $p : N^2 \to M^2$ is orientable, then so is the covering manifold $N^2$.

THEOREM 6.6.5. *Let* $M_g^2$ *be the sphere with* $g$ *handles. Then there exists a branched covering* $p : M_g^2 \to S^2$.

*First proof.* Consider a copy of the sphere with $g$ handles with an axis of symmetry $l$ (Fig.??). Identify all pairs of points symmetric with respect to $l$. The resulting quotient space is (homeomorphic to) the ordinary sphere $\mathbb{S}^2$. The natural projection $p : M_g^2 \to \mathbb{S}^2$ is a 2-fold branched covering which has $2g + 2$ branch points. $\square$

FIGURE ?? Branched covering of the 2-sphere

*Second proof.* Consider a *triangulation* of the manifold $M_g^2$. (This means that $M_g^2$ is cut up into (curvilinear) triangles, any two of which either intersect along a common side, or intersect in a common vertex, or have no common points.) Let $A_1, \dots, A_n$ be the vertices of the triangulation. On the sphere $\mathbb{S}^2$ choose $n$ points $B_1, \dots, B_n$ situated in general position in the following sense: no three of them lie on one and the same great circle and no two are antipodes. Then any three points $B_i, B_j, B_k$ uniquely determine a spherical triangle $\Delta_1$. Suppose $\Delta_2$ is the closure of its complement $S^2 - \Delta_1$; then $\Delta_2$ is also homeomorphic to the triangle. Therefore there exist homeomorphisms

$$f_1 : A_iA_jA_k \to \Delta_1, \qquad f_2 : A_iA_jA_k \to \Delta_2$$

that are linear in the following sense. We can assume that the length of a curve is defined both on the manifold $M_g^2$ and on the sphere $\mathbb{S}^2$; we require that an arbitrary point $X$ divide the arc $A_pA_q$ in the same ratio as the point $f_r(X), r = 1, 2$, divides the arc $B_pB_q$.

Let us fix orientations of $M_g^2$ and $S^2$. The orientations of the triangles $A_iA_jA_k$ and $B_iB_jB_k$ induced by their vertex order may agree with or be opposite to that of $M_g^2$ and $\mathbb{S}^2$. If both orientations agree, or both are opposite, then we map $A_iA_jA_k$ onto $\Delta_1 = B_iB_jB_k$ by the homeomorphism $f_1$. If one orientation agrees and the other doesn't, we map $A_iA_jA_k$ onto $\Delta_2$ (the complement to $B_iB_jB_k$) via $f_2$. Defining such maps on all the triangles of the triangulation of $M_g^2$, we obtain a map $f : M_g^2 \to \mathbb{S}^2$. We claim that this map is a branched covering.

For each interior point $x_0$ of triangle $A_iA_jA_k$ there obviously exists a neighborhood $U(x_0)$ mapped homeomorphically onto its image. Let us prove that this is not only the case for interior points of the triangles, but also for inner points of their sides. Indeed, let the side $A_iA_j$ belong to the two triangles $A_iA_jA_k$ and $A_iA_jA_l$. On the sphere $\mathbb{S}^2$, the great circle passing through the points $B_i$ and $B_j$ may either separate the points $B_k$ and $B_l$ or not separate them. In the first case we must have used the maps $f_1$ and $f_1$ (or the maps $f_2$ and $f_2$), in the second one $f_1$ and $f_2$ (or $f_2$ and $f_1$). In all cases a sufficiently small neighborhood of a point chosen inside $A_iA_j$ will be mapped bijectively (and hence homeomorphically) onto its own image (Fig.??). So at all points except possibly the vertices $A_1, \dots, A_n$ we have a covering, so $f$ is a branched covering. $\square$

FIGURE ?? Structure of $f$ at inner points of the sides

In fact not all the points $B_1, \dots, B_n$ are necessarily branch points, although some must be when $g > 0$. How few can there be? The answer is contained in the next statement, and surprisingly does not depend on $g$.

THEOREM 6.6.6. *Let $M_g^2$ be the sphere with g handles, where $g \geqslant 1$. Then there exists a branched covering $p : M_g^2 \to \mathbb{S}^2$ with exactly three branch points.*

PROOF. Choose an arbitrary triangulation of the manifold $M_g^2$ and take its baricentric subdivision, i.e., subdivide each triangle into 6 triangles by its three medians. To the vertices of the baricentric subdivision assign the numbers $0, 1, 2$ as follows:

$0$ to the vertices of the initial triangulation;
$1$ to the midpoints of the sides;
$2$ to the baricenters of the triangles.

If the orientation of some triangle $012$ induced by its vertex order agrees with that of the manifold $M_g^2$, then we paint the triangle black, otherwise we leave it white (Fig.??).

FIGURE ?? Black and white coloring of the baricentric subdivision

We can regard the sphere $\mathbb{S}^2$ as the union of two triangles glued along their sides; denote their vertices by $0', 1', 2'$ and paint one of these triangles black, leaving the other white. Now we map each white triangle $012$ from the baricentric subdivision in $M_g^2$ to the white triangle $0'1'2'$ in $\mathbb{S}^2$, and each black one to the black $0'1'2'$. In more detail this map was described in the second proof of Theorem ??, where it was established that this map is a branched covering. The three branch points are of course $0', 1', 2'$. $\square$

## 6.7. Riemann–Hurwitz formula

The formula that we prove in this section relates the topological properties of the base and covering manifolds in a branched covering with the branching indices. The main tool involved is the Euler characteristic, and we begin by recalling some of its properties.

**6.7.1. Some properties of the Euler characteristic for surfaces.** The properties that we need were proved in the previous chapter, but we restate them here for completeness as exercises.

Recall that a triangulation $K'$ is said to be a *subdivision* of the triangulation $K$ if any simplex of $K'$ is the union of simplices from $K$. Two triangulations $K_1$ and $K_2$ of a two-dimensional manifold are called *transversal* if their edges intersect transversally at a finite number of points. Any two triangulations of a 2-manifold can be made transversal by a small move (see ??)

EXERCISE 6.7.1. (a) Verify that the Euler characteristic of a 2-manifold does not change when we pass to a subdivision of its triangulation.

(b) Prove that any two transversal triangulations of a 2-manifold have a common subdivision. Using (a) and (b), show that the Euler characteristic for 2-manifolds does not depend on the choice of the triangulation.

EXERCISE 6.7.2. Suppose a surface $M$ is cut into two components $A$ and $B$ by a circle. Prove that in this case $\chi(M) = \chi(A) + \chi(B)$.

EXERCISE 6.7.3. Show that if the surface $M$ can be obtained from the surface $F$ by adding a handle (see Fig.??), then $\chi(M) = \chi(F) - 2$

FIGURE 21.1 Adding a handle

EXERCISE 6.7.4. Prove that if $M_g^2$ is the sphere with $g$ handles, then $\chi(M_g^2) = 2 - 2g$.

Thus the topological type of any oriented compact 2-manifold without boundary is entirely determined by its Euler characteristic.

**6.7.2. Statement of the Riemann-Hurwitz theorem.** The main result of this section is the following

THEOREM 6.7.1 (Riemann-Hurwitz formula). *Suppose $p : M^2 \to N^2$ is an $n$-fold branched covering of compact 2-manifolds, $y_1, \dots, y_l$ are the preimages of the branch points, and $d_1, \dots, d_l$ are the corresponding branching indices. Then*

$$\boxed{\chi(M^2) + \sum_{i=1}^{l}(d_i - 1) = n\chi(N^2).}$$

The proof of Theorem ?? will be easier to understand if we begin by recalling its proof in the particular case of non-branched coverings (see Theorem ?? in the previous section) and work out the following exercise.

EXERCISE 6.7.5. a) Let $p : M_g^2 \to N_h^2$ be the covering of the sphere with $h$ handles by the sphere with $g$ handles. Prove that $g - 1$ is divisible by $h - 1$.

b) Suppose that $g, h \geqslant 2$ and $g - 1$ is divisible by $h - 1$. Prove that there exists a covering $p : M_g^2 \to N_h^2$.

**6.7.3. Proof of the Riemann–Hurwitz formula.** First let us rewrite formula ?? in a more convenient form. After an appropriate renumbering of the branching indices, can assume that $d_1, \dots, d_{a_1}$ are the branching indices of all the preimages of one branch point, $d_{a_1+1}, \dots, d_{a_1+a_2}$ are the branching indices of all the preimages of another branch point, and so on. Since (see ??)

$$d_1 + \cdots + d_{a_1} = d_{a_1+1} + \cdots + d_{a_1+a_2} = \cdots = n,$$

we obtain

$$\sum_{i=1}^{l}(d_i - 1) = (n - a_1) + (n - a_2) + \cdots = kn - a_1 - a_2 - \cdots - a_k,$$

where $k$ is the number of branch points and $a_i$ is the number of preimage points of the $i$th branch point. Hence formula ?? can be rewritten as

$$\chi(M^2) = n(\chi(N^2) - k) + a_1 + \cdots + a_k.$$

It will be more convenient for us to prove the Riemann–Hurwitz formula in this form.

The manifolds $M^2$ and $N^2$ may be presented as follows

$$M^2 = A_M \cup B_M \quad \text{and} \quad N^2 = A_N \cup B_N,$$

where $A_N$ is the union of the closures of small disk neighborhoods of all the branch points, $A_M$ is the inverse image of $A_N$, while $B_M$ and $B_N$ are the closures of the complements $M^2 - A_M$ and $N^2 - A_N$. The sets $A_M \cap B_M$ and $A_N \cap B_N$ consist of nonintersecting circles, and so we can use formula from Exercise ??. As the result we get

$$\chi(M^2) = \chi(A_M) + \chi(B_M), \quad \chi(N^2) = \chi(A_N) + \chi(B_N).$$

The restriction of the map $p$ to the set $B_M$ is a (nonbranched) covering, so by Theorem ?? we have

$$\chi(B_M) = n\chi(B_N).$$

The set $A_N$ consists of $k$ disks, while $A_M$ consists of $a_1 + \cdots + a_k$ disks. Therefore

$$\chi(A_M) = a_1 + \cdots + a_k, \quad \chi(A_N) = k.$$

Combining the displayed formulas, we get

$$\chi(M^2) = a_1 + \cdots + a_k + n\chi(B_N) = a_1 + \cdots + a_k + n(\chi(N^2) - k),$$

which is the required formula ??. This completes the proof of the Riemann–Hurwitz formula. □

**6.7.4. Some applications.** The Riemann–Hurwitz formula has numerous applications. The first one that we discuss has to do with Theorem ??, which asserts the existence of a branched covering $p : M_g^2 \to \mathbb{S}^2$ with exactly 3 branch points (when $g \geqslant 1$). Can this number be decreased? The answer is 'no', as the next statement shows.

THEOREM 6.7.2. *If $g \geqslant 1$, there exists no branched covering $p : M_g^2 \to \mathbb{S}^2$ of the sphere by the sphere with $g$ handles having less than 3 branch points.*

PROOF. By formula ??, we have

$$2-2g=\chi(M_g^2)=n(\chi(S^2)-k)+a_1+\cdots+a_k=n(2-k)+a_1+\cdots+a_k.$$

If $k\leqslant 2$, then $n(2-k)\geqslant 0$ and hence $n(2-k)+a_1+\cdots+a_k>0$, because the relations $n(2-k)=0$ and $a_1+\cdots+a_k=0$, i.e., $k=0$, cannot hold simultaneously. Thus $2-2g>0$, hence $g=0$, contradicting the assumption of the theorem. Therefore $k>2$ as asserted. □

EXERCISE 6.7.6. Prove that if $p:\mathbb{D}^2\to\mathbb{D}^2$ is a branched covering of the disk by the disk with exactly one branch point, then the preimage of the branch point consists of one point.

**6.7.5. Genus of complex algebraic curves.** The Riemann–Hurwitz formula can be applied to the computation of the genus of algebraic curves in $\mathbb{C}P^2$. We begin with the necessary background material. An *algebraic curve* of degree $n$ in $\mathbb{C}P^2$ is the set of points satisfying the homogeneous equation

$$F(x,y,z)=\sum_{i+j+k=n}a_{ij}x^iy^jz^{n-i-j},\quad x,y,z\in\mathbb{C}P^2.$$

When $z=1$ and $x,y\in\mathbb{R}$, we get a *plane algebraic curve*:

$$\sum_{i+j\leqslant n}a_{ij}x^iy^j=0.$$

If the gradient

$$\mathrm{grad}F=\left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right)$$

does not vanish at all points of an algebraic curve in $\mathbb{C}P^2$, then the curve is called *nonsingular*. If the polynomial $F$ is irreducible, i.e., cannot be represented as the product of two homogeneous polynomials of lesser degree, then the curve is said to be *irreducible*. It can be proved that any nonsingular irreducible algebraic curve in $\mathbb{C}P^2$ is homeomorphic to the sphere with $g$ handles for some $g\geqslant 0$; the nonnegative integer $g$ is called the *genus* of the curve.

PROPOSITION 6.7.3. *Fermat's curve* $x^n+y^n+z^n=0$ *is of genus* $(n-1)(n-2)/2$.

PROOF. First note that Fermat's curve $\Gamma\subset\mathbb{C}P^2$ is nonsingular, because

$$\mathrm{grad}F=n\left(x^{n-1},y^{n-1},z^{n-1}\right)\neq 0\quad\forall(x:y:z)\in\mathbb{C}P^2.$$

Now consider the map $p:\mathbb{C}P^2-\{(0:0:1)\}\to\mathbb{C}P^1$ that takes the point $(x:y:z)\in\mathbb{C}P^2$ to $(x:y)\in\mathbb{C}P^1$. The point $(0:0:1)$ does not lie on the curve $\Gamma$, therefore the map $p$ induces the map $p':\Gamma\to\mathbb{C}P^1=\mathbb{S}^2$. The inverse image of the point $(x_0:y_0)\in\mathbb{C}P^1$ consists of all points $(x_0:y_0:z)\in\mathbb{C}P^2$ such that $z^n=-(x_0^n+y_0^n)$. When $x_0^n+y_0^n\neq 0$, the inverse image consists of $n$ points, when $x_0^n + y_0^n = 0$, of only one. Hence $p'$ is an $n$-fold branched covering with branch points $(1 : \varepsilon_n) \in \mathbb{C}P^1$, where $\varepsilon_n$ is a root of unity of degree $n$. So there are $n$ branch points, and the inverse image of each consists of one point. According to formula ??, we get

$$\chi(\Gamma) = n(\chi(S^2) - n) + n = n(2-n) + n = -n^2 + 3n.$$

Therefore,

$$g = \frac{2-\chi(\Gamma)}{2} = \frac{n^2 - 3n + 2}{2} = \frac{(n-1)(n-2)}{2}.$$

□

PROPOSITION 6.7.4. *The hyperelliptic curve $y^2 = P_n(x)$, where $P_n$ is a polynomial of degree $n \geqslant 5$ without multiple roots, is of genus*

$$\left[\frac{n+1}{2}\right] - 1.$$

REMARK 6.7.5. This statement is also true for $n < 5$. When $n = 3, 4$ the curve $y^2 = P_n(x)$ is called *elliptic*.

PROOF. Let $P_n(x) = a_0 + \cdots + a_n x^n$. Then the hyperelliptic curve $\Gamma$ in $\mathbb{C}P^2$ is given by the equation

$$y^2 z^{n-2} = \sum_{k=0}^{n} a_k x^k z^{n-k}.$$

For the curve $\Gamma$, we have grad$F = 0$ at the point $(0:1:0) \in \Gamma$, so the hyperelliptic curve is singular.

Consider the map

$$p : \mathbb{C}P^2 - \{(0:1:0)\} \to \mathbb{C}P^1, \quad \mathbb{C}P^2 \ni (x:y:z) \mapsto (x:z) \in \mathbb{C}P^1.$$

Let $p' : \Gamma - \{(0:1:0)\} \to \mathbb{C}P^1$ be the restriction of $p$. We claim that for $x = 1$ the preimage of the point $(x : z)$ tends to the singular point $(0:1:0) \in \Gamma$ as $z \to 0$. Indeed, by ?? we have $y^2 \approx a_n z^{2-n} \to \infty$, so that

$$(1:y:z) = (1/y : 1 : z/y) \to (0:1:0).$$

Therefore the map $p'$ can be extended to a map of the whole curve, $p' : \Gamma \to \mathbb{C}P^1$, by putting $p'((0:1:0)) = (1:0)$.

In order to find the inverse image under $p'$ of the point $(x_0 : z_0) \in \mathbb{C}P^1$ when $z_0 \neq 0$, we must solve the equation

$$y^2 = z_0^2 \sum_k a_k \cdot \left(\frac{x_0}{z_0}\right)^k = z_0^2 \; P_n\left(\frac{x_0}{z_0}\right).$$

If $x_0/z_0$ is not a root of the polynomial $P_n$, then this equation has exactly two roots. Therefore the map $p' : \Gamma \to \mathbb{C}P^1$ is a double branched covering, the branch points being $(x_0 : z_0) \in \mathbb{C}P^1$, where $x_0/z_0$ is a root of $P_n$ and, possibly, the point

$(1:0)$. We claim that $(1:0)$ is a branch point iff $n$ is odd. To prove this claim, note that for small $z$ the preimage of the point $(1:z)$ consists of points of the form $(1:y:z)$, where $y^2 \approx a_n z^{2-n}$. Let $z = \rho e^{i\phi}$. When $\phi$ varies from $0$ to $2\pi$, i.e., when we go around the point $(1:0) \in \mathbb{C}P^1$, the argument of the point $y \in \mathbb{C}$ changes by

$$(2-n)2\pi/2 = (2-n)\pi.$$

Therefore for odd $n$ the number $y$ changes sign, i.e., we switch to a different branch, while for $n$ even $y$ does not change, i.e., we stay on the same branch.

Thus the number of branch points is $2[(n+1)/2]$. Let $g$ be the genus of the curve $\Gamma$. Then, according to ??,

$$2-2g = 2\left(2-2\left[\frac{n+1}{2}\right]\right) + 2\left[\frac{n+1}{2}\right], \quad \text{i.e.,} \quad g = \left[\frac{n+1}{2}\right] - 1.$$

□

## 6.8. Problems

EXERCISE 6.8.1. Give an example of the covering of the wedge sum of two circles which is not normal.

EXERCISE 6.8.2. Prove that any nonorientable surface possesses a double covering by an oriented one.

EXERCISE 6.8.3. Prove that any subgroup of a free group is free by using covering spaces.

EXERCISE 6.8.4. Prove that if $G$ is a subgroup of a free group $F$ of index $k := [F : G] \le \infty$, then its rank is given by $\operatorname{rk} G = k(\operatorname{rk} F - 1) + 1$

REMARK 6.8.1. Note that the purely group-theoretic proofs of the two theorems appearing in the previous two exercises, especially the first one, are quite difficult and were hailed, in their time, as outstanding achievements. As the reader should have discovered, their topological proofs are almost trivial.

EXERCISE 6.8.5. Prove that the free group of rank 2 contains (free) subgroups of any rank $n$ (including $n = \infty$)

EXERCISE 6.8.6. Give two examples of nonisomorphic three-sheeted coverings of the wedge of two circles,

EXERCISE 6.8.7. Suppose the graph $G''$ covers the graph $G'$. What can be said about their Euler characteristics?

EXERCISE 6.8.8. Prove that any double (i.e., two-sheeted) covering is normal. To what group-theoretic statement does this fact correspond?

EXERCISE 6.8.9. Prove that any three-sheeted covering of the sphere with two handles cannot be normal (i.e., regular in the traditional terminology).

EXERCISE 6.8.10. Construct a double covering of $n$-dimensional projective space $\mathbb{R}P^n$ by the sphere and use it to prove that

(i) $\mathbb{R}P^n$ is orientable for even $n$ and orientable for odd $n$.

(ii) $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ and $\pi_k(\mathbb{R}P^n) \cong \pi_k(\mathbb{S}^n)$ for $n \geq 2$.

EXERCISE 6.8.11. Prove that all the coverings of the torus are normal (i.e., regular in the traditional terminology) and describe them.

EXERCISE 6.8.12. Find the universal covering of the Klein bottle $K$ and use it to compute the homotopy groups of $K$.

EXERCISE 6.8.13. Can the torus double cover the Klein bottle?

EXERCISE 6.8.14. Prove that any deck transformation of an arbitrary (not necessarily normal) covering is entirely determined by one point and its image.