Hyperbolic Dynamics, Invariant Geometric Structures and Rigidity of Abelian Group Actions

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These notes written by V. Nitica are based on the course given by A. Katok at the Pennsylvania State University in the Spring of 1994.

1 Survey of hyperbolic dynamics

1.1 Definitions and Hadamard-Perron Theorem

Definition 1 *Let $0 < \lambda < \mu$. A sequence of invertible linear maps $L_m : \mathbb{R}^l \to \mathbb{R}^l$, $m \in \mathbb{Z}$, is said to* admit a (λ, μ)-splitting *if there exist decompositions $\mathbb{R}^l = E_m^+ \oplus E_m^-$ such that $L_m E_m^{\pm} = E_{m+1}^{\pm}$ and*

$$\|L_m|_{E_m^-}\| \leq \lambda, \|L_m^{-1}|_{E_{m+1}^+}\| \leq \mu.$$

Consider now M a l-dimensional smooth manifold, $U \subset M$ an open set, $f : U \to M$ a C^1 diffeomorphism onto its image, $\Lambda \subset U$ a compact f-invariant set. We identify $T_x M$ with $\mathbb{R}^l$ via the Riemannian metric.

Definition 2 *The set Λ is called a* hyperbolic set *for f if there exists a Riemannian metric (called a* Lyapunov metric*) in an open neighborhood of Λ and $0 < \lambda < 1 < \mu$ such that for any point $x \in \Lambda$ the sequence of differentials $(Df)_{f^n(x)} : T_{f^n(x)} M \to T_{f^{n+1}(x)} M$, $n \in \mathbb{Z}$, admits a (λ, μ)-splitting.*

Definition 3 *A C^1-diffeomorphism $f : M \to M$ of a compact manifold M is called an* Anosov diffeomorphism *if M is a hyperbolic set for f.*

Let $A \in SL(n, \mathbb{Z})$ be an $n \times n$ matrix with determinant one and integer entries. A is called hyperbolic if it has no eigenvalue of absolute value one. In this case, the automorphism of the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ induced by A is an Anosov diffeomorphism.

We want to introduce now the stable and unstable manifolds which play a central role in the theory of hyperbolic dynamical systems.

For $\mathbb{R}^l = E^k \oplus F^{l-k}$ define the standard horizontal γ-cone by:

$$H_\gamma = \{(x,y) \in E \oplus F | \|y\| \leq \gamma \|x\|\}$$

and the standard vertical γ-cone by:

$$V_\gamma = \{(x,y) \in E \oplus F | \|x\| \leq \gamma \|y\|\}.$$

Proposition 1 *A compact f-invariant set Λ is hyperbolic if there exist $0 < \lambda < 1 < \mu$ such that for every $x \in \Lambda$ there is a decomposition $T_xM = S_x^k \oplus T_x^{n-k}$, a family of horizontal cones $H_x \supset S_x$ and a family of vertical cones $V_x \supset T_x$ associated with that decomposition such that*

$$Df_x H_x \subset Int H_{f(x)}, \ Df_x^{-1} V_x \subset Int V_{f^{-1}(x)}$$

and

$$\|Df_x \xi\| \geq \mu \|\xi\|, \ \xi \in H_x, \ \|Df_x^{-1} \xi\| \geq \lambda \|\xi\|, \ \xi \in V_x.$$

Remark 1 *Actually, the decomposition $\{T_xM = S_x \oplus T_x\}_{x \in M}$ and the fields of cones $H = \{H_x\}_{x \in M}$ and $V = \{V_x\}_{x \in M}$ are continuous in an appropriate sense.*

Consider now the continuous fields $\{E_x\}_x$ of k-dimensional subspaces inside H. Then f acts on these fields by:

$$(f_* E)_x = Df_{f^{-1}(x)}(E_{f^{-1}(x)}).$$

Theorem 1 *The action of f has a unique fixed point, denoted by $E^+ = \{E_x^+\}_x$. E^+ is called the unstable distribution.*

The idea of the proof is the following: consider each subspace $E_x \subset H_x$ as a graph of some linear map $\phi_x : S_x \to T_x$, with $\|\phi_x\| \leq \gamma$. Define $\|E\| = \sup_{x \in \Lambda} \|\phi_x\|$. Then the action of f_* is a contraction!

Consider the decreasing sequence of cones: $H_{x,n}^+ = Df^n(C_{f^{-n}(x)}^+)$ in T_xM. Then $E_x^+ = \cup_{n=0}^{\infty} C_{x,n}^+$.

One can show that E^+ is a Hölder continuous distribution. The idea is to show that, for some $0 < \alpha < 1$, the space of bounded α-Hölder fields of k-dimensional subspaces inside H is invariant under f_*.

Something similar about E^-.

We can state now the stable and unstable manifolds theorem (or Hadamard-Perron Theorem).

Theorem 2 *Let Λ be a hyperbolic set for a C^1-diffeomorphism $f : U \to M$ such that Df admits on Λ a (λ, μ)-splitting with $0 < \lambda < 1 < \mu$. Then for each $x \in \Lambda$ there is a pair of embedded C^1-disks $W^s(x)$, $W^u(x)$, called the local stable and local unstable manifolds of x, respectively, such that:*

1. $T_xW^s(x) = E_x^-$, $T_xW^u(x) = E_x^+$;

2. $fW^s(x) \subset W^s(f(x))$, $f^{-1}W^u(x) \subset W^u(f^{-1}(x))$;

3. *For every* $\delta > 0$ *there exist* $C(\delta)$ *such that for* $n \in \mathbb{N}$

   $dist(f^nx, f^ny) < C(\delta)(\lambda+\delta)^n dist(x, y)$ for $y \in W^s(x)$, $dist(f^{-n}x, f^{-n}y) < C(\delta)(\lambda-\delta)^{-n} dist(x, y)$ for

4. *There exists* $\beta > 0$ *and a family of neighborhoods* O_x *containing the ball around* $x \in \Lambda$ *of radius* β *such that*

   $W^s(x) = \{y | f^n(y) \in O_{f^nx}, n = 0, 1, 2, \ldots\} W^u(x) = \{y | f^{-n}(y) \in O_{f^{-n}x}, n = 0, 1, 2, \ldots\}$;

5. *If* $y \in W^s(x)$ *and is closed to* x*, then* $W^u(x) \cup W^u(y)$ *is open in both.*

1.2 Structural stability and topological classification

Definition 4 *A hyperbolic set* Λ *is called* locally maximal *if there exists an open set* V*,* $\Lambda \subset V$*, such that*

$$\Lambda = \cup_{n=-\infty}^{\infty} f^n V.$$

Open problem 1 *Assume that* $f : U \to M$ *is as before and* $\Lambda \subset U$ *is hyperbolic set. Does there exist a locally maximal hyperbolic set* Λ' *such that* $\Lambda \subset \Lambda'$*?*

Locally maximal hyperbolic sets are important in several places: to define the local product structure, for Closing lemma, specification, Gibbs measures etc.

Theorem 3 *Anosov diffeomorphisms are structurally stable via a uniquely defined (bi-Hölder) homeomorphism close to identity.*

Idea of proof Want $h^{-1}fg = g$, or a fixed point for the equation $h = ghf^{-1}$. This leads to a hyperbolic linear operator in a Banach space with an almost invariant point. Hence it has a fixed point.

Theorem 4 *Any topological conjugacy between Anosov diffeomorphisms is Hölder.*

Proof

Theorem 5 *Anosov flows are structurally orbit stable via a bi-Hölder homeomorphism close to identity, unique up to a time change.*

Problem 1 *Construct (or prove existence) of a* C^∞*-diffeomorphism* $f : \mathbb{T}^2 \to \mathbb{T}^2$*, area preserving, which is topologically conjugate to* $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$*, but the conjugacy is not Hölder.*

Hint$x \to 2x$ and $x \to x + x^3$ on $\mathbb{R}$ are conjugated around zero, but not Hölder.

Conjecture 1 *If f is C^∞ and Hölder conjugate to* $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, *then f is Anosov.*

General problem 1 *Classify Anosov diffeomorphisms up to topological (hence Hölder) conjugacy.*

The most important results toward a solution of this problem were obtained (after the work of Smale) by Franks and Manning between 1968-1974.

Theorem 6 *(Franks&Manning) Suppose M is a compact manifold which is known to have an Anosov diffeomorphism. Then any two Anosov diffeomorphisms which induce the same map on the fundamental group $\pi_1(M)$ are topologically conjugate by a unique homeomorphism homotopic to identity.*

A nilmanifold is the quotient of a connected, simply connected nilpotent Lie group N by a cocompact lattice Γ (i.e. N/Γ is a compact manifold). For example H/Γ where H is the Heisenberg group and Γ is a lattice in H is a nilmanifold.

An infranilmanifold is finitely covered by a nilmanifold. More precisely, consider N a connected simply connected nilpotent Lie group and C a compact group of isometries of N. Let Γ be a torsion free cocompact discrete subgroup of the semidirect product CN. By a result of Auslander [?], $\Gamma \cap N$ is a cocompact discrete subgroup of N and $\Gamma \cap N$ has finite index in Γ. An element of NC is a pair (x, c) with $x \in N, c \in C$ and it act on N by first applying a and then left translating by x. Γ acts freely on N. Indeed, if $\gamma \in \Gamma, x \in N$, $\gamma(x) = x$ implies $\gamma^n(x) = x$, for all n. But for some n, γ^n is left translation by an element of N. Hence $\gamma^n(x) = x$ implies γ^n is the identity of Γ. But Γ is torsion free. Thus the quotient space N/Γ is a compact manifold called an infranilmanifold. Let $\bar{f} : CN \to CN$ be an automorphism for which $\bar{f}(\Gamma) = \Gamma$, $\bar{f}(N) = N$. Then it induces a diffeomorphism $f : N/\Gamma \to N/\Gamma$. If the derivative of $\bar{f}|_N$ is hyperbolic, then f will be an Anosov diffeomorphism.

The only known examples of manifolds supporting Anosov diffeomorphisms are the tori, nilmanifolds and infranilmanifolds. It is now an outstanding conjecture that these are the only ones.

1.3 Differentiable conjugacy and moduli

Since Anosov systems are structurally stable (via Hölder homeomorphisms), it would be an interesting problem to classify a small neighborhood in $\mathrm{Diff}^1(M)$ of an Anosov system up to smooth conjugacy.

Definition 5 *A map $\phi : \mathrm{Diff}(M) \to \mathbb{R}$ such that $\phi(hfh^{-1}) = \phi(f)$ for any $h \in \mathrm{Diff}(M)$ is called a* modulus *of smooth conjugacy.*

Open problem 2 *Suppose g is C^r or C^∞ close to f and $g = hfh^{-1}$ for some $h \in \mathrm{Diff}^r(M)$, $r \geq 1$. Is h close to identity in C^k, $1 \leq k \leq r$?*

Open problem 3 *Are there always enough moduli of smooth conjugacy to classify the smooth conjugacy classes?*

This is true in some cases, and in those cases the answer is YES for the previous question. The known examples are C^∞ Anosov diffeomorphisms on two dimensional torus and C^∞ volume preserving Anosov flows on three dimensional manifolds. This follows from the next result.

Theorem 7 *Suppose f and g are Anosov and have invariant Hölder splitting of TM into 1-dimensional subbundles, which are C^∞-integrable and h carries the corresponding 1-dimensional foliations over. Then, if h matches the eigenvalues at the periodic points, h is C^∞.*

We discuss an example given by de la Llave which shows that the previous theorem does not have an immediate generalization in higher dimension.

Assume that $A \in SL(2,\mathbb{Z})$ and $B \in SL(d,\mathbb{Z})$ are hyperbolic matrices such that B has real eigenvalues in the interval $(1,\infty)$. Let $0<\lambda<1$ be one of the eigenvalues of A (hence the other one is λ^{-1}), and $\mu>0$ be an eigenvalue of B. We denote the corresponding normalized eigenvectors as follows:

$$Av_- = \lambda v_- Av_+ = \lambda^{-1} v_+ Be_\mu = \mu e_\mu.$$

Consider the actions on $\mathbb{T}^2 \times \mathbb{T}^d$

$$f(x,y) = (Ax, By)\widetilde{f}(x,y) = (Ax, By + \varphi(x)e_\mu).$$

(Wesee the quantity $\varphi(x)e_\mu$ as its image in $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, which is an abelian group.)

Note that f is hyperbolic, hence for C^1-small φ $\widetilde{f}$ is hyperbolic too, and there is a homeomorphism $h \in Homeo(\mathbb{T}^2 \times \mathbb{T}^d)$ close to identity such that:

$$h\widetilde{f} = fh \qquad (*).$$

Moreover, this homeomorphism is unique among the homeomorphisms which are homotopic to identity. The unique solution to (*) homotopic to identity is given by:

$$h(x,y) = (x, y + \psi(x)e_\mu),$$

provided that ψ satisfies the equation:

$$\mu\psi(x) - \psi(Ax) = \varphi(x). \qquad (**)$$

Equation (**) admits a unique bounded solution, namely:

$$\psi(x) = \mu^{-1}\sum_{k=0}^{\infty} \mu^{-k}\varphi(A^k x).$$

It remains to find how smooth is ψ, assuming φ is C^∞.

By choosing φ to be a trigonometric polynomial and using Fourier series, R. de la Llave shows that:

$$\psi \in C^{r-\varepsilon}, \text{ forany } r < \alpha_c$$

and

$$\psi \notin C^{r+\varepsilon}, \text{ forany } r > \alpha_c$$

where

$$\alpha_c = \frac{\ln \mu}{\ln \lambda^{-1}}.$$

Note that α_c can take a dense set of values in $(0, \infty)$ if we replace A and B by powers in the definitions of f and $\widetilde{f}$. In particular, if $\alpha_c < 1$, ψ is only Hölder and so h is only Hölder.

2 Commuting Anosov diffeomorphisms and rigidity

2.1 Rigidity of $\mathbb{Z}^{n-1}$ actions on $\mathbb{T}^n$

One can study similar regularity questions for the action generated by a family of commuting Anosov diffeomorphisms. A first result in this direction was obtained by Katok and Lewis. Before stating the result, we give a few more definitions which will put the result in a more general perspective.

If Γ is a finitely-generated discrete group and G is any topological group, we denote by $R(\Gamma, G)$ the space of homomorphisms of Γ into G with the compact open topology. A homomorphism $\rho_0 \in R(\Gamma, G)$ is said to be locally rigid if there exist a neighborhood U of ρ_0 in $R(\Gamma, G)$ such that for every $\rho \in U$ there exists $g \in G$ such that $\rho(\gamma) = g\rho_0(\gamma)g^{-1}$ for every $\gamma \in \Gamma$.

In our case Γ will be the group $\mathrm{Diff}^\infty(\mathbb{T}^n)$ of C^∞ diffeomorphisms of $\mathbb{T}^n$ under the C^1 topology.

Theorem 8 *Suppose $\mathcal{A} \subset SL(n, \mathbb{Z})$, $n \geq 3$, is a free abelian group of rank $n-1$, generated by $n-1$ hyperbolic matrices. Then the standard action of $\mathcal{A}$ on $\mathbb{T}^n$ is locally rigid on $R(\mathcal{A}, \mathrm{Diff}^\infty(\mathbb{T}^n))$.*

2.2 Non-stationary normal forms for one-dimensional contractions

One of the main tool in the proof of Theorem is the following generalization of Sternberg linearization lemma for contracting diffeomorphisms. (Define first the topology on C^∞ and C^ω.)

Theorem 9 *(Nonstationary Poincare-Sternberg Linearization)Let X be a compact metric space, $f : X \to X$ homeomorphism and let $F : X \times \mathbb{R} \to X \times \mathbb{R}$ given by $F(x, t) = (f(x), F_x(t))$ such that:*

(i) F_x is a C^∞-diffeomorphism (C^ω-diffeomorphism) of $\mathbb{R}$ for each $x \in X$;

(ii) $F_x(0) = 0$ *for each* $x \in X$ *(F preserves the zero section);*

(iii) $0 < F'_x(t) < 1$*, for each* $x \in X$*,* $t \in \mathbb{R}$*;*

(iv) $x \to F_x$ *is a continuous map* $X \to C^\infty(\mathbb{R})$ $(X \to C^\omega(\mathbb{R}))$*.*

Then there exists a unique reparametrization $G : X \times \mathbb{R} \to X \times \mathbb{R}$*,* $G(x,t) = (x, G_x(t))$ *such that:*

(v) each G_x *is a* C^∞*- diffeomorphism (*C^ω*- diffeomorphism) of* $\mathbb{R}$*;*

(vi) $G_x(0) = 0$*,* $G'_x(0) = 1$ *for every* $x \in X$*;*

(vii) $x \to G_x$ *is a continuous map* $X \to C^\infty(\mathbb{R})$ $(X \to C^\omega(\mathbb{R}))$*;*

(viii) $GFG^{-1}(x,t) = (f(x), F'_x(0)t)$ *for every* $x \in M$*,* $t \in \mathbb{R}$*.*

Furthermore, such G is unique even among continuous maps $X \to C^1(\mathbb{R})$*.*

We show uniqueness of G first. If G_1 and G_2 are like in Theorem, then $G_1FG_1^{-1}$ is linear, $G_2G_1^{-1}$ satisfies (v)-(vii) in Theorem and

$$(G_2G_1^{-1})G_1FG_1^{-1} = G_1FG_1^{-1}(G_2G_1^{-1}).$$

So is enough to prove:

Lemma 1 *(Lemma 1) Assume that* $A : X \times \mathbb{R} \to X \times \mathbb{R}$ *is linear, i.e.* $A(x,t) = (f(x), \alpha_x t)$ *where* $x \to \alpha_x$ *is a continuous map* $X \to (0,1)$ *and suppose that* $GA = AG$*. Then* G *is the identity map on* $X \times \mathbb{R}$*.*

Proof For this argument it is sufficient to assume that the map $x \to G_x$ is only C^1-continuous.

The condition $GA = AG$ becomes:

$$G_{f(x)}(\alpha_x t) = \alpha_x G_x(t)$$

so

$$G_x(t) = \alpha_x^{-n} G_{f^n(x)}(\alpha_x^n t), \mathrm{for} n \geq 1.$$

Since $x \to \alpha_x$ is continuous and X is compact, there exists $\varepsilon > 0$ with $\alpha_x \leq 1 - \varepsilon$ for $x \in X$. Since G_x varies continuously with x in C^1-topology, one has that $G_x(\delta)/\delta$ converges uniformly in x to $G'_x(0) = 1$. Hence

$$G_x(t) = \lim_{n\to\infty} \frac{G_{f^n(x)}(\alpha_x^n t)}{\alpha_x^n} = t.$$

Remark 2 *The previous argument does not work if* $x \to G_x$ *is continuous only in the Lipschitz topology.*

We establish now the existence of G, solving the problem for formal power series at the zero section.

Lemma 2 *(Lemma 2) Suppose $F:(x,t)\to(f(x),F_x(t))$, $F_x(t)=\sum_{i=1}^{\infty}a_i(x)t^i$ is a formal power series based at the zero section in $X\times\mathbb{R}$, with $a_i:X\to\mathbb{R}$ continuous for each i and $0<a_1(x)<1$. Then there exists a formal power series*

$$G:(x,t)\to(x,G_x(t)),\ G_x(t)=t+\sum_{i=2}^{\infty}b_i(x)t^i$$

with $b_i:X\to\mathbb{R}$ continuous for each i, such that:

$$GFG^{-1}(x,t)=(f(x),a_1(x)t) \qquad (*)$$

Proof It follows from equation (*) that

$$G_{f(x)}(F_x(t))=a_1(x)G_x(t). \qquad (**)$$

Now we find b_i's inductively. For b_2 one has the equations:

$$b_2(f(x))a_1(x)^2+a_2(x)=a_1(x)b_2(x)b_2(x)=\frac{a_2(x)}{a_1(x)}+a_1(x)b_2(f(x)).$$

Since X is compact, there exists $\varepsilon>0$ such that $\varepsilon<a_1(x)<1-\varepsilon$ for every $x\in X$, and the series

$$b_2(x)=\frac{a_2(x)}{a_1(x)}+\sum_{i=1}^{\infty}\Big(a_1(x)a_1(f(x))\cdots a_1(f^{i-1}(x))\frac{a_2(f^i(x))}{a_1(f^i(x))}\Big)$$

converges uniformly to a continuous solution for b_2.

Assume now that we have continuous solutions for b_2 through b_{n-1} such that the first $n-1$ coefficients in (**) agree. Then, from the n-th term in (**) it follows

$$b_n(f(x))a_1^n(x)+r(x)=a_1(x)b_n(x),b_n(x)=\frac{r(x)}{a_1(x)}+a_1^{n-1(x)}b_n(f(x)),$$

where $r:X\to\mathbb{R}$ is a polynomial in $b_i(f(x))$, $2\le i\le n-1$ and $a_i(x)$, $1\le i\le n$. Since r is continuous and X is compact, r is uniformly bounded, so

$$b_n(x)=\frac{r(x)}{a_1(x)}+\sum_{i=1}^{\infty}\Big(a_1(x)a_1(f(x))\cdots a_1(f^{i-1}(x))\frac{r(f^i(x))}{a_1(f^i(x))}\Big)$$

converges uniformly to a continuous solution for b_n.

Remark 3 *Finding b_i's is equivalent to solving twisted cocycle equations of the type*

$$\phi(x)=\psi(x)+\lambda(x)\phi(f(x)) \qquad (***)$$

where ψ, λ and f are known, $0<\lambda(x)<1$, and ϕ is unknown. If f is Anosov, (***) is much easy to solve when $0<\lambda(x)<1$, then when $\lambda(x)=1$. The last case is solved by the Livsic's theorem. Anyhow, here we get only that ϕ is Hölder, if f is Hölder, but $f\in C^1$ does not imply $\phi\in C^1$. A counterexample can be found in de la Llave paper [5].

We prove now the existence of G as a C^∞ function (respectively real analytic). The majorisation method used in the analytic case was introduced by David De Latte. See also [3], Proposition 2.1.3.

Case I. $F_x : X \to \mathrm{Diff}^\omega(\mathbb{R})$.

We show first that the formal power series obtained in Lemma 2 is uniformly convergent on $X \times \mathbb{R}$. Since $F_x : X \to \mathrm{Diff}^\omega(\mathbb{R})$ is continuous, there exist $0 < a < 1$, $c > 0$ and a positive integer n_0 such that:

$$\|a_n(x)\| \leq ca^n, \text{andfor } n \geq n_0 \text{ also} \|a_n(x)\| \leq a^n.$$

One determines now the coefficients $b_i(x)$ inductively using (*). Denote $A_n = \sup_{x\in X} \|a_n(x)\| < 1$ and $B_n = \sup_{x\in X} \|b_n(x)\|$. The following inequalities are similar with the formulas for $b_i's$ and follows easily:

$$B_2 \leq \frac{A_2}{A_1 - A_1^2} B_3 \leq \frac{A_3 + 2B_2A_1A_2}{A_1 - A_1^3} \cdots\cdots\cdots B_n \leq (A_1 - A_1^n)^{-1}(A_n + \sum_{k=2}^{n-1} B_k k! F_n^k(A_1, A_2, \ldots, A_{r(n,k)})$$

where F_n^k is the sum of all the monomials of the form

$$A_1^{s_1} A_2^{s_2} \ldots A_r^{s_r} \qquad (*)$$

such that

$$s_1 + 2s_2 + \cdots + rs_r = ns_1 + s_2 + \ldots + s_r = k. \qquad (**)$$

The value of any product (*) satisfying (**) does not exceed $c^{n_0}a^n$.

Fix now $b \in (0, 1)$. We will show inductively that there exists a number d such that for all n

$$B_n \leq db^n.$$

Let N_n be the total number of monomials (*) in the expression (*) for B_n taken with multiplicities. By the inductive assumption we have $\|B_k\| \leq d$ for $k < n$ and thus

$$\|B_n\| < (A_1 - A_1^n)^{-1} d(1 + N_n)a^n.$$

To estimate N_n note that $N_n = \sum_{k=0}^{n-1} C_k$, where C_k is the coefficient of x^n in

$$\Big(\sum_{i=1}^{\infty} x^i\Big)^k = \Big(\frac{x}{1-x}\Big)^k = \frac{x^k}{(k-1)!}\frac{d^{k-1}}{dx^{k-1}}\Big(\frac{1}{1-x}\Big) = = \frac{x^k}{(k-1)!}\Big(\sum_{i=0}^{\infty} \frac{(k+i-1)!x^i}{i!}\Big),$$

which is the coefficient corresponding to $i = n - k$ and thus we have:

$$C_k = \frac{(k+n-k-1)!}{(n-1)!(n-k)!} = \binom{n-1}{k-1}.$$

Using $\sum_{j=0}^{l} \binom{i}{j} = 2^l$ yields

$$N_n = \sum_{k=2}^{n-l} C_k = \sum_{k=2}^{n-l} \binom{n-1}{k-1} = 2^{n-1} - 2 \leq 2^n.$$

Thus $\|h_n\| \leq \|\lambda - \lambda^n\|^{-1} d 2^n \lambda^{-n}$. This is bounded by d if we take n_0 such that $\|\lambda - \lambda^{n_0}\|^{-1} 2^{n_0} \lambda^{-n_0} \leq 1$ and $d = \max_{n<n_0} \|h_n\|$.

Case II. $F_x : X \to \mathrm{Diff}^\infty(\mathbb{R})$

Lemma 3 *(Lemma 3) For F as in Theorem 2, there exists $H : X \times \mathbb{R} \to X \times \mathbb{R}$, $(x,t) \to (x, H_x(t))$, such that*

1. *each H_x is a C^∞-diffeomorphism of $\mathbb{R}$;*
2. *$H_x(0) = 0$, $H_x'(0) = 1$ for each $x \in X$;*
3. *$x \to H_x$ is a continuous map $x \to C^\infty(\mathbb{R})$;*
4. *$H_{f(x)} F_x H_x : \mathbb{R} \to \mathbb{R}$ has a tangency of infinite order with the identity map on $\mathbb{R}$ at 0 for every $x \in X$.*

Proof Fix $\alpha \in C^\infty(\mathbb{R}, [0,1])$ such that $\alpha(t) = 1$ for $|t| < 1/4$ and $\alpha(t) = 0$ for $|t| > 3/4$. Let $\sum_{i=1}^\infty a_i(x) t^i$ be the Taylor series expansion for F_x at 0. Then take:

$$H_x(t) = t + \sum_{i=2}^\infty b_i(x) t^i \alpha(i! b_i(x) t),$$

with b_i as in Lemma 1.

So far, we can assume in Theorem 2 that

$$F_x(t) = F_x'(0) t + \beta(x,t)$$

with $\frac{\partial^n}{\partial t^n} \beta(x,t)|_{t=0} = 0$, for any $n \geq 0$.

Because $0 < F_x'(0) < 1$, we can choose $\varepsilon_0 > 0$ small enough such that:

$$\sup_{|t|\leq\varepsilon} \left|\frac{\partial F_x(t)}{\partial t}\right| < \lambda_1 < 1.$$

Consider now the space $\mathcal{S}_k$ of functions $\alpha : X \times (-\varepsilon, \varepsilon) \to X \times \mathbb{R}$, $(x,t) \to (x, \alpha_x(t))$, where α_x is C^∞ and $D_2^i \alpha|_{t=0} = 0$, for any $0 \leq i \leq k$. (Here $D_2 = \frac{\partial}{\partial t}$.)

On $\mathcal{S}_k$ consider the operator $\Phi : \mathcal{S}_k \to \mathcal{S}_k$, $\Phi(\alpha) = DF^{-1} \circ \alpha \circ F$, where $DF : X \times \mathbb{R} \to X \times \mathbb{R}$, $DF(x,t) = (f(x), F_x'(0) t)$. For any $k \geq 0$, $\varepsilon > 0$ introduce on $\mathcal{S}_k$ the norm:

$$\|\alpha\|_{C^k,\varepsilon} = \max_{x\in X, |t|\leq\varepsilon} \max_{0\leq i\leq k} \|D_2^i \alpha(x,t)\|.$$

Lemma 4 *(Lemma 4) If $k \geq 2$, there exist $\varepsilon > 0$, $0 < \lambda < 1$ such that:*

$$\|\Phi(\alpha)\|_{C^k,\varepsilon} \leq \lambda \|\alpha\|_{C^k,\varepsilon}.$$

Proof Assume first that $\phi : \mathbb{R} \to \mathbb{R}$, $\phi^{(r)}(0) = 0$ for $r = 0, 1, \ldots, k$. Then

$$\|\phi^{(i)}(t)\| = \| \int_0^t \phi^{(i+1)}(s)ds\| \leq t \max_{s\in[0,t]} \|\phi^{(i+1)}(s)\|.$$

Hence, if $\varepsilon < 1$

$$\max_{|t|\leq\varepsilon} \max_{0\leq i\leq k} \|\phi^{(r)}(t)\| \leq \varepsilon \max_{|t|\leq\varepsilon} \|\phi^{(k+1)}(t)\|.$$

Consider now

$$\Phi(\alpha)(x,t) = (x, D_2^{-1}F(f(x),0)(\alpha_{f(x)} \circ F_x)(t)).$$

Then

$$\frac{\partial^k}{\partial t^k}(\alpha_{f(x)} \circ F_x) = \frac{\partial^k}{\partial t^k \alpha_{f(x)}}|_{F_x}(\frac{\partial F_x}{\partial t})^k + P_k(x,t),$$

with $P_k(x,t)$ polynomial in $\frac{\partial^l}{\partial t^l}\alpha_{f(x)}|_{F_x}$, $1 \leq l \leq k-1$, and in the first k-derivatives of F_x.

Now, since $F_x(0) = 0$ and $0 < F_x'(0) < 1$, for $\varepsilon > 0$ small enough we have:

$$\sup_{|t|\leq\varepsilon} \|\frac{\partial^k}{\partial t^k}\alpha_{f(x)}|_{F_x}\| \leq \sup_{|t|\ \leq\varepsilon} \|\frac{\partial^k}{\partial t^k}\alpha_{f(x)}\| \sup_{|t|\leq\varepsilon} \|\frac{\partial F_x(t)}{\partial t}\| \leq \lambda_1 < 1 \sup_{x\in X, |t|\leq\varepsilon} \|P_k(x,t)\| \leq C(k,F)\|\alpha\|_{C^{k-1},\varepsilon} \leq C(k,$$

So

$$\|\Phi(\alpha)\|_{C^k,\varepsilon} \leq \|\alpha\|_{C^k,\varepsilon} \frac{[\max_{x\in X,|t|\leq\varepsilon} \|D_2F(x,\varepsilon)\|^k + C(k,F)]}{\min_{x\in X} \|D_2F(x,0)\|}.$$

The last quotient is less than 1 for $k \geq 2$ and ε small enough.

We can find now the reparametrization G in a small neighborhood $X \times (-\varepsilon, \varepsilon)$ of the zero section. For H as in Lemma 3, $H - \Phi H$ is in $\mathcal{S}_k$, for $k \geq 2$ and ε small enough. The, by Lemma 4, the sequence of maps

$$G_n = \Phi^n H = \sum_{i=1}^{n-1} \Phi^i(\Phi H - H) + H$$

converges uniformly on $X \times (-\varepsilon, \epsilon)$ and the limit G, which is a C^k-function, satisfies (vi)-(viii) in Theorem. Uniqueness of G follows from Lemma 1.

To finish the proof of Theorem, observe that the C^k-reparametrization G can be extended from $X \times (-\varepsilon, \varepsilon)$ to $X \times \mathbb{R}$. Indeed, since F is contraction fiberwise, for any compact neighborhood K of the zero section $F^r(K) \subset X \times (-\varepsilon, \varepsilon)$ for large enough r, so we can extend G to K by setting $\bar{G} = \Phi^r G$.

Since the C^k solution is unique, it coincide with the solution in C^{k+1} for each $k \geq 2$, hence G_x is C^∞ for each $x \in X$. $\square$

2.3 Proof of the rigidity theorem

We start now the proof of Theorem ,i.e., the C^∞-rigidity of the the standard linear action of $\mathbb{Z}^{n-1}$ on $\mathbb{T}^n$. First we introduce some algebraic preliminaries.

Lemma 5 *Let $\Gamma = SL(n, \mathbb{Z})$, $n \geq 2$. Then there exists a Cartan subgroup H of $SL(n, \mathbb{R})$ such that the quotient $H/(H \cap \Gamma)$ is compact. In particular, there exists a subgroup $\mathcal{A} \subset \Gamma$ such that*

1. *the elements of $\mathcal{A}$ are simultaneously diagonalizable over $\mathbb{R}$;*
2. *$\mathcal{A}$ is isomorphic to a free abelian group of rank $n - 1$.*

This follows from a more general result of Prasad and Ragunathan.

Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be a basis of simultaneously eigenvectors for the group $\mathcal{A}$, and $\lambda_i : \mathcal{A} \to \mathbb{R}^0$ the character on $\mathcal{A}$ defined via $Av_i = \lambda_i(A)v_i, A \in \mathcal{A}$. To simplify the notation we pass to a subgroup of finite index and assume that each λ_i takes values in $\mathbb{R}^+$. Then H^0=the identity component in H, above, is a maximal $\mathbb{R}$-split torus in $SL(n, \mathbb{R})$ with eigenvectors v_i and λ_i extendes to $\lambda_i : H^0 \to \mathbb{R}^+$ so that

$$\lambda_1 \times \cdots \times \lambda_n : H^0 \to \{(x_1, \ldots, x_n) \in (\mathbb{R}^+)^n | x_1 \ldots x_n = 1\}$$

is an isomorphism of analytic groups and $H/\mathcal{A}$ is compact. From the compactness of $\mathcal{A}$ in H^0, we have that for each i, $1 \leq i \leq n$, there exists $A_i \in \mathcal{A}$ such that $\lambda_i(a_i) < 1, \lambda_j(A{-}i) > 1$ for each $j \neq i$.

By standard results, the torus H^0 is $\mathbb{Q}$-anisotropic. Equivalently, none of the eigenspaces $\mathbb{R}v_i$ is a rational line. If $\pi : \mathbb{R}^n \to \mathbb{T}^n$ denote the natural projection, then $\pi(\mathbb{R}v_i)$ is the stable manifold through 0 for the hyperbolic diffeomorphism A_i on the torus. In particular, $\pi(\mathbb{R}v_i)$ is dense in $\mathbb{T}^n$ for each i.

Lemma 6 *Suppose $A \in \mathcal{A}$, $A \neq 1$. Then $\lambda_i(A) \neq 1$ for each $1 \leq i \leq n$. In particular, $\mathcal{A} - \{1\}$ contains only hyperbolic elements.*

Proof Otherwise $Av_i = v_i$. So A fixes each point of $\pi(\bar{\mathbb{R}}v_i) = \mathbb{T}^n$. So $A = 1$. $\square$

Lemma 7 *Fix generators $B_1, \ldots, B_{n-1}$ for $\mathcal{A} \simeq \mathbb{Z}^{n-1}$. Then there is no nontrivial relation of the form $\lambda_i(B_1)^{p_1} \ldots \lambda_i(B_{n-1})^{p_{n-1}} = 1$, with $1 \leq i \leq n$, $p_j \in \mathbb{Z}$, and at least one $p_j \neq 0$.*

In particular, for each $1 \leq i \leq n$, $\{\lambda_i(A) | A \in \mathcal{A}\}$ is a dense subgroup in $\mathbb{R}^+$.

Proof Otherwise $B_1^{p_1} \ldots B_{n-1}^{p_{n-1}} \neq 1$ and $\lambda_i(B_1^{p_1} \ldots B_{n-1}^{p_{n-1}}) = 1$, contradicting the previous lemma. $\square$

For each $1 \leq i \leq n$, fix $A_i \in \mathcal{A}$ such that $\lambda_i(A_i) < 1, \lambda_j(A_i) > 1$ for $j \neq i$. Let W_i^s, W_i^u denote the stable and unstable foliations, respectively, for the hyperbolic diffeomorphism A_i

of $\mathbb{T}^n$. The leaves of W_i^s are the images under π of lines in $\mathbb{R}^n$ parallel to v_i, those of W_i^u the images of hyperplanes parallel to the span of the remaining $v_j, j \neq i$.

For suitable ρ, each $\rho(A_i)$ is Anosov and $\widetilde{W}_i^s = h(W_i^s)$ and $\widetilde{W}_i^u = h(W_i^u)$ are the stable and unstable foliations of $\rho(A_i)$.

Fix an index i_0, $1 \leq i_0 \leq n$. Set $A = A_{i_0}$, $\mathcal{F} = W_A^s$, $\widetilde{\mathcal{F}} = \widetilde{A}^s$. The leaves of both $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ inherit natural Riemanian metrics as submanifolds of $\mathbb{T}^n$. For each $x \in \mathbb{T}^n$, denote by $\phi_x : \mathbb{R} \to \mathcal{F}(x)$ the arc length parametrization based at x, oriented so that v_{i_0} points in the positive direction. I.e., $\phi_x(0) = x$, the distance along $\mathcal{F}(x)$ betwwen x and $\phi_x(t) \in \mathcal{F}(x)$ is $\langle v_{i_0}(\phi_x)_*(\frac{d}{dt}\rangle > 0$ (standard inner product on $T_x\mathbb{T}^n = \mathbb{R}^n$). Define $\widetilde{\phi}_x : \mathbb{R} \to \widetilde{\mathcal{F}}(x)$ similarly, oriented so that $\widetilde{\phi}_{h(x)}^{-1} \circ h \circ \phi_x : \mathbb{R} \to \mathbb{R}$ is an orientation preserving (monotone increasing) homeomorphism. We want to show that h is smooth along the leaves of $\mathcal{F}$. More precisely we show that $x \to \widetilde{\phi}_{h(x)}^{-1} \circ h \circ \phi_x$ is a continuous map $M \to C^\infty(\mathbb{R})$.

Let f be the automorphism of $\mathbb{T}^n$ induced by A. Extend f and h to transformations on $\mathcal{L}$, namely, define

$$F : \mathcal{L} \to \mathcal{L},\ (x, t) \to (f(x), F_x(t)), H : \mathcal{L} \to \mathcal{L},\ (x, t) \to (h(x), H_x(t))$$

so that

$$\widetilde{\phi}(F(x, t)) = f(\widetilde{\phi}(x, t)) \text{ and } \widetilde{\phi}(H(x, t)) = f(\phi(x, t)).$$

Then F and H are continuous, $F_x \in C^\infty(\mathbb{R})$ and each $x \in \mathbb{T}^n$, $0 < F_x'(x) < 1$ for every $x \in \mathbb{T}^n$, $t \in \mathbb{R}$, and $x \to F_x$ is a continuous map $\mathbb{T}^n \to C^\infty(\mathbb{R})$. We must show that $H_x \in C^\infty(\mathbb{R})$ and $x \to H_x$ is continuous.

By construction, $\phi_x : \mathbb{R} \to \mathcal{F}(x)$ and $\widetilde{\phi}_x : \mathbb{R} \to \widetilde{\mathcal{F}}(x)$ are diffeomorphisms for every $x \in M$. Let $\mathcal{L} = \mathbb{T}^n \times \mathbb{R}$ denote the trivial line bundle over $\mathbb{T}^n$. It follows easily from (?) that $\phi : \mathcal{L} \to \mathbb{T}^n$, $(x, t) \to \phi_x(t)$ and $\widetilde{\phi} : \mathcal{L} \to \mathbb{T}^n$, $(x, t) \to \widetilde{\phi}_x(t)$ are continuous, and that $x \to \phi_x$, $x \to \widetilde{\phi}_x$ are continuous maps $\mathbb{T}^n \to C^\infty(\mathbb{R}, \mathbb{T}^n)$.

By the nonstationary Sternberg linearization described above, there exist a unique continuous linearization

$$G : \mathcal{L} \to \mathcal{L},\ (x, t) \to (x, G_x(t))$$

such that

1. $G_x \in C^\infty(\mathbb{R})$ with $G_x'(0) = 1$ for all $x \in \mathbb{T}^n$,
2. $\mathbb{T}^n \to C^\infty(\mathbb{R})$, $x \to G_x$ is continuous,
3. $GFG^{-1}(x, t) = (f(x), F_x'(0)t)$ for every $x \in \mathbb{T}^n$ and $t \in \mathbb{R}$.

Lemma 8 *Suppose $p \in \mathbb{T}^n$ is rational, and hence a periodic point for the standard action of every matrix in $SL(n, \mathbb{Z})$. Then $G_{h(p)} \circ H_p|_{\mathbb{R}^+} : \mathbb{R}^+ \to \mathbb{R}^+$ has the form $G_{h(p)} \circ H_p(t) = c_p t^{\mu_p}$ for some c_p, $\mu_p > 0$.*

Proof Since $\mathcal{A}$ is abelian, it follows from the uniqueness (?) that G simultaneously linearizes the transformations on $\mathcal{L}$ corresponding to $\mathcal{A}$ for each $a \in \mathcal{A}$. By (?), we acn find

$B, C \in \mathcal{A}$ such that $\lambda_{i_0}(B) = \beta$, $\lambda_{i_0}(C) = \gamma$ with$\beta, \gamma > 1$ such that β and γ generate a dense subgroup in $\mathbb{R}^+$. By replaceing B and C with appropriate powers, we may assume that p is a fixed point for the action of both B and C.

Denote by r and s the the automorphisms induced by B, respectively C, on $\mathbb{T}^n$, and define $R, S : \mathcal{L} \to \mathcal{L}$ as above, so that $\widetilde{\phi} R = r\widetilde{\phi}$ and $\widetilde{\phi} S = s\widetilde{\phi}$. Then

$$GRG^{-1}(x,t) = (r(x), \widetilde{\beta}_x t), GSG^{-1}(x,t) = (s(x), \widetilde{\gamma}_x T),$$

where $\widetilde{\beta}_x = R'_x(0)$, $\widetilde{\gamma}_x = S'_x(0)$. In particular, since $h(p)$ is fixed by r and s,

$$G_{h(p)} \circ R_{h(p)} = \widetilde{\beta} G_{h(p)} \text{ and } G_{h(p)} \circ S_{h(p)} = \widetilde{\gamma} G_{h(p)},$$

with $\widetilde{\beta} = \widetilde{\beta}_{h(p)}$, $\widetilde{\gamma} = \widetilde{\gamma}_{h(p)}$. Also, since h intertwines ρ and the standard action,

$$R_{h(p)} \circ H_p(t) = H_p(\beta t) \text{ and } s_{h(p)} \circ H_p(t) = H_p(\gamma t).$$

Let

$$\psi = G_{h(p)} \circ H_p|_{\mathbb{R}^+} : \mathbb{R}^+ \to \mathbb{R}^+.$$

Then we have shown that for every $t \in \mathbb{R}^+$,

$$\psi(\beta t) = \widetilde{\beta}\psi(t) \text{ and } \psi(\gamma t) = \widetilde{\psi}(t).$$

By construction ψ is an orientation preserving homeomorphism.

Let $c = \psi(1)$. Then $\psi(\beta^k \gamma^l) = c\widetilde{\beta}^k \widetilde{\gamma}^l$ for every $k, l \in \mathbb{Z}$. Hence

$$\{\beta^k \gamma^l | k, l \in \mathbb{Z}\} \to \{\widetilde{\beta}^k \widetilde{\gamma}^l | k, l \in \mathbb{Z}\},\ \beta^k \gamma^l \to \widetilde{\beta}^k \widetilde{\gamma}^l$$

is an order preserving map between these two subsets of $\mathbb{R}^+$. It follows easily that

$$\frac{\log \beta}{\log \gamma} = \frac{\log \widetilde{\beta}}{\log \widetilde{\gamma}},$$

hence

$$\psi(t) = ct^\mu \text{ forevery } t \in \{\beta^k \gamma^l\},$$

where

$$\mu = \frac{\log \widetilde{\beta}}{\log \beta} = \frac{\log \widetilde{\gamma}}{\log \gamma}.$$

But this set is dense in $\mathbb{R}^+$ and ψ is continuous, hence $\psi(t) = ct^\mu$ for every $t \in \mathbb{R}^+$. $\square$

Now for each $x \in \mathbb{T}^n$, set $\psi_x = G_{h(x)} \circ H_x|_I : I \to \mathbb{R}^+$, $I = [0,1]$. Since I is compact and $G \circ H|_{\mathbb{T}^n \times I}$ is continuous it follows that $\mathbb{T}^n \to C^0(I)$, $x \to \psi_x$ is continuous with respect to the uniform topology on $C^0(I)$. By (?), $\psi_p(t) = c_p t^{\mu_p}$ is a dense set of indexed by $p \in \mathbb{T}^n$. It follows that $p \to c_p$ and $p \to \mu_p$ can be extended to continuous functions $\mathbb{T}^n \to \mathbb{R}^+$ such that $\psi_x(t) = c_x t^{\mu_x}$ for every $x \in \mathbb{T}^n$. An analogous argument works with $-I = [-1,0]$ in place of I and $\mathbb{R}^-$ in place of $\mathbb{R}^+$. Also, we can replace I with any compact interval $[0,T]$.

Thus we have proved the following:

Lemma 9 *There exist continuous functions $c^{\pm}, \mu^{\pm} : \mathbb{T}^n \to \mathbb{R}^+$ such that for every $x \in \mathbb{T}^n$, $G_{h(x)} \circ H_x : \mathbb{R} \to \mathbb{R}$ has the form*

$$G_{h(x)} \circ H_x(t) = \begin{cases} c_x^+ t^{\mu_x^+} & t \geq 0 \\ -c_x^- |t|^{\mu_x^-} & t \leq 0. \end{cases}$$

For each $x \in \mathbb{T}^n$, $G_{h(x)} \circ H_x$ is smooth away from 0, and $G_{h(x)}$ is a C^∞ diffeomorphism, hence H_x is smooth away from 0. But ϕ maps $\mathbb{T}^n \times (\mathbb{R} - \{0\})$ onto $\mathbb{T}^n$, so this implies that h is C^∞ along each leaf of $\mathcal{F}$, more precisely, $h|_{\mathcal{F}(x)} : \mathcal{F}(x) \to \widetilde{\mathcal{F}}(h(x))$ is C^∞ for every $x \in \mathbb{T}^n$. Thus $G_{h(x)} \circ H_x$ must be smooth at 0 as well, hence $c_x^+ = c_x^-$, and $\mu_x^+ = \mu_x^- = 1$ for every $x \in \mathbb{T}^n$. We have shown that $x \to G_{h(x)} \circ H_x$ defines a continuous map $\mathbb{T}^n \to C^\infty(\mathbb{R})$. The same is true for $x \to G_{h(x)}$, and each $G_{h(x)}$ is a diffeomorphism. Since the diffeomorphisms of $\mathbb{R}$ form a topological group with respect to the subspace topology inherited from $C^\infty(\mathbb{R})$, we conclude that $\mathbb{T}^n \to C^\infty(\mathbb{R})$, $x \to H_x = G_{h(x)}^{-1} \circ (G_{h(x)} \circ H_x)$ is continuous.

The foliation $\mathcal{F}$ is smooth (in fact the leaves of W_i^s, $1 \leq i \leq n$, constitute a smooth parallelism on $\mathbb{T}^n$) and the smooth foliation charts determine a uniformly C^∞ structure long the leaves. For each $x \in \mathbb{T}^n$, we can construct a continuous foliation chart for $\widetilde{\mathcal{F}}$ centered at x as follows. First fix a small transverse slice with continuous coordinates centered at x, e.g. a smooth coordinate chart centered at x in $\mathcal{F}_{i_0}^u(x)$. Then extend along the leaves of $\widetilde{\mathcal{F}}$ via the arc length parameter to obtain a continuous foliation chart centered at x with C^∞ leaves and such that the C^∞ coordinate charts along the leaves vary continuously with respect to the transverse coordinate. In particular, This determines a uniform C^∞ structure along the leaves of $\widetilde{\mathcal{F}}$. We summarize the proceding discussion as follows, again making use of the fact that inversion defines a continuous involution on the C^∞ diffeomorphisms $\mathbb{R}$.

Lemma 10 *For each i, $1 \leq i \leq n$, the one dimensional foliations $W_i^s, \widetilde{W}_i^s$ have uniformly C^∞ leaves, and $h, h^{-1} : \mathbb{T}^n \to \mathbb{T}^n$ are uniformly C^∞ along the leaves of $W_i^s, \widetilde{W}_i^s$, respectively.*

We are now in position to apply Journe theorem and deduce the regularity of h.

For $1 \leq j \leq n$, define C^∞ j-dimensional foliations $\mathcal{G}_j$ of $\mathbb{T}^n$ as follows. The leaves of $\mathcal{G}_j$ are the images under π of j-planes in $\mathbb{R}^n$ parallel to the span of the first j basis vectors v_i, $1 \leq i \leq n$, so that $\mathcal{G}_1 = W_1^s$ and $\mathcal{G}_n$ is the trivial foliation with one leaf. Then $\mathcal{G}_{j-1}$ ans W_j^s restrict to transverse foliations on each leaf of $\mathcal{G}_j$. Let $\widetilde{\mathcal{G}}_j = h(\mathcal{G}_j)$. Since $\mathcal{A}$ is cocompact in H^0, there exists $C_i \in \mathcal{A}$, $1 \leq j \leq n-1$, such that $\lambda_i(c_i) < 1$, $i \leq j$, and $\lambda(C_j) > 1$, $i > j$, i.e., so that $\mathcal{G}_i$ is the stable foliation of the standard action of C_i on $\mathbb{T}^n$. For ρ C^1-close enough to the linear representation, each of the diffeomorphisms $\rho(C_i)$ is Anosov, and $\widetilde{\mathcal{G}}_j$ is the stable foliation for the Anosov diffeomorphism $\rho(C_j)$. Thus we can apply the stable manifold theorem to conclude that the foliation $\widetilde{\mathcal{G}}_j$ is Hölder continuous with continuously varying C^∞ leaves.

Now apply Journe theorem inductively. Suppose that we have shown that h^{-1} is uniformly C^∞ along the leaves of $\widetilde{\mathcal{G}}_j$. Then the restrictions of $\widetilde{\mathcal{G}}_j, \widetilde{W}_{j+1}^s$, and h^{-1} to the leaves of $\widetilde{\mathcal{G}}_{j+1}$ satisfy the hypotheses of (?), and we can conclude that h^{-1} is C^∞ along the leaves of $\widetilde{\mathcal{F}}_{j+1}$. Also Journe's argument yields uniform bounds on the derivatives of h^{-1} along the leaves of $\widetilde{\mathcal{G}}_{j+1}$ which depends only on the bounds on the derivatives of h^{-1} along the leaves of $\widetilde{\mathcal{G}}_{j+1}$ and $\widetilde{W}^s_{j+1}$ and the uniform Hölder constants associated to the foliations. Thus h^{-1} is uniformly C^∞ along the leaves of $\widetilde{\mathcal{G}}_{j+1}$, the induction go through, and we conclude that h^{-1} is C^∞. A similar argument shows that h is C^∞. Thus h is a C^∞ diffeomotphism and the proof of Theorem (?) is complete. □

3 Invariant measures and conditional measures

Let $(X, \mathcal{B}, \mu)$ be a complete measure space, i.e. $B \in \mathcal{B}$, $A \subset B$ and $\mu(B) = 0$ implies $A \in \mathcal{B}$. $(X, \mathcal{B}, \mu)$ is called *separable* if the metric $d(A, B) = \mu(A\Delta B)$ defined on $\mathcal{B}/\{setsofmeasure0\}$ makes it a separable metric space. This is equivalent with $L^1(X, \mathcal{B}, \mu)$ separable. Define $\Omega := \{0, 1\}^{\mathbb{N}}$. A family $\{A_k\}_k$ of subsets in X is called a *basis* if for any sequence $\{\omega_k\}_k \in \Omega$, there is at most one point in $\cap_{k\geq 1} A_k^{\omega_k}$, where $A_k^0 = A_k$, $A_k^1 = X\diagdown A_k$.

We recall that an isomorphism mod 0 of two measurable spaces X and Y is a bijective map $F : X' \to Y'$, $X' \subset X$, $Y' \subset Y$, $X = X'(\text{mod}0)$, $Y = Y'(\text{mod}0)$, which is bi-measurable and F and F^{-1} are measure preserving.

$(X, \mathcal{B}, \mu)$ is called a *Lebesgue space* if μ is a probability measure and if X is isomorphic mod 0 with $(I, \mathcal{C}, \lambda)$, where I consists of an interval $[0, \alpha]$, $\alpha \leq 1$, and at most a countable number of atoms, $\mathcal{C}$ is generated by the Borel sets in $[0, \alpha]$ and the atoms, λ is the Lebesgue measure on $[0, \alpha]$ and $\lambda(I) = 1$.

Theorem 10 *(Rokhlin 1947-1948) A probability measure space $(X, \mathcal{B}, \mu)$ is Lebesgue if and only if:*

1. *$(X, \mathcal{B}, \mu)$ is complete;*

2. *$(X, \mathcal{B}, \mu)$ is separable;*

3. *$(X, \mathcal{B}, \mu)$ has a basis and the image HX of the map $H : X \to \Omega$ given by $x \to (\omega_1, \omega_2, \dots)$ with $\cap_{k\geq 1} A_k^{\omega_k} = \{x\}$, is measurable.*

Theorem 11 *(Rokhlin) If $(X, \mathcal{B}, \mu)$ satisfies (3) for some basis, it satisfies it for any basis.*

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue space and ξ be a partition of X with measurable equivalence classes. ξ is called a *measurable partition* if the measure space X/ξ is a Lebesgue space. The measure on X/ξ is the induced measure and is denoted by μ_ξ. Denote by $\mathcal{B}(\xi)$ the σ-algebra generated by ξ.

Assume now that on each element C of the partition we introduce a measure μ_C. We say that $\{\mu_C\}$ is a system of conditional measures with respect to ξ if:

1. μ_C is a Lebesgue measure for every mod) point C of the factor space X/ξ;

2. for every measurable set $A \subset X$ a) $A \cap C$ is measurable in C for every mod 0 point $C \in X/\xi$; b) $\mu_C(A \cap C)$ is a measurable function of the point $C \in X/\xi$; c) $\mu(A) = \int_{X/\xi} \mu_C(A \cap C) d\mu_{xi}$.

Remark 4 *A system of conditional measures with respect to ξ is unique mod 0.*

Theorem 12 *(Rokhlin) A partition ξ has a system of conditional measures if and only if it is measurable.*

Remark 5 *For measurable partition the correspondence between the decomposition ξ and the σ-algebra $\mathcal{B}(\xi)$ is bijective. More precisely, if $B(\xi_1) = B(\xi_2)$ for two measurable partitions ξ_1 and ξ_2, then, up to a set of measure zero, the partitions ξ_1 and ξ_2 are identical. Furthermore, for each subalgebra $\mathcal{B}$ there exist a measurable decomposition ξ such that $\mathcal{B} = \mathcal{B}(\xi)$.*

For an arbitrary partition ξ, not necessarily measurable, we can form the σ-algebra $\mathcal{B}(\xi)$ and find a measurable partition ξ' such that $\mathcal{B}(\xi) = \mathcal{B}(\xi')$. The partition ξ' is called the measurable hull *of ξ. For example, if we have a measure preserving group action and consider the partition of the measurable space given by orbits, then the measurable hull coincides with the ergodic decomposition.*

Consider now X and F complete metric spaces, W a continuous foliation of X with leaves modeled on F. The partition into the leaves of W is not a measurable partition in general. We denote by $m(W)$ the measurable hull of W. It is the finest measurable partition whose elements consist a.e. from the entire leaves of W.

We call a measurable partition ξ *subordinate* to W if for μ-a.e. X we have $\xi(x) \subset W(x)$ and $\xi(x)$ contains a neighborhood of x open in the submanifold topology of $W(x)$. Two different partition subordinate to the same foliation determine conditional measures that are scalar multiples when restricted to the intersection of an element of one partition with an element of the other partition. Hence there is a locally finite measure μ_x^W on $W(x)$ uniquely defined up to rescaling that restricts to a scalar multiple of a conditional measure for each partition subordinate to W. The measures μ_x^W form the system of conditional measures on the leaves of W.

4 Invariant geometric structures on continuous foliations

4.1 Invariant measures

Let M be a smooth Riemannian compact manifold, let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of M. Assume the following:

1. there exists a continuous f-invariant f^{-1}-contracting C^1-foliation W of M, i.e. the leaves are C^1-submanifolds of M, the distribution $TW(x)$ depends continuously on the point $x \in M$ and $\|Df^{-1}|_{TW}\| < \lambda < 1$;

2. the distribution $TW(x)$ is Hölder.

A measure μ is called W-absolutely continuous if the conditionals on the leaves of W are absolutely continuous w.r.t. the Lebesgue measure.

Theorem 13 *(Pesin-Sinai) There exists a Borel probability f-invariant measure μ whose conditional measures on the leaves of W are absolutely continuous w.r.t. the Lebesgue measure. Moreover, the density on $W(x)$ is given by*

$$y \to C(x,y) = \prod_{n=0}^{\infty} \frac{J_W(f^{-n}(x))}{J_W(f^{-n}(y))}, \qquad (*)$$

which is a continuous function in x and y and as smooth as f and the leaves.

Theorem 14 *(Theorem 2) The density of any Borel probability f-invariant measure that has absolutely continuous conditional measures on the leaves of W is given by (*).*

Theorem 15 *(Theorem 3) Let f and g be Anosov maps. Denote by J_f^u (respectively J_g^u) the Jacobians of f and g along their unstable foliations. Suppose h is a homeomorphism such that $g = h^{-1}fh$. Suppose also that for any g-periodic point p $J_g^u(p) = J_f^u(h(p))$.*

Then for any g-invariant W_g^u-absolutely continuous measure, $(h_*)\mu$ is a W_f^u-absolutely continuous measure.

Remark 6 *Assume that $dimW = 1$. Then the invariant measure which is absolutely continuous on W gives a smooth parametrization of W. One can show that this is an affine parametrization.(Compare with the previous section.)*

Theorem 16 *(Theorem 4)Let f and g be two Anosov C^∞-diffeomorphisms of $\mathbb{T}^2$ and let h be a homeomorphism such that $g = hfh^{-1}$. Suppose that for any g-periodic point p ($g^n p = p$), $(Dg^n)_p$ has the same eigenvalues as $(Df^n)_{h(p)}$. Then h is C^∞.*

Theorem 17 *(Theorem 5)Let f and W be like above. Let g be a diffeomorphism which preserves W and such that $fg = gf$. Then there exists a W-absolutely continuous measure, invariant for both f and g.*

We start now the proof of Theorem 1.

Lemma 11 *For any $x \in M$, $y \in W(x)$ define*

$$C_n(x,y) = \prod_{k=0}^{n-1} \frac{J_W(f^{-k}(x))}{J_W(f^{-k}(y))}.$$

Then the following limit exists

$$C(x,y) = C_n(x,y) = \prod_{k=0}^{\infty} \frac{J_W(f^{-k}(x))}{J_W(f^{-k}(y))}.$$

Moreover, the function $C(x,y)$ is continuous and for any $x \in M$, $y,z \in W(x)$

$$C(x,y)C(y,z) = C(x,z).$$

Proof The Jacobian is a uniformly bounded Hölder function on M, so:

$$|\frac{J_W(f^{-k}(x))}{J_W(f^{-k}(y))} - 1| \leq Cd(x,y)^{\alpha}\lambda^{\alpha k}.$$

Then lemma follows using a well known criterion for the convergence of an infinite product. $\square$

Let $d\nu$ be the Riemannian volume on M. We define a sequence of functionals on $C(M)$ by

$$\mu_n(h) = 1/n \sum_{k=0}^{n-1} \int_M h(f^k(x))d\nu(x), \text{ for } h \in C(M).$$

The sequence $\{\mu_n\}_n$ is precompact in weak*-topology, so there exists a weak*-limit μ which is normalized and f-invariant.

We define a rectangle to be an open set in M bounded by

Consider now Π a rectangle and define $\Pi_n := f^{-n}(\Pi)$. Denote by η the partition of Π into local leaves of W, by η_n the partition of Π_n into leaves and by X_n (respectively X) the measurable space Π_n/η_n (respectively Π/η). Let $\nu_{n,y}$ be the measure induced by the Riemannian metric on the leaf $C_{\eta(y)}$ and $d\bar{\nu}_n$ be the induced measure on X_n. Take now a continuous function h with the support in Π. Then

$$\int_M h(f^n(x))d\nu = \int_{\Pi_n} h(f^n(x))d\nu = \int_{X_n} d\bar{\nu}_n(y) \int_{C_{\eta_n(y)}} h(f^n(z))d\nu_{n,y}(z). \qquad (*)$$

We have $d\bar{\nu}_n(y) = J_X^{(n)}(y)d\bar{\nu}_X(y)$ where $J_X^{(n)}(y)$ is the Jacobian of the map $f^n|_X$ at y, $\bar{\nu}_X(y)$ is the measure on X, induced by the Riemannian metric, and

$$d\nu_{n,y}(z) = \prod_{k=0}^{n-1} J_W(f^{-k}(z))d\nu_y(z),$$

where $\nu_y(z)$ is the measure on $C_{\eta(y)}$ induced by the Riemannian metric. Then the last integral in (*) becomes

$$\int_X J_X^{(n)}(y)d\bar{\nu}_X(y) \int_{C_{\eta(y)}} h(z) \prod_{k=0}^{n-1} J_W(f^{-k}(z))d\nu_y(z) =$$

$$\int_X J_X^{(n)}(y) \prod_{k=0}^{n-1} J^u(f^{-k}(y))d\bar{\nu}_X(y) \int_{C_{\eta(y)}} h(z) \prod_{k=0}^{n-1} \frac{J_W(f^{-k}(z))}{J_W(f^{-k}(y))} d\nu_y(z).$$

To finish the proof of Theorem 1 we use now the following lemma.

Lemma 12 *Let ν_n be a sequence of measures in Π with the following properties:*

1. *if $(\delta_n, \nu_n(y,z))$ is the system of conditional measures for ν_n with respect to the partition η so that $\delta_n(y)$ is the measure on X and $\nu_n(y,z)$ is the measure on $C_{\eta(y)}$, then*

$$d\nu_n(y,z) = P_n(y,z)d\nu_y(z)$$

where $P_n(y,z)$ is a continuous function on Π;

2. *the sequence of functions $P_n(y,z)$ converges uniformly in Π to a continuous function $P(y,z)$;*

3. *the sequences of measures $\mu_{n_i} = 1/n_i \sum_{k=0}^{n_i-1} \nu_k$ converges weakly in Π to a measure μ.*

Then the system of conditional measures for the measure μ in Π with respect to the partition η has the form $(\delta(y), \nu(y,z))$, where δ is the measure on X and $\nu(y,z)$ is the measure on C_η for which

$$d\nu(y,z) = P(y,z)d\nu_y(z).$$

Proof For any continuous function h supported on Π we have that

$$\int_\Pi h(w)d\mu_{n_i}(w) = 1/n_i \sum_{k=0}^{n_i-1} \int_X d\delta_k(y) \int_{C_{\eta(y)}} h(z)P_k(y,z)d\nu_y(z) =$$

$$\int_X d\psi_{n_i} \int_{C_{\eta(y)}} h(z)P(y,z)d\nu_y(z) + \varepsilon_{n_i}, \qquad (*)$$

where $\psi_{n_i} = 1/n_i \sum_{k=0}^{n_i-1} \delta_k$ and

$$\varepsilon_{n_i} = 1/n_i \sum_{k=0}^{n_i-1} \int_X d\delta_k(y) \int_{C_{\eta(y)}} h(z)(P_k(y,z) - P(y,z))d\nu_y(z).$$

Denote $c = \max_{z\in\Pi} \|h(z)\|$ and $c_k = \max_{(y,z)\in\Pi} \|P_k(y,z) - P(y,z)\|$, and using condition (2) we have

$$\|\varepsilon_{n_i}\| \le \frac{c}{n_i} \sum_{k=0}^{n_i-1} c_k \to 0 \text{ as } n_i \to \infty.$$

Consider now on Π the sequence of measures $\tilde{\mu}_{n_i}$, where

$$\tilde{\mu}_{n_i}(A) = \int_X d\psi_{n_i} \int_{C_{\eta(y)}} \chi_A(y,z)P(y,z)d\nu_y(z),$$

for $A \subset \Pi$ Borel set.

It follows from condition 3 and (*) that $\{\tilde{\mu}_{n_i}\}_i$ converges weakly to the measure μ. So the family $\{\psi_{n_i}\}_i$ is weakly compact. So $\{\psi_{n_i}\}_i$ has a subsequence $\{\psi_{n_i'}\}_i$ weakly convergent to a measure δ in X. For any continuous function h supported on Π we have

$$\int_\Pi h(w)d\tilde{\mu}_{n_i'}(w) = \int_X d\psi_{n_i'}(y) \int_{C_{\eta(y)}} h(z)P(y,z)d\nu_y(z) \to$$

$$\to_{n'_i\to\infty} \int_X d\delta(y) \int_{C_{\eta(y)}} h(z)P(y,z)d\nu_y(z)$$

and

$$\int_\Pi h(w)d\tilde{\mu}_{n'_i}(w) \to \int_\Pi h(w)d\mu(w).$$

So the result follows. $\square$

We will call an f-invariant measure with absolutely continuous conditionals along the leaves a SBR-measure.

We start now the proof of Theorem 2. Let μ be any SBR-measure.

Denote by Λ^+ the set of all points $x \in M$ such that there exists the limit:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)).$$

By Birkhoff theorem, $\mu(\Lambda^+) = 1$.

Let η be the partition of Λ^+ defined by the following condition: points x, y belong to the same element of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(y)).$$

Denote by $C_\eta(x)$ the element of η which contains x.

Lemma 13 *$W(x) \subset C_\eta(x)$.*

ProofIf $y \in W(x)$, then $d_m(f^{-n}x, f^{-n}y) \le C\lambda^{-n}$. So for any continuous function φ

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(y))$$

and $y \in C_\eta(x)$.

Lemma 14 *Let ξ be the partition of M into ergodic components w.r.t. μ. Then $\xi = \eta$ (mod 0).*

Proof This follows easily from Birkhoff theorem.

Lemma 15 *Let μ_1, μ_2 be two f-invariant SBR-measures. Let F be an element of η such that $\mu_1(F) > 0$ and $\mu_2(F) > 0$. Then, up to a scalar normalization, $\mu_1|_F = \mu_2|_F$.*

Proof We renormalize $\mu_1|_F$ and $\mu_2|_F$ to be probability measures. Then, from Lemma (?) they are both ergodic. If $\mu_1|_F$ has a singular component w.r.t. $\mu_2|_F$ there exists $A \subset F$ such that $\mu_1(A) = 0$, $\mu_2(A) > 0$. Define $\widetilde{A} = \cup_{n=-\infty}^{\infty} f^n(A)$. Then $\widetilde{A}$ is f-invariant, $\mu_1(A) = 0$ and $\mu_2(A) > 0$. Consider now the measure $\mu = (\mu_1 + \mu_2)/2$, which is also an f-invariant ergodic SBR-measure (see Lemma). So $0 < \mu(\widetilde{A}) < 1$ is impossible. We conclude that $\mu_1|_F$ is absolutely continuous w.r.t. $\mu_2|_F$. Since $\mu_2|_F$ is ergodic it follows that $\mu_1|_F = \mu_2|_F$.

Using now Lemma and Lemma it follows that nontrivial conditionals along the leaves of W are uniquely determined, finishing the proof of Theorem 2.

4.2 Invariant affine structures for contracting foliations with pinching

Local statement 1 *Let U be a neighborhood of 0 in $\mathbb{R}^n$, $f : U \to \mathbb{R}^n$ be a diffeomorphism onto its image, $f(0) = 0$. Let $\{\lambda_1, \dots, \lambda_n\}$ be the spectrum of $D_0 f$, and $\chi_i = log|\lambda_i|$ the corresponding Lyapunov exponent. Assume that $|\lambda_i|$ and $\chi_i < 2\chi_j$ for all i, j. Then there exists a C^∞ local coordinate change $h : U_0(\subset U) \to \mathbb{R}^n$ which linearizes f at the origin, i.e.:*

$$f = h^{-1} D f_0 h, \ h(0) = 0.$$

Remark 7 *There are finite smoothness and analytic counterparts.*

Remark 8 *The pinching condition is sharp. Consider the following example:*

$$f(x, y) = (\frac{x}{2}, \frac{y}{4} + ax^2)$$

which has $\chi_2 = 2\chi_1$.

Assume by contradiction that there is a local coordinate change h which linearize f at the origin. It is easy to see that Df_0 has an infinite family of invariant C^1-curves which are tangent in 0 to the x-axis, namely $y = cx^2$. We will show that the image of any of these curves under h can not be invariant under f. Indeed, let $(x_0, y_0) = h(x', y')$ be a point on such a curve, distinct from the origin. Denote $(x_n, y_n) := f^n(x_0, y_0)$. Then $x_n \to 0$ and $y_n = O(x_n)$ (because $(x_n, y_n) = h(Df_0(x', y'))$. On the other hand, we have the following recurrence relation:

$$x_n = \frac{x_0}{2^n}, \ y_n = \frac{y_{n-1}}{4} + \frac{a x_0}{2^{n-1}},$$

so

$$y_n = n x_n^2 + \frac{y_0}{4^n}.$$

Hence y_n is not $O(x_n)$ and we have a contradiction. The problem appears from the existence of the resonance $\chi_2 = 2\chi_1$.

We show now a global version of the previous theorem.

Let M be a smooth Riemannian compact manifold, $f : M \to M$ a C^∞ diffeomorphism of M. Let W be a continuous foliation of M with C^∞ leaves, such that $f(W) = W$. Assume also that W is a contracting foliation for f, i.e.

$$\|Df|_{TW}\| < \lambda < 1.$$

Definition 6 *The function f satisfies the 1/2-pinching condition along W if for any u, $v \in TW$:*

$$\|Df(v)\|^2 < \|Df(u)\|.$$

It is obvious that a 1-dimensional foliation satisfies the 1/2-pinching condition.

Take now $\delta > 0$ small enough and consider the canonical germ extension $F : T_\delta W \to T_\delta W$ of f. In local coordinates the formula for F is

$$F(x, t) = (f(x), (\varphi_{f(x)^{-1}} \circ f \circ \varphi_x)(t)),$$

where $\varphi_x : (T_\delta W)_x \to W(x)$ is the exponential map.(Maybe picture.)

Theorem 18 *Let f, F and W be as before and assume that f satisfies the 1/2-pinching conditions along W. Then there exists $H : T_\delta W \to T_\delta W$ a germ extension of the identity map of M such that:*

1. *H is identity along the zero section of $T_\delta W$;*
2. *H is C^∞ along the fibers of $T_\delta W$;*
3. *$D_t H|_{(x,o)} = Id$;*
4. *H is continuous on $T_\delta W$;*
5. *H is C^∞ along the leaves of W;*
6. *$H \circ F \circ H^{-1}$ is linear in the fibers.*

Moreover, the affine connection induced on W by the linearization in the fibers is integrable. So we have a flat affine connection on TW.

Proof The plan of the proof is similar with the proof of Theorem(?). First we find a formal power series linearization, then we show the C^∞-linearization up to a C^∞-flat error, and finally eliminate the C^∞-flat error.

Step I. We want to conjugate F formally to elliminate the nonlinear terms. We procced by induction.

Assume that for $2 \le k < n$ there are $H_{(k)}$ such that

$$F_{(n)} := H_{(2)}^{-1} \circ H_{(3)}^{-1} \circ \cdots \circ H_{(n-1)}^{-1} \circ F \circ H_{(n-1)} \circ \cdots H_{(2)}$$

is linearized up to order n along the zero section, so the Taylor series expansion of $F_{(n)}$ is:

$$F_{(n)}(x,t) = (f(x), L_x t + F_n(x,t) + F_{n+1}(x,t) + \cdots)$$

where $F_k(x,t)$, $k \ge n$, are homogenous of order k in t, and L_x is the linear part along the zero section.

We want to find $H_{(n)}(x,t)$ with Taylor series expansion:

$$H_{(n)}(x,t) = (x, t + H_n(x,t))$$

such that $F_{(n)} \circ H_{(n)} - H_{(n)} \circ D_t F_{(n)}$ has the n-homogenous term zero.

So the equation for the n-homogenous term is:

$$L_x H_n(x,t) + F_n(x,t) = H_n(f(x), L_x t).$$

Denote

$$L_x^m := L_{f^{m-1}(x)} \circ \cdots \circ L_{f(x)} \circ L_x.$$

Using the compactness of M and making $v = L_x^{-1}u / \|L_x^{-1}\|$ in (*), the pinching condition can be restated as

1. " " there is $0 < \gamma < 1$ such that $\|L_x\|^2 < \|L_x^{-1}\|^{-1}\gamma$.

Hence

$$\|L_x^m\|^2\|(L_x^m)^{-1}\| \leq \prod_{k=0}^{m-1} \|L_{f^k(x)}\|^2\|L_{f^k(x)}^{-1}\| \leq \gamma^m \qquad (**)$$

for any positiv integer m.

From (*) follows:

$$H_n(x,t) = -L_x^{-1}F_n(x,t) + L_x^{-1}(H_n(f(x), L_x t)),$$

so

$$H_n(x,t) = -\sum_{m=0}^{\infty}(L_x^m)^{-1}F_n(f^m(x), L_x^m t).$$

The last sum is uniformly convergent in norm because F_n is homogenous of degree n in the second variable.

The formal power series linearization H is given by:

$$H = \cdots \circ H_{(n)} \circ \cdots \circ H_{(3)} \circ H_{(2)}.$$

Note that for each n, the n-homogenous term in the second variable of H is determined as a finite sum.

Remark As the reader can see from the last proof, in order to elliminate the n-homogenous part of F we need the following condition, which is weaker then 1/2-pinching for W contracting foliation:

$$\|L_x\|^n < \|L_x^{-1}\|, \quad \text{forall} x \in M. \qquad (*)$$

Since the foliation W is contracting, this condition is true for some n big enough. So, even if we can not linearize F along the zero section, at least we can reduce F to a normal form.

Theorem 19 *Assume that (*) is true. then there exists H formal power series such that $F \circ H - H \circ \mathcal{F}_{nCF-1}$ is zero up to arbitrary order along the zero section, where:*

$$\mathcal{F}_{n-1}(x,t) = L_x t + F_2(x,t) + \cdots F_{n-1}(x,t).$$

H can be choosen such that $D_t H|_{(x,0)} = 1$, $D_t^l H|_{(x,0)} = 0$ for $2 \leq l \leq n-1$ and any $x \in M$.

Step II. The linearization H exists now as a formal power series:

$$H(x,t) = (x, t + \sum_{k=2}^{\infty} K_k(x,t)).$$

But the formal series might not converge. We use now a cut-off function to construct a C^∞-linearization up to a C^∞ flat part.

Proposition 2 *Let $b : [0,1] \to [0,1]$ be a C^∞ function, such that $b(t) = 1$ for $t \in [0, 1/2]$ and $b(0) = 0$. Consider the function*

$$\widetilde{K}(x,t) = (x, t + \sum_{k=2}^{\infty} K_k(x,t) b(k! \|t\|^2)).$$

Then $\widetilde{K}(x,t)$ satisfies the conditions in Theorem(?) except (6) which is true up to a C^∞-flat term.

Proof Note that $b(k!\|t\|^2) \neq 0$ implies that $\|t\|^2 < 1/k!$. Hence for each $t \neq 0$ only finitely many terms are non-zero and the series which defines $\widetilde{K}(x,t)$ is convergent. Same is true for the derivatives, so $\widetilde{K}(x,t)$ is a C^∞ function. It is also easy to check that $\widetilde{K}(x,t)$ has the right derivatives at $t = 0$.

Step III. The ellimination of the C^∞-flat term follows from the following theorem.

Theorem 20 *Let F, $G : T_\delta M \to T_\delta M$ be two C^∞ germ extensions of f and let $\alpha : T_\delta M \to T_\delta M$ be a C^∞ germ extension such that $D_t^k \alpha|_{(x,0)} = 0$, $k \geq 1$ and such that $G + F + \alpha$. Assume that F is a linear contraction fiberwise. Then there is $H : T_\delta M \to T_\delta M$ C^∞ germ extension of Id_M such that $D_t^k H|_{(x,0)} = 0$, $k \geq 1$, and $G = H \circ F \circ H^{-1}$.*

Proof We use here the homotopy method. For any $\tau \in [0,1]$, consider F_τ the germ extension of f:

$$F_\tau = F + \tau\alpha,$$

and try to find H_τ the germ extension of Id_M such that:

$$F = H_\tau^{-1} \circ F_\tau \circ H_\tau. \qquad (*)$$

To find the family $\{H_\tau\}_\tau$ is equivalent to find the nonstationary vector field:

$$t \to v_\tau = \frac{d}{d\sigma}(H_\sigma \circ H_\tau^{-1})|_{\sigma=\tau}.$$

Using (*) we have

$$(H_\sigma \circ H_\tau^{-1}) \circ F_\tau = F_\sigma \circ (H_\sigma \circ H_\tau^{-1})$$

and differentiating w.r.t. σ at $\sigma = \tau$ we find:

$$v_\tau \circ F_\tau - D_t F_\tau(v_\tau) = \frac{dF_\tau}{d\sigma}|_{\sigma=\tau},$$

or equivalently

$$v_\tau = (D_t F_\tau)^{-1}(v_\tau \circ F_\tau) - (D_t F_\tau)^{-1}\alpha. \qquad (**)$$

Formally, (**) is solved by:

$$v_\tau = \sum_{n=0}^{\infty} (D_t F_\tau)^{-n-1}(\alpha \circ F_\tau^n). \qquad (***)$$

Because F is contraction fiberwise, there is $0 < \lambda < 1$ such that

$$\|D_\tau F_\tau\| < \lambda.$$

Also, using Taylor formula and the fact that $D_T^n \alpha|_{(x,0)} = 0$, $n \geq 0$, we find positive constants C_n such that:

$$\|\alpha(x,t)\| \leq C_n \|t\|^n, \text{ forall } (x,t) \in T_\delta M.$$

Now, the last two estimates and the fact that α is C^∞ fiberwise can be used to show that the sum *** gives a C^∞-function in all the variables x, t, τ. So we can integrate $\{v_\tau\}$ to obtain $\{H_\tau\}$ which is C^∞-fiberwise, flat and $G = H \circ F \circ H^{-1}$.

5 Cocyles

Let G be a group and let $\alpha : G \times M \to M$ a G action on a space X. Let Γ be a group with unit 1_Γ. To simplify the notation we write gx instead of $\alpha(g)x$. A cocycle over α is a function $\beta(g,x) : G \times M \to M$ such that:

$$\beta(g_1 g_2, x) = \beta(g_1, g_2 x)\beta(g_2, x),$$

for all g_1, $g_2 \in G$, $x \in M$.

Two cocycles β_1 and β_2 are called cohomologous if there exists a function $P : M \to \Gamma$, called coboundary, such that:

$$\beta_1(g,x) = P(gx)\beta_2(g,x)P(x)^{-1}.$$

A cocycle β is called cohomologous to a constant cocycle if there is a representation $\pi : G \to \Gamma$ which is cohomologous with β, i.e.

$$\beta(g,x) = P(gx)\pi(x)P(x)^{-1}.$$

We say that a cocycle β satisfies the closing conditions if $gx = x$ for some $x \in X$ and $g \in G$ implies that $\beta(g,x) = 1$.

If π is the trivial morphism, then we say that P trivializes β, or β is null-cohomological.

In the case $G = \mathbb{Z}$ the cocycle is determined by the function $\bar{\beta}(x) := \beta(1,x)$.

We discuss first some facts about continuous cocycles.

Proposition 3 *Let $f : X \to X$ be a homeomorphism of a complete metric space and $\varphi : \mathbb{Z} \times X \to \mathbb{R}$ a continuous real valued cocycles over the $\mathbb{Z}$-action induced by F. Suppose that there is $C > 0$ such that for all n and x:*

$$|\varphi(x,n)| < C.$$

Then $\phi(x) := -\sup_{n\in\mathbb{Z}} \varphi(x,n)$ trivializes φ. The set of discontinuities of ϕ is a countable union of nowhere dense f-invariant closed sets. In particular, is f is minimal (i.e. any orbit of f is dense), then ϕ is continuous.

More interesting for us are Hölder cocycles over hyperbolic actions on compact manifolds. The first important result belongs to Livsic:

Theorem 21 *(Livsic) Let f be a topologically transitive Anosov C^1 diffeomorphism of a compact manifold M. Let $\bar{\beta} : M \to \mathbb{R}$ be an α-Hölder function. Then the $\mathbb{Z}$- cocycle $\beta(g,x)$ over α determined by $\bar{\beta}$ is null-cohomological if and only if β satisfies the closing conditions. The trivialization P is α-Hölder.*

The definitive results about the regularity of P belong to de la Llave, Marco and Moryion.

Theorem 22 *(de la Llave-Marco-Moryion) If β is C^∞ (analytic), then P is C^∞ (analytic).*

The proof of this theorem is done in two steps. One shows first the regularity along the stable and unstable foliations. Second, one shows the that the regularity of P along stable and unstable foliations implies the regularity of P.

The first step is standard and not very difficult. For the second step, which can be viewed as a regularity result independent of the theory of hyperbolic systems, there are, by now, several proofs. The original proof in [4] is done using elliptic theory. Necessary assumptions are the regularity of the Jacobian along the stable and unstable foliations and the continuity of the foliations. Same assumptions are used by Hurder and Katok in [1], but the proof, which uses now only Fourier analysis, is simplified. Yet other proof is presented by Journe in [2]. His proof uses the Hölder continuity of the foliations, but makes no assumption on the Jacobian.

Theorem 23 *(Journe) Let M be a C^∞ manifold and $\mathcal{F}$ and $\mathcal{F}'$ be two Hölder foliations, transverse and with uniformly C^∞ leaves. If a function f is uniformly C^∞ along the leaves of the two foliations, then it is C^∞ on M.* □

Livsic proved also an analog of Theorem 22 for small $\mathbb{Z}$-cocycles with values in finite dimensional Lie groups. Small means that the function $\beta(1,\cdot)$ has the range in a small enough neighborhood of the identity. Regularity results for cocycles with values in finite dimensional Lie groups which are cohomologous to identity are proved in [6].

Proving an analog of Livsic theorem for general $\mathbb{Z}$-cocycles with values in Lie groups is an outstanding open problem. We note also that, as was remarked by A. Katok, de la Llave conterexample mentioned in Chapter 1 provide a couterexample for the folowing regularity problem:

Problem 2 *Let G be a finite dimensional Lie group and $\beta : \mathbb{Z} \times M$ be a C^∞ cocycle over an Anosov C^∞. Assume that β is cohomologous to a constant cocycle and let $P : M \to G$ be the coboundary. Assume also that P is continuous. Does it follow that P is C^∞?*

Indeed, let $Aff(\mathbb{R}^d)$ be the Lie group of affine autuomorphisms of $\mathbb{R}^d$. Then f and $\widetilde{f}$ define $Aff(\mathbb{R}^d)$-valued cocycles over the hyperbolic action $x \in \mathbb{T}^2 \to Ax \in \mathbb{T}^2$. f corresponds to $\beta(1,x)(y) = By$ and $\widetilde{f}$ to

$$\widetilde{\beta}(1,x)(y) = By + \varphi(x)e_\mu.$$

Since h^{-1} is a bundle map, it corresponds to a cohomology between β and $\widetilde{\beta}$ given by

$$x \in \mathbb{T}^2 \to P(x)(\cdot) = \cdot - \psi(x)e_\mu \in Aff(\mathbb{R}^d).$$

If $\alpha_c < 1$, ψ is only Hölder and so P is only Hölder.

6 Totally nonsymplectic abelian actions

In this section we will continue the study of Anosov $\mathbb{Z}^k$-actions. We already proved in section (?) the C^∞-structural stability of a $\mathbb{Z}^{n-1}$-action generated by $n-1$ commuting hyperbolic matrices acting on the torus $\mathbb{T}^n$. In order to obtain further rigidity results, we need the action to be rich enough. This is described in a formal way below.

As we already remarked in Chapter 1, the only manifolds which are known to admit Anosov diffeomorphisms are tori, nilmanifolds and infranilmanifolds. In these chapter we consider only Anosov $\mathbb{Z}^k$-actions on these manifolds. We call the action α linear if it can be lifted to a $\mathbb{Z}^k$-action on the universal cover of M (which is an euclidean space) induced by linear hyperbolic automorphisms. For a linear hyperbolic automorphism a, we denote by E_a^- and E_a^+ the stable and unstable subspaces of a.

Because the linear automorphisms which implement the $\mathbb{Z}^k$-action commute one with each other, they have the same invariant subspaces. Let m be the dimension of M. So we can write $T_xM = \mathbb{R}^m = \oplus_{i=1}^{\ell} E_i$, where each E_i's is included into a stable or unstable subspaces for each hyperbolic element in $\mathbb{Z}^k$. Moreover, we assume that E_i's are maximal with this property, meaning that if $E_i \subset F$, F linear subspace and if for each hyperbolic element E_i and F are both included either in the stable, or unstable subspace, then $E_i = F$. We call E_i's minimal subspaces. Our richness condition, called <u>totally nonsymplectic</u> (TNS) is the following:

1. "(TNS)" for any two minimal subspaces there is a hyperbolic element a in $\mathbb{Z}^k$ such that both minimal subspaces are included in the stable subspace of a.

Lyapunov subspaces determine integrable distributions. We cal the corresponding foliations on M minimal foliations and denote them by $\mathcal{F}_i$. We can restate (TNS) condition using minimal foliations:

1. "(TNS)" for any two minimal foliations there is a hyperbolic element a in $\mathbb{Z}^k$ such that both minimal foliations are included in the stable foliation of a.

The minimal foliations and the notion of (TNS) action can actually be introduced for general hyperbolic $\mathbb{Z}^k$-actions.

6.1 Cocyles Over $\mathbb{Z}^k$-Actions

Assume now that $\beta : \mathbb{Z}^k \times M \to \mathbb{R}$ is a real valued cocycle over α

$$\beta(n_1 + n_2, x) = \beta(n_1, n_2x) + \beta(n_2, x) \quad , \quad \text{for all } n_1, n_2 \in \mathbb{Z}^k, \;\; x \in M.$$

We want to show that under certain regularity condition on β, β is cohomologous with a constant cocycle, i.e. there is $P : M \to \mathbb{R}$ and a morphism $\pi : \mathbb{Z}^k \to \mathbb{R}$ such that

$$\beta(n, x) = P(nx) + \pi(n) - P(x).$$

Theorem 24 *Let M be a torus, a nilmanifold or an infranilmanifold. Let $\alpha : \mathbb{Z}^k \times M \to M$ be a linear (TNS) $\mathbb{Z}^k$-action. Let $\beta : \mathbb{Z}^k \times M \to \mathbb{R}$ be a C^∞ cocycle over α. Then β is cohomologous with a constant cocycle and the coboundary $P : M \to \mathbb{R}$ is C^∞.*

Proof The idea is to construct first a C^∞ form on M, which is invariant under the group action. Then we show that the form is closed and we are able to construct a foliation of $M \times \mathbb{R}$ with leaves of dimension m. Using a holonomy argument and the hyperbolicity of the action follows that the leaves of the foliation are closed. We can then deduce the morphism π and the coboundary P using the invariance of the foliation.

Assume now that $x \in M$, $a \in \mathbb{Z}^k$ hyperbolic and $y \in W_a^s(x)$ the stable leaf of a though x. Then the following sum is convergent

$$P_a^-(y; x) = -\sum_{n=0}^{\infty} [\beta(a, (na)y) - \beta(a, (na)x)],$$

and we can define a form ω_a^- on $E_a^s(x)$ taking the derivative of P_a^- along the stable leaf. Note that the sum is convergent even if β is only Hölder. This fact will be used later when we show an alternative proof of Theorem 1 which works even for Hölder cocycles.

Similarly, we can define, for $x, z \in W_a^u(x) = W_{-a}^s(x)$

$$P_{-a}^-(z; x) = -\sum_{n=0}^{\infty} [\beta(-a, (-na)z) - \beta(-a, (-na)x)]$$

and obtain a form ω_a^+ on $E_a^u(x)$. So we have a form on T_xM given by $\omega_a = \omega_a^+ \oplus \omega_a^-$.

We show now that the form is the same for a big set of hyperbolic elements in $\mathbb{Z}^k$ used to construct it, that is smooth and closed.

Lemma 16 *There is a subset $S \subset \mathbb{Z}^k$, which contains elements from each Weil chamber and which generates $\mathbb{Z}^k$, such that if $a, b \in S$ are both hyperbolic, then*

$$\omega_a|_{E_a^s(x) \cap E_b^s(x)} = \omega_b|_{E_a^s(x) \cap E_b^s(x)}.$$

Proof We assume first that λ_b, the contraction coefficient along W_b^s, is made smaller than the norm of $\alpha(a - b)$, and prove lemma in this case.

Take now $z \in W_a^s(x) \cap W_b^s(x)$. Using the cocycle relation we find that:

$$\sum_{k=0}^{n-1} \beta(a, (ka)z) = \beta(na, z)$$

and

$$\beta(na,z)-\beta(nb,z)=\beta(n(a-b),(nb)z),$$

and similar for x instead of z.

So, in order to show $P_a^-(z;x)=P_b^-(z;x)$, and consequently $\omega_a|_{E_a^s(x)\cap E_b^s(x)}=\omega_b|_{E_a^s(x)\cap E_b^s(x)}$, is enough to show that

$$\lim_{n\to\infty}\left(\beta(n(a-b),(nb)z)-\beta(n(a-b),(nb)x)\right)=0.$$

But

$$\begin{aligned}
&|\beta(n(a-b),(nb)z)-\beta(n(a-b),(nb)x)|= && (1)\\
&=\left|\textstyle\sum_{k=0}^{n-1}\left(\beta(a-b,[nb+k(a-b)]z)-\beta(a-b,[nb+k(a-b)]x)\right)\right|\le && (2)\\
&\le\|\beta(a-b,\cdot)\|_{\text{Hoelder}}\left[\textstyle\sum_{k=0}^{n-1}d_M(\alpha(nb+k(a-b))(z),\alpha(nb+k(a-b))(x))^\alpha\right] && (3)\\
&\le\|\beta(a-b,\cdot)\|_{\text{Hoelder}}\cdot\lambda_b^{n\alpha}\cdot C\cdot(d_M(z,x))^\alpha\textstyle\sum_{k=0}^{n-1}\ \|\alpha(a-b)\|^{k\alpha}. \qquad (") && (4)\\
&*" && (5)
\end{aligned}$$

Here C is a constant independent of n and d_M is the Lyapunov metric for b on M.

Since $\|\alpha(a-b)\|\ge 1$, it follows that $\sum_{k=0}^{n-1}\|\alpha(a-b)\|^k\le C'\cdot\|\alpha(a-b)\|^n$ where C' is also a constant independent of n. So (*) is convergent to 0 when $n\to\infty$ because λ_b is much smaller than $\|\alpha(a-b)\|$.

We show now how to construct the set $S\subset\mathbb{Z}^k$. We consider first a finite set F of elements in $\mathbb{Z}^k$ close to the origin, which contains a motion in each direction. Obviously, there is $M>0$ such that $\|\alpha(c)\|\le M$, for all $c\in F$. Consider now $\lambda_j:\mathbb{R}^r\to\mathbb{R}$ be the j's Lyapunov exponent and denote by $\mathcal{H}_j$ the hyperplane in $\mathbb{R}^k$ determined by the kernal of λ_j. Then there exists a cone $C(\mathcal{H}_j)$ in $\mathcal{H}_j^-$, which does not exclude any Weil chamber in $\mathcal{H}_j$, and a ball B around the origin, of size independent of j, such that for any element $b\in C(\mathcal{H}_j)\cap(\mathbb{R}^k-B)$ we have

$$\lambda_j(b)<M^{-1}. \qquad (*)$$

Consider two elements a,b in $C(\mathcal{H}_j)\cap(\mathbb{R}^k-B)$. We can join a and b by a sequence of elements in $C(\mathcal{H}_j)\cap(\mathbb{R}^k-B)$ adding at each step an element from F. Formula (*) allows us to apply the first part of the proof recurently and deduce that

$$\omega_a|_{F_j}=\omega_b|_{F_j}. \qquad (**)$$

Because of the definition of ω, (**) is also true if a and/or b are in the opposite cone $-C(\mathcal{H}_j)$.

Define the set S to be

$$S=\left[\cap_{j=1}^m C(\mathcal{H}_j)\cup(-C(\mathcal{H}_j))\right]\cap(\mathbb{R}^k-B).\Box$$

Choose now an element $a\in S$, an open set U in M and a C^∞-coordinate system in U such that in each point $dx_1,\dots,dx_m$ is the dual basis of a basis in T_xM consisting of vectors along the Lyapunov directions. We can do this because the action α is linear. So, in local coordinates $\omega := \omega_a$ becomes

$$\omega(x) = \sum_{i=1}^{m} f_i(x) dx_i$$

where f_i are obviously continuous functions on U and C^∞ along the stable direction of the hyperbolic element used to define the form. Using the (TNS) and Lemma 1 condition it follows that each f_i is C^∞ along a full set of directions so is C^∞ (see [1] or [3]). So ω is a smooth form.

To show that ω is closed, we use again (TNS) condition and Lemma 1. Any two directions i, j are included into the stable subspace of some hyperbolic element a. But then $f_i(x) = \frac{\partial P_a^-(y;x)}{\partial y_i}|_{y=x}$ and $f_j(x) = \frac{\partial P_a^-(y;x)}{\partial y_j}|_{y=x}$ and because $P_a^-(\cdot;x)$ is smooth we have $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$, so ω is closed.

We construct a foliation of $M \times \mathbb{R}$ with m-dimensional leaves . Since ω is closed, we can find locally a C^∞ function $F : U \to \mathbb{R}$ such that $dF = \omega$. Then the graphs of the functions $\{F + t : U \to \mathbb{R}\}_{t \in \mathbb{R}}$ define a smooth foliation $\mathcal{F}$ of $U \times \mathbb{R}$ and using a cover of M by open disks we find a smooth foliation of $M \times \mathbb{R}$.

We show that the foliation $\mathcal{F}$ is invariant under the $\mathbb{Z}$- action

$$\widetilde{\alpha} : \mathbb{Z}^k \times (M \times \mathbb{R}) \mapsto M \times \mathbb{R}, \;\; \widetilde{\alpha}(a, (x, t)) = (ax, \beta(a, x) + t).$$

Consider first $a \in S$ hyperbolic. Let L be an open set in a leaf of $\mathcal{F}$ which sits over an open set U in M, $x \in U$, $(x, t) \in L$. Then the lift of $W_a^s(x)$ to L is the graph of $P_a^-(\cdot, x) + t$. The following computation shows that the image of the graph of $P_a^-(\cdot, x)$ under $\widetilde{\alpha}(a)$ is the graph of $P_a^-(\cdot, ax) + t + \beta(a, x)$ restricted to $W_a^s(ax)$, so belongs to the leaf of $\mathcal{F}$ passing through $\widetilde{\alpha}(a)(x, t)$:

$$\widetilde{\alpha}(a)(y, P_a^-(y, x) + t) = \tag{6}$$
$$= \widetilde{\alpha}(a)(y, -\textstyle\sum_{n=0}^{\infty}[\beta(a, (na)y) - \beta(a, (na)x)] + t) = \tag{7}$$
$$= (ay, \beta(a, y) - \textstyle\sum_{n=0}^{\infty}[\beta(a, (na)y) - \beta(a, (na)x)] + t) = \tag{8}$$
$$= (ay, P_a^-(ay, ax) + t + \beta(a, x)). \tag{9}$$

Similar is true for the lift of $W_a^u(y)$, for any $y \in U$.

Consider the leaf L foliated by the lifts of $W_a^u(y)$, $y \in W_{a(x)}^s$. The image of each of them belongs to the leaf of $\mathcal{F}$ passing through $\widetilde{\alpha}(a)(x, t)$, so $\widetilde{\alpha}(a)(L)$ is part of a leaf of $\mathcal{F}$, so $\mathcal{F}$ is invariant under $\widetilde{\alpha}(a)$.

The fact that $\mathcal{F}$ is invariant under the full $\mathbb{Z}^k$ action follows from the fact that S contains a set of generators of $\mathbb{Z}^k$.

We show now that $\mathcal{F}$ has closed leaves. Take a fixed point $x \in M$ for the action of a hyperbolic element a and consider the holonomy map $H : \pi_1(M) \to \mathbb{R}$ of the foliation $\mathcal{F}$. Because $\mathcal{F}$ is invariant under $\widetilde{\alpha}$, the holonomy map is invariant under the action of a on $\pi_1(M)$, i.e. $H(a\gamma) = H(\gamma)$, for any $\gamma \in \pi_1(M)$. The foliation $\mathcal{F}$ has all leaves closed manifolds if H is the trivial morphism. We prove that H is the trivial morphism in three steps: we consider first that M is a tori, than a nilmanifold and finally an infranilmanifold.

Assume first that $M = \mathbb{T}^m$. Then $\pi_1(M) \simeq \mathbb{Z}^m$ and H is a morphism from $\mathbb{Z}^m$ into $\mathbb{R}$, so is linear. There is $\psi = (\psi_1, \ldots, \psi_m) \in \mathbb{R}^m$ such that

$$H((\gamma_1, \ldots, \gamma_n)) = \psi_1\gamma_1 + \cdots + \psi_m\gamma_m, \quad \text{where}(\gamma_1, \ldots, \gamma_m) \in \mathbb{Z}^m.$$

We extend H by linearity to $\mathbb{R}^m$. Then $H(x) = \langle x, \psi\rangle$, whose $\langle\cdot,\cdot\rangle$ is the standard scalar product on $\mathbb{R}^m$, and H is still a invariant. So

$$\begin{aligned} H(ax) = H(x) \quad (\forall)x \Longrightarrow \langle (a - Id)x, \psi\rangle = 0 \quad (\forall)x \Longrightarrow \qquad (10) \\ \Longrightarrow \langle x, (a - Id)^t\alpha\rangle = 0 \quad (\forall)x, \qquad (11) \end{aligned}$$

i.e. $(a - Id)^t\alpha = 0$. But a is hyperbolic and does not have 1 in the spectrum, so $\alpha = 0$ and H is trivial.

Assume now that M is a nilmanifold. Then $M = N/\Gamma$, where N is a connected simply connected nilpotent Lie group and Γ is a cocompact lattice, $\Gamma = \pi_1(M)$. The rigidity theorem of Malcev allows us to extend $H : \Gamma \to \mathbb{R}$ to a homomorphism $\bar{H} : N \to \mathbb{R}$. The action of a on $\pi_1(M)$ can also be extended to an automorphism of N. Then, by the uniqueness of the extension $\bar{H}$, which follows also from Malcev theorem, the extension $\bar{H}$ is also invariant under the action of a, i.e. $\bar{H}(ax) = \bar{H}(x)$, for all $x \in N$. Consider $N' = [N, N]$ the commutator group of N. Then $a(N') = N'$ and $N' \subset Ker\bar{H}$, so $\bar{H}$ induces a homomorphism on the abelianization $N/[N, N]$ of N which is still a invariant to the action induced by a on the abelianization. But now we can apply the same method used when M was a tori to deduce that H is trivial.

Finally, assume that M is an infranilmanifold. Then $M = G/\Gamma$, where $G = NC$, N a connected simply connected nilpotent Lie group, C a compact group of isometries of N and Γ a cocompact lattice. It is well known that $\Omega := \Gamma \cap G$ is a cocompact lattice in N and a subgroup of finite index in Γ. The homomorphism $H|_\Omega$ is trivial because of the discussion done when M was a nilmanifold. So $\Omega \subset KerH$. But the morphism induced by H on Γ/Ω is trivial, because Γ/Ω is finite, so H is trivial.

So $\mathcal{F}$ has all leaves closed manifolds and there is a function $F : M \to \mathbb{R}$ such that the leaves of $\mathcal{F}$ are the graphs of the functions $\{F + t\}_{t\in\mathbb{R}}$.

We show that the cocycle β is cohomologous to a constant cocycle. This will follow from the invariance of $\mathcal{F}$ under $\widetilde{\alpha}(a)$, for any $a \in \mathbb{Z}^k$.

Take $(x, F(x) + t)$, $(y, F(y) + t)$ two points on the same leaf of $\mathcal{F}$. Then

$$\widetilde{\alpha}(a)(x, F(x) + t) = (ax, \beta(a, x) + F(x) + t)$$

and

$$\widetilde{\alpha}(a)(y, F(y) + t) = (ay, \beta(a, y) + F(y) + t)$$

are again on the same leaf of $\mathcal{F}$, or there is $t' \in \mathbb{R}$ such that

$$\begin{aligned} \beta(a, x) + F(x) + t &= F(ax) + t' \qquad (12) \\ \beta(a, y) + F(y) + t &= F(ay) + t' \qquad (13) \\ & \qquad (14) \end{aligned}$$

So

$$\beta(a,y) + F(y) - F(ay) = \beta(a,x) + F(x) - F(ax).$$

Since the last relation is true for any a, x and y, it follows that

$$\pi(a) := \beta(a,x) + F(x) - F(ax)$$

does not depend on x. We show now that $a \to \pi(a)$ is a morphism. Indeed

$$\pi(a+b) = \pi(a) + \pi(b) \Longleftrightarrow \tag{15}$$
$$\Longleftrightarrow \beta(a+b,x) + F(x) - F((a+b)x) = \tag{16}$$
$$= \beta(a,x) + F(x) - F(ax) + \beta(b,x) + F(x) - F(bx) \Longleftrightarrow \tag{17}$$
$$\Longleftrightarrow \beta(a,bx) - F(a(bx)) + F(bx) = \beta(a,x) + F(x) - F(ax) \tag{18}$$

and the last equality follows because $\pi(a)$ is independent of x.

So $\beta(a,y) = F(ay) - F(y) + C(a)$ is cohomologous with a constant. □

We show now an alternativ method for constructing the invariant foliation which can be applied even to Hölder cocycles. We present the proof only for linear (TNS) actions, but the arguments can be applied to more general hyperbolic $\mathbb{Z}^k$-actions, and even to non-uniformly hyperbolic $\mathbb{Z}^k$-actions.

Theorem 25 *(Theorem 2) Let M be a torus, a nilmanifold or an infranilmanifold. Let $\alpha : \mathbb{Z}^k \times M \to M$ be a linear (TNS) $\mathbb{Z}^k$-action. Let $\beta : \mathbb{Z}^k \times M \to \mathbb{R}$ be a Hölder cocycle over α. Then β is cohomologous with a constant cocycles and the coboundary $P : M \to \mathbb{R}$ is Hölder.*

The idea of the proof is similar with the C^∞ case. The difference appears only at the construction of the invariant foliation.

For $x \in M$, $a \in \mathbb{Z}^k$ hyperbolic and $y \in W_a^s(x)$ we define $P_a^-(y;x)$ using again formula (?). Similarly, for $z \in W_a^u(x)$ we can define $P_{-a}^-(z;x)$.

Lemma 17 *(Lemma 2) Let $a \in \mathbb{Z}^k$ be a hyperbolic element. Let $x \in M$ and $x_1, x_2, \dots, x_n \in W_a^s(x)$. Then:*

$$P_a^-(x_n;x_1) = \sum_{k=1}^{n-1} P_a^-(x_{k+1};x_k).$$

Proof This follows immediately from formula (?), the definition of $P_a^-(y;x)$. □

Consider U an open set included into a coordinate chart. Then U is foliated by the local unstable manifolds of a hyperbolic element a and for any $x \in U$, the local stable manifold $W_a^s(x)$ intersects any local unstable manifold foliating U (not necessary in a point in U). This is just the local product structure determined by the stable and the unstable foliations.

We define a function $F_x : U \to \mathbb{R}$. If $z \in U$, then there is a unique point $u := W_a^s(x) \cap W_a^u(z)$, and we define:

$$F_x(z) = P_a^-(u; z) + P_{-a}^-(z; u).$$

Note that F is continuous and even Hölder.

We want to construct a continuous foliation of $M \times \mathbb{R}$ such that the leaves are determined locally by the graphs of the functions $\{F_x + t\}_{t \in \mathbb{R}}$. Let $y \in U$, $y \neq x$. It is enough to show that if the graphs of the functions $F_x + t_1$ and $F_y + t_2$ has a common point, then the functions coincide on U. Then the definition of the local leaves does not depend on x and we can extend them to a global foliation. We will assume, without lossing from generality, that $t_1 = 0$ and $t_2 = F_x(y)$, i.e. the common point of the local leaves is $(y, F_x(y))$.

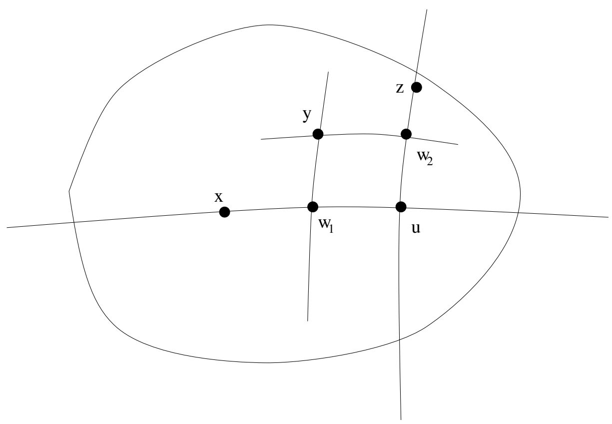

fig. 3

Denote $w_1 := W_a^u(y) \cap W_a^s(x)$ and $w_2 := W_a^s(y) \cap W_a^u(x)$. See fig. (?) above. Then, using Lemma 2 we have:

$$F_x(z) = P_a^-(u; x) + P_{-a}^-(z; u) = P_a^-(w_1; x) + P_a^-(u; w_1) + P_{-a}^-(w_2, u) + P_{-a}^-(z, w_2).$$

and, by definition of F_x:

$$F_x(y) = P_a^-(w_1; x) + P_{-a}^-(y; w_1), F_y(z) = P_a^-(w_2; y) + P_{-a}^-(z; w_2).$$

We see now that in order to have $F_x = F_y + F_x(y)$ is enough to prove:

$$P_{-a}^-(y; w_1) + P_a^-(w_2; y) = P_a^-(u; w_1) + P_{-a}^-(w_2; u). \qquad (?)$$

We prove formula (?), and therefore the existance of a global foliation, in Lemma 7 below. First we prove some necessary technical lemmas. The proof of the following lemma is already done in the proof of Lemma 1.

Lemma 18 *(Lemma 3) If a, b are hyperbolic elements from S and $z \in W_a^s \cap W_b^s$, then $P_a^-(z; x) = P_b^-(z; x)$.* □

Notation Let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k$ be k distinct minimal foliations such that their distributions generate a distribution integrable to a foliation $\mathcal{F}$. Because the action is linear, this is true for any family of minimal foliations. Then we write:

$$\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k\}.$$

Is $\mathcal{F}_i$ is a minimal foliation, denote by $\mathcal{F}_i^{loc}(x)$ the intersection of $\mathcal{F}_i(x)$ with a ball of size r around x in M. Because we can choose the size of the stable and unstable manifolds to be uniform w.r.t. all elements $a \in \mathbb{Z}^k$ and $x \in M$, it follows that, if r is small enough, $\mathcal{F}_i^{loc}$ is included in any local stable or unstable manifold which contains $\mathcal{F}_i$.

The following lemma follows easily because the action α is linear.

Lemma 19 *(Lemma 4) Take now two disjoint families $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k$ and $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_l$ of minimal foliations such that $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k\}$ and $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_l\}$. Then the following local product structure property is true:*

1. *”” for any $x \in M$, $y \in \mathcal{F}^{loc}(x)$, $z \in \mathcal{G}^{loc}(x)$, there is a unique $w := \mathcal{F}^{loc}(x) \cap \mathcal{G}^{loc}(x)$.*

□

Lemma 20 *(Lemma 5) Let $a, b, c \in \mathbb{Z}^k$ be hyperbolic elements. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be minimal foliations such that $\mathcal{F}_1 \subset W_a^s$, $\mathcal{F}_2 \subset W_b^s$, $\mathcal{F}_1 \subset W_c^s$, $\mathcal{F}_2 \subset W_c^s$. Then for any $x \in M$, $y \in \mathcal{F}_1^{loc}(x)$, $z \in \mathcal{F}_2^{loc}(x)$ and $w = \mathcal{F}_2^{loc}(y) \cap \mathcal{F}_1^{loc}(z)$ we have:*

$$P_c^-(w; x) = P_a^-(y; x) + P_b^-(w; y) = P_b^-(z; x) + P_a^-(w; z).$$

Proof Apply first Lemma 2 for the families of points $\{x, y, w\}$ and $\{x, z, w\}$ and then use Lemma 3. □

Lemma 21 *(Lemma 6) Let $a \in \mathbb{Z}^k$ be a hyperbolic element, $x \in M$ and $z \in W_a^s(x)$. Assume that $W_a^s = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k\}$. Then there exist $y_1 \in \mathcal{F}_1^{loc}(x)$, $y_2 \in \mathcal{F}_2^{loc}(y_1)$, $\dots$, $y_{k-1} \in \mathcal{F}_{k-1}^{loc}(z)$ such that $z \in \mathcal{F}_k(y_{k-1})$.*

Proof We find the points y_i recurrently. Assume that we know $y_1, y_2, \dots, y_{i-1}$. The families of foliations $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_i$ and $\mathcal{F}_{i+1}, \mathcal{F}_{i+2}, \dots, \mathcal{F}_k$ are both integrable and let $\mathcal{G}_1 = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k\}$ and $\mathcal{G}_2 = \{\mathcal{F}_{i+1}, \mathcal{F}_{i+2}, \dots, \mathcal{F}_k\}$. Then $\mathcal{G}_1^{loc}(x)$ and $\mathcal{G}_2^{loc}(z)$ are transverse submanifolds of complementary dimension in $W_a^s(x)$, so they intersect in a unique point y_i.

To see that $y_i \in \mathcal{F}_i^{loc}(y_{i-1})$, note first that $\mathcal{F}_i^{loc}(y_{i-1})$ is included in $\mathcal{G}_1^{loc}(x)$ and second that $\mathcal{F}_i^{loc}(y_{i-1})$ and $\mathcal{G}_2^{loc}(z)$ intersect in a unique point by a similar transversality argument. □

We are now in position to prove relation (*) and finish the proof of Theorem 2.

Lemma 22 *(Lemma 7) Let $a \in \mathbb{Z}^k$ be a hyperbolic element, $y, u \in M$, $w_1 = W_a^u(y) \cap W_a^s(x)$ and $w_2 = W_a^s(y) \cap W_a^u(x)$. Then:*

$$P_{-a}^-(y; w_1) + P_a^-(w_2; y) = P_a^-(u; w_1) + P_{-a}^-(w_2; u).$$

Proof Let $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k\}$ and $\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_l\}$ be two disjoint families of minimal foliations such that $W_a^s = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k\}$ and $W_a^u = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_l\}$.

Use now Lemma 6 to find $x_{11} = w_1, x_{12}, \ldots, x_{1k+1} = u$ such that

$$x_{12} \in \mathcal{F}_1^{loc}(w_1), x_{13} \in \mathcal{F}_2^{loc}(x_{12}), \ldots, x_{1k+1} \in \mathcal{F}_k^{loc}(x_{1k}).$$

Use again Lemma 6 to find $x_{21}, \ldots, x_{2l} = y$ such that

$$x_{21} \in \mathcal{G}_1^{loc}, x_{31} \in \mathcal{G}_2^{loc}, \ldots, x_{l+11} \in \mathcal{G}_l^{loc}(x_{l1}).$$

We define now recurrently the points x_{ij} for all $1 \leq i \leq l+1$ and $1 \leq j \leq k+1$. See figure (?) below which shows what happens for $l = 2$ and $k = 3$.

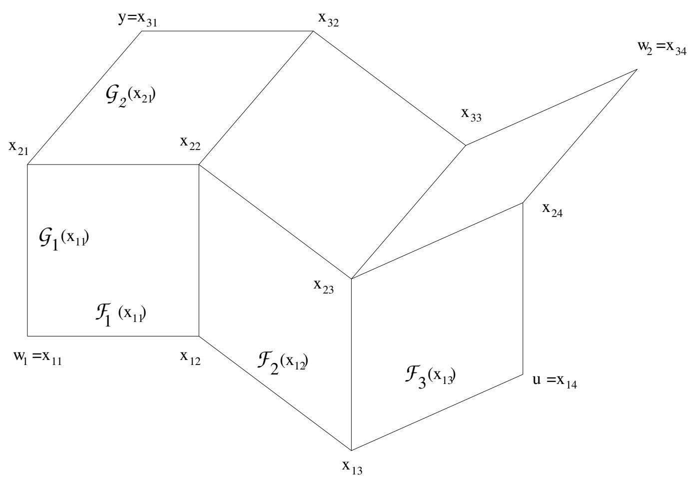

fig. 4

If we know the points x_{ij}, x_{i+1j} and x_{ij+1}, then we can apply Lemma 4 to find $x_{i+1j+1} := \mathcal{F}_j^{loc}(x_{i+1j}) \cap \mathcal{G}_i^{loc}(x_{ij+1})$.

We explain why $x_{l+1k+1} = w_2$. The family of local leaves

$$\mathcal{F}_1(x_{l+11}), \mathcal{F}_2(x_{l+12}), \ldots, \mathcal{F}_k(x_{l+1k})$$

are all contained in $W_a^s(y)$ and the family of local leaves

$$\mathcal{G}_1(x_{1k+1}), \mathcal{G}_2(x_{2k+1}), \ldots, \mathcal{G}_l(x_{lk+1})$$

are all contained in $W_a^u(u)$. It is clear that the families determine two pathes which have nontrivial intersection x_{l+1k+1}. But $W_a^s(y)$ and $W_a^u(u)$ has a unique point of intersection w_2, so $x_{l+1k+1} = w_2$.

Because the action is (TNS), for each pair of foliations $\mathcal{F}_j$, $\mathcal{G}_i$ there is a hyperbolic element containing both of them in its stable manifold. So each quadruple $\{x_{ij}, x_{i+1j}, x_{ij+1}, x_{i+1j+1}\}$ satisfies the hypotesis in Lemma 5, so we have:

$$P_a^-(x_{ij+1}; x_{ij}) + P_{-a}^-(x_{i+ij+1}; x_{ij+1}) = P_{-a}^-(x_{i+1j}; x_{ij}) + P_a^-(x_{i+1j+1}; x_{i+1j}). \qquad (*)$$

We consider now formula (*) for all $1 \leq i \leq l$ and $1 \leq j \leq k$. We add member by member all resulting relations and simplify all equal terms. The terms which can be simplified correspond to the interior segments in fig. 4. The simplified formula is:

$$\sum_{i=1}^{k} P_a^-(x_{1i+1}; x_{1i}) + \sum_{j=1}^{l} P_{-a}^-(x_{j+11}; x_{j1}) = = \sum_{i=1}^{k} P_a^-(x_{l+1i+1}; x_{l+1i}) + \sum_{j=1}^{l} P_a^-(x_{j+1k+1}; x_{jk+1}). \qquad (*)$$

We use Lemma 2 to compute each sum in (*). Then (*) becomes:

$$P_a^-(x_{1k+1}; x_{11}) + P_a^-(x_{l+11}; x_{11}) = P_a^-(x_{l+1k+1}; x_{l+11}) + P_a^-(x_{l+1k+1}; x_{1k+1}),$$

and Lemma 7 follows. □

The only thing left to finish the proof of Theorem 2 along the line used in the proof of Theorem 1 is to show that the foliation of $M \times \mathbb{R}$ is invariant under the full group action. It is enough to check that it is invariant under the action of all element in S. We show that in formula (?), which defines F_x, we obtain the same result, independent of the element a.

6.2 Invariant measures for abelian actions

In this section we show that invariant Borel probability measures are quite rare. This phenomenon is somehow related with the following famous conjecture.

Conjecture 2 *(Furstenberg's Conjecture) The only ergodic invariant measures for the semigroup of circle endomorphisms generated by multiplications by p and q where $p^n \neq q^m$ unless $n = m = 0$ are Lebesgue measure and atomic measures concentrated on periodic orbits.*

Rudolph and Johnson proved this result under the additional assumption that some element of the action has positive entropy.

Consider now a simple model case of (TNS) action. Let A, $B \in SL(3, \mathbb{Z})$ be two hyperbolic matrices, $AB = BA$, which generate a genuine $\mathbb{Z}^2$-action on $\mathbb{T}^3$, i.e. $A^n \neq B^m$ unless $n = m = 0$. Denote the action of $A^k B^l$ by $F_{A^k B^l} : \mathbb{T}^3 \to \mathbb{T}^3$.

Lemma 23 *The $\mathbb{Z}^2$ action F satisfies the (TNS) condition.*

Theorem 26 *Let f be a C^1 Anosov diffeomorphism of M. μ Borel invariant probability measure. Then the metric entropy $h_\mu(f) = 0$ if and only if μ-a.e. x, the conditional measures on W^s are nonatomic.*

Theorem 27 *If $a \in \mathbb{R}^k$ is Anosov, then $\xi(W_a^+) = \xi(W_a^-)$.*

Theorem 28 *(Hopf argument)Let f be a C^1 diffeomorphism, μ a Borel invariant probability measure and W a contracting foliation which is invariant by f. Then $\xi_f \leq \xi(W)$.*

Proof If ψ is a continuous function on M, then the forward averages

$$\Psi^+(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(a^n x),$$

which exist from Birkhoff theorem, are constant along the stable manifolds of f as they contract exponentially under f. The continuous functions are dense in $L^2(m, \mu)$, so any invariant L^2-function is constant a.e. on $W(x)$ with respect to the conditional measure induced by μ.

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