MATH 527: GEOMETRY/TOPOLOGY I.

FALL 2006

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HOMEWORK #1; September 11, 2006

Topological Spaces

Due on Monday September 18

1. Write complete proof of Proposition 1.1.13: For any set X and any collection $\mathcal{C}$ of subsets of X there exists a unique weakest topology for which all sets from $\mathcal{C}$ are open.

2.(Ex. 1.1.1.) How many different (non-homeomorphic) topologies are there on the 2–element set and on the 3–element set?

3.(Ex. 1.3.2.)Prove that the sphere $\mathbb{S}^2$ with two points removed is homeomorphic to the infinite cylinder $C := \{(x, y, z) \in \mathbb{R}^3 | x^2+y^2 = 1\}$.

4.(Example 1.3.3.+ Ex. 1.3.4 & 1.3.5.) Prove that the following three constructions of the n-torus $\mathbb{T}^n$ produce homeomorphic topological spaces:

- Product of n copies of the circle
- The following subset of $\mathbb{R}^{2n}$:

  $$\{(x_1, \dots x_{2n}) : x_{2i-1}^2 + x_{2i}^2 = 1,\ i = 1, \dots, n.\}$$

  with the induced topology.
- The identification space of the unit n–cube I^n:

  $$\{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \le x_i \le 1,\ i = 1, \dots n\}.$$

  where any two points are identified if all of their coordinates but one are equal and the remaining one is 0 for one point and 1 for another.

5.(Ex. 1.10.4 & 1.10.5.) Consider the *profinite* topology on $\mathbb{Z}$ in which open sets are defined as unions (not necessarily finite) of non-constant arithmetic progressions

a) Prove that this defines a topology.

b) Let $\mathbb{T}^\infty$ be the product of countably many copies of the circle with the product topology. Define the map $\varphi : \mathbb{Z} \to \mathbb{T}^\infty$ by

$$\varphi(n) = (\exp(2\pi i n/2), \exp(2\pi i n/3), \exp(2\pi i n/4), \exp(2\pi i n/5), \dots)$$

Show that the map φ is injective and that the pullback topology on $\varphi(\mathbb{Z})$ coincides with its profinite topology.