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THE FORNI COCYCLE

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1. INTRODUCTION

The present note is occasioned by the award to Giovanni Forni of the inaugural Michael Brin Prize in Dynamical Systems. The award reflects the profound contributions to dynamical systems by Giovanni Forni. The *existence* of the award reflects the extraordinary generosity of Michael and Eugenia Brin, who have provided funds for many mathematical and scientific activities, including the Brin Prize.

Our goal is to indicate in outline, if not in detail, some of the many ideas that Forni has introduced in two remarkable papers that have appeared in the *Annals of Mathematics,* in 1997 and 2002. The default reference will be to the latter, with references to the former specified as they arise. References listed in braces, *e.g.,* {1}, may be found in the bibliography of [3], which is appended to this note for the convenience of the reader.

2. THE KONTSEVICH–ZORICH COCYCLE

Forni is concerned with the Teichmüller geodesic flow in genus p, $p \geq 2$, and with a cocycle extension thereof, the *Kontsevich–Zorich cocycle.* The Kontsevich–Zorich cocycle, which is introduced in the seminal paper {38}, is reminiscent of a matrix-valued cocycle that arose in early studies of the measure theory of interval exchange maps. The latter cocycle served as a tool for establishing facts about measure-theoretically generic interval exchanges and about the Teichmüller geodesic flow. In [3] Forni exhibits the Kontsevich–Zorich cocycle as an object of intrinsic beauty, develops its analytic properties and applies the analysis to obtain striking new results for the ergodic theory of the Teichmüller flow.

The Teichmüller geodesic flow has as its phase space the unit cotangent bundle of the Teichmüller moduli space. The flow is the restriction to the positive diagonal subgroup, A, of a natural action of $G = SL(2,R)$. The phase space is partitioned into strata determined by the pattern of zeros of a holomorphic quadratic differential and the property of being or not being the square of a holomorphic 1-form. A natural complete metric turns each stratum into a finitely connected, clopen G-invariant set, with each component supporting a unique everywhere positive, absolutely continuous G-invariant probability measure ({43},{65},{66}).

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For want of a better term, we call this measure a Liouville measure. There are more G-invariant ergodic probability measures on each stratum, *e.g.,* those supported on G-homogeneous subspaces of the stratum ({62},{67}), and still more A-invariant measures supported on closed A-trajectories. Since an ergodic (A- or G-) invariant probability measure is supported on a component of a stratum, one is free to restrict attention to a single stratum when an ergodic measure is at issue. Also, consideration of branched double covers enables one to restrict attention to components of strata of squares, as Forni does.

One of many subtle insights in this paper is the author's discovery of the dynamical significance of the "determinant locus". This is a real-codimension two, real analytic sub-hypersurface of the cotangent bundle whose usable definition will be given later but which can be defined as follows: Associate to a nonzero holomorphic quadratic differential q above a point $X \in \mathcal{T}_p$ (Teichmüller space) the Teichmüller disc D through X that q determines. The "Riemann matrix"is a holomorphic complex $p \times p$ matrix-valued function R on $\mathcal{T}_p$. The restriction of R to D is also holomorphic. While the derivative of R on D requires a choice of parameter, the locus of zeros of the determinant of this derivative is independent of this choice of parameter. If X (below q) belongs to the zero locus, one says that q belongs to the *determinant locus*. The determinant locus is invariant under the mapping class group and therefore well-defined on the moduli level. Forni's strongest results are obtained for ergodic measures which are not supported on the determinant locus. A rather intricate analysis is used to establish Theorem 4.5, the statement that the determinant locus does not contain a component of any stratum. In particular, the strongest statements apply to the Liouville measures. Forni raises the intriguing question (Question 9.8) of whether it is possible for a closed G trajectory to be contained in the determinant locus.

For a description of the Kontsevich–Zorich cocycle, begin with the unit cotangent bundle, Q_p^1, of Teichmüller space and the trivial bundle

$$Q_p^1 \times H^1(M_p, \mathbb{R}).$$

Endow each fiber with the Hodge inner product and norm. The G action on Q_p^1, here denoted $q \to hq$, is extended to the product to be trivial (but not isometric) in the second coordinate. The mapping class group acts on the product, commuting with the G action and preserving the fiberwise Hilbert norm. The quotient is a G action on a Hilbert bundle over the moduli space of unit norm quadratic differentials. This is the *Kontsevich–Zorich cocycle*. In terms of the Teichmüller flow one may think of the cocycle as a function $L(h, q)$ whose value at $(h, q) \in G \times Q_p^1$ is a linear operator from the (finite-dimensional real) Hilbert space above q to the one above hq. (A measurable section of the orthonormal frame bundle can be used to convert L to a matrix-valued cocycle.) Let the subgroup A be identified with $\mathbb{R}$, and for each q consider the function $(L^*(t, q)L(t, q))^{1/2t}$ as $t \to \infty$. For each ergodic A-invariant probability measure the limit does exist a.e. and is a positive definite operator whose $2p$ eigenvalues, counting multiplicities, form a set that is closed under reciprocals and is a.e. independent of q. The logs of these eigenvalues comprise the Lyapunov spectrum.

One Kontsevich–Zorich conjecture is that for every Liouville measure the Lyapunov spectrum is simple and, in particular, does not contain zero. Forni's main results in this regard are the statements that (a) the top element (which is 1) of the Lyapunov spectrum is simple for any ergodic A-invariant probability measure and (b) for each Liouville measure the Lyapunov spectrum does not contain 0. Part (a) was previously known in the Liouville measure case ({66}). The statement (b) does not hold for arbitrary, A- (or G-) invariant ergodic probability measures. Indeed, for some time after the original, 1976, version of {62}, the known examples of pseudo-Anosov maps had quadratic expansion coefficients (cf. {14}, p. 250), and many of these give rise to A-invariant measures with Lyapunov spectrum $\{1,0,-1\}$. Recently, the full Kontsevich–Zorich Conjecture has been established by Avila–Viana [1].

3. THE FORNI ORDINARY DIFFERENTIAL EQUATION

What follows is a rough description (and paraphrase) of one of Forni's basic ideas. Each nonzero holomorphic quadratic differential on a closed Riemann surface determines a volume form on the surface. Therefore, there is a Hilbert space, $\mathscr{H}(q)$, associated to each $q \in Q_p^1$. If $h \in G$, then q and hq are initial and terminal differentials of a Teichmüller map which, among other things, is measure–preserving relative to the two volume forms. This means $\mathscr{H}(q)$ and $\mathscr{H}(hq)$ are identified by the Teichmüller map. If q is a square, Forni uses [2] to associate to each hq an $\mathbb{R}$-linear isometry, $U(hq)$, on $\mathscr{H}(hq)$. (I shall say more about the definition later on.) Assuming the indicated identification of $\mathscr{H}(hq)$ with $\mathscr{H}(q)$, the function $h \to U(hq)$ takes values in $\mathbb{R}$-linear isometries of $\mathscr{H}(q)$. Now identify the subgroup A with $\mathbb{R}$, denoting the A-action by $q \to tq$, and consider the ordinary differential equation for an $\mathscr{H}(q)$-valued function $f(t)$,

$$\frac{df}{dt}(t) = U(tq)f(t).$$

This equation carries much information about the Kontsevich–Zorich cocycle. Most importantly, Forni proves that if the initial condition $f(0) \in \mathscr{H}(q)$ is a meromorphic function on the surface for q, then

1. for each t, $f(t)$ corresponds, under the identification above, to a meromorphic function $f^*(t) \in \mathscr{H}(tq)$ on the surface for tq
2. the cohomology class, c, of the tq-harmonic 1-form $Re(f^*(t)(tq)^{1/2})$ is constant in t.

Using p for genus, let $\mathscr{M}_t^+$ and $\mathscr{M}_t^-$ be the complex p-dimensional subspaces of $\mathscr{H}(q)$ that are pullbacks of the spaces of meromorphic and conjugate meromorphic square integrable functions on the surface for tq. Let $\pi_t^{\pm}$ denote the orthogonal projections of $\mathscr{H}(q)$ onto $\mathscr{M}_t^{\pm}$. The Hodge norm squared of c at time t is the $\mathscr{H}(q)$ norm squared of $f(t)$; the differential equation enables Forni to compute the first two derivatives, along Teichmüller trajectories, of the function "Hodge norm squared". These calculations are of fundamental importance for the main results of the paper. The equations for the first and second derivatives take the shape

$$
c\frac{d}{dt}\|f(t)\|_q^2 = -2\mathrm{Re}\left(f(t)^2,1\right)_q \tag{3.1}
$$
$$
\frac{d^2}{dt^2}\|f(t)\|_q^2 = 4\left\{\|\pi_t^- f(t)\|_q^2\right\} - \mathrm{Re}\left(g_t^+, g_t^-\right)_q
$$

In (3.1), $(\cdot,\cdot)_q$ denotes the inner product with respect to the volume form subordinate to q and $g_t^\pm$ are given by

$$
g_t^+ = f(t) - \pi_t^- f(t) \tag{3.2}
$$
$$
g_t^- = \frac{d}{dt}f(t) + \overline{\pi_t^- f(t)}
$$

(Forni gives a more direct definition of $g_t^\pm$ in terms of Cauchy–Riemann operators, $\partial_t^\pm$, associated to tq. The operators will be recalled in Section 5 below.)

Using (3.1), one finds that

$$
\frac{d}{dt}\log\|f(t)\|_q = -\frac{\mathrm{Re}\left(f(t)^2,1\right)_q}{\|f(t)\|_q^2} \tag{3.3}
$$

When the initial condition, $f(0)$, is the constant function 1 (respectively, i), the solution $f(\cdot)$ realizes the top (resp. bottom) Lyapunov exponent, which is 1 (resp. -1). One is led therefore to consider the space $E_0(q) = \left\{f(0)\,\middle|\,\left(f(0),1\right)_q = 0\right\}$. When $f(0) \in E_0$, (3.3) is used to imply

$$
\frac{1}{T}\log\frac{\|f(T)\|_q}{\|f(0)\|_q} \le \frac{1}{T}\int_0^T \Lambda^+(tq)\,dt
$$

where

$$
\Lambda^+(tq) = \max\left\{\frac{-\mathrm{Re}\left(f^2,1\right)_q}{\|f\|_q^2}\,\middle|\, f \in E_0(tq) \setminus \{0\}\right\}
$$

If μ is an ergodic invariant measure, the Multiplicative Ergodic Theorem implies the second largest Lyapunov exponent satisfies

$$
\lambda_2^\mu \le \int \Lambda^+(q)\mu(dq) \le 1 \tag{3.4}
$$

The right hand inequality of (3.4) must be strict. Otherwise, a compactness argument shows that there exist q and $f \in E_0(q) \setminus \{0\}$ such that

$$
\left|\left(f^2,1\right)_q\right| = \|f\|_q^2.
$$

The Schwarz inequality then implies $\overline{f} = cf$ for some constant c. That is, f is both meromorphic and conjugate meromorphic, thus f is constant. Since $f \in E_0, f = 0$. That is,

$$
\lambda_2^\mu < 1 = \lambda_1^\mu \tag{3.5}
$$

4. SIGNIFICANCE OF THE DETERMINANT LOCUS

Let $\mathscr{M}^+ = \mathscr{M}_0^+$ (resp. $\mathscr{M}^- = \mathscr{M}_0^-$) be the $2p$ real-dimensional subspaces of $\mathscr{H}(q)$ consisting of meromorphic (resp. conjugate meromorphic) functions and let $J\colon \mathscr{M}^+ \to \mathscr{M}^-$ be the restriction of the orthogonal projection of $\mathscr{H}(q)$ onto $\mathscr{M}^-$. That is,

$$J = \pi_0^-|_{\mathscr{M}^+} \tag{4.1}$$

Let μ_q be the canonical Beltrami differential associated to q,

$$\mu_q = \frac{|q|}{q}$$

Forni uses the Rauch formula for the derivative of the Riemann matrix, R, on the Teichmüller disc D determined by q, to establish, in effect, the relation

$$|\det(Q)| = \left|\frac{\det(\frac{dR}{d\mu_q})}{\det(\mathbf{Im}(R))}\right|^2 \tag{4.2}$$

at the point $X \in D$ which corresponds to q. This establishes, for example, that the determinant locus, which he defines as the locus

$$\det\left(\frac{dR}{d\mu_q}\right) = 0, \tag{4.3}$$

is coincident with the locus of q such that $\ker(J) \neq \{0\}$. (We remark that the complex structure on a stratum may be realized in terms of cohomological coordinates, {43}, {65,66,68}, and that, locally, the G-action is given by the coordinatewise $\mathbb{R}$-linear action. If $\sigma_t w = \cosh(t)\,w + \sinh(t)\overline{w}$, $t \in \mathbb{R}$, then for smooth S

$$\frac{d}{dt}(S \circ \sigma_t)(w)|_{t=0} = \sum_{k=1}^{N}\left(\frac{\partial S}{\partial w_k}(w)\,\overline{w_k} + \frac{\partial S}{\partial \overline{w_k}}(w)\,w_k\right). \tag{4.4}$$

In particular, if S is the composition of R with the canonical holomorphic projection of the stratum into the Teichmüller space of genus p unpunctured surfaces, S is real analytic. So long as the stratum is not contained in the determinant locus, the locus is a real analytic, real-codimension two hypersurface.)

Forni's formula (3.1) for the first two derivatives is combined with the analysis of the determinant locus to obtain the statement that at least $\left[\frac{p+1}{2}\right]$ nonnegative Lyapunov exponents must be positive for any Liouville measure. An interesting aspect of this argument is that it turns in part on an extension of a Kontsevich–Zorich integral formula {38} for the sum of the nonnegative Lyapunov exponents for a Liouville measure. At the same time, Forni derives an integral formula for the sum of the first $k < p$ nonnegative Lyapunov exponents for Liouville measures as a consequence of his still-to-be-proved nonvanishing theorem for all of the Lyapunov exponents. The reason the argument (at this stage) captures positivity of only $\left[\frac{p+1}{2}\right]$ Lyapunov exponents is that it turns on a derivation from the integral formula that if r of the nonnegative exponents are zero, then for a.e. q

some complex r-dimensional subspace of $\mathcal{M}^+$ is mapped by the *injection* J inside some complex $(p-r)$-dimensional subspace of M^-. Therefore $2r \le p$. I shall return to this point below.

What follows is an indication of a computational device that Forni employs to derive various integral equalities and inequalities. Let $K = SO(2)$. If Gq is a G trajectory in Q^1_p, then $K\backslash Gq$ is identified with the Teichmüller disc D_q that q determines. D_q is, among other things, a hyperbolic disc with invariant Laplacian Δ. If f, g are (appropriate and) related by the Poisson equation $\Delta f = g$ on D_q and if $A_q(f,t)$ (resp. $B_q(g,t)$) is the average of f (resp. g) over the circle of hyperbolic radius t (resp. hyperbolic disc of radius t) centered at X (beneath q), then Forni establishes the relation

$$\frac{d}{dt} A_q(f,t) \sim B_q(g,t),\ t \to \infty. \tag{4.5}$$

Upstairs in Q^1_p, let m_q be the normalized Haar measure on K transported to Kq, *i.e.,* to the circle above the center, X, of D_q. Let $a_t = \operatorname{diag}(e^{\frac{t}{2}}, e^{-\frac{t}{2}})$, and let $\nu_{q,t} = a_t m_q$. If F and G are functions on Q^1_p that are constant on K orbits, then F and G are lifts of functions f and h, respectively, on D_q. One has

$$\begin{gathered} (F, \nu_{q,t}) = A_q(f,t),\ t > 0 \\ \frac{1}{\cosh t - 1} \int_0^t (G, \nu_{q,s}) \sinh(s)\, ds = B_q\left(g,t\right) \end{gathered} \tag{4.6}$$

Forni's differential equation is used to compute derivatives of specially chosen F along A trajectories, and these derivatives are used in turn to compute g in $\Delta f = g$ on D_q. In the application G turns out to be uniformly bounded and, in particular, integrable with respect to the normalized Liouville measure, $\mu_{\mathscr{E}}$, on a component, $\mathscr{E}$, of a stratum. One knows that

$$\lim_{t\to\infty} (G, \nu_{q,t}) = \int_{\mathscr{E}} G\left(q'\right) \mu_{\mathscr{E}}(dq') \tag{4.7}$$

in $\mu_{\mathscr{E}}$-mean, *e.g.,* by {69}, Theorem 2.6. It follows then that in mean,

$$\begin{gathered} \lim_{t\to\infty} B_q\left(g,t\right) = \int_{\mathscr{E}} G\left(q'\right) \mu_{\mathscr{E}}(dq') \\ \lim_{t\to\infty} \frac{A_q(f,t) - A_q(f,0)}{t} = \int_{\mathscr{E}} G\left(q'\right) \mu_{\mathscr{E}}(dq') \end{gathered} \tag{4.8}$$

The function F, from which the functions f, g and G, are determined (almost) has the form

$$F = \log\left|\det\left(A(q)\right|\right. \tag{4.9}$$

for a certain matrix-valued function of q. If dk is the normalized Haar measure on K, then in mean

$$\lim_{t\to\infty}\frac{A_q(f,t)-A_q(f,0)}{2t}=\lim_{t\to\infty}\int_K \log\left(\left|\det\left(A(a_tkq)\right|\right)^{\frac{1}{2t}}dk \tag{4.10}$$

Actually, the function A depends upon an additional parameter, a point in a Grassmanian, and for each q one must also average (4.8)-(4.10) over the Grassmanian. The Multiplicative Ergodic Theorem is finally brought into play to evaluate the limit on the right.

To indicate the most elementary example, let N be a Lagrangian subspace of $H^1(M_p,\mathbb{R})$, and let v be a volume form for N. Define $M(hq)$ to be the Hodge norm of v on the surface for hq. For fixed N and q, $F_N(hq)=\log(M(hq))$ is well-defined in h, up to an additive constant. Moreover,

$$F_N(khq)=F_N(hq),\ k\in K$$

Therefore, on $K\backslash Gq$

$$f_N(Khq):=F_N(hq)$$

is also defined up to an additive constant. Forni proves

$$\Delta f_N=2\mathrm{tr}(Q)$$

where $Q(hq)$, the hermitian form which was defined earlier above each hq, is independent of the Lagrangian subspace. Endow $H^1(M_p,\mathbb{R})$ with the Hodge metric over hq and average $\frac{1}{t}\left(A_q(f_N,t)-A_q(f_N,0)\right)$ over N in the Grassmanian, $G_p\left(M_p,\mathbb{R}\right)$, of Lagrangian subspaces, with respect to normalized Haar measure, μ_q^p. The result is

$$\frac{A_q\left(f^p,t\right)-A_q\left(f^p,0\right)}{t}=\int_{G_p(M_p,\mathbb{R})}\left(\frac{A_q\left(f_N,t\right)-A_q\left(f_N,0\right)}{t}\right)\mu_q^p(dN) \tag{4.11}$$

where

$$f^p\left(Khq\right)=\int_{G_p(M_p,\mathbb{R})} f_N(Khq)\mu_q^p(dN)$$

Next, integrate with respect to the normalized Liouville measure $\mu_{\mathscr{E}}$ on a component $\mathscr{E}$ of a stratum. The Multiplicative Ergodic Theorem yields the Kontsevich–Zorich integral formula

$$\lim_{t\to\infty}\int_{\mathscr{E}}\frac{A_q\left(f^p,t\right)-A_q\left(f^p,0\right)}{2t}\mu_{\mathscr{E}}\left(dq\right)=\sum_{i=1}^{p}\lambda_i=\int_{\mathscr{E}}\mathrm{tr}\left(Q\left(q\right)\right)\mu_{\mathscr{E}}\left(dq\right).$$

More generally, let $\mathscr{I}=\left\{I_1\subset I_2\subset\cdots\subset I_p\right\}$ be a flag of isotropic subspaces of $H^1(M_p,\mathbb{R})$ such that

$$\dim I_k=k,\ 1\le k\le p$$

Choose a set $C=\left\{c_1,c_2,\cdots,c_p\right\}$ of vectors such that for each k, $\{c_1,c_2,\cdots,c_k\}$ is a basis for I_k. Identify $H^1(M_p,\mathbb{R})$ with the ($\mathbb{R}$–linear) space $\mathscr{M}^+$ at q via the map

$m^+ \to \mathrm{Re}\left(m^+ q^{\frac{1}{2}}\right)$, and let $\left\{m_1^+, \cdots, m_p^+\right\}$ correspond to $\{c_1, c_2, \cdots, c_p\}$. For each k define the $k \times k$ matrix $A^k_{q,C}$ by

$$\left(A^k_{q,C}\right)_{ij} = \left(m_i^+, m_j^+\right)_q.$$

The isotropy condition implies that $A^k_{q,C}$ is real and symmetric. Recalling that C moves trivially by the Kontsevich–Zorich cocycle, one may view the matrices as defining a cocycle

$$B^k_C(h,q) = A^k_{hq,C}\left(A^k_{q,C}\right)^{-1}.$$

Moreover, the function

$$F^k_{I_k}(hq) = \log\left|\det B^k_C(h,q)\right|, \quad F^k_C(q) = 0$$

depends only upon the k^{th} element of the flag $\mathscr{I}$ and is left K−invariant. Therefore the function

$$f^k_{I_k}(Khq) = F^k_{I_k}(hq), \quad f^k_{I_k}(Kq) = 0$$

is defined on the Teichmüller disc D that q determines. For each k the Laplacians

$$\Delta f^k_{I_k} := 2\Phi^k_{I_k} \tag{4.12}$$

also depend only upon the k^{th} element of the *flag* $\mathscr{I}$ and not upon the choice of basis $\{c_1, c_2, \cdots, c_k\}$ for it. In particular, one may as well choose $\{c_1, c_2, \cdots, c_p\}$ to be orthonormal at q. This implies $\left\{m_1^+, \cdots, m_p^+\right\}$ is orthonormal in $\mathscr{H}(q)$. An elaborate calculation produces the remarkable formula

$$\begin{aligned}\Phi^k_{I_k}(q) &= 2\sum_{i=1}^{k} \|Jm_i^+\|_q^2 - \sum_{i,j=1}^{k}\left|\left(m_i^+, \overline{m_j^+}\right)_q\right|^2 \\ &= \sum_{i=1}^{p} \|Jm_i^+\|_q^2 - \sum_{i,j=k+1}^{p}\left|\left(m_i^+, \overline{m_j^+}\right)_q\right|^2 \\ &= \mathrm{tr}\left(Q(q)\right) - \sum_{i,j=k+1}^{p}\left|\left(m_i^+, \overline{m_j^+}\right)_q\right|^2\end{aligned} \tag{4.13}$$

(Recall that $J = \pi_0^-|_{\mathscr{M}^+}$.) When $k = p$,

$$\Phi^p_{I_p}(q) = \mathrm{tr}\left(Q(q)\right) \tag{4.14}$$

as above.

Let $G_k\left(M_p, \mathbb{R}\right)$ be the Grassmanian of isotropic k-planes, and let μ^k_q be the normalized Haar measure on $G_k\left(M_p, \mathbb{R}\right)$ relative to the Hodge metric above q. Define

$$f^k(Khq) = \int_{G_k(M_p,\mathbb{R})} f^k_{I_k}(Khq)\mu^k_{hq}(dI_k).$$

Let the nonnegative Lyapunov spectrum be ordered as

$$1 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_p \geq 0. \tag{4.15}$$

Since $A_q(f^k, 0) = 0$, the Multiplicative Ergodic Theorem implies

$$\lim_{t\to\infty} \int_{\mathscr{E}} \frac{A_q(f^k,t)}{2t} \mu_{\mathscr{E}}(dq) = \sum_{i=1}^{k} \lambda_i, \quad k < p,\ \lambda_k > \lambda_{k+1} \geq 0$$

Moreover, if Φ^k is defined by

$$\Phi^k\left(Kq\right) = \int_{G_k(M_p,\mathbb{R})} \Phi^k_{I_k}(Kq)\mu^k_q(dI_k),$$

one finds that

$$\sum_{i=1}^{k} \lambda_i = \int_{\mathscr{E}} \Phi^k\left(Kq\right)\mu_{\mathscr{E}}(dq), \quad k < p,\ \lambda_k > \lambda_{k+1} \geq 0 \tag{4.16}$$

For one application of (4.16), suppose for some $k < p$ that $\lambda_k > 0$ and $\lambda_{k+1} = 0$. From (4.13),

$$\Phi^k\left(Kq\right) \leq \operatorname{tr}\left(Q(q)\right)$$

Since $\lambda_{k+1} = 0$, it is true that

$$\sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{p} \lambda_i$$

Therefore, by the Kontsevich–Zorich formula, one has for $\mu_{\mathscr{E}}$-a.e. q that

$$\Phi^k\left(Kq\right) = \operatorname{tr}\left(Q(q)\right)$$

For each such q there exists an orthonormal basis $\left\{m_1^+, \cdots, m_p^+\right\}$ such that

$$\sum_{i,j=k+1}^{p} \left|\left(m_i^+, \overline{m_j^+}\right)_q\right|^2 = 0.$$

Therefore, $J\colon \mathscr{M}^+ \to \mathscr{M}^-$ sends the $\mathbb{C}$-linear span of $\left\{m_{k+1}^+, \cdots, m_p^+\right\}$ into the $\mathbb{C}$-linear span of $\left\{\overline{m_1^+}, \cdots, \overline{m_k^+}\right\}$. Since J is injective for $\mu_{\mathscr{E}}$-a.e. q, it must be that $2k \geq p$. It follows that the top $\left[\frac{p+1}{2}\right]$ elements of the Lyapunov spectrum are positive.

More can be obtained from the asymptotic (4.7). For a.e. q there is a flag $\mathscr{I}\left(q\right) = \left(I_1^+(q), I_2^+(q), \cdots, I_p^+(q)\right)$ corresponding to the nonnegative Lyapunov spectrum. If $k < p$ and $\lambda_k > \lambda_{k+1}$, define

$$\Psi^k\left(q\right) = \Phi^k_{I_k^+(q)}(q)$$

Since $I_k^+\left(q\right) = I_k^+\left(tq\right)$, $t \in A$, we have as a limit in mean

$$\begin{aligned}\left\|\Psi^k - \Phi^k\right\|_{1,\mu_{\mathscr{E}}} &= \lim_{t\to\infty}\left(\left|\Psi^k - \Phi^k\right|, \nu_{q,t}\right)\\ &= \lim_{t\to\infty}\left(\left|\int\limits_{G_k(M_p,\mathbb{R})}\left(\Psi^k\left(q\right) - \Phi^k_{I_k}(q)\right)\mu_q^k\left(dI_k\right)\right|, \nu_{q,t}\right)\\ &\le \overline{\lim_{t\to\infty}} \int\limits_{G_k(M_p,\mathbb{R})}\int\limits_K \left|\Psi^k\left(tkq\right) - \Phi^k_{I_k}(tkq)\right| dk\mu_q^k\left(dI_k\right)\\ &= \overline{\lim_{t\to\infty}} \int\limits_{G_k(M_p,\mathbb{R})}\int\limits_K \left|\Phi^k_{I_k^+(kq)}(tkq) - \Phi^k_{I_k}(tkq)\right| dk\mu_q^k\left(dI_k\right)\\ &= 0.\end{aligned}$$

The last line is a consequence of the continuity of $\Phi^k_{I_k}(q)$ on $G_k\left(M_p,\mathbb{R}\right) \times Q_p^1$ and the Multiplicative Ergodic Theorem. It follows now that

$$\sum_{i=1}^k \lambda_i = \int\limits_{\mathscr{E}} \Phi^k_{I_k^+}(Kq)\mu_{\mathscr{E}}(dq), \quad k < p,\ \lambda_k > \lambda_{k+1} \ge 0,$$

which is Forni's integral formula.

5. Invariant Distributions and Cohomology

A reader of Sections 6 and 7 of [3] will be well-advised to keep handy copies of [2], de Rham {11} and Schwartz {56}, because these sections attach/analyze a daunting array of spaces of foliation invariant distributions/currents to the (say) horizontal foliation of a square q. It is worth the effort to follow the constructions/analyses, some of which are indicated here. There are parallel tracks throughout, one for cohomology and one for relative cohomology. I shall focus on the former.

First, one has to decide on a space of test forms. The space Ω_q is defined in terms of the "atlas"of natural parameters, understood here to mean a collection of local branched covering "charts"under which dz^2 pulls back to $q = \omega^2$ (branched covering occurs only at zeros of q). The space Ω_q is the set of pullbacks of smooth forms on the plane under these "charts". $\Omega_q^d \subset \Omega_q$ corresponds to homogeneous forms of dimension d. The dual S_q' of Ω_q (resp. $\left(S_q^d\right)'$) is the space of *q-tempered currents (resp. homogeneous q-tempered currents* of dimension d). Also, since $q = \omega^2$, we may define ∂ to be the meromorphic vector field such that $\omega(\partial) \equiv 1$ on $X \setminus \omega^{-1}(0)$. The "charts"above send ∂ (resp. ∂^*) to $\frac{\partial}{\partial z}$ (resp. $\frac{\partial}{\partial \bar{z}}$) on the plane. Let $S = (\partial + \partial^*)$, $T = i(\partial - \partial^*)$. S and T are singular vector fields that are pointwise $|q|$-orthonormal and tangent to the horizontal and vertical foliations away from the zeros of q. One sees readily that Ω_q and Ω_q^d are closed under ∂ and ∂^* and (the natural interpretation of) the Lie derivative ($\mathscr{L}_S$) and contraction (i_S) with respect to S (and T). Therefore, it makes sense to define a *q-tempered invariant current C* of dimension d for the horizontal foliation by

$$i_S C = 0 = \mathscr{L}_S C \text{ in } \left(S_q^d\right)' \tag{5.1}$$

When $d = 1$, one speaks of *basic currents.* When $d = 2$, or by duality $d = 0$, one speaks of *$q-$tempered invariant distributions. The spaces of basic currents are denoted* $\mathscr{B}_{\pm q}(X)$, where $+q, -q$ correspond to S, T respectively.

This is a good place to indicate how the isometry $U(q)$, mentioned in Section 3, is defined. A nice fact from [2] is an identity for the space $\mathscr{H}_0^1(q)$ of *mean zero* $u, v \in \mathscr{H}^1(q)$, the Sobolev space relative to the q-volume form, which states

$$P(u,v) := (Su, Sv)_q + (Tu, Tv)_q = (\partial u, \partial v)_q) = (\partial^* u, \partial^* v)_q, \tag{5.2}$$

where $u, v \in \mathscr{H}_0^1(q)$. The identity is not purely formal. It strengthens the statement that the operators commute on the next higher level Sobolev space. With the help of a Poincaré inequality

$$(u,u) \le cP(u,u), \ u \in \mathscr{H}_0^1(q) \tag{5.3}$$

from [2], (5.2) sets up an isometry ∂ (or ∂^*) between $\mathscr{H}_0^1(q) \subset \mathscr{H}^1(q)$ and a codimension p (= genus) subspace R^+ (or R^-) of $\mathscr{H}^0(q) = \mathscr{H}(q)$. The operator

$$V = \partial^* \partial^{-1} \colon R^+ \to R^- \tag{5.4}$$

is an isometry. The orthocomplements of R^+ and R^- are, respectively, the spaces $\mathscr{M}^+$ and $\mathscr{M}^-$ from the previous section. Therefore V admits an $\mathbb{R}$-linear extension to an isometry, $U(q)$, of $\mathscr{H}^0(q) = \mathscr{H}(q)$. For example, conjugation is an $\mathbb{R}$-linear isometry of $\mathscr{M}^+$ onto $\mathscr{M}^-$, and this particular choice of extension is useful for certain calculations.

I shall next indicate how Forni attaches a cohomology class $c \in H^1(X,\mathbb{R})$ to a q-tempered invariant distribution D of dimension 2. Given D, define $C = i_S D$. Then C is a q-tempered basic current of dimension one. Forni proves a basic current satisfies $dC = 0$ in $\left(S_q^1\right)'$. In particular, C may be assigned by Poincaré duality a class in $H^1(X \smallsetminus q^{-1}0,\mathbb{R}) \cong H_c^1(X \smallsetminus q^{-1}0,\mathbb{R})^*$ (dual of cohomology with compact supports). The q-tempered condition is used to show this class actually lies in $H^1(X,\mathbb{R})$. More precisely, if N is the kernel of the surjective forgetful map

$$H_c^1(X \smallsetminus q^{-1}0,\mathbb{R}) \to H^1(X,\mathbb{R})$$

he proves that the image of C lies in the annihilator of N. The trick is that if $c \in N$, then $c = [dv]$ for some function v that is constant on each component of a neighborhood of $q^{-1}0$. The constants are not necessarily zero, but in any case $v \in \Omega_q$ and $dv \in \Omega_q^1$. Since $dC = 0$ in $\left(S_q^1\right)'$, $C(dv) = dC(v) = 0$. The correspondence $D \to C = i_S D$ above is also shown to be invertible, and therefore the spaces of q-tempered invariant distributions and q-tempered basic currents are naturally isomorphic.

Forni also deals with Sobolev spaces of q-tempered currents of dimension 1. If q is a square, then, with notations as above, the operators $S = (\partial + \partial^*)$ and

$T = i(\partial - \partial^*)$ are used to define Sobolev spaces $\mathscr{H}^s(q)$ relative to the q-volume form. The test forms are q-tempered 1-forms whose contractions with respect to the vertical and horizontal vector fields belong to Sobolev function spaces associated to q and its volume form. That is,

$$\mathscr{H}^s_q = \left\{\alpha \in S'_q \,|\, (i_S\alpha, i_T\alpha) \in \mathscr{H}^s(q) \times \mathscr{H}^s(q)\right\}.$$

Now one speaks of basic currents and cohomology of order $s > 0$.

The final part of Section 6 is concerned with improvement of the Poincaré inequality (5.3). The goal is to incorporate into the constant inverse dependence on the shortest geodesic length of a connection between cone points for the metric $|q|$. The proof parallels the proof of Cheeger's lower bound in {9} for the first eigenvalue of the Laplacian. The Dirichlet form is $P(u,v) = (Su, Sv)_q + (Tu, Tv)_q$, and the Poincaré inequality is, for mean zero u,

(5.5) $$||u||^0_q < C\lambda(q)^{-1}P(u,u)^{1/2},\ u \in \mathscr{H}^1_0(q),$$

where $\lambda(q)$ is the shortest geodesic length between cone points for $|q|$. The improved inequality is important for Section 8.

Section 7 is interesting not only for the new results it contains but for its connection to the horse and buggy days of this subject. The first bounds for the dimension of the linear space of finite invariant signed measures for a quasiminimal flow with hyperbolic saddles were obtained, by Katok {29}, by injecting the space into a cohomology space and observing the image to be isotropic. Therefore the cone of invariant measures has dimension at most one-half the rank of the associated cohomology (or form). (The corresponding bound for interval exchanges was obtained, independently, by the author in a later paper, but the result is equivalent to Katok's. The bounds are known to be sharp.)

In Section 6 Forni has shown how to assign cohomology classes to invariant distributions, but there are some key differences from the invariant measure case. First, the cohomology class of an invariant distribution may be zero. Second, there is no reason to expect the image in cohomology to be isotropic. Forni proves two theorems which address these issues (in reverse order).

The first theorem is an application of an existence theorem from [2]. For a fixed square q Forni proves that for Lebesgue a.e. θ the distributional cohomology of order $s \gg 1$ for the horizontal foliation for $e^{i\theta}q^{1/2}$ is the nonisotropic space of real cohomology classes c such that $c \wedge [\mathbf{Im}(e^{i\theta}q^{1/2})] = 0$. As above, I shall focus on $H^1(X,\mathbb{R})$, although he also deals with $H^1\left(X \setminus q^{-1}0, \mathbb{R}\right)$. The proof is a variation on a construction of invariant distributions in [2]. Represent a given class $c \in H^1(X,\mathbb{R})$ as a real harmonic differential, $u = \mathbf{Re}(fq^{1/2})$. The meromorphic function f has its poles in the zero set $\left(q^{1/2}\right)^{-1}0$. At each zero the order of the zero is at least as large as the order of the pole. This makes it possible to construct a smooth real function g such that for $S = (\partial + \partial^*)$, $\mathbf{Re}(f) - Sg$ has compact support in $X \setminus q^{-1}0$. The condition $c \wedge [\mathbf{Im}(e^{i\theta}q^{1/2})] = 0$ implies $\mathbf{Re}(f) - Sg$ has integral zero relative to the q-volume. Assume that the equation $Sh = \mathbf{Re}\left(f\right) - Sg$ admits a real distributional solution h of order s, and express

$F = g + h$ as $F = F^*\omega_q$. where F^* is a current of dimension 2. Define a current C by $dF^* + \mathbf{Re}(fq^{1/2}) = C$. Then C is a q-tempered basic current of order $s+1$, and C is cohomologous to $u = \mathbf{Re}(fq^{1/2})$. While F may not exist for the given q, Theorem 4.1 of [2] implies that for ($s \gg 1$ and) Lebesgue a.e. θ, F does exist when S corresponds to the horizontal foliation for $e^{2i\theta}q$.

The second theorem in Section 7 characterizes the basic currents of order s which are cohomologous to zero in terms of a series of exact sequences

$$0 \to \mathbb{R} \xrightarrow{i} \mathscr{B}_q^{s-1} \xrightarrow{\delta_s} \mathscr{B}_q^s \xrightarrow{j} H_q^{1,s}(X,\mathbb{R}), \quad s \geq 1, \tag{5.6}$$

in which $\mathscr{B}_q^s$ denotes basic currents of order s, $i(a) = a\eta_S, a \in \mathbb{R}$, j is the assignment "cohomology class" and δ_s is defined by

$$\delta_s(C) = d\left(-C \wedge \mathbf{Im} q^{1/2}\right)^*.$$

Exactness of the sequence (5.6) is valid for all s and is used for $s = 1$ in Section 8. The important conclusion is: For Lebesgue a.e. θ, the cohomology map, from the space of invariant distributions of order one for the horizontal foliation of $e^{2i\theta}q$ into $H^1(X,\mathbb{R})$, is injective. Notwithstanding its dependence on θ, this result extends, in a nontrivial way, the corresponding injectivity result for invariant measures for minimal foliations and interval exchanges mentioned above. Curiously, the proof uses only the existence of the exact sequence (5.6) and not the form of the map for the third arrow of (5.6). The space of basic currents of order zero is known (by Section 6) to be isomorphic to the space of invariant distributions of order zero. The latter are finite invariant measures. Therefore, the Kerckhoff–Masur–Smillie Theorem, {35}, implies for Lebesgue a.e. θ that the space of basic currents of order zero is one-dimensional. Since (5.6) is exact, it follows for the same θ that the space of invariant distributions of order one injects into $H^1(X,\mathbb{R})$, under the cohomology map.

6. THE FORNI COCYCLE

In my opinion, the greatest of the many achievements of [3] is the establishment of a deep connection between the Lyapunov spectrum and natural bundles of (horizontal-, vertical-) invariant distributions of order one. This is the subject of Section 8. (I shall restrict attention to Liouville measures.) By now we know there is a Kontsevich–Zorich cocycle, that it has a Lyapunov spectrum symmetric about zero and that at least half of this spectrum is nonzero. The ranks of the stable/unstable bundles, denoted S/U, of the Kontsevich–Zorich cocycle are the (equal) cardinalities of the positive/negative subsets, with multiplicities, of the Lyapunov spectrum. Each of these bundles has Lagrangian fibers in the bundle, denoted $B = S + U$, whose fiber at q is the sum of the S and U fibers at q. The nonvanishing of the Lyapunov spectrum is equivalent to

$$B = \mathbf{B}, \tag{6.1}$$

where $\mathbf{B}$ is the full cohomology bundle. This is also proved in Section 8.

Forni introduces new bundles and cocycles of which I shall indicate but one. Let Z' be the bundle whose fiber at q is the infinite-dimensional space of closed currents of dimension 1. Forni observes that Z' admits a natural G-action which is a cocycle extension of the Teichmüller flow and that the map $R\colon Z' \to \mathbf{B}$, defined on each fiber to be the cohomology map, is a bundle map which is equivariant with respect to this the new cocycle and the Kontsevich–Zorich cocycle on the image. Denote by H and V the finite-dimensional subbundles of Z' whose fibers at q are the spaces of basic currents of dimension and order one for the horizontal and vertical foliations, respectively. Forni proves that H and V have a.e. constant rank and that R maps the bundles H and V a.e. isomorphically to the bundles U and S. Moreover, the fibers of H and V are a.e. independent in the fibers of Z'. Therefore, $R\colon H+V \to B$, also isomorphically. (I shall mention the important structure theorem for Z' later.)

The proof that, say, $R\colon H \to U$ isomorphically has two parts, "into"and "onto". The "into"part involves a proof that the Lyapunov spectrum is nonzero for $H+V$ and that $H+V$ is the splitting into unstable/stable subbundles. For the proof the author employs a device of Burns–Katok ({31}). He defines a.e. on H a function which is homogeneous of degree two on the fibers, positive on nonzero elements and monotone on A-trajectories. (The construction, which is not at all obvious, involves another application of Section 7.) The proof of monotonicity involves an operator ordinary differential equation similar to the one which was discussed in Section 3 above. Once the constructions are complete, the nonvanishing of Lyapunov spectrum for H (and V) then follows from Burns–Katok. In particular, $RH \subseteq U$ and $RV \subseteq S$.

For the existence ("onto") part of the theorem, one should first refer to a strategy which was employed above. I shall paraphrase the strategy here: Represent a given class $c \in U$ as a real harmonic differential, $u = \mathbf{Re}(fq^{1/2})$. Assume that for $S = (\partial + \partial^*)$, the equation $SF = \mathbf{Re}(f)$ admits a real distributional solution F of order one, and express F as $F = F^*\omega_q$, where F^* is a current of dimension 2. Define a current C by $dF^* + \mathbf{Re}(fq^{1/2}) = C$. Then C is a basic current of order one for the horizontal foliation, and C is cohomologous to $u = \mathbf{Re}(fq^{1/2})$. The existence proof (using [2]) above was hard enough, but because the conclusion required less of F, the proof could be carried out for Lebesgue a.e. θ and the horizontal foliation for $e^{2i\theta}q$. This time it will be valid only for a.e. q relative to a fixed (say) Liouville measure. More precisely, q must be a generic point for the Multiplicative and Birkhoff Ergodic Theorems.

To get started, we recall the idea, from Section 3 above, that one can study the A-trajectory of (q,c) in terms of an operator ordinary differential equation which is defined on the surface for q. (Recall (a) the $A = \mathbb{R}$-action on q is denoted $q \to tq$ and (b) there is fixed a $c \in U$ above q and (c) $c = [u] = [\mathbf{Re}(fq^{1/2})]$.) First, solve the Forni equation

$$\frac{d}{dt}f(t) = U(t)f(t), \quad f(0) = f$$

Next, let $v(t)$ be the pullback of the form $(tq)^{1/2}$ from the surface for tq to the surface for q under the Teichmüller map. Then $\mathbf{Re}(f(t)v(t))$ is a closed form (away from the zeros of q) whose cohomology class is also c. It follows that the form $\mathbf{Re}(f(t)v(t)) - \mathbf{Re}(f(0)v(0))$ is exact. Let F_t be the unique solution to $dF_t = \mathbf{Re}(f(t)v(t)) - \mathbf{Re}(f(0)v(0))$ such that F_t has integral zero with respect to the q-volume form. The role of the continuing assumption $c \in U$ is to guarantee, for generic q, that

$$\text{Hodge norm of } c \text{ at time } t = \|f(t)\|_{2,q}$$

decreases exponentially to zero as $t \to -\infty$. The PoincarŐ inequality combined with a theorem by Masur will provide the *coup de grace*.

The first step of the analysis of F_t is to use the special form of the differential equation to prove that

$$\frac{d}{dt} F_t = 2\mathrm{Re}\,(v_t) \tag{6.2}$$

for a function $v_t \in \mathscr{H}^1(q)$ of the following description: $\partial_t^* v_t$ is the first summand in the representation of $f(t)$ with respect to the splitting $\mathscr{H}(tq) = R_t^- + (R^-)_t^\perp$ which is described for q following (5.4) above. Here $\mathscr{H}(tq)$ is identified with $\mathscr{H}(q)$ and $\mathscr{H}(tq) = R_t^- + (R_t^-)^\perp$ is carried along by the identification. Notice that

$$||\partial_t^* v_t||_{2,q} \le ||f(t)||_{2,q}$$

and v_t can be taken to have integral zero. By the improved Poincaré inequality,

$$\begin{aligned} ||v_t||_{tq} &\le C\lambda(tq)^{-1} Q(v_t, v_t)^{1/2} \\ &= C\lambda(tq)^{-1} ||\partial_t^* v_t||_{2,q} \\ &\le C\lambda(tq)^{-1} ||f(t)||_{2,q} \\ &\le C\lambda(tq)^{-1} e^{at}, \quad a > 0,\ t \to -\infty, \end{aligned}$$

where $\lambda(tq)$ is the shortest geodesic length function. The last inequality is valid if q is generic for the Multiplicative Ergodic Theorem. A theorem of Masur implies the shortest geodesic length factor $\lambda(tq)^{-1}$ satisfies

$$\lambda(tq)^{-1} = O(|(\log|t|)|).$$

Using (6.2), and expressing F_t as the integral of its t derivative, it follows that

$$\|F_t\|_{2,q} = O(1), \quad t \to -\infty.$$

Let F be a weak cluster point. It is then an easy matter to conclude that in the distributional sense (a) $SF = -Re(f(0))$ and (b) $dF + \mathbf{Re}(f(0)q^{1/2}) = C$ is a basic current of order 1 for the horizontal foliation. This establishes the existence half of the isomorphism between the fibers of U and of H, above a generic q.

A good bit of work is still required in order to complete the proof of the identity (6.1). In the context of (4.5)-(4.9), the statement $\ker(J) = \{0\}$ at q is equivalent to the statement that q does not lie in the determinant locus, *i.e.,* that a certain real analytic function is not zero at q. The proof that

$$\mathrm{rank}(U) = \mathrm{rank}(S) = k \ge \left[\frac{p+1}{2}\right]$$

for Liouville measures then requires “only”the fact that this function is nonzero somewhere on each component of a stratum. This is the role of Forni’s theorem that the determinant locus contains no component. His proof that rank(U) = rank$(S) = p$ for Liouville measures is technically much more difficult. It involves a proof that for any $k < p$ the function

$$\operatorname{tr}\left(Q\left(q\right)\right) - \Phi^{k}_{I^+_k(q)} = \sum_{i,j=k+1}^{p} \left|\left(m_i^+, \overline{m_i^+}\right)_q\right|^2 \tag{6.3}$$

is nonzero on a set of positive (Liouville) measure. This proof does not turn on a property of the real analytic determinant function. The function on the left-hand side of (6.3) is only measurable, defined a.e. The proof that it does not vanish a.e. with respect to Liouville measure is probably the most difficult of many difficult arguments in [3].

Section 8 ends with an important theorem on the structure of the bundle of closed currents or order one. This is a three-fold sum, $Z' = H + V + E$, where H and V are as above and E is the bundle of exact currents of order one. Forni proves the Lyapunov spectrum for the Forni cocycle on Z' is contained in $[-1,1]$ and for the invariant infinite-dimensional bundle E it is $\{0\}$.

7. DEVIATION OF ERGODIC AVERAGES

Forni’s cocycle and the important structure theory he develops for it were probably motivated, as the title indicates, by the problem of deciding what, if any, Denjoy–Koksma theory is available for generic measured flows.

For a Liouville measure-generic q, Forni obtains the estimate $O(T^{\lambda-1})$, with a sharp lower bound for λ, for a.e. ergodic *averages* of mean zero, order one Sobolev functions f along horizontal trajectories. If there are order one invariant distributions which do not vanish at f, then the largest corresponding value of the Lyapunov spectrum is the sharp lower bound for λ; when all order one invariant distributions vanish at f, then

$$\inf\{\lambda | \lambda > 0 \text{ and ergodic averages of } f \text{ are } O(T^{\lambda-1})\} = 0$$

The strategy is first to make a dual formulation and then to use his infinite-dimensional Oseledec Theorem for $Z' = H + V + E$, mentioned at the end of the last section, as indicated below.

Let $\mathbf{C}$ be the bundle whose fiber at a square q is the (Hilbert) space of currents (in general nonclosed) of order one and dimension 1 on the surface for q. There is a bundle map $\mathbf{C} \to Z' = H + V + E$ (above) which is an orthogonal projection on each fiber. There are also bundle maps $Z' \to H$, V, E to the various subbundles for the Forni cocycle. Considering the test functions which are to be used, the Forni Theorem amounts to a growth estimate for the (order one, dimension-one) currents in q-fibers of $\mathbf{C}$ which are represented by segments of q-horizontal trajectories of length T, $T \to \infty$. Using the projection $\mathbf{C} \to Z'$ and further projecting (not orthogonally) into H, V and E, the desired growth estimate can be seen to be equivalent to corresponding estimates in H, V and E. As one does in Denjoy–Koksma arguments, one approaches the problem with an attempt to decompose a trajectory of length T into a sum of trajectories of varying length in which one has control of individual summands. The bound for the full sum then becomes a problem of the "number theory"of q. ("Number theory"is not such a bad analogy since there are continued fraction-like expansions in the background.)

What follows is a crude idea of how a problem about a long trajectory is converted to one about a sum of trajectories, in particular how one decides to cut a trajectory up. Recall that the trajectory Gq has been studied in terms of Teichmüller maps and objects on the surface for q. An additional property of a Teichmüller map from q to hq is that it is real affine in the affine structure determined by the atlases of natural parameters for q and hq (away from the zero sets). Therefore the notion of straight line is independent of location in Gq. At the same time, the notions of "horizontal"and "vertical"straight lines are not independent of location because G contains K. It is an important feature of the A-action that it preserves horizontal and vertical lines. A horizontal (resp.vertical) line segment of length α for $|q|$ is a segment of the same kind of length $e^t\alpha$ (resp.$e^{-t}\alpha$) for $|tq|$, $t \in A$, $A \approx \mathbb{R}$.

A related notion, which is also independent of location in Aq, is the notion that a point x in the surface for q be *regular*, meaning that x is not contained in a horizontal or vertical separatrix. If x is a regular point, Forni attaches to each tq the vertical segment $I(tq,x)$ with center x and length $\lambda\left(tq\right)$ (the shortest $|tq|$-geodesic length between zeros of tq). Let $\gamma\left(tq,x\right)$ be the horizontal segment which begins at x and ends with the first return to $I(tq,x)$. (I'll call $\gamma\left(tq,x\right)$ a *coil* at x.) The $|tq|$–length of $\gamma\left(tq,x\right)$ is uniformly bounded for the set of tq which project to a fixed compact set in the moduli space. He fixes x and $I(q,x)$ with q in a fixed "large"compact cross-section Ξ to the Teichmüller flow and then considers $tq, t<0$, along the sequence of times t of backward returns (of the projection of tq) to Ξ. If $t<0$ is a return time, the forward (time $-t$) image of the coil $\gamma\left(tq,x\right)$ is not in general the coil $\gamma\left(q,x\right)$ because the image has length $e^{-t}\left|\gamma\left(tq,x\right)\right|_{tq} \gg \left|\gamma\left(q,x\right)\right|_q$ for large $-t$. (That is, while the image coil ends in $I(q,x)$, it may visit $I(q,x)$ at many times before $-t$.) Call this length, *i.e.,* $e^{-t}\left|\gamma\left(tq,x\right)\right|_{tq}$, a *principal return time* and the image, *i.e.,* $(-t)\left(\gamma\left(tq,x\right)\right)$, a *principal return trajectory* (for (q,x) and $I(q,x)$). Forni partitions a horizontal segment of q-length $T \gg 0$ into one segment of bounded length and a sequence of segments, each one of which is a principal return trajectory for some point $x' \in I(q,x)$. Since the principal return trajectories are images of coils $\gamma\left(tq,x\right)$, $tq \in \Xi$ (compact) under time $-t \gg 0$ maps, he is able to control their current norms by (careful) application of the analysis of the Forni cocycle, on H, V and E above. Having thus controlled the individual summands, the remainder of the estimate of the current norm of a long horizontal trajectory from x is governed by the "number theory"of q.

I shall conclude with mention of one more result because the argument is such a pretty application of the theory of the Forni cocycle. Forni proves for the generic q that there exists a C_∞ function f with integral zero and compact support (away from the zeros of q) such that the equation $Su = f$, S the horizontal vector field, has no distributional solution $u \in L^2$. The proof is by contradiction: He shows that if the statement is false, then for each square integrable meromorphic function F with integral zero the equation $Sg = \mathbf{Re}(F)$ admits an L^2 solution. The idea is to first solve the problem in a neighborhood of the zero set of q and then to use the partial solution to reduce the equation to $Sg' = h$ with h smooth, compactly supported, with integral zero. Under the assumption that the latter equation admits an L^2 solution it follows that $Sg = \mathbf{Re}(F)$ also admits an L^2 solution. Using a construction which has been indicated above, it follows that $[\mathbf{Re}(Fq^{1/2}]$ is the class of a basic current of order one for the horizontal foliation. This implies the image of a.e. fiber of H is a codimension-one subspace of the corresponding fiber of the cohomology bundle. This contradicts Forni's theorem that a.e. q the image of the H-fiber at q under the cohomology map is Lagrangian.

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