SMOOTH GROUP ACTIONS AND RIGIDITY

A.Katok

PROBLEM SET # 1

1. Let n be a natural number and $E_n : S^1 \to S^1$ be the endomorphism of of degree $n, E_n x = nx \pmod{1}$. Denote by $\mathcal{E}$ the action of the multiplicative semigroup of positiove integers $\mathbb{Z}_+$ by these endomorphisms. Prove that the only invariant Borel probability measures for $\mathcal{E}$ are the combinations of Lebesgue and the δ–measure at zero.

2. Find all invariant Borel probability measures for the action $\mathcal{E}^{'}$ of $\mathbb{Z}_+$, $\mathcal{E}^{'}(n) = E_{n^2}$. Consider possible generalizations of your argument to other semi-groups of circle endomorphisms.

3. Prove that the standard projective action $\mathcal{P}$ of the group $SL(n, \mathbb{Z})$ on $S^{n-1}, n \geq 2$ does not have any Borel probability invariant measure.

4. Find all Borel probability ergodic invariant measures for the standard action $\mathcal{A}_n$ of the group $SL(n, \mathbb{Z})$ by automorphisms of the torus $\mathbb{T}^n, n \geq 2$. Extend your argument to any subgroup $\Gamma \subset SL(n, \mathbb{Z})$ of finite index.

5. Construct uncountably many non-conjugate smooth actions of $SL(2, \mathbb{Z})$ on the two-torus close to the standard action $\mathcal{A}_2$.

6. Prove that for an open dense set of actions of $SL(2, \mathbb{Z})$ on the two-torus close to the standard action $\mathcal{A}_2$ there is no Borel probability invariant measure.

Hint to Problems 5 and 6: You may first consider actions of a particular subgroup of finite index.

7. Prove that $SL(n, \mathbb{Z})$ does not contain a subgroup isomorphic to $\mathbb{Z}^n$ which consists of diagonalizable matrices.

8. Classily up to the real-analytic conjugacy all effective real-analytic actions of $\mathbb{Z}^2$ on the unit interval such that the endpoints are the only fixed points of the action. Consider a generalization of your argument to arbitrary effective real-analytic actions.

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PROBLEM SET # 2

9. Let G be a discrete abelian group, $\mathcal{R}$ be its *regular* representation in $l^2(G)$ ie the representation induced by the action of G on itself by translations. Describe the decomposition of $\mathcal{R}$ into irreducible representations.

10. Prove that the topological conjugacy with a linear map E_n holds for any C^0 *expanding map* of the circle onto itself, ie a map f such that for some $\epsilon > 0$, $0 < dist(x, y) < \epsilon$ implies $dist(fx, fy) > dist(x, y)$.

11*). Prove that every C^2 non-singular map of the circle of degree m, $|m| \geq 2$ has an expanding periodic point, ie a point for which the derivative over the period has absolute value greater than one.

Remark: I do not know a proof of the similar statement for C^1 maps.

12. Prove that any C^∞ action on the circle generated by maps of degree two and three which is topologically conjugate to the linear action is in fact C^∞ conjugate to it. *Hint:* Use previous problem.

13. Construct an example of a C^∞ action on the circle generated by non-singular maps of degree two and three which is not topologically conjugate to the linear action.

14. The same statement as in the previos problem but for real-analytic actions.

15. Let $A \in SL(n, \mathbb{Z})$ be any non-hyperbolic matrix. Prove that there exists an arbitrary small C^∞ perturbation f of the automorphism F_A such that there is no continuous map $h : \mathbb{T}^n \to \mathbb{T}^n$ homotopic to identity and satisfying $F_A \circ h = h \circ f$.

16. Suppose that $A \in SL(n, \mathbb{Z})$ is such that no positive power of the automorphism F_A preserve any rational subtorus of positive dimension. Prove that all eigenvalues of A are simple and non of them is a root of unity.

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PROBLEM SET # 3

17.Construct an example of a totally irreducible free action of $\mathbb{Z}^2$ on the four-dimensional torus $\mathbb{T}^4$ which contains a non-hyperbolic (but naturally partially hyperbolic) element.

18.Construct an example of a totally irreducible free action of $\mathbb{Z}^2$ on the four-dimensional torus $\mathbb{T}^4$ by hyperbolic symplectic automorphisms.

19.Prove that no hyperbolic automorphism of the three-dimensional Heisenberg group preserving Haar measure preserves a lattice.

20.Construct am example of an irreducible $\mathbb{Z}^2$ action by hyperbolic aoutomorphisms of a compact nilmanifold. *Hint:* Use a proper modificaton of a construction described in [KH,Section 17.3]

21.Let$A \in SL(n, \mathbb{Z})$. Show that the following four conditions are equivalent:

(1) A is ergodic with respect to Lebesgue measure;
(2) The set of periodic points of the toral automorphism defined by A coincides with the set of points in $\mathbb{T}^n$ with rational coordinates;
(3) None of the eigenvalues of the matrix A are roots of unity;
(4) A has at least one eigenvalue of absolute value greater than one and has no eigenvectors with rational coordinates.

22. Let $\mathcal{A} \in SL(n, \mathbb{R})$ be a maximal connected (non-split) Cartan subgrop generated by matrices with k positive real eigenvalues and l pairs of complex-conjugate eigenvalues, $\mathcal{A}^{'} \subset \mathcal{A}$, the subgroup of matrices with positive real eigenvalues, K a maximal compact subgroup in the centralizer of $\mathcal{A}^{'}$, Γ a cocompact lattice in $SL(n, \mathbb{R})$. Consider the rigth action of $\mathcal{A}^{'}$ on the double coset space $\Gamma \backslash SL(n, \mathbb{R})/K$. Show that it is not an Anosov action and find the number of zero Lyapunov exponents.

23. Prove that the Weyl chamber flow on a (double) coset space of a semi-simple Lie group G has simple Lyapunov exponents if and only if G is *split*, ie if the dimension of the maximal split Cartan subgroup is equal to the comlex rank of the complecification of G.

24. Consider an action of $\mathbb{Z}^k$ by partially hyperbolic (ie ergodic, cf Problem 21) automorphisms of $\mathbb{T}^n$. Calculate the specrtum of the corresponding group of unitary operators in $L_2(\mathbb{T}^n), \lambda$ where λ is Lebesgue measure.