MATH 527: GEOMETRY/TOPOLOGY I

FALL 2006

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MIDTERM EXAM; Thursday October 19, 2006

In order to receive perfect score you need to solve two problems from each section.

SECTION 1

1.1 Let X be a compact path-connected space. Is it true that X is *locally connected*, i.e. for any point $x \in X$ and any open set $U \ni x$ there exists a connected open set V such that $U \supset V \ni x$?

1.2 Consider the map $f \colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x/2, 2y)$ and the restriction of this map to the positive quadrant plus the origin:

$$\mathcal{Q} := x > 0, y > 0 \cup \{(0, 0)\}.$$

Let F and F_+ be the factor spaces of $\mathbb{R}^2$ and $\mathcal{Q}$ obtained by identifying orbits of f.

Which of the separation properties $(T_0), (T_1), (T_2)$ (Hausdorff), (T_4) (normal) does each of the spaces F and F_+ satisfy?

1.3 Prove that the unit interval can be represented as uncountable union of disjoint subsets each of which is homeomorphic to the Cantor set.

1.4 Prove that the factor space of the complex projective plane $\mathbb{C}P(2)$ by the complex conjugation is homeomorphic to the sphere $\mathbb{S}^4$.

SECTION 2

2.1 Find the fundamental group of the surface of the tetrahedron with interiors of all edges removed, i.e. the space which consists of the vertices and interiors of the faces of the tetrahedron.

2.2 Prove that the universal cover of the open Møbius strip $\mathcal{M}$ is homeomorphic to $\mathbb{R}^2$, i.e. construct a covering map $p\colon \mathbb{R}^2 \to \mathcal{M}$.

2.3 Consider the wedge of three circles W whose fundamental group F_3, the free group with three generators. Let G be the subgroup of F_3, which consists of all elements for which the shortest representation as a word composed of generators and their inverses has even length. Describe the corresponding covering space over W.

2.4 Prove that the following two spaces are homotopy equivalent:

- *Sierpinski carpet*, i.e the subset of the unit square which consists of all points at least one of whose coordinates belongs to the Cantor set.
- Subset of the square $[-\sqrt{2}, \sqrt{2}] \times [-\sqrt{2}, \sqrt{2}]$ which consists of all points at least one of whose coordinates is irrational.