CHAPTER 10

SMOOTH MANIFOLDS REVISITED

We return to the general discussion of differentiable manifolds started in Chapter 4.

10.1. Basics, notation

Here, for the reader's convenience, we recall the main notions related to smooth manifolds that we studied in Chapter 4, but only to refresh the terminology and fix the notation.

Let M be a smooth manifold of dimension n. Then by

$$A_M = \Big\{ \big(U_i;\ \ h_i = (x^1, \dots, x^n) : U_i \to \mathbb{R}^n\big) \Big\}$$

we denote some atlas of M; here the h_i are the charts (local coordinates) of the atlas (we assume that each h_i is a homeomorphism of U_i *onto* $\mathbb{R}^n$); by $\varphi_{i,j}$ we denote the transition functions (coordinate transformations) given by $\varphi_{i,j} = h_j \circ h_i^{-1}$.

Further we denote the tangent bundle of M by $\tau : TM \to M$; if $p \in M$, then $T_pM = \tau^{-1}(p)$ is the tangent space at p. The space T_pM is an n-dimensional vector space with basis

$$\partial_{p,1} = \frac{\partial}{\partial x^1}\Big|_p, \dots, \partial_{p,n} = \frac{\partial}{\partial x^n}\Big|_p;$$

we will use the shorter notation $\partial_{p,i}$ rather than the cumbersome (but more often used) notation $(\partial/\partial x^i)\big|_p$. Recall that any element (vector) v_p of T_pM is a derivation, i.e., an $\mathbb{R}$-valued linear functional defined on functions (given near p) and satisfying the *local Leibnitz rule*:

$$v_p(f \cdot g) = f(p) \cdot v_p(g) + v_p(f) \cdot g(p).$$

The value of a given vector on a given function can be calculated as the linear combination of the partial derivatives of the function $f \circ h^{-1} : \mathbb{R}^n \to \mathbb{R}$ at the point $h^{-1}(p)$.

The set of smooth sections of the tangent bundle $\tau : TM \to M$, i.e., smooth maps $V : M \to TM$ such that $\tau \circ X = \mathrm{Id}_M$, is denoted by $\Gamma(M)$; its elements are called (*smooth*) *vector fields*. The value of the vector field V at the point $p \in M$ is a tangent vector at p that we will denote by V_p (rather than $V(p)$). The set $\Gamma(M)$ of all smooth vector fields has the natural structure of a module over $C^\infty(M)$. Locally, in a fixed coordinate system $\big(U, (x^1, \dots, x^n)\big)$, the module

$\Gamma(U)$ is finitely generated with basis

$$\partial_1 = \frac{\partial}{\partial x^1}, \dots, \partial_n = \frac{\partial}{\partial x^n}.$$

Of course, as a topological space, $\Gamma(M)$ (and even the space $\Gamma(\mathbb{R}^n)$ is infinite-dimensional.

The *trajectory* of a vector field V is a smooth map $\alpha : \mathbb{R} \to M$ whose tangent vector at each point $p \in M$ coincides with V_p.

We denote the algebra of smooth functions on a manifold M by F_M. Recall that in the local coordinates $(U,\ h = x^1, \dots, x^n)$ any function $f \in F$ can be expressed locally (in the neighborhood U) in coordinate form; we write $f(q) = f\big(x^1(q), \dots, x^n(q)\big)$ for any $q \in U$. For any $f \in F$, the function $f \circ h^{-1} : \mathbb{R}^n \to \mathbb{R}$ is infinitely differentiable.

The algebra F_M is a commutative associative infinite-dimensional (provided $n = \dim M \geq 1$) algebra over $\mathbb{R}$ with unit. It has numerous zero divisors: the product of two nonzero functions f and g can be zero (this occurs if their supports do not intersect: $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$).

10.2. Vector fields, flows and differential operators

By the theorems of existence, uniqueness, and smooth dependence for solutions of ordinary differential equations a C^1 vector field on M induces a local flow, that is, for every $p \in M$ there is a curve $c_{v,p}\colon (-\varepsilon, \varepsilon) \to M$ such that $c_{v,p}(0) = p$ and $\dot{c}_{v,p}(t) := \dfrac{d}{dt} c_{v,p}(t) = v(c_{v,p}(t))$. Here ε can be chosen to depend continuously on p. Where defined the map $\varphi_v\colon (p,t) \mapsto \varphi^t(p) := c_{v,p}(t)$ is as smooth as v. By continuity of ε it is bounded on any compact manifold and hence by the group property $c_{v,p}(t+s) = c_{v(c_{v,p}(t)), c_{v,p}(t)}(s)$ (which follows from uniqueness) every vector field on a compact manifold induces a *complete* flow, that is, φ_v^t is defined for *all* times. If φ_v^t and φ_w^s are the flows for vector fields v and w, respectively, then usually the diffeomorphisms φ_v^t and φ_w^t do not commute, that is, $\varphi_v^t \circ \varphi_w^s \neq \varphi_w^s \circ \varphi_v^t$. If they do, the vector fields v and w are said to *commute*. The extent to which two vector fields v, w fail to commute is measured by their Lie bracket $[v,w]$ which can be computed as $[v,w](p) = \lim_{t \to 0} \big(w - d\varphi_v^t w\big)(\varphi_v^t(p))/t$.

Let us now show briefly how these invariant notions appear in local coordinates. If (U,h) is a chart then we say that we have coordinates $(x^1, \dots, x^n)$ on U. For $p \in U$ the canonical basis of T_pM is the set of *derivations* $\partial/\partial x^i$ induced by the curves $c_i(t) := h^{-1}(h(p) + te_i)$, where e_i is the ith standard basis vector in $\mathbb{R}^n$. A tangent vector $v \in T_pM$ can then be written as $v = \sum_{i=1}^n v^i \partial/\partial x^i$ and if $f\colon M \to \mathbb{R}$ is smooth then $vf = \sum_{i=1}^n v^i \partial(f \circ h^{-1})/\partial x^i$. Thus the induced coordinates of TM are $(x^1, \dots, x^n, v^1, \dots, v^n)$, where the v^i are the components we just defined. Likewise a vector field is locally given by a representation $v(p) = \sum_{i=1}^n v^i(p)\partial/\partial x^i$ and it is smooth if and only if the v^i are. To see that the Lie bracket of two vector fields v, w defines a derivation, that is, a vector field, we calculate in local coordinates. Namely, write $v = \sum_{i=1}^n v^i \partial/\partial x^i$,

$w = \sum_{i=1}^n w^i \partial/\partial x^i$ and for convenience write f for $f \circ h$. Then using the theorem of H. A. Schwarz that second partial derivatives commute we obtain

$$\begin{aligned}
(vw - wv)f &= v \sum_{i=1}^n w^i \frac{\partial f}{\partial x^i} - w \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i} \\
&= \sum_{i,j=1}^n v^j \frac{\partial w^i}{\partial x^j} \frac{\partial f}{\partial x^i} + \sum_{i,j=1}^n v^j w^i \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{i,j=1}^n w^j \frac{\partial v^i}{\partial x^j} \frac{\partial f}{\partial x^i} + \sum_{i,j=1}^n v^i w^j \frac{\partial^2 f}{\partial x^j \partial x^i} \\
&= \sum_{i,j=1}^n \Big(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \Big) \frac{\partial f}{\partial x^i},
\end{aligned}$$

that is, $[v, w]$ is indeed a vector field given locally by $v^j \dfrac{\partial w^i}{\partial x^j} - w^i \dfrac{\partial v^j}{\partial x^i}$. In particular $[\partial/\partial x^i, \partial/\partial x^j] = 0$. There are several important properties of Lie brackets that are not hard to check in local coordinates. By definition we obviously have $[v, w] = -[w, v]$ and $[\cdot, \cdot]$ is $\mathbb{R}$-bilinear, that is, $[\alpha v + \beta w, z] = \alpha[v, z] + \beta[w, z]$ for $\alpha, \beta \in \mathbb{R}$. Next observe that for functions as coefficients we get $[fv, gw] = fg[v, w] + f(vg)w - g(wf)v$ by a coordinate calculation similar to the preceding one. This means in particular (for $f \equiv 1$) that the Lie derivative is a derivation, that is, satisfies the product rule $\pounds_v(gw) = g\pounds_v w + \pounds_v g\, w$. Furthermore there is the fundamental *Jacobi identity*

$$[v, [w, z]] + [w, [z, v]] + [z, [v, w]] = 0. \tag{10.2.1}$$

This is straightforward in coordinates. Namely, we know that only first-order derivatives occur, so we may simplify the calculation by discarding all higher-order derivatives. The symmetry then makes the remaining terms cancel. Alternatively write $[v, w] = vw - wv$ and expand (10.2.1) accordingly to see that all terms cancel.

Differentiating differentiable maps between manifolds is also straightforward calculus on local coordinates: If $f\colon M \to N$ and (U, h), (V, k) are local charts around $p \in M$ and $f(p) \in N$, respectively, then the differential of f at p is represented by the matrix of partial derivatives of the map $k \circ f \circ h^{-1}$ in Euclidean space with respect to the standard bases.

10.3. Tensor bundles

The tangent bundle is an example of the following:

DEFINITION 10.3.1. A differentiable *vector bundle* with *structure group* G, a subgroup of $GL(m, \mathbb{R})$, over M (the *base space*) is a manifold P, called the *total space* or *bundle space*, such that the projection $\pi\colon P \to M$ is differentiable and furthermore locally $P = M \times \mathbb{R}^m$, that is, every $x \in M$ has a neighborhood U such that there is a diffeomorphism $h\colon \pi^{-1}(U) \to U \times \mathbb{R}^m$, $u \mapsto (\pi(u), \varphi(u))$ and such that for any point x in the intersection $U_1 \cap U_2$ of two such neighborhoods the trivialization differs by an element of G. A *subbundle* or *distribution* is a bundle whose fibers are contained in those of P. For two distributions E, F we define the *Whitney sum* $E+F$ to be the distribution with $(E+F)_p = E_p + F_p$. We use "$\oplus$" if the sum is (pointwise) direct, that is, $E_p \cap F_p = \{0\}$ for all $p \in M$. A *section* of P is a map $v\colon M \to P$ such that $\pi \circ v = \mathrm{Id}_M$.

EXAMPLE 10.3.2. The tangent bundle TM of M is of this form: Here m is the dimension of M and $G = GL(m,\mathbb{R})$ acts by the linear coordinate changes in the tangent fibers induced by coordinate change in the base. The sections are the vector fields. If there is a nonvanishing vector field on M then the one-dimensional subspaces it spans at every point define a one-dimensional distribution.

Note that the differentiable manifold TM has in turn a tangent bundle TTM. This is an important object. On one hand it allows us to differentiate vector fields. On the other hand classical mechanics involves second-order differential equations and the natural setting for second derivatives is the second (or double) tangent bundle TTM.

The second tangent bundle TTM is obviously a vector bundle over TM, but it is, in fact, a vector bundle over M as well. To that end notice that coordinate changes in M change coordinates in TTM by a coordinate change determined again by the linear part of the coordinate change in M. We will return to this in the setting of Riemannian manifolds.

From the linear structure in the tangent spaces arise linear objects other than vectors and linear maps (for example, differentials). Namely, it is often important to consider *multilinear* maps. The easiest examples, and a building block, are *1-forms*.

DEFINITION 10.3.3. We denote by T^*M the *cotangent bundle* consisting of the spaces $T_p^*M = (T_pM)^*$ of linear maps (covectors) $T_pM \to \mathbb{R}$. A section of T_p^*M is called a *1-form*. A multilinear map $\underbrace{T_p^*M \oplus \cdots \oplus T_p^*M}_{k \text{ times}} \oplus \underbrace{T_pM \oplus \cdots \oplus T_pM}_{l \text{ times}} \to \mathbb{R}$ (that is, linear in each entry independently) is called a (k,l)-*tensor*. A section of the bundle $TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M = (TM)^{\otimes k} \otimes (T^*M)^{\otimes l}$ is a (k,l)-*tensor field* (or tensor). A tensor is called *smooth* if its values on smooth vector and covector fields define a smooth function. (Alternatively, if its coefficients in local coordinates are smooth.)

Thus a vector is a $(1,0)$-tensor, a 1-form is a $(0,1)$-tensor, and the Riemannian metrics defined in Definition 13.2.1 are $(0,2)$-tensors. A basis for the space of 1-forms on T_pM is given by the forms dx^i which are given by the derivatives of the coordinate functions x^i, that is, $dx^i(\partial/\partial x^j) = \delta^i_j := \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$ The derivative of a function f is a 1-form $Df(v) := vf = \sum_{i=1}^n \partial f/\partial x_i \, dx^i$. If T is a (k,l)-tensor then $T = T^{j_1,\dots,j_k}_{i_1,\dots,i_l} \partial/\partial x^{j_1} \otimes \cdots \otimes \partial/\partial x^{j_k} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_l}$ with $T^{j_1,\dots,j_k}_{i_1,\dots,i_l} = T(dx^{j_1},\dots,dx^{j_k},\partial/\partial x^{i_1},\dots,\partial/\partial x^{i_l})$. There is a natural way to extend the Lie derivative to tensors. Namely, note first that for $(1,0)$-tensors (vector fields) it is already defined and that for $(0,0)$-tensors (functions) we can define $\pounds_v f := vf$. Now extend to $(0,1)$-tensors ξ by setting $\pounds_v(\xi(w)) = \pounds_v(\xi)(w) + \xi(\pounds_v w)$. Likewise one can extend $\pounds_v$ to any tensor field by postulating the product rule $\pounds_v(\xi \otimes \eta) = \pounds_v \xi \otimes \eta + \pounds_v \eta \otimes \xi$. If ω is a $(0,1)$-tensor on N and $f \colon M \to N$ differentiable then we can define the *pullback* $f^*\omega$ of ω on M by $f^*\omega(v) := \omega(Dfv)$. This, of course, works for $(0,k)$-tensors just as well. Likewise one can send vectors from M to N via Df, but this can be expected to send vector fields to vector fields only if f is injective (if $f(p) = f(q)$ and v is a vector field such that $Dfv(p) \neq Dfv(q)$ then there is no well-defined vector field "f_*v" on $f(M)$). If f is a diffeomorphism then this is no problem, however. Using pullbacks the Lie derivative of a $(0,k)$-tensor can be computed by using the flow φ^t defined by the vector field v to write

$$\pounds_v \omega = \lim_{t\to 0} (1/t)((\varphi^t)^*\omega - \omega).$$

The Lie derivative of any (k,l)-tensor can be computed similarly.

An important special class of tensors is that of alternating ones:

DEFINITION 10.3.4. A $(0,k)$-tensor ω on a linear space is said to be an *alternating tensor* or an *(exterior) form* if $\omega(v_1, \dots, v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$. A $(0,k)$-tensor field is said to be *alternating* if it is alternating at every point. Alternating $(0,k)$-tensor fields are called *k-forms*, and the space of k-forms is denoted by $\Gamma(\bigwedge^k T^*M)$. In analogy to the asymmetric part of a matrix the alternating part $\mathcal{A}\eta$ of a $(0,k)$-tensor η is defined by $\mathcal{A}\eta = 1/k! \sum_{\pi\in S_k} \operatorname{sgn}\pi\, \eta \circ \pi$, where π permutes the entries and $\operatorname{sgn}\pi$ is its sign, that is, -1 if π is odd, 1 otherwise. Thus $\mathcal{A}$ is a projection of $(T^*M)^{\otimes k}$ to $\bigwedge^k T^*M$. We define the *wedge product* or *exterior product* of $\omega \in \bigwedge^k T^*M$ and $\eta \in \bigwedge^l T^*M$ by

$$\omega \wedge \eta := \frac{(k+l)!}{k!\, l!} \mathcal{A}(\omega \otimes \eta) \in \bigwedge^{k+l} T^*M.$$

Nonzero elements of $\Gamma(\bigwedge^n T^*M)$ are called *volume elements* and two volume elements Ω, Ω' are said to be *equivalent* if $\Omega' = f\Omega$ for some $f \in C^\infty(M)$, $f > 0$. An equivalence class of volume forms is called an *orientation* of M and M is called *orientable* if there exists an orientation on M.

With these definitions one gets the following standard facts: $\omega \wedge \eta$ is $\mathbb{R}$-bilinear in ω and η, $\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta$ (hence $\omega \wedge \omega = 0$ for odd k), $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$, and $\omega \wedge (\eta \wedge \lambda) = (\omega \wedge \eta) \wedge \lambda =: \omega \wedge \eta \wedge \lambda$. A basis for $\bigwedge^k T_p^*M$ is given by $\{dx^{i_1} \wedge \cdots \wedge dx^{i_k} \mid 1 \leq i_j \leq n\}$, where $\{dx^i \mid 1 \leq i \leq n\}$ is the dual basis for $\{\partial/\partial x^i \mid 1 \leq i \leq n\}$. Thus $\dim \bigwedge^k T_p^*M = \binom{n}{k}$. In fact, $\beta^1 \wedge \cdots \wedge \beta^k \neq 0$ if and only if $\{\beta^1, \dots, \beta^k\} \subset T_p^*M$ is linearly independent.

A manifold is orientable if and only if $\Gamma(\bigwedge^n T^*M)$ is one-dimensional over $C^\infty(M)$. (Namely, there exists a volume, hence the dimension is at least one, and for two volumes Ω and Ω' the function $\varphi := \Omega'/\Omega$ is well defined, since $\Gamma(\bigwedge^n T_p^*M)$ is one-dimensional, and smooth as well.) One can also check that orientability is equivalent to the existence of an oriented atlas, that is, an atlas where $h \circ h'$ preserves the orientation of $\mathbb{R}^n$ for any two charts h, h'. On a compact manifold a volume form can be integrated to give the total volume. This is done via charts as follows. In $\mathbb{R}^n$ we define $\int \Omega := \int \Omega_{1,\dots,n} dx^1 \cdots dx^n$ for any volume $\Omega = \Omega_{1,\dots,n} dx^1 \wedge \cdots \wedge dx^n$. For orientation-preserving diffeomorphisms f we get $\int f^*\Omega = \int \Omega$. Thus we can define $\int \Omega$ for a manifold M by taking a partition of unity $\{U_i, \psi_i\}$ subordinate to a covering by charts (V_i, h_i) and define $\int \Omega := \sum_i \int (h_i)_*(\psi_i \Omega)$, and this definition via charts is coordinate independent.

10.4. Exterior calculus and de Rham differential

Next we want to study the calculus of exterior forms, also called exterior calculus.

DEFINITION 10.4.1. The *exterior derivative* $d\colon \Gamma(\bigwedge^k T^*M) \to \Gamma(\bigwedge^{k+1} T^*M)$ (for any k) is defined by the following axioms (which uniquely determine d): $df = Df$ for functions, d is $\mathbb{R}$-linear and $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$, $d \circ d = 0$, and d is locally defined, that is, if two forms coincide on an open set O then their derivatives coincide on O as well.

By induction on dimension one sees that this is well defined. Namely, if $\omega = \varphi d\psi^1 \wedge \cdots \wedge d\psi^k$ then necessarily $d\omega = d\varphi \wedge d\psi^1 \wedge \cdots \wedge d\psi^k$. The last property is also satisfied inductively since it holds for functions: $dd\varphi = \sum_{i,j=1}^n (DD\varphi)_{ij} dx^i \wedge dx^j = \dfrac{\partial^2 \varphi}{\partial x^i \partial x^j} dx^i \wedge dx^j = 0$. Furthermore d commutes with pullback and the Lie derivative: $f^* d\omega = d(f^*\omega)$ (and $f_* d\omega = d(f_*\omega)$ if f is a diffeomorphism) and $\pounds_v(\omega^1 \wedge \cdots \wedge \omega^k) = \pounds_v \omega^1 \wedge \cdots \wedge \omega^k + \cdots + \omega^1 \wedge \cdots \wedge \pounds_v \omega^k$, whence $d\pounds_v = \pounds_v d$.

We occasionally use the convenient notation of the *contraction* of ω with a vector v defined by $v \lrcorner \omega := \omega(v, \cdot, \dots, \cdot)$. This is $\mathbb{R}$-linear and $C^\infty(M)$-linear in v. Furthermore $v \lrcorner (\omega \wedge \eta) = (v \lrcorner \omega) \wedge \eta + (-1)^k \omega \wedge (v \lrcorner \eta)$ and $v \lrcorner df = \pounds_v f$ and

$$\pounds_v \omega = v \lrcorner d\omega + d(v \lrcorner \omega). \tag{10.4.1}$$

Finally $f^* v \lrcorner f^* \omega = f^*(v \lrcorner \omega)$ and $f_* v \lrcorner f_* \omega = f_*(v \lrcorner \omega)$ for any diffeomorphism f.

10.5. De Rham cohomology

Associated with forms is a cohomology theory which is based on the following notion and theorem:

DEFINITION 10.5.1. $\omega \in \Gamma(\bigwedge^k T^*M)$ is said to be *closed* if $d\omega = 0$ and *exact* if $\omega = d\eta$ for some $\eta \in \Gamma(\bigwedge^{k-1} T^*M)$.

Since $d^2 = 0$ every exact form is closed. Locally the converse holds:

THEOREM 10.5.2 (Poincaré Lemma). *If ω is closed then for all $p \in M$ there is a neighborhood U of p on which ω is exact.*

PROOF. We use the homotopy trick (see **??**, **??**): Assume $p = 0 \in \mathbb{R}^n$ and let $v_t(x) = x/t$. v_t generates the flow $\varphi^t(x) = tx$ for $t > 0$, so $d/dt(\varphi^t)^*\omega = (\varphi^t)^* \pounds_{v_t} \omega = (\varphi^t)^*(d(v_t \lrcorner \omega)) = d((\varphi^t)^*(v_t \lrcorner \omega))$ since $d\omega = 0$, and $\omega - (\varphi^{t_0})^*\omega =$