# THEORY OF DYNAMICAL SYSTEMS AND GENERAL TRANSFORMATION GROUPS WITH INVARIANT MEASURE

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UDC 513.83:513.88+517.9+519.4

## INTRODUCTION

The theory of dynamical systems with invariant measure, or ergodic theory, is one of those domains of mathematics whose form changed radically in the last 15-20 years. This has to do both with the internal problems of ergodic theory and with its connections with other parts of mathematics. In ergodic theory itself, there arose the theory of entropy of dynamical systems, whose origin was in the papers of A. N. Kolmogorov. Recently, remarkable progress has been made by Ornstein and his collaborators in the problem of metric isomorphism of Bernoulli automorphisms and K-automorphisms, i.e., dynamical systems with very strong mixing properties. Another important event of recent times is a new, profound connection of ergodic theory with statistical mechanics, not only enriching ergodic theory itself, but also leading to new progress in the mathematical problems of statistical mechanics. Both of the circles of problems mentioned occupy a significant place in this survey. On the other hand, a series of applications of ergodic theory is intentionally excluded from our survey. This has to do in the first place with physics papers, which do not contain strictly mathematical results. Also, we shall not dwell on many mathematical papers connected in one way or another with ergodic theory, but not relating directly to it. Such, for example, are the papers of G. A. Margulis and Mostow on quasiconformal mappings of manifolds of negative curvature, in which the ergodicity of flows on such manifolds is used, or the papers of Glimm and Jaffe, which can be partially interpreted as investigations of the mixing properties of some dynamical systems which arise in quantum field theory.

The development of ergodic theory up to 1967 is reflected in the survey of A. M. Vershik and S. A. Yuzvinskii [48], published in "Itogi Nauki" for 1967. In this connection, in the present survey, one considers basically papers that have appeared since 1967, but in connection with earlier papers, give reference to the survey of Vershik and Yuzvinskii. However, where this requires substantial exposition, we shall deviate from this rule.

Various results and even entire directions in ergodic theory are considered by us with quite different degrees of detail, while the distribution of volume among sections is far from proportional to the number of published papers. We realize that here there are necessarily assumptions in such situations, and possibly, a quite considerable subjectivity.

Since 1967 a series of more or less systematic accounts of various parts of ergodic theory have appeared. Relevant here are the books of Billingsley [22] and Friedman [351] and the lectures of Ya. G. Sinai [167], devoted to the foundations of ergodic theory, the lectures of Ya. G. Sinai [670], devoted to a wider circle of questions, the monograph of Parry [597], devoted to a systematic account of the theory of generators and entropy theory, the fundamental monograph of Ornstein [584], in which he summarizes the recent progress in the problem of metric isomorphism, the book of V. I. Arnol'd and Avez [221], on the applications of ergodic theory to problems of classical mechanics, and also the surveys of Ornstein [578] and Weiss [719].

The progress in ergodic theory in the last 20 years was greatly promoted by the small book of Halmos "Lectures on Ergodic Theory," which at the time of its appearance could serve simultaneously as an elementary textbook on ergodic theory and a monograph (although not en-

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Translated from Itogi Nauki i Tekhniki, Matematicheskii Analiz, Vol. 13, pp. 129-262, 1975.

tirely exhaustive). At the present time, a book of this character apparently could not be written. Unfortunately, there are also no textbooks in which the basic ideas and methods of ergodic theory in its contemporary state are expounded intelligibly and sufficiently systematically.

We have tried to give in the text complete definitions of the most important of those concepts which have appeared in ergodic theory in the last decade. The basic, long established concepts are not defined in the survey. The reader can find the corresponding definitions in the already mentioned book of Billingsley [22] or in the first paragraphs of the paper of Rokhlin [145], published in "Uspekhi Matematicheskii Nauk." One can also recommend the already mentioned book of Halmos in combination with the survey of Vershik and Yuzvenskii [48].

# CHAPTER 1

# ENTROPY AND THE ISOMORPHISM PROBLEM

The entropy theory of metric automorphisms and flows showed and continues to show a stimulating influence on the development of ergodic theory on the whole. The reader can acquaint himself with the foundations of entropy theory in the lectures of V. A. Rokhlin [145] and the book of Billingsley [22]. The development of this theory in the period from 1957 to 1967 is completely reflected in the survey of A. M. Vershik and S. A. Yuzvinskii [48].

## §1. Generating Partitions and Abstract Entropy Theory

1.1. A. A. Kirillov [96] formulated a general approach to the definition of entropy of group actions. In connection with the problem of statistical mechanics, the entropy of an action of the group $\mathbf{Z}^n$ was considered by Robinson and Ruelle [626]. In connection with the problem of classification of decreasing sequences of partitions (cf. [173]), B. S. Pitskel' and A. M. Stepin [140] considered the entropy of actions of commutative groups. For the action of countable periodic subgroups on circles, in [140] the existence almost everywhere of the speed of refinement of subdivisions with finite entropy was proved, and for ergodic actions of the group $\mathbf{Z}^n$ the asymptotic property of uniform distribution was proved. The latter is also proved in [429, 692].

Föllmer [342] proved Breiman's theorem for Gibbsian stationary fields on $\mathbf{Z}^n$ (cf. Chap. 8). In the general case of stationary fields on $\mathbf{Z}^n$, $n>1$, this question remains open. The paper of Conze [302] is devoted to the entropy of an action of a commutative group with a finite number of generators. In it is introduced the concept of a group action with the K-property (cf. Paragraph 3), and it is proved that such an action has completely positive entropy, and a multidimensional analogue of Abramov's formula for the entropy of a flow is obtained.

We shall call a subset P of the group G informationally past if for each action T of the group G, $h(T)=H\left(\xi \,\middle|\, \bigvee_{g\in P} T_g\xi\right)$, where $\xi$ is a generator of a subdivision. B. S. Pitskel' [139] proved that the set of negative elements of an ordered amenable group is informationally past; for commutative groups, the converse of this fact is also proved.

1.2. It is known (cf., for example, [145, 597]) that each aperiodic automorphism T admits a representation by a shift in the space of infinite sequences $A^{\mathbf{Z}}$ with a no more than countable set $A^{\mathbf{Z}}$. This representation is connected with the existence of a partition $\xi$ of the space X with $|A|$ elements, all motions of which generate a $\sigma$-algebra of measurable sets; such a partition is called generating for the automorphism T. The entropy of a transformation does not exceed the entropy of a generating partition. V. A. Rokhlin [145] proved that an aperiodic automorphism with finite entropy has a generating partition with finite entropy. For ergodic automorphisms, Krieger [472] strengthened this result, proving the existence of finite generating partitions. The condition of ergodicity here is essential.

The number of elements of a partition, which is generating for T, must satisfy the inequality $|\xi|\geqslant 2^{h(T)}$. It turns out that there are no other entropy restrictions on the number of elements of a generating partition. Krieger [475] proved that a partition $\xi$, generating with respect to T and satisfying the condition $|\xi|\leqslant 2^{h(T)}+1$, can be chosen to be measurable with respect to a previously given subalgebra which is exhausting with respect to T. By other methods, A. N. Livshits [116] and Smorodinsky [678] constructed generating partitions with no more than $2^{h(T)}+2$, elements. Finally, Denker [312] strengthened Krieger's theorem in the following way: For an ergodic automorphism T with finite entropy and each probability distribution $(p_1,\ldots,p_k)$, $k=[2^{h(T)}]+1$, whose entropy is greater than h(T), $\varepsilon>0$, there exists a generating partition $\xi=(C_1,\ldots,C_k)$, satisfying the condition $|\mu(C_i)-p_i|<\varepsilon$, $i=1,\ldots,k$. The paper of B. A. Rubenshtein [149] is devoted to generating partitions of Markov endomorphisms.

The case of flows was considered by Krengel [459, 461]. He proved for an ergodic flow $\{T_t\}$ with exhaustive σ-algebra $\mathfrak{A}$, the density in $\mathfrak{A}$ of the collection of sets A, for which the family $\{T_tA,\ t\geqslant 0\}$ generates a full σ-algebra.

1.3. The entropy of generalized powers (i.e., transformations of the form $T^{n(x)}$) of an automorphism T was calculated by R. M. Belinskii [19] and Neveu [553]. For example, if $T^{n(x)}$ is an ergodic automorphism, and the function n(x) is positive and bounded, then $h(T^{n(x)})=h(T)\int n(x)\,d\mu(x)$. R. M. Belinskii [21] and Newton [560] calculated the entropy of a skew product with fibers $\{T^{\theta(x)}\}$.

B. S. Pitskel' [136] established a connection of A-entropy with the spectrum of the automorphism. Newton [561, 562] proved that for an ergodic automorphism the A-entropy is completely determined by the entropy if h(T) ≠ 0. This dependence was made more precise in [483, 484]. On the connections of entropy with other invariants, cf. [389, 238].

## §2. Entropy and Bernoulli Shifts

2.1. The question of isomorphism of Bernoulli shifts with the same entropy arises at the very beginning of the development of entropy theory (cf., for example, the survey of V. A. Rokhlin [144]). The first result in this direction belongs to L. D. Meshalkin. He constructed an explicit code, establishing the isomorphism of Bernoulli shifts with equal entropy and probability states of the form $k/p^l$, where p is a prime (cf. [48]).

Ya. G. Sinai [157] proved that for each ergodic automorphism with positive entropy, there eixsts a Bernoulli factor-automorphism with the same entropy. In particular, Bernoulli shifts with the same entropy are homomorphic images of one another (weak isomorphism).

2.2. The complete solution of the problem was obtained by Ornstein [573, 574]: Bernoulli shifts with the same entropy are isomorphic. He proved a strengthened variant of the theorem on weak isomorphism, which allows a weak isomorphism to be rebuilt as an isomorphism.

In order to formulate the basic approximation lemma of Ornstein, we introduce an appropriate metric in the set of finite measurable partitions. The distance d(ξ, η) between the finite (ordered) partitions $\xi=(A_1,\ldots,A_k)$ and $\eta=(B_1,\ldots,B_k)$ is defined by the formula

$$d(\xi,\ \eta)=\sum_{i=1}^{k}|\mu(A_i)-\mu(B_i)|.$$

We shall call the distance $\bar{d}$ between sequence of finite partitions $\{\xi_i\}_1^\infty$ and $\{\eta_i\}_1^\infty$ of the space $(X,\ \mu)$, the number

$$\lim_{n\to\infty}\frac{1}{n}\inf_U\sum_{i=1}^{n}\rho(\xi_i,\ U\eta_i),$$

where $\rho(\xi, \eta)=H(\xi/\eta)+H(\eta/\xi)$ and the infimum is taken over all automorphisms of the space $(X, \mu)$. a pair $(T, \xi)$, consisting of an automorphism T and a finite partition $\xi$. As the distance between processes $(T, \xi)$ and , we take the number $\bar{d}(\{T^i\xi\}_1^\infty, \{S^i\eta\}_1^\infty)$.

In a free account, Ornstein's approximation lemma appears like this. Let T be a Bernoulli shift, $\xi$ be its Bernoulli generator, S be an ergodic automorphism, and $h(T, \xi) \leqslant h(S)$. If for a partition $\eta$ the distance between the processes $(T, \xi)$ and $(S, \eta)$ is sufficiently small, then close to $\eta$ (with respect to the metric $\rho$) one can find a partition $\eta'$, such that $\rho((T, \xi), (S, \eta'))$ is less than a previously given positive number.

Inductive application of the approximation lemma leads to the following fundamental proposition: If the distance between processes $(T, \xi)$ and $(S, \eta)$ is sufficiently small, then close to $\eta$ one can find a partition $\zeta$ such that the processes $(T, \xi)$ and $(S, \zeta)$ are isomorphic. This is a strengthened variant of the theorem of Ya. G. Sinai formulated at the beginning of this paragraph.

2.3. In the general case, there are no methods for calculating (or estimating) the distance between processes. Ornstein isolated a class of processes, which he called finitely determined, for which the distance to any other process depends on a finite number of parameters. More exactly, a process $(T, \xi)$ is finitely determined if for any $\varepsilon>0$ there exists a $\delta>0$ and a natural number n such that each process $(S, \eta)$, satisfying the conditions $|h(S, \eta)-h(T, \xi)|<\delta$ and $d\left(\bigvee_0^{n-1} S^k\eta, \bigvee_0^{n-1} T^k\xi\right)<\delta$, is $\varepsilon$-close to $(T, \xi)$.

A Bernoulli process has this property for n = 1. Whence it follows that the fundamental proposition can be formulated in this way: Let T be a Bernoulli shift, $\xi$ be its Bernoulli generator, S be an ergodic automorphism, and $h(T, \xi) \leqslant h(S)$; if the entropy of the processes $(T, \xi)$ and $(S, \eta)$ is sufficiently close and the distance $d(\xi, \eta)$ is small, then close to $\eta$ one can find a partition $\zeta$, such that the processes $(T, \xi)$ and $(S, \zeta)$ are isomorphic. The fundamental proposition in this formulation allows a weak isomorphism of Bernoulli shifts to be rebuilt as an isomorphism.

Lemma on reconstruction. Let $(T, \xi)$ be a Bernoulli process, $\eta$ be a partition with independent shifts with respect to T, and $h(T, \xi)=h(T, \eta)$. Then for any $\varepsilon>0$ there exists a partition $\eta$, such that $\rho(\eta, \bar{\eta})<\varepsilon$, the processes $(T,\eta)$ $(T, \bar{\eta})$ are isomorphic, and the $\sigma$-algebra of $\xi$-measurable sets with exactness to $\varepsilon$ is contained in the $\sigma$-algebra corresponding to the partition $\bigvee_{-\infty}^{\infty} T^i\bar{\eta}$.

In view of the exceptional importance of this assertion in the theory of metric isomorphism, we shall outline its proof. We choose k so that the partition $\eta$ is sufficiently well approximated by some enlargement of the partition $\bigvee_{-k}^{k} T^i\xi$, which we shall denote by $L\left(\bigvee_{-k}^{k} T^i\xi\right)$.* In addition, we choose a $k_1$ such that the partition $\eta$ is approximated by some enlargement of $\bigvee_{-k_1}^{k_1} T^i\xi$ with considerably greater precision. Applying the Halmos–Rokhlin lemma (cf. Paragraph 2.1 of Chap 5), we choose a set F, measurable with respect to the partition $\bigvee_{-\infty}^{\infty} T^i\eta$ and such

*Here L denotes some operation of enlargement of the lexicographically ordered partition $\bigvee_{-k}^{k} T^i\xi$. (We recall that the partition $\xi$ is ordered.)

that the sets $T^iF$, $i=-n,\dots,n$, are pairwise disjoint (here the ratio k/n is small) and the measure of $X\setminus\bigcup_{-n}^{n}T^iF$ is sufficiently small. There exists a partition $\xi'<\bigvee_{-\infty}^{\infty}T^i\eta$ such that the partitions $\left(\bigvee_{-n}^{n}T^i(\xi'\vee\eta)\right)\Big|_F$ and $\left(\bigvee_{-n}^{n}T^i(\xi\vee\eta)\right)\Big|_F$ have the same distribution of measures of elements. Whence it follows that the partition η is sufficiently well approximated by the enlargement $L\left(\bigvee_{-k}^{k}T^i\xi'\right)$ of the partition $\bigvee_{-k}^{k}T^i\xi'$. Here and in what follows the symbol L denotes the enlargement operation as above, but applied to other partitions. In addition, the $h(T,\eta)$ is bounded above with sufficient precision by $h\left(T,\bigvee_{-k_1}^{k_1}\xi'\right)$, which is equal to $h(T,\xi')$. Thus, the entropies $h(T,\xi)$ and $h(T,\xi')$ are close. Since in the construction of ξ' one can guarantee the equality $d(\xi,\xi')=0$, according to the fundamental proposition, close to ξ' one can find a partition $\bar{\xi}<\bigvee_{-\infty}^{\infty}T^i\eta$ such that the processes $(T,\bar{\xi})$ and $(T,\xi)$ are isomorphic. Since the partitions $\bar{\xi}$ and ξ' are sufficiently close, the distance between the partition η and $L\left(\bigvee_{-k}^{k}T^i\bar{\xi}\right)$ is small.

Now we choose a number $l$ such that the partition $\bar{\xi}$ is sufficiently well approximated by an enlargement of the partition $\bigvee_{-l}^{l}T^i\eta$. Then we choose a sufficiently large m, a set G, measurable with respect to the partition $\bigvee_{-\infty}^{\infty}T^i\bar{\xi}$ and such that $T^iG\cap T^jG=\varnothing$, $i,j=-n,\dots,n$, $i\neq j$, and the measure of $X\setminus\bigcup_{-n}^{n}T^iG$ is sufficiently small. We construct a partition η' such that the partitions $\left(\bigvee_{-m}^{m}T^i(\xi\vee\eta')\right)\Big|_G$ and $\left(\bigvee_{-m}^{m}T^i(\bar{\xi}\vee\eta)\right)\Big|_G$ have the same distribution of measures of elements. The parameters of this construction can be chosen so that the entropies $h(T,\eta)$ and $h(T,\eta')$ will be close (cf. above). Since one can assume that $d(\eta',\eta)=0$, according to the fundamental proposition, close to the partition η' one can find a partition $\bar{\eta}$ such that the the process $(T,\bar{\eta})$ is Bernoulli, while $d(\bar{\eta},\eta)=0$. Further, from the choice of the partition η' it follows that the partitions η' and $L\left(\bigvee_{-k}^{k}T^i\xi\right)$ are close, and hence $\bar{\eta}$ and η are also close. Finally, from the fact that the partition ξ is approximated by some enlargement of $\bigvee_{-l}^{l}T^i\eta'$, it follows in our situation that ξ is also approximated by an enlargement of the partition $\bigvee_{-l}^{l}T^i\bar{\eta}$.

Inductive application of the lemma on reconstruction leads to the following important result: Finitely determined processes with the same entropy are metrically equivalent. In particular, Bernoulli shifts with the same entropy are isomorphic. In Ornstein's paper [575] it is proved that the property of being finitely determined is possessed by each process $(T,\zeta)$, where T is a Bernoulli shift and ζ is a finite partition. Whence it follows that a factor-automorphism of a Bernoulli shift is a Bernoulli shift.

2.4. In the paper of Friedman and Ornstein [353], for a stationary process with a finite number of states, a condition is found on the character of the mixing, called by the authors weak Bernoullianness, under which the process is metrically equivalent to a sequence of independent stochastic quantities.

Partitions $\xi=\{A_i\}$ and $\eta=\{B_j\}$ are called ε-independent if $\sum_{i,j}|\mu(A_i\cap B_j)-\mu(A_i)\mu(B_j)|<\varepsilon$. A partition ζ is called weakly Bernoulli with respect to the automorphism T if for each $\varepsilon>0$ there exists a $k(\varepsilon)$, such that for all $n\geqslant 0$ the partitions $\bigvee_{-n}^{0} T^i\zeta$ and $\bigvee_{k(\varepsilon)}^{n+k(\varepsilon)} T^i\zeta$ are ε-independent. A partition on the states in an aperiodic, ergodic Markov chain with a finite nubmer of states serves as an example of a weak Bernoulli partition.

The basic result of Friedman and Ornstein is stated as follows: An automorphism that has a weak Bernoulli generating partition is isomorphic to a Bernoulli shift with the same entropy.

In [576] Ornstein proved that a Bernoulli shift can be included in a flow. For this he introduced a special class of finitely determined partitions. A partition ξ is called very weakly Bernoulli with respect to T if for each $\varepsilon>0$ there exists an $N_0$ such that for all $N'\geqslant N\geqslant N_0$, all $n>0$ and ε-almost all elements of A, the partition $\bigvee_{N}^{N'} T^k\xi$ satisfies the condition

$$\bar{d}(\{T^{-i}\xi\}_1^n,\ \{T^{-i}\xi|A\}_1^n)<\varepsilon.$$

The question of isomorphism of some Bernoulli shifts or others reduces, thus, to the verification of the property of being weakly Bernoulli or very weakly Bernoulli. The first criterion is more effective for Markov automorphisms, and the second for application to flows.

2.5. Using the Friedman–Ornstein theorem [353] on the isomorphism of mixing Markov shifts and Bernoulli shifts, Adler, Shields, and Smorodinsky [196] proved that the shift transformation corresponding to a Markov chain with n subclasses is isomorphic with the direct product of a Bernoulli shift and a shift on $\mathbf{Z}_n$. Ornstein and Shields [585] considered Markov chains having transient density with respect to a stationary distribution. If the shift transformation corresponding to such a chain is mixing, then it is isomorphic with a Bernoulli shift. Azencott [237] proved that the Markov partition of a $\mathscr{Y}$-diffeomorphism is weakly Bernoulli with respect to the measures $\mu_+$ and $\mu_-$ (cf. Paragraphs 4.2 and 6.3 of Chap. 2).

In [529] Maruyama considered the relations among various regularity properties of stationary stochastic processes: The condition of being weakly Bernoulli occupies an intermediate position between the conditions of being mixing proposed by Ibragimov and Rozenblatt (cf., for example, [80]). It was proved by him that if the Markov operator T in the separable space $L_1(\Omega,\mu)$ satisfies the condition $\|T-P\|<1$, where P is the averaging operator with respect to μ, then the corresponding Markov chain has the property of being uniformly strongly mixing, and consequently, is metrically equivalent with a sequence of independent stochastic quantities (result obtained earlier by McCabe and Shields [531]).

In connection with the fact that Markov shifts are isomorphic with Bernoulli shifts, the question arises: Do shifts that are well approximated by Markov shifts with finite memory have this property? Suitable candidates for this are Gibbsian stochastic fields (cf. Chap. 8). Gallavotti [359] proved that one-dimensional Gibbsian fields, generated by potentials with finite first moment (as was proved by Ruelle and R. L. Dobrushin, this property guarantees the absence of phase transitions) are Bernoulli. In [512] the same thing was proved in a special case when the Gibbs distribution is unique. In [515], Markovian stochastic fields corresponding to the Ising model with attraction were considered. It was proved that for some values of the thermodynamic variables, including as cases uniqueness of the Gibbs distribution as well as coexistence of phases, ergodic fields are isomorphic with Bernoulli ones.

In connection with the consideration of stochastic fields on n-dimensional lattices arising from statistical mechanics, and also in connection with problems of metric classification of decreasing sequences of partitions, a series of papers is devoted to Bernoulli shifts with the same entropy on a countable periodic subgroup of the circle are isomorphic. Katznelson and Weiss [429] proved this for Bernoullian actions of the group $Z^n$ (cf. also [720, 699]). Then A. M. Stepin [177] proved a theorem about isomorphisms of Bernoulli shifts for groups with elements of infinite order, and also for periodic groups, having a countable locally finite subgroup. For countable periodic groups with a finite number of generators, the question remains open.

2.6. Ornstein [576] proved that for special flows $\{S_t\}$, constructed from Bernoulli shifts, and functions with incommensurable values, depending only on a finite number of coordinates, there exists a partition $\xi$, very weakly Bernoulli with respect to $S_t$ for all t and generating for small t. Whence it follows that all $S_t$ are Bernoulli automorphisms. Such flows are called Bernoullian. L. A. Bunimovich [34, 36] proved that the special flows constructed from shift automorphisms in spaces realizing stationary stochastic processes, well approximated by Markov chains and by functions depending only on the past and well approximated by functions of a finite number of coordinates are Bernoullian. In this class appear the transitive $\mathscr{Y}$-flows with Gibbsian measures (cf. Secs. 3, 4 of Chap. 2). Ornstein and Weiss [588] established that the geodesic flows on compact manifolds of negative curvature are Bernoullian. Feldman and Smorodinsky [333] proved that a Bernoulli shift with finite entropy is included in a flow. This flow is generated by shifts in the realization space of a stationary Markov process with a finite set of states and an irreducible matrix of transition probabilities. Ornstein [584] proved that Bernoullian flows with the same entropy are isomorphic.

Katznelson [427] proved that for an ergodic automorphism T of a finite-dimensional torus a partition of the cube is very weakly Bernoullian, and hence, T is isomorphic with a Bernoulli shift. For an automorphism of the torus satisfying condition $\mathscr{Y}$, the latter follows from the existence of a Markovian partition (cf. Chap. 2). We note that the first result in this direction was obtained by Adler and Weiss [197]. They proved that the entropy is a complete invariant for ergodic automorphisms of the two-dimensional torus. Lind [517] proved that ergodic automorphisms of an infinite-dimensional torus are isomorphic with Bernoulli shifts.

Using the technique of [576], Adler and Shields [195] proved that the skew product U: $(x, y) \to (Tx, S_{\alpha(x_0)}y)$, where T is a Bernoulli shift with states i = 0, 1, $S_{\alpha(i)}$ is rotation of the circle by $\alpha_i$, while $(\alpha_0 - \alpha_1)$ is an irrational number, is Bernoullian. An automorphism derived from a Bernoulli shift may not be a Bernoulli shift; however, the collection of those sets for which this is so is everywhere dense in the $\sigma$-algebra of measurable sets (cf. [351, 352]). Salesky [639] gave explicit conditions for the derived shift and special automorphisms constructed from Bernoulli shifts to be Bernoullian (cf. also Paragraph 1.5, Chap. 6).

Methodical refinements of the proof of Ornstein's theorem about isomorphisms are presented in the papers of Maruyama [528], Smorodinsky [680], and Ito, Murata, and Totoki [407].

## §3. K-Automorphisms and K-Flows

3.1. A. N. Kolmogorov [100] introduced a class of transformations T, now called K-automorphisms, for which each process (T, $\xi$) is regular. This class contains the Bernoulli shifts, and also many transformations arising in applications of ergodic theory to algebra, probability theory, and mechanics.

Ornstein [581] showed by example that the class of K-automorphisms is wider than the class of Bernoullian shifts. His example is itself a shift in the space of sequences of four symbols. The invariant measure in this space is such that some sequential combinations of symbols (n-blocks) almost uniquely determine the position of the first symbol of this sequence in the full trajectory. Simplifying the construction of Ornstein, S. A. Yuzvinskii [189] proved for any completely ergodic automorphism T the existence of an ergodic automorphism R such that there exists a K-automorphism S, whose derivative is $T \times R$.

Ornstein and Shields [586] constructed a continuum of pairwise nonisomorphic K-automorphisms with the same entropy, each of which is not isomorphic with its inverse. In [582, 583] Ornstein modified his construction of a K-automorphism which is not a Bernoulli shift, and constructed a counterexample to the conjecture of Pinsker on the decomposition of any ergodic automorphism into a direct product of a K-automorphism and an automorphism with zero entropy.

3.2. A series of papers is devoted to discovering conditions under which automorphisms or flows possess the K-property. B. M. Gurevich [61] gave conditions for the existence of a K-partition for special flows, constructed from K-automorphisms and functions f, in terms of the incommensurability of its values in the discrete case and its smoothness in the case of a continuous set of values. Totoki [701] considered special automorphisms S, constructed from Bernoulli shifts and functions f, depending only on the zero coordinates. It turned out that the automorphisms S has the K-property if and only if $f \neq$ const $n(x)$, where $n(x)$ is an integer-valued function. In the proof, probability theory is used.

A transformation $Y_n = f(X_n)$ of a stationary Markov chain with finite or countable set of states is considered by Robertson [625]. He proved that a shift in the realization space of $Y_n$ has the K-property if it is completely ergodic.

It is known (cf., for example, Parry [597]) that if a stationary process $X_t$ with discrete time and a finite number of states is regular, then the process $X_t$ has the same property. In other words, the partitions $\pi_+ = \bigwedge_n \bigvee_{i=n}^{\infty} T^i\xi$ and $\pi_- = \bigwedge_n \bigvee_{i=-\infty}^{n} T^i\xi$ coincide for each finite partition $\xi$. B. M. Gurevich [66] proved that the partition $\bigwedge_n \bigvee_{|i|>n} T^i\xi$, in general, is different from $\pi_+$. Krengel [461] discovered that for processes with continuous time the equation $\pi_+ = \pi_-$ can be violated. More exactly, for each K-flow, there exists a representation by shifts in the realization space of a stationary process which is deterministic forward and absolutely indeterministic backward.

Kolmogorov automorphisms in spaces with $\sigma$-measures were considered by Dugdale [318].

## §4. Endomorphisms and Decreasing Sequences of Partitions

We shall now consider the isomorphism problem for endomorphisms of a Lebesgue space. Here there arises a new metric invariant in comparison with the case of automorphisms. Such an invariant of an endomorphism T is the sequence of partitions $\{T^{-n}\varepsilon\}$ on the pre-image of a point under the map $T^n$. V. K. Vinokurov [49] proved that this invariant is not complete in the class of strict endomorphisms.

4.1. A. M. Vershik considered the problem of metric classification of decreasing sequences of partitions $\{\xi_k\}_0^\infty$, possessing the following homogeneity property: Elements of the partition $\xi_k$ consist (mod 0) of the same number $n_k$ of points with uniform conditional measure. We shall call the numerical sequence $\{n_k\}$ the type of the sequence $\{\xi_k\}$. The basic result of A. M. Vershik is the theorem on lacunary isomorphisms: If sequences $\{\xi_k\}$ and $\{\eta_k\}$ have the same type and $\bigwedge_k \xi_k = \bigwedge_k \eta_k = \nu$, then one can find a sequence of natural numbers $\{k_i\}$, such that the sequences $\{\xi_{k_i}\}$ and $\{\eta_{k_i}\}$ are isomorphic.

It turned out that lacunary isomorphisms, in general, do not extend to isomorphisms. A. M. Vershik [42] constructed an example of two nonisomorphic homogeneous sequences of partitions and on the basis of this example obtained a negative solution to the problem of Levi–Rozenblat on the existence of a Markov shift which is not representable as a factor of a Bernoulli shift. Later, A. M. Vershik [44] and A. M. Stepin [173] independently introduced for homogeneous sequences of partitions invariants of entropy type. These invariants allowed one to distinguish a continuum of pairwise nonisomorphic sequences of partitions. The approach to the construction of metric invariants of decreasing sequences of measurable partitions proposed in [173], consists of the following. For each homogeneous decreasing sequence of partitions $\{\xi_k\}$ one can construct an action of the inductive limit G of finite groups $G_k$, such that $\xi_k$ is partition on the orbits of the group $G_k$. It is defined in a nonunique way. We denote by $\mathscr{T}(\Xi)$ the class of actions of the group G, generated by sequences $\Xi = \{\xi_k\}$. If the value of some metric invariant of an action of the group G does not coincide on the classes $\mathcal{T}(\Xi)$ and $\mathcal{T}(\Xi')$, then the sequences $\Xi$ and $\Xi'$ are not isomorphic. The character of the spectrum of the action changes strongly on the class $\mathcal{T}(\Xi)$. In contrast to this for some numbers $n_k$, bounded in growth, the entropy of the action is constant on the class $\mathcal{T}(\Xi)$, and this gives the possibility of defining the entropy of a decreasing sequence of partitions. The approach of A. M. Vershik consists of considering the group of automorphisms of the Lebesgue space, leaving the partitions $\xi_k$ invariant and some fixed partition $\eta$ fixed.

4.2. Decreasing sequences of partitions with finite positive entropy are not generated by endomorphisms (cf. [173]). S. A. Yuzvinskii [188] proved that the entropy of the sequence of partitions $\{T^{-n}\varepsilon\}$ is equal to zero if T is an endomorphism with finite entropy. A. M. Stepin [173] proved that sequences of partitions with the same entropy, induced by Bernoullian actions, are isomorphic and each decreasing sequence with positive entropy has a Bernoullian factor-sequence with the same entropy.

The problem of metric classification of inhomogeneous decreasing sequences of partitions was considered by V. G. Vinokurov and B. A. Rubshtein [50, 146]. They isolated the class of completely inhomogeneous sequences of partitions and obtained a complete description of the sequences of this class. B. A. Rubshtein [148] gave a complete classification of strict endomorphisms, generated by a given completely inhomogeneous sequence of partitions, and strengthened the result of [49], constructing a continuum of nonisomorphic strict endomorphisms, generated by one and the same homogeneous sequence of partitions. In [147] the two-point extension of an inhomogeneous diadic sequence of partitions $T^{-n}\varepsilon$, where T is a Bernoulli endomorphism is considered.

Using the technique of extension, Parry and Walters [606] constructed an example of two nonisomorphic strict endomorphisms: T and S, for which the sequences $\{T^{-n}\varepsilon\}$, $\{S^{-n}\varepsilon\}$ are isomorphic and $T^2 = S^2$. For an extension of this paper, cf. [712].

The problem of metric classification of endomorphisms of a Lebesgue space can be considered as a problem of classifying automorphisms with respect to a coding, independent of the future. From this point of view the classification was considered by Parry and Walters [606, 712].

V. G. Vinokurov and V. K. Tsipuridu [52] isolated a class of semigroup endomorphisms, for which a complete invariant is given in a natural way. In [498] a class of Markov endomorphisms is described where the conjugacy problem is completely solved.

The question of how one can construct partitions of the form $T^{-1}\varepsilon$, where $\varepsilon$ is a strict endomorphism is studied in the papers of V. G. Sharapov [181, 182]. He proved that a partition with discrete elements, for which the functions of conditional measure are piecewise constant, has the form $T^{-1}\varepsilon$. In the general case the solution of this probelm was obtained by A. M. Vershik in his doctoral dissertation "Approximation in Measure Theory" (Leningrad University, 1974).

4.3. A series of papers is devoted to the conjugacy problem for number-theoretic endomorphisms and their generalizations. We note that one can speak of the property of a partition $\xi$ being weakly Bernoullian with respect to an endomorphism T. If $\xi$ is a weakly Bernoullian generating partition, then a natural extension of the endomorphism T is the Bernoulli shift. In [60], [681] under some conditions on the function f it is proved that the endomorphism $T_f\omega = f(\omega) - [f(\omega)]$ is weakly Bernoullian. Smorodinsky [682] and Takahashi [692] proved that the endomorphism $T_\beta : x \to \{\beta x\}$, $\beta > 1$, has a weakly Bernoullian generating partition. Adler, in the survey [193] established that the last result actually follows from results of Rényi and Deblin.

In [497] it is proved that the endomorphisms $T_\beta$ and $T_{\beta,\alpha}: x \to \{\beta x + \alpha\}$ $(\beta \geqslant 2,\ 0 \leqslant \alpha < 1)$ are not Bernoulli endomorphisms in the case of nonintegral $\beta$. If $\beta$ is an integer, then the transformations $T_\beta$ and $T_{\alpha,\beta}$ are Bernoulli endomorphisms. Takahashi [691] gave an explicit construction of an isomorphism between a $\beta$-automorphism and a mixing Markov shift.

In [496] conditions are found under which a Markovian endomorphism is isomorphic with a Bernoulli endomorphism. N. N. Ganikhodzhaev [54] proved that some power of an expanding group endomorphism of a two-dimensional torus is isomorphic with a Bernoulli endomorphism with equiprobable states.

The ergodic properties of number-theoretic endomorphisms and also their multidimensional and complex analogues were studied in [661, 335, 336, 716]. The multidimensional central limit theorem for number-theoretic endomorphisms was obtained by Dubrovin [74]. In [52, 252, 708] the question of the existence of roots of endomorphisms is considered.

# CHAPTER 2

# ERGODIC THEORY OF DYNAMICAL SYSTEMS OF HYPERBOLIC TYPE

The title of this chapter can serve as an approximate definition for a rich, but not entirely precisely drawn circle of ideas and results, determined over the last 10-15 years. The subject of the theory is continuous and smooth dynamical systems, some of whose trajectories behave asymptotically in an unstable fashion, while this instability in some sense or other is "exponential" in time. Such dynamical systems are important both from the point of view of the general theory (they form an open set in many natural spaces of dynamical systems), and for applications, because they arise in many concrete problems of varied origins — from the theory of numbers to celestial mechanics.

A central place in the circle of questions considered is occupied by the theory of Gibbsian measures for two of the most important classes of dynamical systems — topological Markov chains (Sec. 3) and locally maximal hyperbolic sets of smooth dynamical systems (Secs. 2, 4). The theory of Markovian partitions (Sec. 4) allows one to establish a close connection between these classes and to carry over results from the symbolic case to the smooth. The theory of Gibbsian measures allows one to look in a new way at the question of metric properties of $\mathcal{Y}$-systems with smooth invariant measures [9, 16]. Contiguous to the theory of Gibbsian measures and partially following from it are results on the asymptotic behavior of smooth measures in a neighborhood of a hyperbolic attracting set and on necessary and sufficient conditions for the existence of an absolutely continuous invariant measure for a $\mathcal{Y}$-system (Sec. 6).

At the same time, the indefiniteness of the concept of "dynamical systems of hyperbolic type" should be noted. First of all, the basic results of the theory of Gibbsian measures can be obtained, starting only from a series of axiomatic properties of dynamical systems in which exponential instability does not figure explicitly (Sec. 5). Secondly, the approach, lying at the foundation of the theory of Gibbsian measure, turns out to be partially applicable to arbitrary dynamical systems. We have placed a survey of the corresponding results in Sec. 1 of the present chapter, although here no "hyperbolicity" is implied. Finally, there exists a series of interesting problems in which one has a weaker "hyperbolic" property than that from which the axioms of Sec. 5 follow. Roughly speaking, in these problems the property of exponential instability is either completely nonuniform in time and not for all but only for almost all points (Sec. 8), or one has this instability not in all directions (Sec. 9). We note that for now in the study of ergodic properties with respect to smooth invariant measures in these cases one does not use "Gibbsian methods," but techniques based on the concept of absolutely continuous fibers, generalizing the initial methods of study of $\mathcal{Y}$-systems [9].

## §1. Topological Entropy and the Variational Principle for Dynamical Systems

Ergodic theory applies to the study of continuous and smooth dynamical systems in two ways. First of all, it is used to study the behavior of almost all trajectories with respect to some "good" invariant measure, whose existence is postulated or follows directly from properties of the class of dynamical systems considered (for example, the measure induced by the phase volume in classical mechanical systems). The other direction is the description of some properties of all collections of invariant measures of dynamical systems, finding in this collection of measures those having some remarkable properties. In this paragraph we set forth some general results on properties of collections of invariant measures for continuous dynamical systems.

1.1. Let X be a compact metric space, $f: X \to X$ be a continuous map, U be a finite covering of X by open sets, $U^n$ be the covering whose elements are the nonempty intersections of the form $U_0 \cap f^{-1}U_1 \cap \cdots \cap f^{-n+1}U_{n-1}$, where $U_0, \ldots, U_{n-1} \in U$, and $f^{-k}A$ denotes the complete preimage of the set A.

We denote by $Z_n(U, f)$ the minimal number of elements in a subcovering of the covering $U^n$. Then there exists a finite limit $h(U, f) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(U, f)$. The topological entropy h(f) of the map f is the least upper bound (finite or equal to $+\infty$) of the quantities h(U, f) for all finite coverings U. Let $U^{(n)}$ be a sequence of finite coverings such that the maximal diameter of elements of $U^{(n)}$ tends to zero as $n \to \infty$. Then

$$h(f) = \lim_{n\to\infty} h(U^{(n)}, f).$$

The concept of topological entropy was introduced by Adler, Konheim, and McAndrew in 1965 [194] by analogy with the metric entropy of a transformation with invariant measure.

Goodwyn [366] (cf. also [368]) proved that for any Borel invariant measure the metric entropy does not exceed the topological entropy h(f). A simpler proof of this fact was given later by Denker [310]. E. I. Dinaburg [72, 73], in the case when the topological dimension of X is finite and f is a homeomorphism, proved that h(f) is equal to the least upper bound of $h_\mu(f)$ over all Borel invariant measures $\mu$.

Finally, Goodman [363] proved the equation $h(f) = \sup h_\mu(f)$ without any restriction on the compactum X (cf. also [369]). In connection with these results there is natural interest in the question of the existence and uniqueness of an invariant measure with maximal entropy, i.e., a measure $\mu$, such that $h_\mu(f) = h(f)$. Such a measure does not always exist. An example of a homeomorphism that does not have a measure with maximal entropy was contructed by B. M. Gurevich [63], and the corresponding example of a diffeomorphism of a compact manifold by Misiurewicz [541]. In the latter paper it is also shown that the topological entropy as a function on the space of diffeomorphisms of the compact manifold with the $C^r$ topology is not upper semicontinuous.

However, if the homeomorphism f has the property of separating trajectories, i.e., one can find an $\varepsilon_0$ such that for any two distinct points $x, y \in X$ one can find a integer N for which the distance from $f^N x$ to $f^N y$ is greater than $\varepsilon_0$ then an invariant measure with maximal entropy exists (T. Goodman [346]). A measure with maximal entropy may not be unique even for a topologically transitive system. Examples of this kind with positive topological entropy were found for example by M. S. Shtil'man [185] and I. P. Kornfel'd [101]. We shall speak about those situations in which a measure with maximal entropy exists and is unique in Secs. 3, 4, and 5. We still note that for a $C^1$-diffeomorphism f of a compact n-dimensional manifold M the topological entropy is finite and admits the upper bound

$$h(f) \leqslant n \ln (\max_{x \in M} |\det Df_x|),$$

where the norm is induced by an arbitrary Riemannian metric on M [405]. This result is analogous to the upper bound for the metric entropy with respect to a smooth invariant measure, which was proved earlier by A. G. Kushnirenko [112]. Some other results, connected with topological entropy are in the papers [30, 63, 71, 75, 123, 260-262, 367, 722].

1.2. The construction and extremal properties of topological entropy admit a generalization, expansive statistical mechanics.

1.2.1. Let $\varphi$ be a continuous function on X and $Z_n(U, f, \varphi)$ be equal to the greatest lower bound over all subcoverings $\Gamma$ of the covering $U^n$ of the quantity

$$\sum_{U \in \Gamma} \sup_{x \in U} \exp\left(\sum_{k=0}^{n-1} \varphi(f^k(x))\right).$$

Then there exists a finite limit

$$P(f, \varphi, U) = \lim_{n\to\infty} \frac{1}{n} \lg Z_n(f, \varphi, U).$$

The quantity $P(f, \varphi) = \sup_U P(f, \varphi, U) = \lim_{\operatorname{diam} U \to 0} P(f, \varphi, U)$ is called the topological pressure (although from the point of view of the analogy with statistical mechanics it would be more proper to call it the free energy).

Obviously, h(f) = P(f, 0). The results of Goodwyn, Dinaburg, and Goodman generalize in the following way:

1.2.2. $P(f, \varphi) = \sup_\mu \left(h_\mu(f) + \int \varphi d\mu\right).$

The least upper bound, as before, is taken over the set of all Borel invariant measures of the map f.

The assertion 1.2.2 is called the variational principle for the topological pressure.

The idea of the approach expounded in this point belongs to Ruelle. In [634, 635] Ruelle defined the pressure for homeomorphisms with the property of separating trajectories somewhat differently than this was done above (a similar construction will be set forth in Sec. 5) and proved the variational principle in this case, and also the existence of a measure μ, for which

$$h_\mu(f) + \int \varphi d\mu = P(f, \varphi).$$

The definition of pressure and the proof of the variational principle in the general case was given by Walters [174]. Still another proof, close to the ideas of Sec. 5, is due to Misiurewicz.

1.2.3. Let $f_t$ be a continuous flow on X, i.e., a one-parameter group of homeomorphisms of X. For continuous function $\varphi$, we write $\varphi^1(x) = \int_0^1 \varphi(f_t x)\,dt$ and we define the topological pressure $P(f_t, \varphi)$ of the flow $f_t$ with respect to the function $\varphi$ as $P(f_1, \varphi^1)$. The definition of topological pressure in the form of [634] carries over immediately to the case of continuous time and coincides with that given in [270].

The variational principle 1.2.2 carries over in a natural way to the case of flows. Namely (cf. [270]):

$$P(\varphi, f_t) = \sup_\mu (h_\mu(f_1) + \int \varphi d\mu).$$

## §2. Hyperbolic Sets

2.1. Let M be a smooth (of class $C^\infty$) manifold, $U \subset M$ be open subset, $f: U \to M$ be a map of class $C^1$, mapping the set U diffeomorphically onto f(U). For brevity, we shall call f simply a diffeomorphism. In a series of cases one has to require somewhat more smoothness of f: usually $C^2$ or even $C^{1+\delta}$, where $\delta > 0$ is sufficient. A subset $\Lambda \subset U$ is called invariant with respect to f, if Λ belongs to the domain of definition of $f^{-1}$ and from $x \in \Lambda$ it follows that $f(x) \in \Lambda$, $f^{-1}(x) \in \Lambda$. The differential Df acts on the tangent bundle $Df: TU \to TM$, or (pointwise for $x \in U$), $Df_x: T_xM \to T_{f(x)}M$. An invariant compact set Λ of the diffeomorphism $f: U \to M$ is called a hyperbolic set of f, if at each point $x \in \Lambda$ the tangent space $T_xM$ splits into the direct sum of subspaces $E^s_x$ and $E^u_x$, with the following properties:

2.1.1. The dimensions of the spaces $E^s_x$ and $E^u_x$ depend continuously on x.

2.1.2. $Df_x E^s_x = E^s_{f(x)}$, $Df_x E^u_x = E^u_{f(x)}$.

2.1.3. There exist a Riemannian metric on M and constants c, λ, $c > 0$, $0 < \lambda < 1$, such that for any natural number n and any $x \in \Lambda$, $u \in E^s_x$, $v \in E^u_x$

$$||Df_x^n u|| \leqslant c\lambda^n ||u||, \quad ||Df_x^{-n} v|| \leqslant c\lambda^n ||v||,$$

where the norm of a tangent vector is defined by the Riemannian metric.

The corresponding definition for the case of continuous time looks like this: Let X be a vector field of class $C^1$, defined on an open subset U of the smooth manifold M; $f_t$, $t \in R$, is the map of translation in time t along trajectories of the vector field X. A compact set $\Lambda \subset U$ is called a hyperbolic set of the vector field X if for each point $x \in \Lambda$ and each $t \in R$, $f_t x$ is defined and belongs to $\Lambda$, $X(x) \neq 0$ and the tangent space $T_x M$ splits into a direct sum

$$T_x M = E_x^0 + E_x^s + E_x^u,$$

where $E^0$ is the one-dimensional subspace generated by the vector X(x), and for $E_x^s$ and $E_x^u$ the conditions 1.1.1, 2.1.2 are satisfied.

For $0 \leqslant t < 1$

$$(Df_t)_x E_x^s = E_{f_t x}^s, \quad (Df_t)_x E_x^u = E_{f_t x}^u$$

and 2.1.3 is satisfied with the natural number n replaced by the real number t > 0.

If 2.1.3 is satisfied for one Riemannian metric, then it is satisfied for any other Riemannian metric, possibly with other coefficients c and $\lambda$. Moreover, one can always choose a Riemannian metric for which c = 1. Such a metric is called Lyapunovian.

2.1.4. The subspaces $E_x^s$ and $E_x^u$ depend on the point $x \in \Lambda$ continuously. Moreover, if the diffeomorphism or the vector field belong to the class $C^{1-\varepsilon}$, $\varepsilon > 0$, then these subspaces satisfy a Hölder condition in the following sense. The Riemannian metric induces for any k, $0 \leqslant k \leqslant n$, a metric in the Grassman bundle $G^k(TM)$ of k-dimensional tangent planes to M (this metric is defined by the stationary angle between subspaces). The maps associating with a point x the subspaces $E_x^s$ and $E_x^u$, satisfy a Hölder condition with exponent $\delta(\varepsilon) > 0$ with respect to this metric.

2.2. If the compact manifold M is a hyperbolic set of the diffeomorphism $f: M \to M$ (respectively, of the flow $f_t$, generated by the vector field X), then f (respectively $f_t$) is called a $\mathscr{Y}$-diffeomorphism, or an Anosov diffeomorphism (respectively, a $\mathscr{Y}$-flow, or an Anosovian flow).

All known examples of $\mathscr{Y}$-diffeomorphims act on manifolds of quite special form — tori, nilmanifolds and some of their generalizations (infranilmanifolds) — and are topologically conjugate to diffeomorphims generated by some group automorphisms of the universal covering spaces.

Classical examples of $\mathscr{Y}$-flows are the geodesic flows on manifolds of negative curvature. A hyperbolic set of a vector field remains such under a smooth, nonvanishing change of time. Finally, each $\mathscr{Y}$-diffeomorphism can be made to correspond to a $\mathscr{Y}$-flow with the help of the suspension construction. The complete topological classification of $\mathscr{Y}$-diffeomorphisms and $\mathscr{Y}$-flows is unsolved, and evidently is a quite difficult problem.

There is also interest in the class of irreversible maps, analogous to $\mathscr{Y}$-diffeomorphisms. Namely, a regular map $f: M \to M$ is called an expanding endomorphism if one can find $c > 0, \lambda > 1$ and a Riemannian metric on M such that for any natural number n, $u \in TM$,

$$||Df^n u|| \geqslant c\Lambda^n ||u||.$$

Expanding endomorphisms are in many ways simpler than $\mathcal{Y}$-diffeomorphisms and many important results, for example the existence of Markovian partitions (cf. Sec. 4 of this Chapter), become quite transparent in this case.

A hyperbolic set $\Lambda$ of a diffeomorphism or a vector field is called locally maximal (basic in the terminology of R. Bowen), if one can find an open neighborhood $V\supset\Lambda$, such that $\Lambda=\bigcap_{n\in\mathbf{Z}} f^nV$. In other words, $\Lambda$ is a maximal f-invariant closed set, entirely lying in some open subset of M.

In contrast to $\mathcal{Y}$-systems, of which, in essence, there are known not to be too many (up to topological equivalence), locally maximal hyperbolic sets occur in quite a variety and one meets them in many problems.

2.3. The definition of $\mathcal{Y}$-systems was introduced by D. V. Anosov (cf. [48]) in 1962, starting from properties of the variational equations for geodesic flows on manifolds of negative curvature, which were noted and used in papers of Madamard, Cartan, Morse, Hedlund, and Hopf. Besides the classical papers of these authors, a large role in the development of the theory was played by Thom's conjecture on the structural stability of hyperbolic automorphisms of the torus and especially by the nontrivial example of a hyperbolic set constructed by Smale [170, 676] in 1961 ("horseshoe"). The general definition of hyperbolic set is also due to Smale [675]. Smale considered an important class of dynamical systems on compact manifolds, distinguished by the so-called "axiom A": The set of nonwandering points is hyperbolic and coincides with the closure of the set of periodic points. The definition of a locally maximal set was introduced somewhat later by D. V. Anosov [10]. Expanding endomorphisms were first considered from the point of view of dynamical systems by Shub [664]. The Hölder condition for the distribution $E^s$ and $E^u$ was also proved by D. V. Anosov [9, 11] (cf. also [36]). Rather important examples of locally maximal hyperbolic sets in dynamical systems, connected with the problems of celestial mechanics, were constructed in a series of papers of V. M. Alekseev [2-8]. Consult also the book of Moser [544]. Essential progress in the problem of the topological classification of $\mathcal{Y}$-systems was achieved by Franks [348, 349], Newhouse [556], and Manning [522].

2.4. We shall not, in the present survey, systematically illuminate subsequent progress in the study of hyperbolic sets, since a fundamental role here is played by questions relating to the topological theory of smooth dynamical systems (differential dynamics) and not by ergodic theory. A bibliography and historical remarks can be found in [9], the survey of Smale [675], the lectures of A. B. Katok [86] and V. M. Alekseev [8], and the recent survey of Shub [665]. We shall restrict ourselves to the enumeration of those topological properties of hyperbolic sets which turn out to be useful in the study of properties of invariant measures, concentrated on these sets.

2.4.1. In the interest of brevity, we shall formulate this property only for the case of discrete time. There exists a neighborhood V of the hyperbolic set $\Lambda$ of the diffeomorphism f, such that for any $\delta>0$ one can find an $\varepsilon>0$, for which from

$$x_0,\ x_1,\dots,x_{n-1},\ x_n=x_0\in V,\ d(f(x_i),\ x_{i+1})<\varepsilon,\ i=0,\dots,n-1,$$

follows the existence of a point x', for which

$$f^nx'=x',\ d(f^ix',\ x_i)<\delta,\ i=0,\dots,n-1.$$

In particular, if $\Lambda$ is a locally maximal set, then the neighborhood V can be chosen so that the point x' will belong to $\Lambda$.

2.4.2. The restriction of the diffeomorphism or the flow to a hyperbolic set has the property of separating trajectories.

2.4.3. The periodic points are dense in the set $\Omega(f|\Lambda)$ of nonwandering points of the restriction of the diffeomorphism f to $\Lambda$.

The periodic trajectories are dense in the set $\Omega(f_t|\Lambda)$ of nonwandering points of the flow $f_t$, acting on $\Lambda$.

2.4.4. Let $\Lambda$ be a locally maximal hyperbolic set of the diffeomorphism f or the flow $f_t$. The set $\Omega(f|\Lambda)$ [or $\Omega(f_t|\Lambda)$] can be represented as a union of a finite number of pairwise disjoint subsets $\Omega_1,\ldots,\Omega_k$, on each of which f (or $f_t$) is topologically transitive, i.e., has an everywhere dense trajectory.

2.4.5. In the case of diffeomorphism, each of the sets $\Omega_i$ can be represented as a union of pairwise disjoint subsets

$$\Omega_i=\Omega_i^1\cup\ldots\cup\Omega_i^{n_i}$$

where $f\Omega_i^j=\Omega_i^{j+1}$, $j=1,\ldots,n_i-1, \Omega_i^{n_i+1}=\Omega_i^1$ and the restriction $f^{n_i}|\Omega$ has the property of mixing domains. The latter property for the homeomorphism $f:X\to X$ means that for any open sets $U, V\subset X$ one can find an N such that for $n\geqslant N$ one can find a point $x\in U$ such that $f^n_x\in V$.

2.4.6. If the restriction $f|\Lambda$ has the property of mixing domains, then from 2.4.1 follows the following specification property: For any $\delta>0$ one can find a natural number $p=p(\delta)$ with the following property: If $a_1,\ldots,a_n$ are integers, $a_{i+1}>a_i+p$, $i=1,\ldots,n-1$, then for any $x_1,\ldots,x_{n-1}\in X$ one can find a point $x\in X$ such that $f^{a_n-a_1}x=x$ and $d(f^{a_i+k}x, f^kx_i)<\delta$, $k=0, 1,\ldots,a_{i+1}-a_i-p$, $i=1,\ldots,n-1$.

In other words, the specification means that any finite collection of finite segments of trajectories can be approximated by periodic trajectories so that the size of the "lacunae" will depend only on the exactness of the approximation. We shall omit the precise formulation for the case of continuous time.

2.5. Properties 2.4.1 and 2.4.2 for $\mathscr{Y}$-systems were proved by D. V. Anosov [9]. In the general case, both of these properties are special cases of Anosov's theorem "on families of ε-trajectories" [10, 86]. However the property of separating trajectories for hyperbolic sets first appeared in Smale [675]. There too the spectral decomposition theorem, which is the special case of 2.4.4 when f is a diffeomorphism satisfying axiom A and $\Lambda = \Omega(f)$, was proved. Properties 2.4.3, 2.4.4, 2.4.5 for locally maximal hyperbolic sets appeared in [10]. The specification was introduced for diffeomorphisms [259] and flows [263] by Bowen.

2.6. The stable and unstable spaces $E^s_x$ and $E^u_x$ from the definition of hyperbolic sets can be "omitted" from the tangent bundle of the manifold M. This fact, which carries the appellation "stable manifold theorem" is a natural generalization to the case of hyperbolic sets of the classical theorem of Hadamard–Perron on stable and unstable manifolds of hyperbolic fixed points. We shall present here one of the variants of the formulation of the stable manifold theorem [86].

2.6.1. Let $\Lambda$ be a hyperbolic set of a $C^1$-diffeomorphism $f:U\to M$, $\rho$ be the metric on M induced by a Lyapunovian Riemannian metric. One can find an $\varepsilon_0>0$, such that for $0<\varepsilon<\varepsilon_0$ and $x\in\Lambda$ there exists a map $\varphi_{x,\varepsilon}$ of the ball $\mathbf{D}^{k(x)}$ into M [$k(x) = \dim E^s_x$] with the following property: $\varphi_{x,\varepsilon}$ is a $C^1$-imbedding of $\mathbf{D}^{k(x)}$ into M; $T_xW^s_\varepsilon(x)=E^s_x$, where $W^s_\varepsilon(x)=\operatorname{Im}\varphi_{x,\varepsilon}$, $fW^s_\varepsilon(x)\subset\subset W^s_\varepsilon(fx)$; if $y\in W^s_\varepsilon(x)$, then $\rho(f^nx, f^ny)\to 0$, as $n\to\infty$; $W^s_\varepsilon(x)=\{y\in M:\rho(f^nx, f^ny)<\varepsilon;\ n\geqslant 0\}$; the correspondence $x\mapsto\varphi_{x,\varepsilon}$ determines a continuous map of $\Lambda$ into the space $C^1(\mathbf{D}^{k(x)}, M)$.

The sets $W^s_\varepsilon(x)$ are called the local stable manifolds of the point x, and the analogous sets constructed for the diffeomorphism $f^{-1}$, are denoted by $W^u_\varepsilon(x)$ and are called the local unstable manifolds of x.

By the global stable and unstable manifolds of a point $x\in\Lambda$ are meant the sets

$$W^s(x)=\bigcup_{n\geqslant 0}f^{-n}W^s_\varepsilon(f^nx),\quad W^u(x)=\bigcup_{n\geqslant 0}f^nW^u_\varepsilon(f^{-n}x).$$

Each of these manifolds is the image of a Euclidean space under a $C^1$-immersion, which, as a rule, is not an imbedding.

2.6.2. For any sufficiently small $\delta > 0$, one can find a $\varepsilon(\delta) > 0$ such that for any points $x, y \in \Lambda$, $\rho(x, y) < \varepsilon(\delta)$, the intersection $W^s_\delta(x) \cap W^u_\delta(y)$ is nonempty and consists of a unique point, which we denote by $r_\delta(x, y)$. The property of local maximality of a hyperbolic set is equivalent with the following property, which is called "the local product structure." One can find a $\delta > 0$, such that for $x, y \in \Lambda$, $\rho(x, y) < \varepsilon(\delta)$, it follows that $W^s_\delta(x) \cap W^u_\delta(y) \in \Lambda$. We shall give an almost obvious reformulation of this property: Some neighborhood $U_x$ of the point $x \in \Lambda$ in the set $\Lambda$ has the structure of the canonical direct product of $U_x \cap W^s_\delta(x)$ and $U_x \cap W^u_\delta(x)$.

The literature relating to stable and unstable manifolds is sufficiently extensive, but not in relation to the subject of our survey. We note that the stable manifold theorem for $\mathscr{Y}$-systems was proved by Anosov (cf. [9]), for hyperbolic sets the formulation is due to Smale [675], the first published proof is that of Hirsch and Pugh [398]. We have borrowed the formulation from [86]. The results of Paragraph 2.6.2 are proved in [86, 397].

## §3. Topological Markov Chains and Gibbsian Measures

3.1. We consider an alphabet of k symbols $\mathfrak{A} = \{a_1, \ldots, a_k\}$ and we construct the space $\Sigma_k$ of sequences $\omega = \{\omega_s\}_{-\infty}^{\infty}$, infinite in both directions, where each coordinate $\omega_s$, $s \in \mathbf{Z}$, assumes values from the alphabet $\mathfrak{A}$. In the space $\Sigma_k$ there is a natural topology as the Tikhonov product of a countable set of copies of the k-point space $\mathfrak{A}$. In this topology the space $\Sigma_k$ is compact. It is convenient to introduce in $\Sigma_k$ a family of metrics $\rho_\lambda$, $0 < \lambda < 1$, each of which generates the Tikhonov topology. Namely, we set $d(a_i, a_j) = \delta_{i,j}$, $i, j = 1, \ldots, k$, and for $\omega^1, \omega^2 \in \Sigma_k$ we define the distance

$$\rho_\lambda(\omega^1, \omega^2) = \sum_{-\infty}^{\infty} \lambda^{|s|} d(\omega^1_s, \omega^2_s).$$

In the space $\Sigma_k$, the shift transformation $\sigma_k$, $(\sigma_k \omega)_s = \omega_{s+1}$, $\omega \in \Sigma_k$, $s \in \mathbf{Z}$, acts as a homeomorphism. This transformation is sometimes called a topological Bernoulli scheme with k constants.

The restrictions of the homeomorphism $\sigma_k$ to closed invariant subsets of the space $\Sigma_k$ generate a rich and interesting class of dynamical systems, which are sometimes called symbolic maps. The domain of the theory of dynamical systems devoted to symbolic maps is called symbolic dynamics. In a known sense, symbolic maps serve as models for arbitrary dynamical systems both in topological dynamics as well as in ergodic theory (cf. Chap. 4). In "hyperbolic" theory the role of such models is played by one special class of symbolic maps. Let $A = \|a_{ij}\|$ be a square matrix of the k-th order, whose elements $a_{ij}$ assume the values 1 and 0, while in each column there is only one unit. We write

$$\Sigma_A = \{\omega \in \Sigma_k : a_{\omega_{s-1}\omega_s} = 1, \quad s \in \mathbf{Z}\}.$$

Obviously, $\Sigma_A$ is a closed subset of $\Sigma_k$, , invariant with respect to the shift $\sigma_k$. We write $\sigma_A = \sigma_k / \Sigma_A$. The transformation $\sigma_A$ in the Russian literature is usually called a topological Markov chain (abbreviated as t.m.c.), the term "subshift of finite type" is applied. The elements of the alphabet $\mathfrak{A}$ are usually called states of the t.m.c.

3.1.1. For the states of a t.m.c. one has a classification, analogous to the classification of states of Markov chains in probability theory. Let $A^n=\|a_{ij}^n\|$. The state $a_i$ is called recurrent if for some n, $a_{ii}^n>0$.

We shall call recurrent states $a_i$ and $a_j$ equivalent, if for some n, $a_{ij}^n>0$, $a_{ji}^n>0$. With the help of the concepts of recurrence and equivalence, one can formulate and prove for t.m.c. a "symbolic" analogue of the theorem of 2.4.4 (cf. [7]).

3.1.2. For ergodic theory one can restrict oneself to the case when all states are recurrent and equivalent to one another, which is equivalent to topological transitivity.

If for some n, $A^n$ is a matrix with positive elements, then $\Sigma_A$ has the property of mixing domains (the converse is obvious). In this case the t.m.c. is called transitive and for it one has the property of specification of 2.4.6. In the general case there is a partition $\mathfrak{A}=\mathfrak{A}_1\cup\ldots\cup\mathfrak{A}_m$, such that the corresponding partition

$$\Sigma_A=\Sigma^1\cup\ldots\cup\Sigma^m,\ \Sigma^0=\Sigma^m,$$

where

$$\Sigma^i=\{\omega\in\Sigma_A,\ \omega_0\in\mathfrak{A}_i\},$$

has the following property:

$$\sigma_A\Sigma^i=\Sigma^{i+1},\quad i=1,\ \ldots,\ m,$$

and the restriction of $\sigma_A^m$ to each of the sets $\Sigma^i$ has the property of mixing domains. This is an analogue of the property of 2.4.5.

3.1.3. A natural generalization of the sets $\Sigma_A$ is a subset of the space $\Sigma_k$, , defined by restriction to a fixed finite number of steps. Let B be a subset of $\mathfrak{A}^l$ and $\Sigma_B=\{x\in\Sigma_k:(\omega_s,\ \ldots,\ \omega_{s+l-1})\in B,\ s\in\mathbf{Z}\}$. However, actually, such a generalization does not give anything new, because passing from the alphabet $\mathfrak{A}$ to the alphabet $\mathfrak{A}^{l-1}$, it is easy to represent the restriction $\sigma_k|_B$ as a t.m.c. with $k^{l-1}$ states.

Weiss [721] considered the minimal class of symbolic systems containing t.m.c. and closed with respect to passage to factor-systems. This class, which Weiss called sofic systems, does not reduce to t.m.c., but from the point of view of ergodic theory has just as nice properties.

3.2. There is a rather close connection between topological Markov chains and locally maximal hyperbolic sets of diffeomorphisms.

3.2.1. For points $\omega\in\Sigma_A$ one defines an analogue of the local stable manifold:

$$W_{\text{loc}}^s(\omega)=\{\omega'=(\ldots\,\omega'_{-1}\omega'_0\omega'_1\,\ldots),\ \omega'_k=\omega_k,\ k\geqslant 0\}.$$

3.2.2. For any matrix A of zeros and ones and any manifold M, $\dim M>1$, one can give an open set $U\subset M$ and a diffeomorphism $f:U\to M$ such that the set $\Lambda=\bigcap_{n\in Z}f^nU$ is hyperbolic and the restriction $f|\Lambda$ is topologically conjugate to the t.m.c. $\sigma_A$ (generalization of the "horseshoe" construction of Smale; cf. in this connection [8, 265]).

3.2.3. Any locally maximal invariant subset $\Lambda$ of the shift $\sigma_k$ has the form $\Sigma_B$ from Paragraph 3.1.3 and hence, the restriction $\sigma_k|\Lambda$ is topologically conjugate to some t.m.c.

3.2.4. If $\Lambda$ is a zero-dimensional locally maximal hyperbolic set of the diffeomorphism f, then the restriction $f|\Lambda$ is topologically conjugate to some t.m.c. [265]. A more profound connection between locally maximal hyperbolic sets and t.m.c. is given by the theory of Markov partitions (cf. Sec. 4).

3.3. Let $N_r$ be the number of periodic points of $\sigma_A$ of period r. Obviously, $N_r=\operatorname{tr} A^r$. In the transitive case we can use the known theorem of Frobenius, according to which the matrix A has a unique eigenvector $e = (e_1, \ldots, e_k)$ with positive components and positive eigenvalue $\lambda$. It is easy to show that $\lim\limits_{r\to\infty}\frac{\ln N_r}{r}=\lim\limits_{r\to\infty}\operatorname{tr}\frac{A^r}{r}=\ln\lambda$. It turns out that the periodic points in some sense uniformly fill the space $\Sigma_A$.

The confirmation of this is the following theorem: Let $f(\omega)$ be a continuous function on the space $\Sigma_A$. There exists a measure $\mu_0$ on $\Sigma_A$, such that

3.3.1. $\lim \frac{1}{N_r}\sum\limits_{\omega_i:\sigma_A^r\omega=\omega} f(\omega)=\int f(\omega)\,d\mu_0$.

We shall describe the measure $\mu_0$. Let $e=(e_1, \ldots, e_k)$ be the above-mentioned eigenvector of the matrix A, i.e., $\sum\limits_j a_{ij}e_j=\lambda e_i$. We consider the homogeneous Markov chain in the space $\Sigma_A$ with transition probabilities $p_{ij}=\frac{a_{ij}e_j}{\lambda e_i}$. By virtue of the transitivity, for these transition probabilities the stationary distribution is unique. Then $\mu_0$ is the measure in the space $\Sigma_A$, corresponding to the stationary Markov chain with transition probabilities $p_{ij}$. This measure also has other remarkable properties.

3.4. We denote by $M_n(A)$ the number of distinct words of the alphabet $\mathfrak{A}$ of length n of the form $(\omega_0,\ldots,\omega_n)$, where $a_{\omega_i\omega_{i+1}}=1$, $i=0,\ldots,n-1$. From the definition of $\Sigma_A$ it follows that $M_n(A)=\sum\limits_{i,j=1}^{k} a_{ij}^n$. From the already mentioned theorem of Frobenius it follows that $\lim\frac{\ln a_{ij}^n}{n}=\ln\lambda$ for any i, j; and consequently, $\lim\frac{\ln M_n(A)}{n}=\ln\lambda$. It follows from the form of of the measure that $h_{\mu_0}(\sigma_A)=\ln\lambda$, and from the definition of $M_n(A)$ it is easy to deduce that for any invariant with respect to $\sigma_A$ Borelian measure $\mu$, $h_\mu(\sigma_A)\leqslant\ln\lambda$ and, on the other hand, $h(\sigma_A)=\ln\lambda$. Thus, $\mu_0$ is a measure with maximal entropy for $\sigma_A$. Parry [594] proved that for a transitive t.m.c., $\mu_0$ is the unique measure with maximal entropy. Another proof of the theorem of Parry was given by Adler and Weiss [197].

3.5. Gibbsian measure in the space $\Sigma_A$. Some mathematical methods of statistical mechanics can be applied to obtain other invariant measures for the shift $\sigma_A$ in the space $\Sigma_A$ [671, 164]. The space $\Sigma_A$ will be represented as the space of states of a one-dimensional lattice model of statistical mechanics. Each point $s\in\mathbf{Z}$ can be found in one of k states, while two states: $\omega_{s-1}$, $\omega_s$, can be met by a row only if $a_{\omega_{s-1},\omega_s}=1$.

We denote by $\mathcal{F}_A$ the class of continuous functions on $\Sigma_A$, satisfying a Hölder condition with some exponent in some metric $\rho_\lambda$ (cf. Paragraph 3.1). It is easy to see that this class is independent of $\lambda$. We fix a function $\varphi\in\mathcal{F}_A$.

From the point of view of statistical physics the function $\varphi(\omega)$ plays the role of the potential of interaction of the point 0 with all other points of the lattice. Having the function $\varphi$, we define the energy of the word $\omega_{-n},\ldots,\omega_0,\ldots,\omega_n$ for fixed coordinates $\omega_s$, $|s|>n$, as $H(\omega_{-n},\ldots,\omega_n)=\sum_{k=-n}^{n}\varphi(\sigma_A^k\omega)$. In statistical mechanics the Gibbs distribution is constructed in the form $\frac{e^{-H}}{\Xi}$, where $\Xi$ is a normalizing factor. We shall also deal in our case with a given probability distribution on the words $\omega'_{-n},\ldots,\omega_n$ for fixed states $\omega_s$, $|s|>n$, in the form

$$\mu^{(n)}(\omega_{-n},\ldots,\omega_n|\omega_s,|s|>n)=\frac{e^{-H(\omega_{-n},\ldots,\omega_n)}}{\Xi_n},$$

where $\Xi_n$ is a normalizing factor. We fix remote coordinates equivalent to given boundary conditions. The probability distribution $\mu^{(n)}$ as $n\to\infty$ converges weakly to a limit, which is independent of the fixed boundary conditions. This limit is an invariant measure for the t.m.c. $\sigma_A$, denoted by $\mu_\varphi$ and called the Gibbsian measure constructed from the function $\varphi$.

The shift $\sigma_A$, considered as an automorphism of the space $\Sigma_A$ with the measure $\mu_\varphi$, is a K-automorphism. For the natural generating partition $\xi=(C_1,\ldots,C_k)$, where $C_i=\{\omega\in\Sigma_A,\ \omega_0=a_i\}$ one has the condition of being Bernoullian, so that $\sigma_A$ is conjugate with a Bernoulli automorphism (cf. Sec. 2 Chap. 1).

In the case of a topological Bernoulli scheme, the existence of the measure $\mu_\varphi$ and the K-property was proved by Ruelle [633]. This proof carries over directly to the case of a transitive t.m.c.; later this was done by Bowen [269], who also verified the condition of being weakly Bernoullian. Ya. G. Sinai [164, 671] introduced the Gibbsian measures for transitive t.m.c. in another way than Ruelle, and proved the existence of these measures and the K-property. Ya. G. Sinai started from a measure with maximal entropy $\mu_0$ and a function $\varphi\in\mathscr{F}_A$ and constructed measures $\mu_{m,n}(\varphi)=\rho_{m,n}\mu_0$, where the density

$$\rho_{m,n}(\omega)=\frac{\exp\sum_{k=-m}^{n}\varphi\left(\sigma_A^k\omega\right)}{\int_M\exp\sum_{k=-m}^{n}\varphi\cdot\sigma_A^k\,d\mu_0}.$$

The weak limit of the measures $\mu_{m,n}(\varphi)$ as $n,m\to\infty$ exists and coincides with the measure $\mu_\varphi$, described above. With the help of the theory of Markovian partitions, Ya. G. Sinai carried over his results to the case of $\mathscr{Y}$-systems (cf. Sec. 4 of the present chapter). Independently of Bowen, the condition of being weakly Bernoullian was verified by Bunimovich [36], Ratner [620], and Ruelle.

3.6. Just as the measure $\mu_0$ is the unique measure with maximal entropy, other measures can also be characterized by their extremal properties from the point of view of the variational principle, (cf. Paragraph 1.2.2 of the present chapter). Namely, $\mu_\varphi$ is the unique $\sigma_A$-invariant Borelian measure $\mu$, for which

$$P(\varphi,\sigma_A)=h_\mu(\sigma_A)+\int\varphi\,d\mu.$$

For topological Bernoulli schemes this fact was proved by Lanford [507]. For transitive t.m.c. there are several proofs. Two of them are due to Bowen: one generalizes the proof of Adler and Weiss of the theorem of Parry [269], the other [266] relates to a wider class of dynamical systems than t.m.c. and we shall speak about it in Sec. 5 of this chapter. Yet another proof was given by Ruelle [634, 635].

3.7. Ya. G. Sinai [164] considered the question of conditions for coincidence of Gibbsian measures constructed from various functions. If $\varphi_1, \varphi_2 \in \mathcal{F}_A$, then $\mu_{\varphi_1}=\mu_{\varphi_2}$ if an only if one can find a function $g \in \mathcal{F}_A$ such that

$$\varphi_1(\omega)=\varphi_2(\omega)+g(\sigma_A\omega)-g(\omega)+P(\varphi_1, \sigma_A)-P(\varphi_2, \sigma_A).$$

This fact is used in the theory of $\mathcal{Y}$-systems (cf. Sec. 6).

3.8. In conclusion, we shall touch on the basic results in the case of continuous time. Let $F \in \mathcal{F}_A$ be a positive function. We denote by $\Sigma_{A,F}$ the suspension over $\Sigma_A$, constructed from the t.m.c., $\sigma_A$ and the function $\varphi$, i.e.,

$$\Sigma_{A,F}=\{(\omega, t)\in\Sigma_A\times\mathbf{R}, \quad 0\leqslant t\leqslant F(\omega);$$
$$(\omega, F(\omega))\sim(\sigma_A\omega, 0)\}.$$

In this space there acts naturally the one-parameter group of homeomorphisms $\sigma^t_{A,F}$. This action consists of moving the point $(\omega, t)$ along the segment $\{\omega\}\times[0, F(\omega)]$ with unit speed, and attaining the end, it turns out in correspondence with the definition at the point $(\sigma_A\omega, 0)$, whence it extends the motion analogously.

One has an obvious correspondence between invariant measures of the flow $\sigma^t_{A,F}$ and homeomorphisms $\sigma_A$.

Let $\varphi$ be a continuous function on $\Sigma_{A,F}$, such that $\Phi\in\mathcal{F}_A$, where $\Phi(x)=\int\limits_0^{F(x)}\varphi(x, t)\,dt$, $x\in\Sigma_A$. Then the measure $\mu_\varphi$ on $\Sigma_{A,F}$ corresponds to the Gibbsian measure for $\sigma_A$, constructed by the function $\Phi-P(\sigma^t_{A,F}, \varphi)F$. Here $\mu_\varphi$ is the unique Borelian, $\sigma^t_{A,F}$, -invariant measure $\mu$, for which $P(\sigma^t_{A,F}, \varphi)=h_\mu(\sigma^t_{A,F})+\int\varphi d\mu$ (cf. Paragraph 1.2.3 of this chapter).

The connection between Gibbsian measures of t.m.c. and suspension over t.m.c. is due to Ya. G. Sinai [164], who used another definition of Gibbsian measure (cf. here Paragraph 3.6) and did not explicitly introduce the topological pressure. The property of uniqueness of Gibbsian measures for flows was proved by Ruelle and Bowen [270].

## §4. Markovian Partitions and Symbolic Representations of Hyperbolic Sets

4.1. Topological Markov chains serve as models for the behavior of diffeomorphisms on locally maximal hyperbolic sets. If the dimension of the hyperbolic set $\Lambda$ is equal to zero, then there exists a homeomorphism $\psi$ of some space $\Sigma_A$ onto $\Lambda$ such that:

4.1.1. $\psi\circ\sigma_A=f|\Lambda\circ\psi$ (cf. Paragraph 3.2.4). In the general case, a homeomorphism satisfying 4.1.1, of course does not exist, but one can construct a continuous map $\psi:\Sigma_A\to\Lambda$, which, from the point of view of ergodic theory, differs slightly from being one-one. In some rough sense even an arbitrary hyperbolic set can be approximated by t.m.c.

4.1.2. Let $\Lambda$ be a hyperbolic set of the diffeomorphism $f: U\to M$. For any neighborhood $V\supset\Lambda$ there exists a t.m.c. and a continuous map $\psi:\Sigma_A\to V$ such that $\varphi\circ\sigma_A=f|\Lambda\circ\psi$ $\operatorname{Im}\psi\supset\Lambda$. In particular, the restriction of a diffeomorphism to a locally maximal hyperbolic set is a continuous image of a t.m.c. (V. S. Alekseev [7]; cf. also [90]).

4.2. A subtler approximation of a diffeomorphism by topological Markov chains is effected with the help of Markovian partitions.

4.2.1. A closed subset R of a locally maximal hyperbolic set $\Lambda$ of a diffeomorphism f is called a parallelogram if the diameter of R is less than $\frac{\varepsilon_0}{2}$ (cf. Paragraph 2.6.1 of this chapter), $\Lambda=\overline{\operatorname{Int}\Lambda}$, and for some $\delta>0$ from $x,y\in R$ and $\rho(x, y)<\varepsilon(\delta)$ follows $r_\delta(x, y)\in R$ (cf. Paragraph 2.6.2). We write for $x\in R$, $W^s(x,R)=W^s_{\varepsilon_0}(x)\cap R$, $W^u(x,R)=W^u_{\varepsilon_0}(x)\cap R$.

4.2.2. By a Markovian partition of a locally maximal hyperbolic set $\Lambda$ is meant a covering of $\Lambda$ by parallelograms $R_1, \ldots, R_k$ such that $\operatorname{Int} R_i \cap \operatorname{Int} R_j = \varnothing$, $i \neq j$; if $x \in \operatorname{Int} R_i$, $fx \in \operatorname{Int} R_j$, then $fW^u(x, R_i) \supset W^u(fx, R_j)$, $f^{-1}W^s(fx, R_j) \supset W^s(x, R_i.)$

4.2.3. For any locally maximal hyperbolic set $\Lambda$ and any $\varepsilon > 0$ there exists a Markovian partition of the diameters of all of whose elements is less than $\varepsilon$.

4.2.4. The first concrete example of a Markovian partition was constructed by Adler and Weiss [197] for hyperbolic automorphisms of the two-dimensional torus. In this example the "parallelograms" from 4.2.2 are projections on the torus of real parallelograms in the plane. The geometric construction of Adler and Weiss does not carry over to other cases, even to hyperbolic automorphisms of tori of higher dimension.

The general definition of Markovian partitions for $\mathscr{Y}$-diffeomorphisms was given by Ya. G. Sinai [159]. This definition differed from that given above in the case when the elements of the Markovian partition are not connected. In [160] Sinai actually constructed partitions satisfying 4.2.2 in the case of transitive diffeomorphims.

Yet another variant of the approach to the definition of Markovian partitions develops in the paper of B. M. Gurevich and Ya. G. Sinai [68], where infinite partitions are admitted. Definition 4.2.2 and Theorem 4.2.3 in complete generality are due to Bowen [257].

4.2.5. Let the diameters of the elements of the Markovian partition $R_1, \ldots, R_k$ be less than the distance between any two points $x, y \in \Lambda$ (cf. 2.4.2). We construct the intersection matrix $A = \|a_{ij}\|$, where

$$a_{ij} = \begin{cases} 1, & \text{if } \operatorname{Int} R_i \cap f^{-1} \operatorname{Int} R_j \neq \varnothing; \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem allows one to carry over the theory of Gibbsian measure to locally maximal hyperbolic sets.

Let $\omega = \{\omega_s\} \in \Sigma_A$. Then the infinite intersection $\bigcap_{s \in \mathbf{Z}} f^s R_{\omega_s}$ is nonempty and consists of precisely one point. The map $\psi: \Sigma_A \to \Lambda$, where $\psi(\omega) = \bigcap_{s \in \mathbf{Z}} f^s R_{\omega_s}$ is continuous and one-one outside some subset of the first Baire category. If $\varphi \in F_A$ and $\mu_\varphi$ is a Gibbsian measure on $\Sigma_A$, and $\psi_* \mu_\varphi$ is the image of the measure $\mu_\varphi$ on $\Lambda$, then $\psi$ is an isomorphism mod 0 of the spaces $(\Sigma_A, \mu_\varphi)$ and $(\Lambda, \psi_* \mu_\varphi)$.

The first results in the direction of the theorem formulated were Theorem 5.1 of the paper of Ya. G. Sinai [159], in which in the case of $\mathscr{Y}$-diffeomorphisms a system of conditional measures on the fibers was constructed, corresponding to the measure $\psi_* \mu_0$. For the measure $\mu_0$ in the case of hyperbolic sets the theorem was proved by Bowen in [257]; the result for the measure $\mu_\varphi$ in the case of $\mathscr{Y}$-diffeomorphisms was proved by Ya. G. Sinai [164], in the general case by Bowen [269]. The existence, uniqueness, and Bernoullianness of the measure with maximal entropy for expanding endomorphisms was proved by Krzyzewski [487, 488].

4.2.6. Let the diffeomorphism $f: U \to M$ belong to the class $C^{1+\varepsilon}$, $\varepsilon > 0$, the restriction $f|\Lambda$ be topologically transitive and the function $g: \Lambda \to \mathbf{R}$ satisfy a Hölder condition with some exponent. It follows from 2.1.4 and 4.2.5 that the functions on $\Sigma_A$ $\varphi = \psi_* g \in \mathscr{F}_A$. Hence, according to 3.6 and 4.2.5, $\mu_g = \psi_* \mu_\varphi$ is the unique Borelian $f|\Lambda$ -invariant measure $\mu$, for which the quantity $h_\mu(f|\Lambda) + \int g d\mu$ achieves its maximum, which is equal to $P(g, f|\Lambda)$ (cf. Paragraph 1.2.2).

For the case g = 0 this fact means the uniqueness of the measure with maximal entropy, which was proved independently for $\mathscr{Y}$-diffeomorphisms by B. M. Gurevich [65] and (in the general case) Bowen [257].

If $f|\Lambda$ has the property of mixing domains, then from 3.5 and 4.2.5 it follows that the map $f|\Lambda$ with the measure $\mu_g$ is metrically isomorphic with a Bernoulli scheme.

4.3. The concept of Markovian partition carries over to the case of continuous time. Here in the theorem analogous to 4.2.5 the question is of the isomorphism with some suspension over a t.m.c. (cf. Paragraph 3.8). Markovian partitions for transitive $\mathscr{Y}$-flows on three-dimensional manifolds were constructed by M. E. Ratner [143]; in the multidimensional case Markovian partitions were constructed independently by Bowen [268] (for locally maximal hyperbolic sets of flows) and Ratner [619] (for transitive $\mathscr{Y}$-flows).

## §5. General Approach to the Construction of Gibbsian Measures

5.1. Gibbsian measures, which were first constructed for topological Markov chains (Paragraphs 3.5–3.7) and were carried over with the aid of the theory of Markovian partitions to locally maximal hyperbolic sets (Paragraph 4.2) can be constructed directly by some other method, applicable to a wider class of dynamical systems, which includes both classes considered in the preceding paragraphs.

Let f be a homeomorphism of the metric space X, which has the property of separating trajectories (Paragraph 1.1) and that of specification of 2.4.6, and $\varphi$ be a continuous function of X. Here it is necessary to consider the following.

5.1.1. For some $\varepsilon>0$ one can find an $M=M(\varepsilon)$ such that if $x, y\in X$ and $d(f^ix, f^iy)\leqslant\varepsilon, 0\leqslant i\leqslant n-1$ (d is the distance in X), then

$$\left|\sum_{i=0}^{n-1}\varphi(f^ix)-\varphi(f^iy)\right|<M.$$

In what follows (for the validity of 5.3.5) some strengthening of this condition is required:

5.1.2. If $\varepsilon\to0$, then one can choose $M(\varepsilon)\to0$.

This property is satisfied in both cases which were considered in the preceding paragraphs: when f is a transitive t.m.c. and $\varphi\in\mathscr{F}^A$, and also when f is the restriction of a diffeomorphism to a locally maximal hyperbolic set and $\varphi$ satisfies a Hölder condition.

5.2. We denote by $P_n$ the set of periodic points of the homeomorphism f with period n. We set

$$Z_n^P(f, \varphi)=\sum_{x\in P_n}\exp\sum_{i=0}^{n-1}\varphi(f^ix).$$

We denote by $\mu_x$ the normalized measure concentrated at the point $x\in X$. Let

$$\mu_{\varphi,n}=Z_n^P(f,\varphi)^{-1}\left(\sum_{x\in P_n}\exp\left(\sum_{i=0}^{n-1}\varphi(f^ix)\right)\mu_x\right).$$

We write

$$B_n(x,f)=\{y\in X: d(f^ix, f^iy)\leqslant\varepsilon,\ i=0,1,\ldots,n-1\}.$$

If the properties of separating trajectories, specification, and 5.1.1 are satisfied, then one has the following estimate, which is a fundamental technical fact in the theory of Gibbsian measures.

5.2.1. There exist an N and constants $d_1, d_2, 0<d_1<d_2$, such that for $n>N$

$$d_1e^{P(f,\varphi)n}\leqslant Z_n^P(f,\varphi)\leqslant d_2e^{P(f,\varphi)n}.$$

5.2.2. For any k and sufficiently small $\varepsilon>0$ one can find a natural number $N(\varepsilon, k)$ and positive constants $A_1(\varepsilon)$, $A_2(\varepsilon)$ such that for $n>N(\varepsilon, k)$ and $x\in X$

$$A_1(\varepsilon)\exp\left(\sum_{i=0}^{k-1}\varphi(f^i x)-kP(f,\varphi)\right)\leqslant\mu_{\varphi,n}(B_k(x,f))\leqslant A_2(\varepsilon)\exp\left(\sum_{i=0}^{k-1}\varphi(f^i x)-kP(f,\varphi)\right).$$

5.3. The estimates given allow one to prove the following facts.

5.3.1. The sequence $\mu_{\varphi,n}$ converges weakly as $n\to\infty$ to some measure which we denote by $\mu_\varphi$.

5.3.2. The map f is ergodic with respect to the measure $\mu_\varphi$.

5.3.3. $\mu_\varphi$ is the unique Borelian invariant measure for which the expression $h_\mu(f)+\int\varphi d\mu$ attains its maximum, which is equal to $P(f,\varphi)$.

5.3.4. The homeomorphism f has the K-property with respect to the measure $\mu_\varphi$.

5.3.5. If the functions $\varphi_1, \varphi_2$ satisfy 5.1.2, then $\mu_{\varphi_1}=\mu_{\varphi_2}$ if and only if one can find a continuous function g such that

$$\varphi_1(x)=\varphi_2(x)+g(f,x)-g(x)+P(f,\varphi_1)-P(f,\varphi_2).$$

From 3.6 and 5.3.3 it follows that in the case of a topologically transitive t.m.c. the measure coincides with the Gibbsian measure $\mu_\varphi$ of Sec. 3.

The approach described in this paragraph to the construction of the measure $\mu_0$ and the proof of 5.3.1 and 5.3.2 in this case are due to Bowen [259]. With this approach, the connection of the measure $\mu_0$ with the asymptotic distribution of periodic points is obvious. Bowen proved analogous results for flows in [263].

The papers of Bowen appeared after the paper of Ya. G. Sinai and G. A. Margulis on the asymptotic number of closed trajectories and the asymptotic distribution of these trajectories for $\mathscr{Y}$-flows, in particular, geodesic flows on manifolds of negative curvature. We shall speak about these papers in Sec. 7.

Bowen's construction is carried over to arbitrary Gibbsian measures by Ruelle [634, 635] (cf. Paragraph 1.2). The inequality 5.2.1, the left-hand side inequality 5.2.2, and assertions 5.3.1-5.3.3 are in the paper of Bowen [266]. The right-hand side of 5.2.2 and assertions 5.3.4 and 5.3.5 were proved by A. B. Katok.

The basic construction of this paragraph carries over in an obvious way to the case of flows.

## §6. Measures Connected with Smoothness and Homological Equations

6.1. Suppose on the smooth compact manifold M there is a fixed Riemannian metric. There is a certain measure connected with it (the Riemannian volume). Let us agree to call a Borelian normalized measure on M absolutely continuous, absolutely continuous positive, Hölderian, or smooth if in relation to the Riemannian volume this measure is respectively absolutely continuous, equivalent, defined by a positive density, satisfying a Hölder condition, or defined by a positive smooth density. All the enumerated classes of measures are independent of the Riemannian metric.

If $f:M\to M$ is a diffeomorphism, then the existence for f of an absolutely continuous invariant measure is equivalent to the existence of a nonnegative integrable function $\rho$, such that for almost all $x\in M$

6.1.1. $\lg|\det Df_x|=\lg\rho(fx)-\lg\rho(x)$, where $|\det Df_x|$ denotes the coefficient of expansion of the Riemannian volume. The function $\rho$ serves as the density for an invariant measure relative to the Riemannian volume.

If the density of the invariant measure is continuous and positive, then the following condition is satisfied.

6.1.2. For any point of period n,

$$|\det Df_x{}^n|=1.$$

6.2. Let $f:M\to M$ be a topologically transitive $\mathscr{Y}$-diffeomorphism of class $C^2$. A. N. Livshits [115] and also D. V. Anosov and A. M. Stepin proved the following theorem.

Let f have an absolutely continuous positive invariant measure and let g be a continuous function with modulus of continuity $\omega_g(s)$ such that $\int\limits_0^1 \frac{\omega_g(s)}{s}\,ds<\infty$. Then if h is a measurable function and almost everywhere

6.2.1. $g(x)=h(fx)-h(x)$,

then there exists a continuous function h', which coincides with h almost everywhere.

For $\mathscr{Y}$-flows, 6.2.1 is replaced by the following.

6.2.2. For any $T>0$, almost everywhere

$$\int\limits_0^T g(f_t x)\,dt=h(f_t x)-h(x).$$

Thus, if f has an absolutely continuous positive measure, then this measure is really given by a continuous density, and consequently, satisfies 6.1.2.

Equations of the form 6.2.1 and some close equations are met with not only in the problem of existence of absolutely continuous invariant measures, but also in a series of other problems of the theory of dynamical systems (cf. above: Paragraphs 3.7 and 5.3.5, and also Chaps. 5 and 6). They are usually called homological equations. The solvability properties of these equations depend very strongly on the properties of the transformation f. In [114, 115] A. N. Livshits studied the homological equation (6.2.1) and similar equations for functions with values in more general groups than the additive group of real numbers for $\mathscr{Y}$-systems and topological Markov chains. For the problem of existence of invariant measures the following result of Livshits is more essential [114, 117]:

6.2.3. Let $f:M\to M$ be a topologically transitive $\mathscr{Y}$-diffeomorphism of class $C^2, U\subset M$ be an open set, g be a function satisfying a Hölder condition. Equation (6.2.1) has a solution satisfying a Hölder condition if and only if for any periodic point $x\in U$ $\sum\limits_{i=0}^{n-1} g(f^i x)=0,$ where n is the period of the point x. If g is a function of class $C^1$, then h is also a function of class $C^1$.

Combining 6.1.2, the theorem of Paragraphs 6.2 and 6.2.3, we get the following assertion.

6.2.4. The following conditions are equivalent: f has an absolutely continuous invariant measure; for any point x of period n, $|\det Df_x{}^n|=1$; f has a smooth invariant measure.

For flows, we get an analogue of 6.2.3 if in 6.1.2 we replace periodic points by periodic trajectories.

6.3. Condition 6.2.4 is also equivalent to the existence of an absolutely continuous invariant measure. This fact can be obtained in the framework of another approach to the proof of 6.2.4, proposed by Ya. G. Sinai in [671, 164] and complete in the joint paper of A. N. Livshits and Ya. G. Sinai [177]. This less direct approach arises in working out the program of applying the ideas of statistical physics to the ergodic theory of smooth systems, to which the two above-mentioned papers of Ya. G. Sinai are devoted.

6.3.1. If the $\mathscr{Y}$-diffeomorphism f of class $C^{1+\varepsilon}$ has a smooth invariant measure $\mu$, then it is a K-automorphism with respect to this measure [9, 16], and consequently for any absolutely continuous measure $\nu$, the measure $(f_*)^n$ converges weakly to $\mu$ as $n \to \pm\infty$.

In [158], Ya. G. Sinai with the aid of Markovian partitions constructed for any topologically transitive $\mathscr{Y}$-diffeomorphism invariant measures $\mu_+$ and $\mu_-$ such that for any absolutely continuous measure $\nu$, the measure $(f_*)^n\nu$ converges weakly to $\mu_+$ as $n \to \infty$ and to $\mu_-$ as $n \to -\infty$. Whence, obviously, it follows that the existence for f of an absolutely continuous invariant measure is equivalent with the equation $\mu_+ = \mu_-$. In [158] it is also proved that f is a K-automorphism with respect to the measures $\mu_+$ and $\mu_-$.

6.3.2. Let the Riemannian metric on M be fixed. We write

$$\varphi^{(u)}(x) = \lg \det Df_x \big|_{E_x^u}, \quad \varphi^{(s)}(x) = \lg \det Df_x \big|_{E_x^s}$$

By virtue of 2.1.4 the functions $\varphi^{(u)}$ and $\varphi^{(s)}$ satisfy a Hölder condition. Upon passage to a new metric, to $\varphi^{(u)}$ and $\varphi^{(s)}$ are added a term of the form g(fx)—g(x) with smooth g. In [164], Ya. G. Sinai proved that the measures $\mu_+$ and $\mu_-$ coincide with the Gibbsian measures constructed from the functions $\varphi^{(u)}$ and $\varphi^{(s)}$. This allows one to prove a strengthened variant of 6.2.4, starting from 3.7 and 6.2.3. Analogous facts are valid for flows [164].

6.3.3. We note an interesting fact, proved by B. M. Gurevich and V. I. Oseledets [67] with the aid of 6.3.2. If the topologically transitive $\mathscr{Y}$-diffeomorphism f has no absolutely continuous invariant measure, then with respect to any absolutely continuous measure $\nu$, f is dissipative, i.e., one can find a set A, $\nu(A) > 0$, such that $T^nA \cap T^mA = \varnothing$ for n ≠ m and $\bigcup_n T^nA = M \pmod 0$. Whence it follows that 6.2.1 is solvable in measurable functions for any measurable g.

6.4. Let $\Lambda$ be a hyperbolic attractor of the diffeomorphism f or the flow $f_t$ of class $C^2$, i.e., a hyperbolic set such that for some neighborhood $V \supset \Lambda$ $\bigcap_{n>0} f^nV = \Lambda$ (respectively, $\bigcap_{t>0} f_tV = \Lambda$).

Let g be a continuous function on V. Then, if the restriction $f|\Lambda$ $(f_t|\Lambda)$ is topologically transitive,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(f^kx) \to \int_\Lambda g\, d\mu_{\varphi^{(u)}}$$

(respectively, $\lim_{T\to\infty} \frac{1}{T} \int_0^T g(f_tx)\, dt \to \int_\Lambda g(x)\, d\mu_{\varphi^{(u)}}(x)$). The function $\varphi^{(u)}$ is defined only on $\Lambda$.

If the restriction $f|\Lambda(f_t|\Lambda)$ has the property of mixing domains, then for any absolutely continuous measure $\nu$ with carrier in the neighborhood V one has the weak convergence $(f^*)^n\nu \to \mu_{\varphi^{(u)}}$ as $n \to \infty$ $((f_t^*)\nu \to \mu_{\varphi^{(u)}}$ as $t \to \infty)$. This generalization of a result of Ya. G. Sinai is due to Ruelle and Bowen [270] (the case of flows). The proof for diffeomorphisms is in [268].

6.5. For an expanding endomorphism $f:M\to M$ there always exists a unique absolutely continuous positive invariant measure. With respect to this measure f is a strict endomorphism (Krzyzewski and Szlenk [489]). Krzyzewski proved that this measure is always smooth [485].

L. A. Kosyakin and E. A. Sandler [102], and later, independently, Lasota and Yorke [510] proved that a map of a segment that is twice continuously differentiable everywhere except at a finite number of points has an absolutely continuous invariant measure whose density has bounded variation.

## §7. $\mathcal{Y}$-Flows

7.1. Let $f_t:M\to M$ be a topologically transitive $\mathcal{Y}$-flow of class $C^2$ on the compact manifold M. An invariant measure, which we shall denote by $\mu_0$, for this case was first constructed by G. A. Margulis [121]. The method of Margulis is different from the "Markovian" method of Sinai (Secs. 3 and 4), as well as from the method of Bowen (Sec. 5).

The global integral manifolds of the distributions $E^s+E^0$ and $E^u+E^0$, $E^s$, $E^u$ are called, respectively, contracting and expanding leaves, and contracting and expanding orispheres. Let a Riemannian metric be fixed on M. We shall consider each expanding leaf $\sigma$ as an independent Riemannian manifold and we define linear functionals $l_t$, $t\geqslant 0$, on the space of continuous finitary functions on $\sigma$:

$$l_t(\varphi)=\int_\sigma \varphi(f_t x)\,d\mu_\sigma,$$

where $\mu_\sigma$ is the measure induced by the Riemannian volume on the leaf. Further, one considers the closure of the set of linear combinations of $l_t$ with nonnegative coefficients, satisfying some normalization condition, and with the aid of Tikhonov's theorem in this closure a functional $\tilde{l}$ is sought, for which $\tilde{l}(\varphi_\circ f_t)=d^t l(\varphi)$, where $d>1$ is some constant. The functional $\tilde{l}$ generates a $\sigma$-finite measure $\tilde{\mu}$ on the leaf, and consequently, also a conditional measure on the orispheres that make up the leaf, while it turns out that the measures on the different leaves are well coordinated so that with the aid of these measures and the corresponding measures on the contracting leaves one can define a global invariant measure on M.

7.2 We recall that the geodesic flow on a compact Riemannian (or Finslerian) manifold N is the flow on the manifold $W(N)\subset TN$ of tangent vectors to N of unit length, which can be described as motion of a tangent vector $w\in W(N)$ along the geodesic, which it defines with unit speed with respect to the parameter of arc length. On W(M) there is induced a Riemannian metric and Riemannian volume which is an invariant measure for the geodesic flow.

If the curvature of the manifold N in any two-dimensional direction is negative, then the geodesic flow on N is a $\mathcal{Y}$-flow. The first estimate of the asymptotic number of closed geodesics on a compact n-dimensional Riemannian manifold of negative curvature was given in 1966 by Ya. G. Sinai [168]. Namely, if $K_1^2$ and $K_2^2$ are the infimum and supremum of the curvatures of N in two-dimensional directions, and $\nu(R)$ is the number of closed geodesics of length not more than R, then

$$(n-1)K_2\leqslant \varliminf_{R\to\infty}\frac{\ln\nu(R)}{R}\leqslant\varlimsup_{R\to\infty}\frac{\ln\nu(R)}{R}\leqslant(n-1)K_1.$$

In [120], G. A. Margulis essentially sharpened the result of Ya. G. Sinai, proving that

$$7.2.1\quad \lim_{R\to\infty}\frac{d\cdot R\cdot\nu(R)}{e^{dR}}=1,$$

where d is the constant from Paragraph 7.1. Here, actually, Margulis proved the coincidence of the measure of Paragraph 7.1 with the measure $\mu_0$, which is constructed for flows by the method of Sec. 5 (there is a similar proof in the dissertation of G. A. Margulis, "Some questions of the theory of $\mathcal{Y}$-systems," Moscow University, 1970).

It should be noted that the exact multiplicative asymptotic 7.2.1 for $\nu(R)$ is not obtained by the general methods of Secs. 3, 4, or 5 (the analogous result for discrete time follows from the theory of Markovian partitions). In [120] some other asymptotic characteristics of manifolds of negative curvature are also obtained.

We note, in conclusion, the later paper of Bowen [264] in which it is proved that for the geodesic flow on manifolds of constant negative curvature the measure $\mu_0$ coincides with the natural smooth invariant measure.

7.3. The geodesic flow on a Riemannian manifold N can be a $\mathcal{Y}$-flow not only if the curvature of N in every two-dimensional direction is negative. A series of necessary conditions for this was found by Klingenberg [448]. Among these conditions are the absence of conjugate points on N and the fact that the universal covering of N should be diffeomorphic with Euclidean space. P. Eberlein [323, 324] found some conditions that are necessary and sufficient for the geodesic flow on a manifold without conjugate points to be a $\mathcal{Y}$ -flow.

7.4. If $f_t:M\to M$ is a $\mathcal{Y}$-flow, then for any flow $g_t$ that is close to $f_t$ in the $C^1$-topology there exists a homeomorphism $\varphi_g:M\to M$ carrying the trajectories of $f_t$ into trajectories of $g_t$. This homeomorphism is not uniquely defined. There is interest in the question of necessary or sufficient conditions under which the homeomorphism $\varphi_g$ can be chosen absolutely continuous. A. B. Katok [84] applied the rotation number to this problem. Namely, he proved that if the flows $f_t$ and $g_t$ preserve the smooth measure $\mu$ and the rotation classes $\lambda_\mu(f_t)$ and $\lambda_\mu(g_t)$ are not collinear in the homology group $H_1(M, \mathbf{R})$, then any homeomorphism $\varphi_g$ is singular. If dim M = 3, then one can get another necessary condition, and in the case when $f_t$ and $g_t$ are geodesic flows on surfaces of negative curvature, sufficient conditions also. We shall dwell on the latter case. A necessary condition for the existence of a nonsingular homeomorphism, homotopic to the identity, sending trajectories of $f_t$ into trajectories of $g_t$ is the proportionality of the length of all homotopic closed trajectories of $f_t$ and $g_t$. Now if the Riemannian metrics generating $f_t$ and $g_t$ are homotopic in the class of metrics of negative curvature, then this condition is sufficient for the existence of a homeomorphism preserving a smooth invariant measure, conjugating the flows $f_t$ and $g_{\alpha,t}$, $\alpha$ = const.

## §8. Billiards and Some Other Systems

In this paragraph we shall consider some classes of dynamical systems on manifolds, in which one has hyperbolic behavior of trajectories, i.e., exponential approach in one direction and exponential dispersal in another, but this is not true of all trajectories and the estimates of dispersal are not uniform in time.

8.1. In the case of smooth dynamical systems with smooth invariant measure, a natural generalization of the hypotheses of Paragraph 2.1 is the absence of null characteristic Lyapunov exponents (cf. [40] and Sec. 2 of Chap. 9) on a set of full measure. Ya. B. Pesin constructed a theory of stable and unstable manifolds for dynamical systems with nonzero exponents and with its aid proved the following theorem:

Let $f:M\to M$ be a diffeomorphism with a smooth invariant measure $\mu$, $\Lambda$ be the set of those points at which all characteristic exponents are different from zero. Then almost any ergodic component of f, lying in the set $\Lambda$, has positive measure. Each such component decomposes into a union of a finite number of sets, invariant with respect to some power of f, while this power is a K-automorphism on each of the invariant sets.

8.2. From a general point of view, billiard systems can be considered as a generalization of geodesic flows to the case of Riemannian manifolds with piecewise smooth boundary. Let Q be such a manifold, W(Q), as in Sec. 7, be the manifold (also with boundary) of tangent vectors to Q of unit length.

By billiards in Q is meant a dynamical system in W, generated by motion of a tangent vector w in W along the geodesic defined by it with unit speed, while upon the carrier of w hitting the boundary of Q there occurs instantaneous reflection according to the law "angle of incidence equals angle of reflection." This means that the tangential component of the vector w is preserved but the normal component changes sign, after which the motion can be continued inside Q, provided only the hit did not occur at a "fracture" of the boundary. It follows from paragraph 7.2 that billiards has a natural invariant measure. The practical word in what follows will go only to two-dimensional manifolds Q with metric of zero curvature, i.e., domains with piecewise smooth boundary in the Euclidean plane or on the torus. The ergodic properties of billiards depend essentially on geometric properties of the boundary $\Gamma = \partial Q$. If $\Gamma$ is a convex curve, while the curvature of $\Gamma$ is different from zero and the smoothness is sufficiently high, then billiards in the bounded domain Q is nonergodic. We shall speak about billiards inside polygons in Paragraph 3.3 of Chap. 4. Hyperbolic effects were first discovered in billiards whose boundary turns convexly inside the domain. The role of such boundaries in billiards is similar to the role of regions of negative curvature for geodesic flows.

In the paper of Ya. G. Sinai [161] billiards in domains Q, obtained from an n-dimensional torus or square by the rejection of a finite number of convex domains, was considered. We shall explain how to obtain in such systems exponentially unstable trajectories. Let us assume that n = 2 and we fix a point $q \in Q$ and an angle $\varphi$. We release a pencil of trajectories from the point q at angles close to $\varphi$. As long as this pencil is not reflected from a concave component of the boundary, the separating of the pencil occurs with linear speed, and its front is an arc of a circle, whose radius grows linearly with time, and the curvature decreases with the same speed. It turns out that upon reflection from a concave boundary the convex pencil remains convex in the small, but its curvature increases upon reflection by a quantity depending on the angle of incidence and the curvature of the boundary at the point of reflection. Between two reflections the width of the pencil grows in the small by $(1+\tau k)$ times, where $\tau$ is the interval between the reflections and k is the curvature of the pencil at the initial moment. Hence by n reflections the width of the pencil increases by $\prod_{i=1}^{n}(1+\tau_i k_i)$ times, where $\tau_i$, $k_i$ are the corresponding quantities for the i-th reflection. Since the number of reflections for typical trajectories grows linearly with t, the latter expression grows exponentially with t.

We note two characteristic difficulties arising in the study of the asymptotic behavior of trajectories in such billiards. First of all, some trajectories can in general not hit the boundary (example: a certain family of closed geodesics on the torus from which a small disk has been excised). Such trajectories are small, but others may from time to time approach close to them and because of this some $\tau_i$ will be large.

Secondly, billiards is a discontinuous dynamical system in W. Discontinuities arise upon trajectories hitting fracture points of the boundary (and there always are such points if Q is a domain in the plane) and upon tangency of trajectories with the boundary.

It is proved in [161] that for typical points in phase space one can construct stable and unstable manifolds. The measure of the set of typical points is equal to one. But in contrast to $\mathscr{Y}$-systems the manifolds $W^s(x)$ and $W^u(x)$ are discontinuous functions of x. Moreover, each fiber $W^s(x)$, $W^u(x)$ as a submanifold of M itself has singularities of fracture type. Hence, the general theory of systems with transversal fibers [158] gives for this case positive entropy. In [161] it was proved that nevertheless the systems considered are ergodic and are K-systems. At the basis of the considerations lies a theorem that is naturally called the fundamental theorem of this theory. Its meaning is that in a sufficiently small neighborhood of a point of general form for any p, $0<p<1$, and any C, $0<C<\infty$, if one takes a regular (i.e., without singularities) segment of $W^s$ ($W^u$) of length $C\delta$, then with probability not less than p, through its points one can pass regular segments of $W^u$ ($W^s$) of length not less than $C\delta$.

In the paper of L. A. Bunimovich and Ya. G. Sinai [39] an essentially simpler proof of the fundamental theorem than in [161] was obtained, which is applicable in more general circumstances. Simultaneously, by the same methods Kubo proved the K-property for billiards in domains of more general form [494, 495]. The definitive condition in the two-dimensional case looks like this: The domain Q has a boundary consisting of a finite number of concave curves of class $C^2$, which intersect transversally. Billiards in such domains are called scattering.

8.3. Further progress in the study of billiards is due to L. A. Bunimovich. In his papers [33, 37, 38] he singled out a new class of domains, in which one succeeds in proving the K-property for billiards with the aid of the construction of the stable and unstable manifolds for most points. In the papers [33, 37] this program is realized for domains with boundary consisting of several concave curves, intersecting transversally (scattering components of the boundary) and several arcs of different circles (focusing components or pockets). Additional conditions are that the arcs cannot abut on one another, and also that the disk bounded by each circle belongs to Q. In [38] yet another series of examples is considered in which the boundary consists of scattering components, arcs of circles, and segments of lines (neutral components). There is more interest in the principal point of view of the very simple example: The domain Q is convex and bounded by two simicircles, joined by segments of common exterior tangents (such domains, naturally, are called "stadia"). The value of this example is that it demonstrates a new mechanism in the appearance of hyperbolic properties, connected not with scattering pencils of trajectories but with successive agreements of focusing. It should be noted that the theory here is not complete. In particular, the requirement of constant curvature of the focusing component is nonprinciple. However, in this case it is necessary to introduce restrictions on the integral of the curvature of the focusing components and hence one should not expect elegant general conditions which would contain all the interesting cases.

In [32, 35] for the scattering billiards considered in [37, 39], the central limit theorem is proved.

The ergodicity of billiards is of interest not only for its own sake. As was proved by A. I. Shnirel'man [183], from it follows very interesting information about the asymptotically uniform distribution of the eigenvalues of the Laplace operator with reflection condition on the boundary.

8.4. Kubo (preprint) considered a class of systems close to billiards. Billiards can be represented as a dynamical system generated by the motion of a particle with unit energy in a force field whose potential has the form

$$U(q)=\begin{cases}0, & q\in \operatorname{Int} Q,\\ \infty, & q\in\partial Q.\end{cases}$$

Kubo considered a Hamiltonian system generated by the motion of a particle in a torus in a force field with potential $U(|q-q_0|)$ for which $U(r)\equiv 0$ for $r\geqslant r_0$, $r_0<1/2$, and the derivative U' is negative and sufficiently large in modulus (strongly repellent).

8.5. It is natural to try to seek out conditions on the Riemannian metric weaker than negativity of curvature, which would allow one to investigate the metric properties of the geodesic flow in the spirit of "nonuniform hyperbolicity."

The first steps in this direction were made in the papers of A. Kramli [106, 108]. His result is the following. Let M be a compact orientable surface of genus greater than one with a Riemannian metric without focal points (this condition is somewhat stronger than the condition of absence of conjugate points), $W_-$ be the set of those line elements which do not always remain in domains of zero curvature. Then the measure of $W_-$ is positive and almost all ergodic components of the geodesic flows making up $W_-$ have positive measure and on each of these components the geodesic flow is a K-flow. Ya. B. Pesin strenthened this result, proving the ergodicity of the geodesic flow on $W_-$.

We note that there are interesting papers of P. Eberlein on the topological structure of geodesic flows on Riemannian manifolds with various properties, weaker than negative curvature (cf. [322]). The methods of P. Eberlein apparently are poorly suited to the investigation of the metric properties of these flows.

8.6. Yet another example of systems of discontinuous type with hyperbolic properties was investigated by V. I. Oseledets. The question here is of a model for stochastic acceleration of Fermi, described in the book of G. M. Zaslavskii (cf. Chap. IV of [76]). We consider a transformation of the two-dimensional torus

$$T(x_1, x_2) = (x_1 + \varepsilon N f(x_1) + N x_2 \,(\mathrm{mod}\, 1), \quad x_2 + \varepsilon f x_1 \,(\mathrm{mod}\, 1),$$

clearly preserving Lebesgue measure. The function f is such that

$$f'(x) = \begin{cases} 1, & 0 < x < 1/2, \\ -1, & 1/2 \leqslant x < 1. \end{cases}$$

Hence the transformation T is not smooth. It turns out that as $N \to \infty$, $\varepsilon \to 0$, $N\varepsilon = k$ this transformation becomes ergodic and will even have the K-property for sufficiently large N, if

or $k < -4$. It is interesting to note that the latter result is obtained with the aid of a definition of hyperbolicity in terms of "cones," which is more convenient for many applications (cf. [3-5]).

8.7. There are one-dimensional irreversible maps, which are somewhat similar with the systems considered in this paragraph (cf. 2.2). We shall speak in this connection of the paper of L. A. Bunimovich [31], which proved the existence of an absolutely continuous invariant measure for maps of the circle of the form $Tx = q\pi \sin x (\mathrm{mod}\ \pi)$, where q is an integer not equal to zero. From the presence of critical points of this mapping, the density of the invariant measure has singularities.

## §9. Partially Hyperbolic Dynamical Systems

In this paragraph we shall consider a useful generalization of $\mathscr{Y}$-system (Paragraph 2.1).

9.1. A diffeomorphism f of a compact connected manifold is called partially hyperbolic if at each point $x \in M$ the tangent space $T_xM$ decomposes into the direct sum of three subspaces $E_x^s$, $E_x^0$, $E_x^u$, while the following conditions are satisfied:

9.1.1. The dimensions of $E_x^s$, $E_x^0$, $E_x^u$ are independent of x and at least two of these dimensions are greater than zero.

9.1.2. $DfE_x^r = E_{fx}^r$, $r = s, 0, u$.

9.1.3. For some Riemannian metric on M one can find a number $c > 0$ and $0 < \lambda_1 < \mu_1 \leqslant 1 \leqslant \mu_2 < \lambda_2$ such that for $x \in M$ and a natural number n one has the following inequalities (the norm is induced by the Riemannian metric)

$$\begin{aligned} &\|Df^n u\| \leqslant c\lambda_1^n \|u\|, \quad u \in E_x^s; \\ &\|Df^n u\| \geqslant c\mu_1^n \|u\|, \quad u \in E_x^0; \\ &\|Df^{-n} u\| \geqslant c\mu_2^{-n} \|u\|, \quad u \in E_x^0; \\ &\|Df^{-n} u\| \leqslant c\lambda_2^{-n} \|u\|, \quad u \in E_x^u. \end{aligned}$$

The flow $f_t$ is called partially hyperbolic if $f_1$ is a partially hyperbolic diffeomorphism.

9.2. Conditions of the type given in Paragraph 9.1 are well known in the theory of invariant manifolds for ordinary differential equations. In application to dynamical systems on compact manifolds these conditions first appeared explicitly in Hirsch, Pugh, and Shub [399]. This paper is devoted to basic topological questions. As an example, Anosovian actions of Lie groups are considered in it. A locally free action of group G as diffeomorphims $\{f_g\}$ of a manifold M is called Anosovian if for some element $g \in G$ the diffeomorphism $f_g$ is partially hyperbolic while $E^0$ is the tangent space to the orbits of the action of G. The element g is called an Anosovian element. In [614] Pugh and Shub proved that an Anosovian action with a smooth invariant measure is ergodic if an Anosovian element lies in the contralizer of G.

The first paper on the metric theory of dynamical systems in which nontrivial effects connected with partial hyperbolicity were studied was the paper of A. M. Stepin [171] on spectral properties of some flows on solvable manifolds, generated by groups of motions (cf. Sec. 1 of Chap. 3). Almost simultaneously, there appeared the paper of Sacksteder [638] in which properties of mixing diffeomorphisms with smooth expanding and contracting fibers of nonfull dimension were studied, and in particular it was proved that absolute nonintegrability (cf. Paragraph 9.4) is sufficient for mixing with respect to a smooth invariant measure. In the situations considered by Stepin and Sacksteder, because of the smoothness of the fibers analytic difficulties are easily overcome. The definitions and results given below are due to M. I. Brin and Ya. B. Pesin and are contained in their joint papers [27-29] and in the papers of M. I. Brin [24-26]. The approach of Brin and Pesin to the study of partially hyperbolic systems is a rather extensive development of the approach of Sacksteder. This approach allows one to get a series of interesting, although apparently not definitive, metric and topological results in the case of nonsmooth fibers. M. I. Brin and Ya. B. Pesin also used some results of [399].

The subsequent results on partially hyperbolic systems generated by motions on homogeneous spaces of Lie groups are in the paper of A. M. Stepin [176] (cf. Sec. 1 of Chap. 3). We shall mention some other papers in the course of the exposition.

9.3. As in the case of $\mathcal{Y}$-systems, 9.1.3 is independent of the choice of Riemannian metric, and the subspaces $E^s_x$, $E^u_x$, $E^\circ_x$ depend continuously on x and generate three continuous distributions on M, which are naturally denoted by $E^s$, $E^u$, $E^\circ$. For the first two of them one has an analogue of the Hadamard–Perron theorem, which allows one to construct integral manifolds of these distributions, satisfying a uniqueness condition and generating fibrations, which we denote by $W^s$, $W^u$. These fibrations have the important property of absolute continuity; the map of a smooth area element $U_1$, which is transverse to the fibers into a $C^1$-close area element $U_2$, under which the point $x \in U_1$ corresponds to the unique point of $U_2$ of the local fiber of the point x, has bounded Jacobian with respect to the Riemannian volumes on $U_1$ and $U_2$ [27, 29]. This property allows one to apply to partially hyperbolic systems the theory of transversal fibrations of Ya. G. Sinai [158].

The entropy of a partially hyperbolic system with respect to a smooth invariant measure is positive and bounded by the characteristic exponents [24]. Using his general theory of stable manifolds (cf. Paragraph 8.1), Ya. B. Pesin proved that the entropy of a diffeomorphism with respect to a smooth invariant measure is equal to the integral of the sum of the positive characteristic exponents.

9.4. More subtle metric properties of partially hyperbolic systems, in the first place ergodicity and the K-property, hold under additional restrictions. In order to clarify the meaning of these restrictions, we return to the case when the distributions $E^s$ and $E^u$ have sufficiently high smoothness (this case was studied by Sacksteder [638]).

We call the pair of fibrations $W^s$ and $W^u$ absolutely nonintegrable if the values at each point $x \in M$ of smooth vector fields subordinate to the distributions $E^s$ and $E^u$ and their Lie brackets of those orders for which the bracket operation is defined generate the entire tangent space $T_xM$.

There is a geometric property, derived from absolute nonintegrability, which can be defined in several variants and for nonsmooth fibrations. Namely, the pair of continuous fibrations $W^s$ and $W^u$ is called almost locally transitive (respectively, transitive) if any two sufficiently close points $x, y \in M$ can be joined by a chain of points $x_0=x, x_1, \ldots, x_{N-1}$, (N is independent of x and y) such that for i = 0, 1, ..., N − 1, the points $x_i$ and $x_{i+1}$ lie on one local fiber of one of the fibrations (respectively, on one global fiber but at uniformly bounded distance in the fiber).

9.4.1. The property of almost local transitivity of the pair $W^s$, $W^u$ is preserved under small $C^2$ perturbations of the dynamical system, if in the original dynamical system the distributions $E^s$ and $E^u$ have sufficiently high smoothness [29].

9.4.2. If the pair of fibrations $W^s$, $W^u$ of a partially hyperbolic diffeomorphism is transitive, then the diffeomorphism is topologically transitive [25].

9.5. Starting here we shall always assume that the diffeomorphisms or flows considered have smooth invariant measure μ.

9.5.1. If the distributions $E^s$ and $E^u$ of a diffeomorphism satisfy a Lipschitz condition, and the pair of fibrations $W^s$, $W^u$ is transitive, then f is a K-automorphism with respect to $\mu$ [29].

It is unknown whether this assertion is true if one relinquishes the Lipschitz condition on $E^s$ and $E^u$. The difficulties that arise here are connected with the fact that the "middle" fibration $W^\circ$, corresponding to the distribution $E^\circ$, even if it exists, may not be absolutely continuous. Nevertheless one can successfully replace the condition of 9.5.1 by others which correspond to a series of interesting examples.

9.5.2. Let the distributions $E^\circ$, $E^\circ + E^s$, $E^\circ + E^u$ be integrable, the pair of fibrations $W^s$ and $W^u$ be almost locally transitive, and in addition $W^s$ and $W^u$ satisfy some additional condition which is fulfilled if, for example, $E^s$ and $E^u$ are Lipschitzian along the fibers of the fibration $W^\circ$. Then f is a K-automorphism [27, 29].

9.6. The condition of 9.5.2 turns out to be crude for some classes of dynamical systems. Let M, N be smooth manifolds, where M is a smooth bundle with base N and projection $\pi:M\to N$. A diffeomorphism $f:M\to M$ is called an extension of the diffeomorphism $h:N\to N$ if $\pi f = h\pi$. If, in addition, the bundle M over N is a principal right G-bundle, where G is a Lie group, then the extension f is called a G-extension, if for $g\in G$, $x\in M$ $f(xg)=f(x)g$. The definition for flows is analogous.

If $f:M\to M$ is an extension of a $\mathscr{Y}$-diffeomorphism $h:M\to M$, $E^0=\operatorname{Ker}\pi$ and $\lambda_1\frac{\mu_2}{\mu_1}<1$, $\frac{\mu_2}{\lambda_2\mu_1}<1$, and the pair of bundles $W^s$, $W^u$ is almost locally transitive, then the conditions of 9.5.2 are satisfied, and hence, f is a K-automorphism [29]. If the bundles $W^s$ and $W^u$ are sufficiently smooth, then by virtue of 9.4.1, 9.5.2 is satisfied also for extensions of the diffeomorphism h which are sufficiently close to f in the $C^2$-topology.

9.7. We give more detailed results relating to a special class of G-extensions of $\mathscr{Y}$-flows. By the flow of frames on an n-dimensional compact Riemannian manifold M is meant a flow in the manifold $\Omega_n(M)$ of orthonormal tangent n-frames, under which the first vector of the frame moves with unit speed along a geodesic and the frame itself is parallel transported along this geodesic. Obviously, the frame flow is an SO(n − 1)-extension of the geodesic flow.

9.7.1. In [28, 29], M. I. Brin and Ya. B. Pesin, using 9.4.1 and 9.5.2, proved that frame flow is a K-flow if the Riemannian metric on M is close in the $C^3$-topology to a metric of constant negative curvature. For three-dimensional manifolds, ergodicity in this case was proved independently by A. B. Kramli [107].

9.7.2. There is a conjecture (Green [370]) that the frame flow is ergodic if the curvature on M is negative and the ratio of maximal and minimal curvatures in two-dimensional directions is less than four. In [29] an example of G. A. Margulis is given: If a Riemannian metric is Kahlerian, then the frame flow has a first integral. In such examples, the ratio of greatest curvature and least may be equal to four.

9.7.3. In [25] M. I. Brin proved that in the space of metrics of negative curvature of class $C^r$ $(r\geqslant 3)$ an open dense set is formed by the metrics for which the frame flow is a K-flow.

9.8. In [26] M. I. Brin proved that for any diffeomorphism or flow $f_t$ on M which is a G-extension of a $\mathscr{Y}$-system on N there exist a closed subgroup $H\subset G$ and a map $\sigma:M\to G/H$ of class $C^1$, such that for $x\in M$ we have $\sigma(x)\cdot g=\sigma(x\cdot g)$ and the preimages of left cosets Hg are invariant with respect to $f(f_t)$, and on each such preimage $f(f_t)$ is ergodic with respect to the smooth invariant measure. Whence it is easy to deduce that the frame flow on any three-dimensional manifold of negative curvature is a K-flow. Using some more refined topological considerations, D. V. Anosov proved an analogous fact for five-dimensional manifolds.

CHAPTER 3

# DYNAMICAL SYSTEMS ON HOMOGENEOUS SPACES

## §1. Flows on Homogeneous Spaces of Lie Groups

1.1. Let G be a Lie group, Γ be a closed subgroup of G. Let us assume that on the homogeneous space G/Γ there exists a finite measure, invariant with respect to the left action $\{T_g\}$ of the group G on G/Γ.* The restriction of this action to any one-parameter subgroup of G will be called a G-induced flow. For the study of the metric and topological properties of such flows, one uses algebraic methods and methods of the theory of representations. The first application of these methods appears in the paper of I. M. Gel'fand and S. V. Fomin [56], in which the methods of the theory of representations were used to calculate the spectrum of the geodesic flow on compact manifolds of negative curvature.

The fact that G-induced flows admit extension to the action of a wider class of Lie groups imposes essential restrictions on their properties. For example, there is a conjecture that the spectrum of G-induced flow can only have discrete and countably multiple Lebesgue components [174]. All the results obtained up to this time confirm this conjecture.

1.2. Let $g_t$ be a one-parameter subgroup of G. The expanding orispherical subgroup (cf. [55]) connected with $g_t$ is the collection $H^+$ of elements $h\in G$ such that as $t\to\infty$ $\lim g_{-t}hg_t=e$ (e is the identity of the group). The contracting orispherical subgroup $H^-$ connected with $g_t$ is the subgroup $H^+$ connected with $g_{-t}$. We denote by $\mathfrak{G}^c$ the complexification of the (real) Lie algebra $\mathfrak{G}$ of the group G. We decompose $\mathfrak{G}^c$ into a sum of root subspaces $V_\lambda$ of the operator $\mathrm{ad}\,x\colon y\to[x, y]$, $x\in\mathfrak{G}$. The subspace $V_\lambda+V_{\bar\lambda}$ is invariant with respect to complex conjugation in $\mathfrak{G}^c$ and hence can be represented in the form $\mathfrak{G}_\lambda+i\mathfrak{G}_\lambda$, where $\mathfrak{G}_\lambda\subset\mathfrak{G}$. We write $\mathfrak{G}^s=\sum_{\mathrm{Re}\lambda<0}\mathfrak{G}_\lambda$, $\mathfrak{G}^u=\sum_{\mathrm{Re}\lambda>0}\mathfrak{G}_\lambda$, $\mathfrak{G}^0=\sum_{\mathrm{Re}\lambda=0}\mathfrak{G}_\lambda$. Then $H^-=\exp\mathfrak{G}^u$, $H^-=\exp\mathfrak{G}^s$. We extend the decomposition $\mathfrak{G}=\mathfrak{G}^s+\mathfrak{G}^u+\mathfrak{G}^0$ to a left invariant decomposition of the tangent bundle of G: $TG = E^s + E^u + E^0$. The projection of this decomposition onto T(G/Γ) forms a decomposition of T(G/Γ) into contracting, expanding, and neutral distributions with respect to the flow $T_{\exp t x}$. The fibrations $W^s$ and $W^u$ on the integral manifolds of the first two distributions are fibrations into orbits of the action on G/Γ of the orispherical subgroups $H^-$ and $H^+$. Thus, if the subgroups $H^+$ and $H^-$ are not contained in Γ, then the G-induced flow $T_{\exp t x}$ is a partially hyperbolic system in the sense of Sec. 9 Chap. 2.

Transitivity of the pair of fibrations $W^s$ and $W^u$ in terms of the orispherical subgroups means that the subgroups $H^-$, $H^+$, and Γ in aggregate generate the entire group. Since the fibrations $W^s$ and $W^u$ are analytic, it follows from the transitivity that $T_{\exp t x}$ is a K-flow.

1.3. Basic special cases of dynamical systems on homogeneous manifolds are G-induced flows on factor spaces of semisimple and solvable groups. In the first case, the simplest examples are flows on factor spaces of the group G = SL(2, R) by discrete subgroups. If x is a second-order matrix with trace zero and distinct eigenvalues, the the subgroup $e^{tx}$ induces a $\mathscr{Y}$-flow on $G/\mathbf{Z}_2D$; it admits a geometric realization as the geodesic flow on a surface of constant negative curvature. If x has zero eigenvalues then the subgroup $e^{tx}$ induces a minimal ergodic flow on $G/\mathbf{Z}_2D$; its geometric realization is the oricycle flow. In the case of nonreal eigenvalues of the matrix x, the subgroup $e^{tx}$ gives a periodic flow.

A subgroup H of G is called ergodic if on each factor space G/Γ of finite volume the action of the subgroup is ergodic. The metric properties of group actions on homogeneous spaces of semisimple groups were considered by Moore [542]. He found in this case a geometric description of ergodic subgroups and proved the absolute continuity of the spectrum of the flow generated by a one-parameter ergodic subgroup. The final result in the problem of spectral analysis of such flows is due to A. M. Stepin [176]: A flow on a factor space of finite volume of a connected semisimple group induced by an ergodic subgroup has countably

*Such a measure is unique up to a factor. For a discussion of the question of the existence of an invariant measure, cf. [97].

multiple Lebesgue spectrum. Another basic result of the paper [176]: If R is a representation, which is continuous in the strong topology of a semisimple group G, by isometric operators on a Banach space B, then the subspace of $H^+$-invariant vectors in B coincides with the subspace of $H^-$-invariant vectors. Whence it follows that the measurable (topological) hull of the fibration $W^s$ or $W^u$ coincides with the measurable (topological) hull of the fibration by orbits of some normal subgroup of G. In particular, the flow $T_{\exp tx}$ will be a K-flow on any homogeneous space $G/\Gamma$ with finite left-invariant measure if an only if the operator adx in each simple ideal of the algebra $\mathfrak{g}$ has an eigenvector corresponding to an eigenvalue with nonzero real part.

1.4. The results on the topological properties and ergodicity of G-induced flows on nilpotent and some solvable manifolds obtained up to 1966 are expounded in the book of Auslander, Green, and Hahn [17]. Typical examples of G-induced flows on solvable manifolds are flows on factor spaces of semidirect products of G planes and their one-parameter groups of automorphisms $A_t$. If the group $A_t$ is hyperbolic, then the flows induced by regular elements of the Lie algebra of the group G are $\mathfrak{g}$-flows. Now if the group of automorphisms $A_t$ is compact, then any such flow is isomorphic with a flow induced by a one-parameter subgroup on the three-dimensional torus. Yet another interesting example is the G-induced flow on the factor space of a three-dimensional nilpotent Lie group N by a uniform discrete subgroup D. Such a flow is minimal and ergodic if an only if its factor-flow on the two-dimensional torus N/[N, N]D, where [N, N] is the commutator subgroup of N, has these properties. A G-induced flow on a nilpotent manifold is called a nilflow.

The final solution of the problems of ergodicity and minimality of flows on solvable manifolds was obtained by Auslander in [224, 227, 228]. He found necessary and sufficient conditions for the ergodicity of a G-induced flow on a homogeneous manifold of a solvable group. In [227] the theorem is proved: If the flow $T_t$ on the solvable manifold G/D, induced by a one-parameter subgroup of G, is minimal, then the manifold G/D is homeomorphic with a nilmanifold, and the flow $T_t$ is algebraically conjugate with a nilflow.

Let G be a simply connected solvable group, which is the image of its Lie algebra $\mathfrak{G}$ under the map exp $D\subset G$ be a uniform discrete subgroup. A. M. Stepin [171] proved that the spectrum of an ergodic flow on G/D, generated by a regular element of $\mathfrak{G}$, consists of discrete and countable multiple Lebesgue components. The proof is based on the fact that the measurable hull of the expanding fibration $W^u$ of such a flow coincides with the measurable hull $\xi$ of the pair of fibrations $W^u$ and $W^s$, while the factor-flow $T_{\exp tx}/\xi$ is isomorphic with a nilflow.

Parry [600] proved that metrically isomorphic nilflows are algebraically isomorphic.

1.5. Let G be a simply connected Lie group with Lie algebra $\mathfrak{G}$, while the factor-algebra $\mathfrak{G}$ by its radical contains no compact ideals. Conze [301] considered flows on compact homogeneous spaces of the group G, induced by real regular elements of $\mathfrak{G}$. It turned out that such flows have the K-property if they have continuous spectra.

Le G be a locally compact group; $H_1$, $H_2$ be closed subgroups of G. Using the duality between $H_1$-invariant measures on $G/H_2$ and $H_2$-invariant measures on $G/H_1$, Furstenberg [358] proved the strict ergodicity (cf. Sec. 1, Chap. 4) of the horocycle flow on compact surfaces of constant negative curvature. A generalization of this result was obtained in preprint of Eberlein and Veech.

A. S. Mishchenko [127] considered geodesic flows (with respect to left-invariant Riemannian metrics) on Lie groups and found a series of integrals of motions for geodesic flows on groups of orthogonal matrices with metrics corresponding to motions of n-dimensional rigid bodies.

## §2. Groups of Automorphisms and Affine Transformations

Another class of measure-preserving transformations, whose analysis can be carried out by algebraic means, is formed by the automorphisms and affine transformations of groups and their homogeneous spaces. Let $\Gamma$ be a discrete subgroup of G with factor space of finite volume, A be an automorphism (endomorphism) of the group G, carrying $\Gamma$ into itself, and $h\in G$.

The transformation $T:g\Gamma\to hA(g\Gamma)$ is called an affine transformation of the space $G/\Gamma$; it preserves the left-invariant measure on $G/\Gamma$. Typical examples of transformations of the class considered are automorphisms of tori and nilmanifolds, and also transformations with quasidiscrete spectrum (cf. par. 3).

A cycle of papers of Parry [596, 598-601, 603] is devoted to the analysis of the metric and topological properties of affine transformations of nilmanifolds. An affine transformation T of a nilmanifold $N/\Gamma$ is an extension (cf. Sec. 4) of an affine transformation T' on the torus $N/[N, N]\Gamma$. Necessary and sufficient conditions for ergodicity (strict ergodicity, minimality) of the transformation T are ergodicity* (strict ergodicity, minimality) of the transformation T'. An ergodic affine transformation T of a nilmanifold has countably multiple Lebesgue spectrum in the orthogonal complement of the space generated by the eigenfunctions of T [599]. Let S be an ergodic affine transformation of the nilmanifold X, T be a unipotent† affine transformation of the nilmanifold Y; if the homomorphism $\varphi:X\to Y$ is such that $\varphi S=T\varphi$, then $\varphi$ coincides almost everywhere with an affine homomorphism. In particular, if ergodic unipotent affine transformations of nilmanifolds are metrically isomorphic, then they are algebraically conjugate. In [603] it is shown that the class of ergodic unipotent affine transformations on nilmanifolds is closed with respect to factorization.

Conze [301] calculated a $\pi$-partition of an affine transformation of a compact homogeneous space. His method is analogous to the method of the earlier paper of A. M. Stepin [171]. Conze also proved that an affine transformation of a compact homogeneous space of a solvable group G has a continuous spectrum only in the case when G is nilpotent. The metric properties of endomorphisms on homogeneous spaces of compact groups are studied by S. A. Yuzvenskii in [187]. In [417] criteria are given for when the action of a commutative group of diffeomorphisms of a compact manifold is isomorphic with the action of groups of affine transformations on a factor space of a Lie group.

Walters [709, 710] proved that the problem of topological classification of affine transformations of compact commutative groups and nilmanifolds reduces to the problem of the algebraic similarity of such transformations. Brown [273] constructed a universal object for ergodic automorphisms of commutative metrizable groups.

Berg [244] proved that the metric entropy of a (group) automorphism of a compact metrizable group achieves its maximum on the Haar measure m. Now if the automorphism is ergodic with respect to m and has finite entropy, then m is the unique measure maximizing the entropy. Bowen [260] calculated the topological entropy $h_d(T)$ (with respect to the metric d) for affine transformations of Lie groups and their homogeneous spaces.

A series of papers is devoted to the question of the existence of ergodic automorphisms of noncompact locally compact groups. For commutative or connected groups, a negative answer to this question is due to S. A. Yuzvenskii [186] and Rajagopalan. Rajagopalan and Schreiber [615] announced a negative solution of this problem in the general case. Sato [643, 644] considers the question of the existence of an ergodic affine transformation of a noncompact locally compact group and gives a negative answer for commutative or totally disconnected groups (cf. also [215]). Moreover, for groups of this class Sato [641, 642] proved that compactness follows from the existence of a topologically transitive affine transformation.

## §3. Transformations with Quasidiscrete Spectra

Transformations with quasidiscrete spectra are generalizations of transformations with discrete spectra. An eigenfunction of an automorphism T of the space $(X, \mu)$ is called a quasieigenfunction of the first order. Functions of higher order are defined by induction. Namely, if $\varphi$ is a quasieigenfunction of order n, and $f\in L_2(X, \mu)$, $f\neq 0$, is a function such that $f(Tx)=\varphi(x)f(x)$, then f is called a quasieigenfunction of order n + 1. If the quasi-

*The criterion for ergodicity of an affine transformation of the torus is due to Hahn; for another proof cf. the survey of Auslander [225].

†An affine transformation $T:g\Gamma\to hA(g\Gamma)$ of a factor-space $G/\Gamma$ of a Lie group G is called unipotent if the differential of the automorphism A at the identity of the group G is unipotent.

eigenfunctions form a complete system in $L_2(X, \mu)$, then one says that the transformation T has quasidiscrete spectrum.

3.1. In 1962 L. M. Abramov [1] constructed a metric classification of completely ergodic* transformations with quasidiscrete spectra and proved that such transformations are realized as affine transformations of compact connected commutative groups. The metric properties of transformations with quasidiscrete spectra are considered by Hahn and Parry [383]. A completely ergodic transformation T with quasidiscrete spectrum is disjoint† from any completely ergodic transformation whose spectrum is singular mod{1} with discrete spectrum T. Any invariant partition for a transformation T with quasidiscrete spectrum is a partition on the cosets with respect to some closed subgroup upon realization of T as an affine transformation; whence follows, in particular, the closedness of the class of transformations with quasidiscrete spectra with respect to factorization.

3.2. In the posthumous paper of Hahn [381] interesting examples of one-parameter groups of transformations with quasidiscrete spectra are considered. Let $\Lambda_n$ be a closed subalgebra of the algebra of bounded continuous functions on the line, generated by elements of the form $e^{iq(s)}$, where q is a real polynomial of degree no greater than n. The real line acts on its compactification with respect to $\Lambda_n$. This action is ergodic and minimal. On the basis of this example, Hahn undertook an attempt to construct a theory of flows with quasidiscrete spectrum. He proved the existence of a flow with a given system of quasieigenfunctions. In the remarks [604] on the paper of Hahn, with the supplementary assumption of complete minimality of the flow, Parry proved a uniqueness theorem and a theorem about the representability of flows with quasidiscrete spectra as flows of affine transformations.

3.3. Brown [273] constructed a universal object for ergodic transformations with quasidiscrete spectra. In [212] it is proved that the maximal factor with quasidiscrete spectrum of an ergodic affine transformation of a connected group coincides with the maximal factor of zero entropy. In [209, 210, 213, 642] the indices of commuting for transformations with quasidiscrete spectra are computed. The question of the existence of roots of automorphisms with quasidiscrete spectra is solved by Michel [536-538]. Transformations with discrete spectra are considered in [490, 231, 232, 548, 505].

## §4. G-Extensions

A dynamical system $T_t$ in a space X, commuting with the action R of a compact group G, induces a factor-system $T_t'$ on the space of orbits of the action R and is called a G-extension of the system $T_t'$ (cf. Paragraph 9.6., Sec. 9, Chap. 2). If G acts freely, then each G-extension has the form $S_t: x \to \check{R}_{\theta(x,t)} T_t x$, where $T_t$ is some fixed G-extension. The function $\theta(x, t)$, called the cocycle corresponding to the extension $S_t'$ has the following properties: 1) $\theta(R_g x, t) = \theta(x, t)$; 2) $\theta(x, t+s) = \theta(T_t x, s)\theta(x, t)$. Extensions $S_t^{(1)}$ and $S_t^{(2)}$ are isomorphic if their corresponding cocycles $\theta^{(1)}$ and $\theta^{(2)}$ are homologous, i.e., there exists a measurable function $\varphi(x)$, such that $\theta^{(1)}(x, t) = \varphi(T_t x)\theta^{(2)}(x, t)\varphi^{-1}(x)$.

We note that the nilflows considered in Sec. 1 are obtained by successive application of the procedure of $\mathbf{T}^n$-extension from flows with discrete spectra on tori.

The question arises of under what conditions on the cocycle θ does the system $T_t$ inherit ergodic properties (such as ergodicity, mixing, complete positivity of entropy, minimality, etc.) from the system $T_t'$. In [595] Parry studies topological properties of G-extensions (for commutative G) and gets a structural theorem for a class of minimal transformations, generalizing transformations with quasidiscrete spectra.

Jones and Parry [422] proved that cocycles with values in commutative compact groups, homologous to the cocycle $\theta(x, t) \equiv e$, form a set of the first category in the group of cocycles with metric of the space C or $L_1$. Whence it follows that commutative G-extensions, as a rule, inherit the dynamical properties of the base.

*That is, ergodic along with all its powers.
†Transformations $T_1$ and $T_2$ are disjoint if they are always independent as factor-transformations (cf. [357]).

Thomas [696] considered the class of transformations T, satisfying the condition $T(R_g x)=R_{\sigma(g)}Tx$, where $\sigma$ is an automorphism of the group G. He proved that such extensions, having continuous spectra, inherit complete positivity of entropy. In [697] for such extensions there is proved an addition formula for entropy. Conze [303] made a survey of the metric properties of group endomorphisms and transformations, getting extensions with the help of group endomorphisms. He proved also that uniqueness of measures with maximal entropy is preserved for group extensions (cf. also [713] of Walters).

A student at Moscow university, Morozov, proved that a G-extension of an ergodic shift on a compact group H has quasidiscrete spectrum only in the case when the cocycle giving the extension is homologous to a homomorphism of H into G.

For a smooth flow, which is a $\mathbf{T}^n$-extension of a flow on a torus with discrete spectrum and sufficiently incommensurable eigenvalues, a geometric condition for its realization as a nilflow is obtained in [272].

A. G. Kushnirenko [113] considered an $\mathbf{S}^1$-extension of a rotation of a circle of the form $(x, y)\to(x+y+h(y), y+\alpha)$ and under the additional assumptions $h\in C^1$, $h'+1>0$, he proved that in the orthogonal complement of the subspace of functions depending only on y, this transformation has countably multiple Lebesgue spectrum.

Interesting examples of group extensions of Bernoulli automorphisms were considered by Goldstein, Landford, and Leibowitz [362] in connection with a physical model.

# CHAPTER 4

# DYNAMICAL SYSTEMS ON COMPACT METRIC SPACES

In this chapter we shall consider continuous dynamical systems (homeomorphisms and one-parameter groups of homeomorphisms of compact metric spaces) whose space of Borelian invariant measures is finite-dimensional. We note that much has been said in Chap. 2 (especially Secs. 1, 3, and 5) about dynamical systems on compact metric spaces, but there in the center of attention was the situation (of most interest at least from the point of view of applications) when there are very many invariant measures. On the other hand, in Chap. 3 many strictly ergodic systems of algebraic origin were mentioned, and hence we shall not return to their consideration. We shall dwell on two questions: strict ergodic realization of abstract dynamical systems (existence theorems in Sec. 1 and concrete constructions in Sec. 2) and the properties of some closely connected and completely concrete classes of dynamical systems (Sec. 3), which are interesting from the point of view of applications and at the same time are typical for the problems considered in this chapter.

In writing the first two paragraphs, an acquaintance with Jacobs' survey ([414], preprint), the third paragraph of which is devoted to the same problems, turned out to be very useful.

## §1. Strictly Ergodic Realization of Dynamical Systems

1.1. A homeomorphism f (respectively, a continuous flow $f_t$) of a compact metric space X is called strictly ergodic if it has a unique Borelian normalized invariant measure. This is equivalent with the fact that for any continuous function $\varphi$ on f the mean

$$\frac{1}{n}(\varphi(x)+\varphi(fx)+\ldots+\varphi(f^{n-1}x))$$

$$\left(\text{respectively},\ \frac{1}{T}\int_0^T\varphi(f_tx)\,dt\right)$$

converges uniformly as $n\to\infty$ $(T\to\infty)$ to a constant depending on $\varphi$. The homeomorphism or continuous flow is called minimal if any of its trajectories is everywhere dense. Minimality follows from strict ergodicity. The metric properties of strictly ergodic dynamical systems with respect to the unique invariant measure are, obviously, topological invariants of the dynamical system.

Classical examples of strictly ergodic dynamical systems are natural realizations of ergodic automorphisms and flows with discrete spectra on commutative compact groups, and also the corresponding realizations of ergodic automorphisms with quasidiscrete spectra and nilflows (cf. Chap. 3). These examples give rise to the impression that strict ergodicity may impose some additional restrictions besides erogodicity on the metric properties of a dynamical system with respect to its unique invariant measure. In particular, at the fifth Berkeley symposium on probability theory in 1965, the question was posed of whether a strictly ergodic system can have positive entropy [414].

If one speaks of smooth or even continuous strictly ergodic systems on manifolds, then this question remains open, since in all known examples the entropy is equal to zero. In general, all known examples of strictly ergodic diffeomorphisms of manifolds are connected with more or less special constructions, either algebraic (cf. Chap. 3), or of the character of approximations (cf. Paragraph 1.3, Chap. 5).

1.2. In the case of continuous, and in partituclar symbolic (cf. Paragraph 3.1, Chap. 2) systems, strict ergodicity turned out to be compatible with arbitrary metric properties (in the symbolic case, one has, of course, obvious entropy restrictions).

The first important paper in the direction indicated was the paper of Hahn and Katznelson [382], in which a positive answer is given to the question formulated at the end of the previous point. The example of Hahn and Katznelson is a symbolic system. A decisive step was made by Jewett [416], proving that any weakly mixing automorphism of a Lebesgue space admits a strictly ergodic realization. The homeomorphism obtained with the help of the construction of Jewett, in general, is not symbolic, although the space on which it acts is homeomorphic with a Cantorian perfect set.

The final solution of this question is due to Krieger [480]. He proved that any ergodic automorphism of a Lebesgue space with finite entropy h admits a strictly ergodic realization on a closed subset of the space $\sigma_k$ which is invariant with respect to the shift $\Sigma_k$, if $k > e^h$ (cf. Paragraph 3.1, Chap. 2). There is an analogous result for automorphisms with infinite entropy. Another proof of this theorem was given by Denker [311]. In the proof the theorem of Krieger on generating partitions [472] (cf. Sec. 1, Chap. 1) is used.

Hansel and Raoult [386] proved a theorem from which the possibility of strictly ergodic realization of any ergodic automorphism of a Lebesgue space also follows: If $T:(X, \mu)\to(X, \mu)$ is an ergodic automorphism, then there exists a countable subalgebra $\mathfrak{A}'$, dense in the σ-algebra of all measurable sets, invariant with respect to T and such that for any set $A\subset\mathfrak{A}'$ the average characteristic functions $\frac{1}{n}\sum_{i=0}^{n-1}\chi_A(T^ix)$ converge uniformly to a constant $\mu(A)$.

Another generalization of the theorem on strictly ergodic realization is connected with nonergodic transformations. Hansel [388] proved that for any automorphism of a Lebesgue space one can find a strictly uniform realization. A homeomorphism of a compact metric space X is called strictly uniform if the closure of any trajectory is a minimal set and the averages of any continuous function converge uniformly to a limit. Then the space X is partitioned into the closures of trajectories, on each of which the homeomorphism is strictly ergodic, and any invariant measure decomposes into ergodic components with respect to these strictly ergodic measures.

1.3. One can obtain the theorem on strictly ergodic realizations for ergodic flows more naturally from a theorem on special representations. Krieger's theorem on strictly ergodic realizations for automorphisms and Ornstein's theorem on continuous realization of any measurable change of time (Paragraph 3.2.2, Chap. 6). With this method one gets that any ergodic flow is metrically isomorphic with the suspension of some symbolic strictly ergodic system (cf. Paragraph 3.8, Chap. 2), which is a strictly ergodic flow. However, the original proof of the theorem on strictly ergodic realizations of flows used not a direct reduction to the case of discrete time, but a certain analogy with this case. Namely, Jacobs [413] considered the space L of functions on the line with values in the segment [0, 1], satisfying a Lipschitz condition with constant 1, with uniform metric, and the flow $\Phi_t$ in this space generated by the natural one-parameter group of shifts of the line. The restrictions of this flow to closed invariant subsets are analogues of symbolic systems. Jacobs, following the method of Jewett, proved that any weakly mixing flow admits a strictly ergodic realization in the Tikhonov product of a countable number of copies of the space L with the corresponding flow. The analogue of Krieger's theorem was proved by Denker and E. Eberlein [314]: Any ergodic flow admits a strictly ergodic realization in the space L on a closed subset which is invariant with respect to the flow $\Phi_t$.

## §2. Some Symbolic Systems

The invariant measure of a strictly ergodic system is completely re-established on any trajectory of this system. In this connection there is interest in the following problem: how, for a given ergodic automorphism T, to construct a sequence $\omega \in \Sigma_k$, such that the restriction of the shift $\sigma_k$ to the closed trajectory $\{\sigma_k{}^n\omega\}$ is strictly ergodic and metrically isomorphic with the automorphism T. Here it is assumed that the sequence must be constructed with the help of an algorithm which can be described as a program for a Turing machine.

The inverse problem consists of the following. For a sequence $\omega \in \Sigma_k$, which is defined with the aid of an algorithm, to determine whether the restriction of $\sigma k$ to the closed trajectory $\omega$ will be a strictly ergodic homeomorphism, and in the case of a positive answer to find the metric properties of $\sigma_k$ with respect to the corresponding invariant measure. The direction in ergodic theory which occupies itself with similar questions is called by K. Jacobs [412] the combinatorial approach. The first combinatorial construction of a nontrivial strictly ergodic transformation appeared in the paper of Kakutani [423], but the idea of such constructions goes back to Morse, Gottschalk, and Hedlund.

The majority of the papers here relate to the second of the problems formulated. In Kakutani's examples [423] the automorphism from the metric point of view is a skew product over an automorphism with a discrete spectrum with a two-pointed fiber. The set of eigenvalues coincides with the set of roots of unity of powers $2^n$, $n = 1, 2, \ldots$. In the orthogonal complement of the corresponding eigenfunctions the spectrum is automatically continuous. Kakutani's construction was generalizaed by Keane [432].

Jacobs and Keane [415] considered so-called Toeplitz sequences and proved strict ergodicity and discreteness of the spectrum of symbolic systems connected with these sequences. An interpretation of the results of [415] was given by Neveu [554], who showed how to establish a metric isomorphism between symbolic systems and the corresponding shifts on commutative compact groups. The paper of E. Eberlein [319] is also devoted to Toeplitz sequences.

Generalizations of the constructions of Kakutani and Keane are the so-called sequences (or minimal sets) generated by substitutions (having in mind substitutions of elements of the alphabet). These sequences were considered by Martin [524], Coven and Keane [306], Kamae [426], and Michel [539]. In the latter paper, the strict ergodicity is proved of the restriction of the shift in the space $\Sigma_k$ to the closures of trajectories of sequences generated by substitutions. The remaining papers are devoted to the description of metric properties (the entropy always turns out to be equal to zero), the conditions for topological and metric conjugacy in the class of dynamical systems considered, and the construction of normal forms for these systems. The papers of Veech [704] and Petersen [610] also partially touch on the problems considered.

A step in the direction of the first of the problems of the combinatorial approach (the building of combinatorial constructions realizing automorphisms with given metric properties) appeared in the papers of Grillenberger [372, 373]. In [372] for any h, $0<h<\ln k$, a combinatorial construction is given of a strictly ergodic symbolic system in the space $\Sigma_k$ with entropy h. In [373] for any topological Markov chain $\sigma_A$ (cf. Paragraph 3.1, Chap. 2) and any $\varepsilon>0$ a combinatorial construction is built of a strictly ergodic symbolic system in the space $\Sigma_A$, which is a K-automorphism and has entropy greater than $h_{\text{top}}(\sigma_A)-\varepsilon$ (it is assumed that $h_{\text{top}}(\sigma_A)>0$).

## §3. Exchange of Segments, Flows on Surfaces, Billiards in Polygons

3.1. By exchange of segments is meant a one-to-one map f of the segment [0, 1] onto itself, preserving Lebesgue measure and discontinuous at a finite number of points. We denote the intervals of continuity in the order of their location on the segment from the left end to the right by $\Delta_1, \ldots, \Delta_m$. Sometimes it is also assumed that on each interval of continuity the exchange preserves orientation. We shall call such a map an oriented exchange. An oriented exchange is completely determined (up to the trajectories of the points of discontinuity, which as we shall see below, it is natural to disregard) by a vector $(a_1, \ldots, a_m)$, $a_i > 0$, $i = 1, \ldots, m$, $\sum_{i=1}^{m} a_i = 1$, where $a_i$ is the length of the interval $\Delta_i$, and a permutation $\sigma \in S_m$, which indicates the order in which the images of the intervals $\Delta_i$ are situated on the segment. In the general case it is still necessary to add another vector of $\pm 1$ $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)$, where $\varepsilon_i$ is equal to the derivative of f on the interval $\Delta_i$.

Although exchanges are discontinuous maps, they are of interest from the metric as well as the topological point of view. Here the question will be of the metric properties with respect to Lebesgue measure as well as of properties of the entire collection of invariant measures of an exchange. The following condition allows one to discard some automatically uninteresting cases.

3.1.1. $\xi \wedge f\xi \wedge \ldots \wedge f^n \xi \ldots = \varepsilon$, where $\xi = (\Delta_1, \ldots, \Delta_m)$.

If this condition is not satisfied, then the entire segment is filled by periodic points.

V. I. Oseledets in [130] first systematically cosidered exchanges, in truth only from the point of view of metric properties with respect to Lebesgue measure. He proved the following theorem.

3.1.2. If 3.1.1 is satisifed, then the multiplicity of the spectrum of the exchange with respect to Lebesgue measure does not exceed m − 1.

From this it is easy to deduce the following assertion (or to carry over the proof of 3.1.2).

3.1.3. The number of Borelian, invariant, normalized measures of an exchange satisfying 3.1.1 does not exceed m − 1. The multiplicity of the spectrum with respect to any invariant measure does not exceed m − 1.

The first half of 3.1.3 was also proved by Keane under somewhat stronger restrictions on f [434].

If 3.1.1 is satisfied, one can introduce in the complement of the points of discontinuity of a trajectory a new topology, stronger than the topology of a segment, with respect to which the exchange is a symbolic system in $\Sigma_m$. Keane [434] proved the following theorem.

3.1.4. If the permutation $\sigma$ does not leave invariant any set of the form $\{1, \ldots, j\}$, $j = 1, \ldots, m-1$, the exchange is orientable and between the numbers $a_1, \ldots, a_m$ there are no other rational relations besides $\sum_{i=1}^{m} a_i = 1$, then the symbolic system connected with the exchange is minimal.

Keane stated the conjecture that in the situation 3.1.4 the exchange is strictly ergodic, but as we shall show below (cf. Paragraph 3.2), this is not always so. A. B. Katok proved the following fact.

3.1.5. An exchange is not mixing with respect to any of its invariant measures.

In the ergodic, or what is equivalent the minimal case, 3.1.5 follows from the following assertion.

3.1.6. For any natural number N one can find a partition $\eta_N = (A_1^N, \ldots, A_{(m+1)^2}^N)$ and numbers $n_i(N) > N$, $i = 1, \ldots, (m+1)^2$ such that for any measurable set A as $N \to \infty$

$$\mu\left(A \Delta \left(\bigcup_{i=1}^{(m+1)^2} T^{n_i(N)} \left(A \cap A_i^N\right)\right)\right) \to 0.$$

Selecting the parameters of the exchange in a special way, one can get examples of automorphisms with various interesting metric properties (cf. for this Sec. 3, Chap. 5).

3.2. With exchanges there is closely connected a more "traditional" class of dynamical systems — certain flows on surfaces. Let $f_t$ be a smooth flow on a compact surface $M^2$, havir a finite number of fixed points, which are generalized saddles, and all points of $M^2$ nonwandering with respect to the flow $f_t$. Then the map of following for $f_t$ on any transversal section is topologically conjugate to some exchange, while the function of time of return has singularities at the points corresponding to the appearance of separatrices of fixed points of the flow. We note that in this case an invariant measure for the exchange is not necessarily generated by the invariant measure for the flow, since the time of return function may turn out to be nonintegrable for the relatively invariant measure of the exchange. In order to save oneself from similar effects, A. B. Katok [88] proposed along with finite to consider certain $\sigma$-finite invariant measures of flows of the type described. Namely, an invariant measure of a flow is called nontrivial if the measure of any trajectory is equal to zero and the measure of the complement of any neighborhood of the set of fixed points is finite.

If $M^2$ is an orientable surface of genus p, then in the typical situation on the transversal section there arises an exchange of $4p - 2$ segments, whence, by virtue of 3.1.2, one gets an estimate for the number of nontrivial ergodic measures of the flow $f_t$ up to normalization. However, this estimate can be significantly improved with the aid of additional geometric considerations, connected with the rotation number. In [88] the following assertion is proved.

3.2.1. The number of nontrivial invariant ergodic measures of a flow satisfying the conditions formulated above does not exceed p (the measure is considered up to factors). In [88] it is assumed that the fixed points are nondegenerate saddles, but this additional restriction is absolutely inessential for the proof (cf. [78]).

The analogous property of minimality in the case of flows on surfaces is the following property, called quasiminimality in [78] and equivalent in the case considered to topological transitivity: Any semitrajectory, except for fixed points and their separatrices, is everywhere dense on the manifold. In [88] there is a criterion for quasiminimality of flows, analogous to 3.1.4, which is deduced from some strong results of A. G. Maier. Moreover, in this paper there are found sufficient conditions for topological conjugacy (but for flows with smooth invariant measure and differentiable) up to change of time with flows of the type considered in terms of some finite-dimensional objects, generalizing rotation numbers.

The estimate given by 3.2.1 is unimprovable even in the class of quasiminimal flows. This was proved by E. A. Sataev [151].

3.2.2. For any k, $1 \leqslant k \leqslant p$, on an orientable surface of genus p, there exists a quasiminimal flow (all its fixed points are nondegenerate saddles), having exactly k normalized invariant ergodic measures, whose sum with certain positive coefficients is a smooth measure, and having no other nontrivial invariant ergodic measure.

In [151] the analogous result for exchanges of a special form is first proved, and then a flow is constructed on the surface for which the map of following on transversal sections is an appropriate exchange. In the case k = p, the restriction of the exchange to each ergodic component is isomorphic with some rotation of the circle.

3.3. Certain considerations expounded in the preceding points turned out to be useful in the study of billiards in polygons (for the definition of billiards cf. Paragraph 8.2, Chap. 2), to which the paper of A. N. Zemlyakov and A. B. Katok [78] is devoted. In this paper, billiards in polygons all of whose angles are commensurable with $\pi$ are first considered. In this case the phase space decomposes into invariant two-dimensional manifolds, while after a change of time on all the manifolds, except for a countable set, the flow becomes miminal. The question of the ergodicity with respect to the natural invariant measure still remains open. Then, with the aid of passage to a limit, it is proved that for a typical (from the categorical point of view) polygon, billiards are topologically transitive in the entire phase space.

## CHAPTER 5

## APPROXIMATION METHODS IN THE THEORY OF DYNAMICAL SYSTEMS WITH INVARIANT MEASURE

The approximation approach to the problems of ergodic theory in a quite general way consists of the fact that the action of a noncompact group is represented as a limit in some sense of the actions of compact groups. In the case of a transformation (action of the group

Z), such approximating groups are naturally finite cyclic groups, while a flow (action of R) can be approximated either by finite cyclic groups of by circles $S^1$; for the action of $R^k$ the role of approximating groups can be played either by finite commutative groups with a finite number of generators, or by tori $T^l$, $l\leqslant k$. Approximation assertions are used in the study of general dynamical systems with invariant measure, (cf. Paragraph 2.1); approximation criteria allow one to establish various metric properties of automorphisms and flows (Sec. 2, Sec. 4, Chap. 6) and in particular, verify the typicality of these properties in various spaces of dynamical systems (Sec. 4); finally, approximation constructions allow one to construct a variety of examples of dynamical systems with interesting properties in abstract spaces with measure, as well as in topological and smooth situations (Secs. 1 and 3).

If in the fifties and the first half of the sixties the basic source of new examples for ergodic theory was probability theory (cf. [48]), then beginning with the middle of the sixties, this role passed to "geometric" approximating constructions.

We shall begin (Sec. 1) with the description of a class of such constructions, where smooth dynamical systems on manifolds with smooth invariant measure are constructed (in particular, Hamiltonian systems) and the approximating maps converge to a limit in the $C^\infty$-topology, while the speed of this convergence turns out to be rather high. Then (Sec. 2) general aspects of the approximation approach in the abstract metric situation are considered.

## §1. Smooth Realization of Ergodic Dynamical Systems

1.1. Let M be a compact connected manifold of class $C^\infty$ of dimension not less than two, closed or with boundary, on which there is a nontrivial $C^\infty$-action of the group of rotations of the circle $S^1$. We denote this action by $\{S_t\}$, $0\leqslant t\leqslant 1$, $S_1=\mathrm{id}$. Let $\mu$ be a measure on M, invariant with respect to the action $\{S_t\}$ and defined in local coordinates by a $C^\infty$-density, which can vanish only on the boundary of M. We denote by $\mathrm{Diff}^\infty(M,\mu)$ the space of $C^\infty$-diffeomorphisms of M, preserving the measure $\mu$, endowed with the $C^\infty$-topology. In the papers of D. V. Anosov and A. B. Katok [14, 15] the following theorem is proved.

1.1.1. For any $t_0: 0\leqslant t_0<1$, in any neighborhood of the diffeomorphism $S_{t_0}$ in the space $\mathrm{Diff}^\infty(M,\mu)$, there exist diffeomorphisms which are ergodic with respect to the measure $\mu$, having (according to choice) discrete, continuous, or mixed spectra, admitting cyclic approximation by periodic transformations (a.p.t.) with any preassigned speed f(n) (cf. [93, 48] and Paragraph 2.4.1 below; different sequences f(n) correspond, in general, to different diffeomorphisms), while the discrete components in the spectra can be generated by any preassigned number (finite or infinite) of independent frequencies over the ring of integers.

1.1.2. A diffeomorphism T, satisfying a specific variant of 1.1.1, is constructed as a limit as $n\to\infty$ converging in the $C^\infty$-topology of a sequence of diffeomorphisms $T_n$ of the form $T_n = B_n^{-1}S_{\alpha_n}B_n$, where

$$B_n\in\mathrm{Diff}^\infty(M,\mu),\ \alpha_n=\alpha_{n-1}+\beta_n,\ \alpha_n=\frac{p_n}{q_n},\ \beta_n=\frac{1}{s_nq_{n-1}},$$

$p_n, q_n, s_n$ are integers, $B_n=A_nB_{n-1}$, $S_{\alpha_{n-1}}A_n=A_nS_{\alpha_{n-1}}$. A basic role in the n-th step of the construction is played by the construction of the diffeomorphism $A_n$. This diffeomorphism, in general, has very many derivatives and is constructed so that $T_n$ is cyclically represented by elements of some finite partition $\xi_n$, which is a refinement of a partition $\xi_n'$, constructed so that each of its elements has "nucleus" of small diameter and large conditional measure.

In the construction of $A_n$ it is convenient to use the following assertion proved by A. B. Krygin [109] (in [14, 15] a much weaker assertion is used).

1.1.3. Let $O^m$ be an open m-dimensional connected $C^\infty$- manifold, $\mu$ be a measure $O^m$, given by a locally positive $C^\infty$-density, $F_i$, $G_i$, i = 1, ..., k, be two collections of compact subsets of $O^m$, each of which is $C^\infty$-diffeomorphic with an m-dimensional ball $D^m$, while $F_i\cap F_j=G_i\cap G_j=\varnothing$ for $i\neq j$ and $\mu(F_i)=\mu(G_i)$. Then there exists a $C^\infty$-diffeomorphism $S: O^m\to O^m$, which is the identity outside some compact $N \subset O^m$, preserving the measure $\mu$ and such that $SF_i = G_i$, $i=1,\dots,k$.

1.1.4. The choice of the number $\beta_n$ is made after the construction of $A_n$. It can be chosen so small that first of all, the convergence of the sequence $T^n$ in the $C^\infty$-topology is guaranteed, and secondly, one can construct with the help of the partition $\xi_n$ and diffeomorphism $T_n$ a cyclic a.p.t. limit diffeomorphism with given speed. At each step of the induction the construction of $A_n$ and $\beta_n$ depends on a series of parameters, the special choice of which allows one to get the properties which were spoken about in 1.1.1 for the limiting diffeomorphism. Here it turns out that for "quite natural" choice of the parameters the diffeomorphism T is metrically isomorphic with a rotation of the circle by the angle $2\pi\alpha$, where $\alpha = \lim\limits_{n\to\infty} \alpha_n$.

This circumstance can be interpreted in the following way: Although the sequence of diffeomorphisms $B_n$ does not converge in any sense, there exists a map $\pi$ of the set $M_r$ of regular points of the action of $\{S_t\}$ onto $S^1$, such that for $x \in M_r$ we have $\pi S_t x = \pi x + t \pmod 1$, while $\pi_* \mu = \lambda$ is the Lebesgue measure on the circle, and the sequence of maps $B_n \circ \pi$ converges to a one-to-one mod 0 map, which realizes the isomorphism between T and a rotation of the circle by the angle $2\pi\alpha$.

1.2. There follows immediately from theorem 1.1.1, for example, the existence of ergodic diffeomorphisms with smooth invariant measure on spheres and balls (cf. [48]), and also many "nonstandard" smooth realizations of automorphisms with discrete spectra. Not as immediately, one deduces from this theorem the following assertion [13, 14].

1.2.1. Let M be an arbitrary compact connected $C^\infty$ manifold, closed or with boundary, of dimension greater than two, and $\mu$ be a measure on M, defined by a locally positive density of class $C^\infty$. There exists a flow of class $C^\infty$ on M, preserving the measure $\mu$ and ergodic with respect to this measure.

Reduction of 1.2.1 to 1.1.1 is done as follows [14]. Let $\nu$ be a measure on the ball $\mathbf{D}^{m-1} = \left\{(x_1,\dots,x_{m-1}) : \sum\limits_{i=1}^{m-1} x_i^2 \leqslant 1\right\}$, such that $d\nu = \rho(\Sigma x_i^2)\, dx_1 \dots dx_{m-1}$, while as $r \to 1$, $\rho(r) \to 0$ with sufficiently high speed. With the aid of 1.1.1, one constructs an ergodic diffeomorphism $T \in \mathrm{Diff}^\infty(\mathbf{D}^{m-1}, \nu)$, which is close to the identity. Then one constructs a flow in the direct product $\mathbf{D}^{m-1} \times S^1$ preserving the measure $\nu \times \lambda$ ($\lambda$ is the Lebesgue measure), for which T is the following map on $\mathbf{D}^{m-1}$. Using a $C^\infty$-map of $\mathbf{D}^{m-1} \times S^1$ onto $\mathbf{D}^m$, which is a diffeomorphism off the boundary, and making an appropriate change of time, we get a flow on $\mathbf{D}^m$ that preserves the Lebesgue measure, with preassigned (independent of the choice of $\rho$) speed of decrease of the vector v of the phase space at the boundary. The manifold M is the image of $\mathbf{D}^m$ under a $C^\infty$ map f, which is a diffeomorphism off the boundary.

Let $d(f^*\mu) = \sigma(x_1 \dots x_m)\, dx_1 \dots dx_m$. The flow generated by the vector field $\sigma^{-1}v$ preserves the measure $f^*\mu$, and for sufficiently rapid decrease of v at the boundary, $\sigma^{-1}v$ is a vector field of class $C^\infty$. The vector field $f_*(\sigma^{-1}v)$ generates the desired flow on M.

In the paper of D. V. Anosov [13] some other variants of the reduction are given with all the details. About other metric properties of the flows obtained, besides ergodicity, it is difficult to judge, since in the process of construction a time change has been made. At least, this flow is standard (cf. Sec. 2, Chap. 6 and [91]).

1.2.2. A. A. Blokhin [23] constructed examples of ergodic flows with invariant measure on any surface except the sphere, projective plane, and Klein bottle, on which such flows cannot exist. These flows have closed transversals and hence, are also standard.

1.2.3. The papers of E. A. Sidorov [152-154], which appeared earlier than [14, 15] partially abut on the problems considered in the present paragraph. In these papers, for arbitrary domains of Euclidean space, topologically transitive diffeomorphisms are constructed, which also have some additional properties. In Sidorov's examples, either there is no invariant smooth measure at all or there is a continuum of ergodic components.

1.3. In the constructions of Paragraphs 1.1 and 1.2, the behavior is controlled not of all, but only almost all trajectories T. In a series of cases, on some set of trajectories of measure zero, the behavior is inevitably different; this happens if M has a boundary, or if there is no nontrivial action of $S^1$ on M, or if there is such an action but not all the orbits are regular. However, if the action $\{S_t\}$ is free, then one can realize a uniform variant of the construction of Paragraph 1.1 (cf. [87]).

1.3.1. Let M be a compact closed manifold of class $C^\infty$, $\dim M = m \geqslant 2$ and suppose there is a free $C^\infty$-action $\{S_t\}$, $0 \leqslant t \leqslant 1$, $S_1$-id, of the group $S^1$ on M; in other words, M is a principal $S^1$-bundle of class $C^\infty$ over some manifold of dimension m − 1. Let μ be a smooth $C^\infty$-measure, invariant with respect to the action $\{S_t\}$. Then for any $t_0: 0 \leqslant t_0 < 1$ in any neighborhood of the diffeomorphism $S_{t_0}$ in $\mathrm{Diff}^\infty(\bar{M}, \mu)$ there exists a minimal diffeomorphism T, having any preassigned finite or countable set of ergodic invariant measures. In particular, the diffeomorphism T can be strictly ergodic.

1.3.2. For flows one has an analogue of Theorem 1.3.1, if $M \geqslant 3$ and on M the toru $\mathbf{T}^2$ acts freely. Here the role of the "unperturbed" diffeomorphism $S_{t_0}$ is played by the flow, generated by some one-parameter subgroup of $\mathbf{T}^2$. From this assertion it follows that on some simply connected manifolds (for example, on product of odd-dimensional spheres) there exist minimal flows, which refutes the strong conjecture of Gottschalk.

1.4. In some situations, a construction analogous to the one described above can be carried out in the class of Hamiltonian systems. This is done in the papers of A. B. Katok [85, 89]. The condition of being Hamiltonian essentially decreases the "flexibility" of the construction.

1.4.1. In [85] a $C^\infty$-Hamiltonian system (flow) is constructed on the space $\mathbf{R}^{2m}$, uniformly close in the $C^\infty$-topology to a system generated by m independent oscillators with commensurable frequencies, ergodic on each manifold of constant energy and having for almost all values of the energy a discrete spectrum, generated by k independent frequencies $(2 \leqslant k \leqslant m)$, one and the same for distinct values of the energy (cf. the end of Paragraph 1.1.4).

1.4.2. Another variant of the Hamiltonian construction is given in detail in [89]. In this paper a noncompact symplectic manifold $(M^{2m}, \Omega)$ closed or with boundary (Ω is the canonical 2-form) is considered with additional structure — a complete vector field u, such that the Lie derivative $\mathscr{L}_u\Omega$ is equal to $\lambda\Omega$, where λ is a positive constant. A function H on $M^{2m}$ is called homogeneous if $\mathscr{L}_u H = \lambda H$. It is assumed that on $M^{2m}$ there is given an effective symplectic action of the two-dimensional torus $\mathbf{T}^2$, while each one-parameter subgroup $\gamma \subset \mathbf{T}^2$ correspond to a Hamiltonian vector field $v_\gamma$ with homogeneous Hamiltonian function $H_\gamma$. If for some subgroup γ the manifold $H_\gamma^{-1}(1)$ is compact, then the function $H_\gamma$ can be perturbed in the class of $C^\infty$ homogeneous functions by an arbitrarily small amount so that the perturbed function generates a Hamiltonian vector field, ergodic on each manifold of constant energy. The condition of homogenity allows one to work on all manifolds of constant energy at once, but on the other hand, lessens the arbitrariness in the construction. Hence aside from ergodicity one can only guaranteee the standardness of the flows constructed (cf. Sec. 2, Chap. 6).

This result is also applied in [89] to the case of independent oscillators, and also allows one to construct Finslerian metrics on compact symmetric spaces of rank 1, close to standard Riemannian metrics and such that the geodesic flows have two ergodic components filling the phase space except for a set of arbitrarily small measure.

## §2. Various Types of Approximation

2.1. A classical and extraordinarily useful fact of approximation character is the lemma of Halmos—Rokhlin on uniform approximation: For any aperiodic autormophism T: $(X, \mu)\to(X, \mu)$, any natural number n and $\varepsilon>0$ one can find a set $A\subset X$, such that

$$A\cap T^iA=\varnothing,\ i=1,\ \ldots,n-1,\ \mu\left(\bigcup_{i=0}^{n-1}T^iA\right)>1-\varepsilon.$$

This lemma is used widely in ergodic theory. As examples, we indicate papers on the isomorphism problem (cf. Sec. 2 Chap. 1), time change ([91, 104, 354]; Secs. 1, 2, Chap. 6), the connection of the speed of approximation with entropy [82, 656], and also the paper of A. B. Katok and Foiash [95], in which it is proved that for any aperiodic automorphism and continuous function $\varphi;S^1\to S^1$, from the fact that the operator $\varphi(U_T)$ is generated by some automorphism, it follows that $\varphi(\lambda)=\lambda^n$, $n\in\mathbf{Z}$.

The lemma of Halmos—Rokhlin was generalized in various directions: for subsequences of powers of an automorphism (Keane, Michel [435]), for transformations with quasi-invariant measure, for actions of certain countable groups on which there is an invariant mean (A. M. Vershik [47]). These assertions are connected with the property of approximate finiteness of the operator algebra constructed on a given group of transformations (cf. Chap. 7).

2.2. The further development of approximation methods is connected with the study of metric invariants of approximation character. This approach is based on the refinement of the concept of approximation in two ways, in comparison with approximations, which are given by the lemma of Halmos—Rokhlin. First of all, with an approximating sequence of action of compact groups there is connected an exhausting (tending to ε) sequence of invariant partitions $\xi_n$, such that the factor-space $X|_{\xi_n}$ consists of a finite number of orbits of the approximating action. Secondly, one introduces a certain concept of the speed of convergence of approximating actions to the limit. In what follows, we shall speak almost exclusively of automorphisms.

The program indicated can be realized in many ways which differ essentially (which is connected in the first instance with the desire to cover approximation approaches to dynamical systems with various properties) as well as in many secondary points (which is connected, basically, with the fact that only asymptotic properties of the approximating sequences of transformations are essential and hence there arises a large amount of nonuniqueness in the choice of these sequences).

We note that approximating sequences of transformations do not figure explicitly in all variants of the concept of approximation. A complete analysis of the logical connections between various concepts of approximation which we have is not carried out and apparently this is a thankless problem. It is more important to identity those variants of the definition which are better connected with the basic metric invariants (decomposition into ergodic components, various mixing properties, entropy) and with the basic constructions of ergodic theory (derivative, special automorphism, factor-automorphism, skew product).

2.3. The approach described at the beginning of the previous point was given in the papers of A. B. Katok and A. M. Stepin [92, 93], where some variants of the definition of approximation by periodic transformation (a.p.t.) with given speed were introduced for automorphisms and flows — a.p.t.I, a.p.t.II, cyclic a.p.t. — and the connections of these concepts with the basic metric invariants of automorphisms and flows were established. The basic results of [92, 93] are formulated in the survey of A. M. Vershik and S. A. Yuzvenskii [48], and [93] is devoted to the development of approximation methods up to 1967.

It should be noted that Chacon in [282, 283], which appeared almost simultaneously with [92, 93], used an approximation approach and corresponding techniques for the construction of an example of an automorphism T with continuous spectrum, having no roots (i.e., $T\neq S^n$ for $n>1$). Later [284, 286] Chacon offered his version of a definition of approximation with given speed, based on the concept of tower, which is also used in some papers on the isomorphism problem mentioned in Secs. 2, 3, Chap. 1. We shall consider the more fruitful of the various definitions of approximation which have appeared since 1967.

2.4. The definitions of a.p.t.I and cyclic a.p.t. were generalized and improved by Schwartzbauer [657] (for earlier versions cf. [290, 655]).

2.4.1. Let $\{f(n)\}$, $n=1, 2, \ldots$, be a sequence of positive numbers tending to zero. An automorphism $T:(X, \mu)\to(X, \mu)$ admits approximation with speed f(n), if there exists a sequence of partitions $\xi_n=\{c_i(n), n=1, \ldots, q_n\}$ and rearrangements $T_n:X|_{\xi_n}\to X|_{\xi_n}$, such that $\xi_n\to\varepsilon$ and

$$\sum_{i=1}^{q_n}\mu(Tc_i(n)\cap(X\setminus T_nc_i(n))<f(q_n).$$

If $T_n$ is a cyclic permutation of the elements of the partition $\xi_n$, then the automorphism T admits a cyclic approximation with speed f(n). This definition turns into the definitions of a.p.t.I and cyclic a.p.t. with speed 2f(n), if one requires in addition that $\mu(T_nc_i(n))=\mu(c_i(n))$, $i=1,\ldots, q_n$.

The basic result of [657] consists of the fact that such an approximation "splits into ergodic components" (partial results were obtained earlier by Chacon and Schwartzbauer in [290]): If $\beta$ is a measurable partition, consisting mod 0 of sets which are invariant with respect to T, and T admits approximation (respectively, cyclic approximation) with speed f(n), then there exists a family of sequences $f_B(n)$ $(B\in\beta)$ such that for almost all $B\in X|_\beta$ the automorphism $T|_B$ admits approximation (respectively, cyclic approximation) with speed $f_B(n)$ and

$$\int_{X|\beta} f_B(n)\,d\mu<f(n),\quad n=1, 2, \ldots.$$

With the aid of this theorem, Schwartzbauer [656] refined the results of A. B. Katok [82] on connnections of approximability with entropy of automorphisms. Let b(T) and c(T) denote the infimum of numbers $\lambda$ such that the automorphism T admits, respectively, approximation with speed $\lambda/2\ln n$ and a.p.t.I with speed $\lambda/\ln n$. Then 2h(T) = b(T) = c(T). In [82] and in the dissertation of A. B. Katok, "Some applications of approximation methods of dynamical systems to periodic transformations in ergodic theory" (Moscow University, 1968), the equation 2h(T) = c(T) is proved only for ergodic T, and in the general case, the inqualities $h(T)\leqslant c(T)\leqslant 2h(T)$ are proved.

2.5. The following two versions of approximation are successive weakenings of the property of cyclic a.p.t. with speed O(1/n) [92, 93].

2.5.1. The automorphism $T:(X, \mu)\to(X, \mu)$ admits good approximation if one can find a sequence of partitions

$$\xi_n=(C_1^n, \ldots, C_{q_n}^n=C_0^n, d^n),$$

such that

$$\mu(C_1^n)=\ldots=\mu(C_{q_n}^n),\ \ \xi_n\to\varepsilon,$$

and

$$q_n\sum_{i=0}^{q_n-1}\mu(TC_1^n,\ \Delta C_{i+1}^n)\to 0\quad\text{for}\quad n\to\infty.$$

This definition is contained in [91] in the form given here and in the earlier papers [203, 289, 523]; an equivalent property is called strong approximation by partitions.

2.5.2. An automorphism T admits approximation by partitions if one can find a sequence $\xi_n = (C^n, TC^n, \ldots, T^{q_n-1}C^n, d^n)$ such that $\xi_n \to \varepsilon$ ([203, 654]).

2.5.3. The results of [93] on spectral properties of automorphisms admitting cyclic a.p.t. with sufficiently high speed (absence of mixing, simplicity, and singularity of the spectrum) carry over word for word to the case of automorphisms admitting good approximation [91, 523, 654].

In [91] A. B. Katok proved that a factor-automorphism by any invariant partition with an infinite set of elements of an automorphism admitting good approximation also admits good approximation.

Another result of [91] consists of the fact that for any two automorphisms T, S admitting good approximations, and any number β, 0 < β < 1, one can find a set A of measure β, such that the derivative of the automorphism $T_A$ is metrically isomorphic with S. These results are used in the study of change of time and monotone equivalence of dynamical systems (cf. Sec. 2, Chap. 6).

Any ergodic automorphism with discrete spectrum admits good approximation [91].

2.5.4. A series of interesting papers is devoted to the study of the set C(T) of automorphisms commuting with a given automorphism T. Obviously, this set is a group, closed in the weak topology and containing all powers of T. If the automorphism T admits good approximation, then the weak closure of the set of its powers is a perfect set, and consequently, contains a continuum of elements. On the other hand, if T is mixing, then the set of powers of T is closed. In [289] Chacon and Schwartzbauer proved that for an automorphism T admitting good approximation, the set C(T) coincides with the weak closure of the set of powers of T. The proof is based on an ingenious combinatorial lemma which first appeared in an earlier paper of Chacon [282] in connection with the construction of an example of an automorphism from which no roots can be extracted.

Akcoglu, Chacon, and Schwartzbauer [203] described the set C(T) for an automorphism $T:(X,\mu)\to(X,\mu)$, admitting approximation by partitions. Namely, if S commutes with T, then either $S=T^k$, $k\in\mathbf{Z}$, or one can find a partition $\eta_n=(X_n^1, X_n^2)$ and numbers $j_n^1, j_n^2$ such that $j_n^1, j_n^2\to\infty$ as $n\to\infty$ and for $A\subset X$ $SA=\lim\left(T^{j_n^1}(A\cap X_n^1)\cup T^{-j_n^2}(A\cap X_n^2)\right)$. If, in addition, T is mixing then $S=T^k$.

These results are interesting, in particular, in connection with the fact that Ornstein [572] constructed an example of a mixing automorphism which admits approximation by partitions. Later, a similar construction of Ornstein and Friedman was used to construct a mixing derivative for an arbitrary ergodic automorphism (cf. Sec. 1, Chap. 6).

There are further generalizations in the paper of Akcoglu and Chacon [202], where it is proved that an automorphism commuting with some power of an automorphism T, which admits approximation by partitions, can be approximated by finite combinations of powers of T. From this assertion it follows that a mixing automorphism, which admits approximation by partitions, has no rational powers other than integral ones.

2.6. In [93] the following assertion is proved.

2.6.1. An automorphism that admits cyclic a.p.t. with speed $f(n)=\frac{1/2-\varepsilon}{n}$, $\varepsilon>0$, has simple spectrum.

In the proof one uses certain properties derived from the required cyclicity of the approximation and weaker than approximation by partitions.

2.6.2. The simplicity of the spectra of automorphisms admitting approximation by partitions was reproved in [239, 523]. In the latter paper a useful technical assertion is contained: The sequence of partitions $\xi_n$ in definition 2.5.2 can be chosen to be monotone increasing, and also the following result: Among the elements of maximal spectral type there are characteristic functions of sets of arbitrarily small measure.

Chacon [286] introduced a certain generalization of approximation by partitions (simple approximation with multiplicity n with respect to a sequence of partitions) and proved that in this case the multiplicity of the spectrum of the automorphism does not exceed n. Goddson [365] considered skew products over automorphisms admitting such approximations.

2.6.3. In [175] A. M. Stepin generalized and simultaneously strengthened 2.6.1. He proved that the multiplicity of the spectrum of an automorphism, which admits cyclic a.p.t. with speed $f(m)=\frac{2-2/m-\varepsilon}{m}$ ($\varepsilon>0, m$ is a natural number), does not exceed $m-1$.

2.7. One concept of speed of approximation is insufficient to distinguish dynamical systems with continuous and discrete spectra. Approximation criteria for continuity of spectra (in all of $L^2$ or in some subspace) are based on the fact that the approximating transformation contains some "incongruity," which in the limit destroys the eigenfunctions. This idea is considerably older than contemporary approximation methods in ergodic theory, actually it first appeared in 1932 in the classical paper of von Neumann, "Operator methods in classical mechanics," where with its aid the first examples of dynamical systems with continuous spectra were constructed.

Continuity of spectra for concrete clases of transformations, among which are those admitting cyclic a.p.t. with arbitrarily high speed, was proved in [93, 282]. Actually, in these papers some general approximation criteria for continuity of spectra were used, similar to those which were later proved explicitly in [94, 284, 654]. In [284] such a criterion is approximation in m-pairs — essentially cyclic approximation with sufficiently high speed, where $q_{2n+1}=mq_{2n}+1$, $m$ is a given number; in [94] the existence is required of an a.p.t.I with speed $o(\frac{1}{n})$, such that the approximating automorphism T of the permutation $T_n$ of elements of the partition $\xi_n$ should have two cycles whose periods differ by one, and the measure of the set of elements forming each of the cycles does not tend to zero.

Papers also continued to appear in which, with the aid of considerations of approximation character, concrete examples of automorphisms with continuous spectra are constructed which are not mixing (cf. [441, 425]).

## §3. Some Applications of Approximation Methods

Applications of approximation methods to problems of ergodic theory, basically, are indicated in the following scheme: the construction of examples of dynamical systems with "exotic" properties (here the examples themselves often turn out to be completely natural), the study of typical properties in various spaces of dynamical systems (so-called "category theory"), the study of metric properties of certain concrete classes of dynamical systems. In this paragraph we shall consider applications relating to the first of the directions enumerated. We note that the results considered in Sec. 1 also apply to this direction to some degree, but we have separated them in view of the fact that the basic interest in this case is not so much the metric properties projected by the dynamical system on itself as the appearance of their smooth realizations. The following paragraph is devoted to the second direction, and some results relating to the third direction are in Sec. 3, Chap. 4 and Sec. 4, Chap. 6 (cf. also [48]).

The basic value of the examples of which we shall speak is that they allow one to refute some "natural" conjectures on the structure of the spectra and other metric properties of automorphisms and flows, generated, fundamentally, by analogy with the case of discrete spectra. We note that all the examples from Paragraphs 3.1-3.3 are exchanges of segments (cf. Sec. 3, Chap. 4).

3.1. V. I. Oseledets [130] turned his attention to exchanges as sources of interesting and often unexpected examples. In particular, in [130] examples are constructed of ergodic exchanges of 24x segments, whose maximal spectral multiplicity is not less than two (and not greater than 23x, by virtue of 3.1.2, Chap. 4). To the present time, this is the only known example of an ergodic automorphism with finitely multiple but not simple spectrum.

It is unknown whether there exist ergodic automorphisms with spectra of constant multiplicity not equal to one. It is also unknown whether an automorphism can have finitely multiple absolutely continuous spectrum. A close question, whether there exist automorphisms with simple Lebesgue spectrum, is one of the oldest unsolved problems of ergodic theory.

3.2. S. V. Fomin, in the appendix to the book of Halmos, "Lectures on Ergodic Theory," posed the question of whether automorphisms with identical simple spectra will be metrically isomorphic. A negative answer to this question was given by S. A. Malkin [119], who constructed an automorphism with simple mixed spectrum, which is not isomorphic with its inverse automorphism. Then V. I. Oseledets [133] constructed an analogous example with simple continuous spectrum. Apparently, using the idea of Oseledets one can construct a continuum of pairwise nonisomorphic automorphisms having identical simple continuous spectra.

3.3. An analog of the well-known group property of the set of eigenvalues of an ergodic automorphism is the following property: The maximal spectral type of an automorphism subordinates its convolution. It is satisfied in examples of probability-theoretic origin (cf. [48]), and also, obviously, in the case of Lebesgue spectra. A. M. Stepin [93] constructed an example of an automorphism with mixed spectrum, for which the property indicated is not satisfied. An analogous example with continuous spectrum was constructed by V. I. Oseledets [132]. In these examples the spectra are simple. Later, Stepin and Oseledets introduced the concept of mixing with exponent $\varkappa$ (it would be more exact to call this property weak mixing with exponent $\varkappa$), which allows one to construct similar examples systematically [685, 174]. In particular, they can be realized by diffeomorphisms in the framework of the constructions of [15], described in Paragraph 1.1.

We note that Foias [341] proved a weak analogue of the group property for arbitrary ergodic automorphisms. The results of this paper are used in [95].

3.4. The results of Chacon [282, 283] and A. M. Stepin [93], touching on roots of automorphisms, were generalized by Akcoglu and Baxter [200]. In this paper for any preassigned set Q of prime numbers, an automorphism is constructed with continuous spectrum, which has roots of degree n for precisely those n all of whose prime divisors lie in the set Q.

3.5. R. S. Ismagilov [81] applied the method of approximation in ergodic theory for the construction of inequivalent representations of commutation relations.

## §4. Typical Properties of Dynamical Systems

A set in a Hausdorff topological space L will be called massive if it contains an everywhere dense subset of L of type $G_\delta$. Let L be some space of dynamical systems, equipped with a Hausdorff topology. A property of dynamical systems will be called typical in the space L, if the dynamical systems of L which have this property form a massive set. In ergodic theory, one considers typical properties in the following spaces (we shall restrict ourselves basically to the case of discrete time): the group $\mathfrak{U}$ of all automorphisms of a Lebesgue space with the weak topology; the group $H(X, \mu)$ of all homeomorphisms of a compact metric space X, preserving a continuous Borelian measure $\mu$, with the topology of uniform convergence; the group $\mathrm{Diff}^r(M, \mu)$, of $C^r$-diffeomorphisms of a compact manifold m, r = 1, 2, ..., $\infty$, preserving the smooth measure $\mu$, with the $C^r$-topology; the space of all derived automorphisms of a given automorphism T with metric $d(T_A, T_B)=\mu(A\Delta B)$; the space of all special flows over a given automorphism with $L_1$-metric; the space of all invariant measures of a given homeomorphism of a compact metric space with the weak topology.

4.1. The initial results for the group $\mathfrak{U}$ were obtained in the forties by Halmos and V. A. Rokhlin. The results obtained up to 1967 can be summarized (cf. [48, 93]) in the following way.

4.1.1. A typical automorphism admits cyclic a.p.t. with preassigned speed f(n) ($f(n)>0$, n = 1, 2, ...) and has continuous spectrum.

The latter fact was proved again by Chacon [284] with the aid of his approximation criteria (cf. 2.7). Natarajan [550] proved the typicality of weak mixing for actions of the group $Z^2$, i.e., for pairs of commuting automorphisms.

4.1.2. A. M. Stepin [174, 697] proved that a typical automorphism of maximal spectral type does not subordinate its convolution (cf. Paragraph 3.3).

4.2. A. B. Katok and A. M. Stepin [94] proved an analogue of assertion 4.1.1 for the group $H(X, \mu)$. The precise formulation in this: X is a finite regular connected cellular polyhedron, or a topological manifold; the measure $\mu$ vanishes on the set of nonregular points, and the measure of any open subset of the set of regular points is positive. Moreover, of course, $\mu$ is Borelian and continuous. The paper [94] is a combination of the approaches of Oxtoby and Ulam, who in 1941 proved the typicality of ergodic homeomorphisms in $H(X, \mu)$ under somewhat stronger conditions on X (cf. [48]), and the approximation approach in the spirit of [93] and Sec. 2. To prove the continuity of the spectrum, the approximation criterion mentioned in Paragraph 2.7 is used.

Oxtoby [592] proved, under the assumptions of [94], the following approximation assertion: If $T:(X, \mu)\to(X, \mu)$ is a metric automorphism (not necessarily continuous) and $\varepsilon>0$, then there eixsts a homeomorphism $S\in H(X, \mu)$, which is the identity outside some set D, homeomorphic with an r-dimensional ball (r = dim X) and coinciding with T on a set E such that $\mu(X\setminus E)<\varepsilon$. This theorem of Oxtoby allows one to obtain simply the basic approximation assertions of [94].

4.3. The smooth case is very interesting but very unstudied. It is clear, at least, that the situation is markedly different from the two previous cases. This follows from the fact that $\mathcal{Y}$-diffeomorphisms form an open set ([9, 16]; cf. also Sec. 2, Chap. 2). Any complete category theory in the smooth case, apparently, is a thing of the far future. There is a likely conjecture that the positivity of entropy is a typical property. It is not excluded that the situation for diffeomorphisms will turn out different for dimension 2 and for higher dimensions, and also for $C^1$ smoothness and higher smoothness.

4.4. Some results on derivatives and special automorphisms are presented in Sec. 1, Chap. 6. We shall add to this the almost obvious assertion: If the automorphism T admits cyclic approximation with speed f(n), then a typical derived automorphism admits cyclic approximation with speed cf(n) for some $c>0$. The properties of good approximation and approximation by partitions (cf. Paragraph 2.5) are also preserved upon passage to typical derived automorphisms.

4.5. It makes sense to study typical properties of invariant measures if there are enough such measures, for example, for transitive topological Markov chains (cf. Sec. 3, Chap. 2) or for locally maximal hyperbolic sets of diffeomorphisms (cf. Sec. 2, Chap. 2). Apparently, in these cases the answer will be similar to 4.1.1. Sigmund [666, 668, 669] concerned himself with these problems, considering the basic case of a locally maximal hyperbolic set, on which a diffeomorphism has the property of mixing domains (cf. Paragraph 2.4.5, Chap. 2). In [666] it is proved that for a typical invariant measure one has ergodicity, the absence of mixing, and zero entropy. In [669] the density of those measures with respect to which the diffeomorphism is a Bernoulli automorphism is proved. In [668] some results of [666] are carried over to the case of flows.

# CHAPTER 6

# CHANGE OF TIME IN DYNAMICAL SYSTEMS

## §1. General Questions

1.1. The operation of change of time is well known in the theory of ordinary differential equations. This operation from the dynamical point of view consists of passage from the flow $f_t$ generated by the vector field X to the flow $g_t$ generated by the vector field $\rho X$, where $\rho$ is a scalar, nonnegative (or even positive) differentiable function. If the flow $f_t$ has an invariant measure $\mu$, then the flow $g_t$ has the invariant measure $\rho^{-1}\mu$.

The operation described has natural analogue in abstract ergodic theory. Let $a\in L^1(X,\mu)$ be a function that is positive almost everywhere on the Lebesgue space $(X,\mu)$, $T_t$ be a measurable flow on X. We write $\varphi(t,x)=\int\limits_0^t a(T_u x)\,du$. Let $\tau(t,x)$ be the inverse function with respect to t to $\varphi(t,x)$ for given x. Then the flow $T_t^a x=T_{\tau(t,x)}x$ preserves the measure $a\mu$, equivalent with $\mu$. We write $\frac{d\tau(t,x)}{dt}=\rho(x)$. Obviously, $\rho=a^{-1}$.

Measurable flows $S_t$ and $T_t$ are called monotone equivalent if $S_t$ is metrically isomorphic with some flow $T_t^a$ [91]. An equivalent approach to change of time is connected with special representations of flows. Flows $S_t$ and $T_t$ are monotone equivalent if an only if one can find an automorphism $R:(M,\mu)\to(M,\mu)$ and positive functions $\varphi, \psi \in L^1(M,\mu)$ such that $S_t$ and $T_t$ are metrically isomorphic with the special flows $R_t^{\varphi}, R_t^{\psi}$ constructed from the automorphism R and the functions $\varphi$ and $\psi$, respectively.

1.2. There exists a more general approach to change of time proposed by Maruyama [526] and expounded in detail in Totoki's paper [700]. The basic difference consists, roughly speaking, of the following: change of time for Maruyama and Totoki is not assumed to be absolutely continuous along trajectories, i.e., the function $\tau(t, x)$ (the new "time") for given x may not be the integral of its derivative with respect to t. Because of this, the invariant measure of the new flow too may not be absolutely continuous with respect to $\mu$; hence, there arise difficulties connected with the choice of a sufficiently large $\sigma$-algebra of sets, measurable with respect to both measures. These difficulties are more naturally overcome if one assumes that the flow $T_t$ and the change of time are measurable with respect to some Borelian $\sigma$-algebra in X (a minimal $\sigma$-algebra generated by a countable system of sets).

1.3. In general, monotone equivalent flows are not metrically isomorphic, even up to change of scale in the group $\mathbf{R}$ (cf. below). However, there is an important sufficient condition for the isomorphism of flows $T_t$ and $T_t^a$, when the transformation conjugating these flows preserves each trajectory of $T_t$. This condition can be put in two equivalent forms:

1.3.1. There exists a measurable function $\psi: X\to\mathbf{R}$ such that for each $t\in\mathbf{R}$ and for almost all $x\in X$

$$\psi(\tau(x,t))-\psi(x)=t.$$

A criterion for solvability of such equations is found in Kowada [450].

1.3.2. For almost all $x\in M$, $\left.\frac{d\psi(T_t x)}{dt}\right|_{t=0}=a(x)$ (cf. Sec. 6, Chap. 2).

A corresponding criterion for metric conjugacy of special flows $R_t^{\varphi_1}$ and $R_t^{\varphi_2}$ over automorphism $R: X\to X$ appears thus:

1.3.3. $\varphi_1(x)-\varphi_2(x)=\psi(Rx)-\psi(x)$,

where $\psi$ is a measurable function of X (cf. Sec. 6, Chap. 2).

1.4. In distinction from the continuous and smooth cases the metric concept of monotone equivalence has a natural analogue for discrete time. It is natural to restrict oneself to the ergodic case. Ergodic automorphisms T and S are called monotone equivalent if over them one can construct metrically isomorphic special flows [91].

This definition is equivalent with the following. Let $T: M\to M$ be an ergodic automorphism, $m\in L^1(M,\mu)$ be a function assuming nonnegative integral values and not identically equal to zero. We write $M_{m(\cdot)}=M\setminus m^{-1}(0)$; let $T_{m(\cdot)}$ be the special automorphism from the derived automorphism $T_{M_{m(\cdot)}}$, constructed by restricting the function m to $M_{m(\cdot)}$. Then the automorphism S is monotone equivalent with T, if S is metrically isomorphic with some $T_{m(\cdot)}$ (in [589] this equivalence relation is called local equivalence).

1.5. Of the basic metric invariants, only entropy generates an invariant of monotone equivalence. From the results of L. M. Abramov (cf. [48], Sec. 2) it follows that the property of entropy being zero, a positive number, or infinity is invariant with respect to monotone equivalence.

1.5.1. In 1966, Chacon [281] proved that any ergodic flow can be made weakly mixing by a change of time, and also that for any ergodic automorphism there is a weakly mixing special automorphism. R. M. Belinskaya (cf. [48]) proved an analog of the latter assertion for derived automorphisms.

The results of Chacon and Belinskaya have recently been strengthened in two directions. Conze [305] proved that for an ergodic automorphism T, the sets A, for which the derived automorphism $T_A$ is weakly mixing, form an everywhere dense $G_\delta$ in the space $\mathfrak{A}$ of all measurable sets with the natural metric $d(A, B)=\mu(A\triangle B)$ (cf. also [350]). An analogous result for flows was obtained by A. Tagi-Zade.

1.5.2. More profound are the results of Ornstein and Friedman [354], who proved that an ergodic automorphism T has a mixing derivative, where too the sets A for which $T_A$ is mixing are dense in $\mathfrak{A}$, and of A. V. Kochergin [104], proving that any ergodic flow can be made mixing with the help of a change of time, infinitely differentiable along trajectories and the identity outside a given set of positive measure. In these papers a new construction of "sluggish" mixing is given. The question of the speed of mixing which can be guaranteed with the aid of similar constructions remains open. In particular, it is unknown whether by a change of time one can convert an arbitrary ergodic flow into a flow with absolutely continuous spectrum.

1.5.3. Hansel [384] proved that for any aperiodic automorphism $T:M\to M$ and any $\varepsilon$, $\lambda$, $\varepsilon>0$, $|\lambda|=1$, there exists a set A such that $\mu(A)>1-\varepsilon$ and the derived automorphism $T_A$ has an eigenfunction with eigenvalue $\lambda$. Somewhat earlier Conze [305] proved this for an everywhere dense set of numbers $\lambda$. From the results of A. B. Katok (cf. [91] and Sec. 2, Chap. 5) follows a strengthening of Hansel's result: For any not more than countable subgroup $G\in S^1$ and $\varepsilon>0$ one can find a set A, $\mu(A)>1-\varepsilon$, such that the automorphism $T_A$ has eigenfunctions with all eigenvalues $\lambda\in G$.

1.5.4. Osikawa [589] proved that all Bernoulli automorphisms with finite entropy are monotone equivalent. We note that by virtue of the theorem of Ornstein about isomorphism (cf. Sec. 2, Chap. 1) it was sufficient for any Bernoulli automorphism T and any $\alpha$, $0<\alpha<1$, to construct a set A, $\mu(A)=\alpha$, such that the automorphism $T_A$ is metrically isomorphic with a Bernoulli automorphism. This follows also from the result of Salesky [639] mentioned in Sec. 2, Chap. 1.

1.5.5. The question of monotone equivalence for various classes of automorphisms with positive entropy is not sufficiently studied. Here are some unsolved questions:

1) Is every automorphism with positive entropy monotone equivalent with a K-automorphism?

2) Is every K-automorphism which is monotone equivalent with a Bernoulli automorphism metrically isomorphic with a Bernoulli automorphism?

3) Does there exist a K-automorphism which is not monotone equivalent with a Bernoulli automorphism?

## §2. Standard Dynamical Systems

2.1. An ergodic automorphism is called standard if it is monotone equivalent with an automorphism with discrete spectrum, whose set of eigenvalues coincides with the set of roots of unity of all degrees. An ergodic flow is called standard if it is metrically isomorphic with a special flow over a standard automorphism.

In [91] A. B. Katok proposed a new approach to the study of properties of ergodic dynamical systems that are invariant with respect to change of time and found effective necessary and sufficient conditions for standardness. It turned out that standardness of the automorphism $T:M\to M$ is equivalent with the following weak approximation property of the automorphism: There exists a sequence of sets $F_n\subset M$ and maps $\varphi_n:F_n\to[0,1\times\{0,1,\ldots,q_n-1\}$ such that

2.1.1. For a measurable set $A \subset F_n$ the normalized Lebesgue measure of the set $\varphi_n(A)$ is equal to $\alpha_n\ \mu(A)$, where $\alpha_n$ is independent of A and $\alpha_n \to 1$ as $n \to \infty$.

2.1.2. If $x_1, x_2 \in F_n$, $\varphi_n(x_1)=(t, m)$, $\varphi_n(x_2)=(t, m+k)$, $k>0$, then $x_2=T^l x_1$, where $l>0$.

2.1.3. If $x \in F_n$, $\varphi(x)=(t, m)$, then either $\varphi(T_{F_n}x)(t, m+k)$, $k>0$, or $\varphi(F_n) \cap \{t\} \times \{m+1, \ldots, q_n-1\}=\varnothing$

2.1.4. We write $\xi_n=\{\varphi_n^{-1}([0, 1] \times \{i\}), i=0, \ldots, q_n-1; M \setminus F_n\}$. Then $\xi_n \to \varepsilon$.

There is an analogous criterion for flows which is more difficult to formulate.

2.2. Another criterion for standardness allows one to prove that a factor-automorphism of a standard automorphism by any infinite invariant measurable partition is standard. At the same time, for any ergodic automorphism S, one can find an automorphism $T_1$, monotone equivalent with T, some factor-automorphism of which is metrically isomorphic with S. These facts show that standard automorphisms form the simplest class of monotone equivalent metric automorphisms.

The criterion mentioned for standardness is based on comparison of how various points "wander" through the elements of an arbitrary finite measurable partition. Let $\eta = (A_1 \ldots A_k)$ be such a partition. For a point $x \in M$ we set $T^i x \in A_{k_i(x)}$; then there is defined a map $\varphi_\eta$: $M \to \Sigma_k$, $\varphi_\eta x = (\ldots, k_{-1}(x), k_0(x), k_1(x), \ldots)$ and a measure $((\varphi_\eta)_* \mu)$ in the space $\Sigma_k$, invariant with respect to the shift $\sigma_k$. The projected measures $(\varphi_* \mu)$ on coordinates with numbers 0, 1, ..., n − 1 generate a finite-dimensional distribution $\mu_n(\eta)$ in the space $\Sigma_{k,n}$ of sequences of numbers 1, 2, ..., k of length n. In the spaces $\Sigma_{k,n}$ one can introduce a normalized metric such that the standardness of the automorphism T turns out to be equivalent with the fact that for any partition $\eta$ as $n \to \infty$ the measure $\mu_n(\eta)$ is concentrated in some ball of arbitrarily small diameter.

2.3. This approach also allows one to construct invariants of monotone equivalence connected with the asymptotic, as $n \to \infty$, minimal number of balls of small radius in $\Sigma_{k,n}$ covering sets whose measure is close to one. These invariants are natural analogues of entropy and allow one to prove the nonstandardness and monotone inequivalence of various automorphisms and flows with zero entropy. We note that there is some informal connection between the approach expounded in this paragraph and the approach proposed earlier by A. M. Vershik ([42, 46]; cf. also Chap. 7) in connection with other problems.

## §3. Smooth and Continuous Change of Time

3.1. The classical procedure for change of time (cf. Paragraph 1.1) in the case of smooth flows with smooth invariant measure is a special case of the procedure we consider. The following question is natural: What may the metric properties of the flow $T_t^{\rho^{-1}}$ with respect to the invariant measure $\rho^{-1}\mu$ be in this case. In the following paragraphs we shall consider a simple class of dynamical systems, where the corresponding question has been studied in considerable detail. Here we shall indicate a few of the results which are known in other cases.

3.1.1. $\mathscr{Y}$-flows (cf. Sec. 2, Chap. 2) remain $\mathscr{Y}$-flows under smooth change of time with a positive function $\rho$ [9]. On the other hand, a theorem of A. N. Lifshitz [115] (cf. also Sec. 6, Chap. 2) gives the following necessary and sufficient condition for the conjugacy of the $\mathscr{Y}$-flow $T_t$ and the flow $T_t^{\rho^{-1}}$ by means of a $C^1$-diffeomorphism homotopic with the identity. For any periodic trajectory

$$\gamma: \{T_t x\} \quad \text{period} \quad t_0 \int_0^{t_0} \rho^{-1}(T_t x)\, dt = t_0.$$

3.1.2. A. G. Kushnirenko [113] proved that for the horocyclic flow on a compact two-dimensional Riemannian manifold of constant negative curvature $\varkappa^2$ the flow $T_t^{f^{-1}}$ has countably multiple Lebesgue spectrum if f is a function of class $C^1$ and $\varkappa f+E_2 f>0$, where $E_2$ is the vector field generating the geodesic flow.

3.2. Now let $\{T_t\}$ be a continuous flow on a compact metric space X, preserving a continuous Borelian measure μ positive on open sets, and ergodic with respect to this measure.

3.2.1. Parry [602] proved that continuous change of time in such a flow makes it weakly mixing. The result of Parry was essentially strengthened by A. V. Kochergin [104], who proved that there exists a continuous change of time, the identity outside a previously chosen open set U with $\mu(U)>0$, carrying the flow $\{T_t\}$ into a mixing one. The proof of Kochergin is based on the fact that the construction of a change of time which he used to make an arbitrary ergodic measurable flow mixing can be realized in the class of continuous changes.

3.2.2. The definitive clarity on the question of continuous time change is brought in by Ornstein (cf. [91]). He proved that any flow obtained by a measurable time change from a continuous flow $\{S_t\}$ is metrically isomorphic with some flow obtained from $S_t$ by a continuous time change. The proof of this theorem of Ornstein was also obtained by A. V. Kochergin. In order to elucidate the meaning of Ornstein's theorem, we shall give a result of homological equations, very closely connected with this theorem and being its analogue for special flows.

3.2.3. Let $(M, \mu)$ be a Lebesgue space, K be some cone in $L^1(M, \mu)$ closed with respect to the uniform norm: $\|f\|=\operatorname{ess\,sup}|f(x)|$, consisting of positive functions and dense in the norm of $L^1$ in the cone of all positive functions. Then for any positive function $f\subset L^1(M, \mu)$ and any ergodic automorphism $T:M\to M$ one can find functions $g\in K$ and $h\in L^1(M, \mu)$ such that

$$f(x)=g(x)+h(Tx)-h(x).$$

Whence it follows that any special flow over T is metrically isomorphic with a special flow with a function from the cone K.

## §4. Special Flows over Rotations of the Circle and Some Other Flows

4.1. In this paragraph we shall consider results on metric properties of special flows over the transformations of rotations of a circle and exchanges of segments (cf. Sec. 3, Chap. 4) with various functions (analytic, smooth, continuous, piecewise smooth with singularities of special form).

The classes of flows indicated are closely connected with smooth flows on two-dimensional surfaces. In the case of a rotation of the circle by an angle α, the special flow with constant function β, obviously, is isomorphic with a linear flow on the torus and has discrete spectrum, generated (over Z) by the numbers α and β. Any smooth flow on the torus without fixed points with a positive smooth invariant measure is metrically isomorphic with a special flow $T_t^{\alpha,f}$ over a rotation of the circle by some angle α with a smooth positive function f (cf., for example, [83]). The converse is also true; an arbitrary flow without fixed points on the torus with irrational rotation number, satisfying a Lipschitz condition, is strictly ergodic and metrically isomorphic with a flow $T_t^{\alpha,f}$, where f is a function of bounded variation [103]. Finally, a smooth flow with smooth invariant measure on an orientable surface of genus $p\geqslant 1$, with a finite number of fixed points, among which none are elliptic, is metrically isomorphic with a special flow over an exchange of segments with a function, differentiable everywhere except at a finite number of singularities. The form of these singularities depends on the local structure of the flow in a neighborhood of the fixed points.

4.2. We consider conditions under which the flow $T_t^{\alpha,f}$ is metrically isomorphic with a flow $T_t^{\alpha,f_0}$, where the constant $f_0$ is equal to $\int f(x)\,dx$. For this it suffices that the equation

4.2.1. $f(x)-f_0=\psi(x+\alpha)-\psi(x)$

have a measurable solution $\psi$ (cf. Paragraph 1.3.3). If the function $\psi$ has integrable square modulus, then expanding f in a Fourier series: $f=\sum_n f_n \exp 2\pi i x$, we get

4.2.2. $\psi=\sum_{n\neq 0}\frac{f_n}{\exp 2\pi i\alpha-1}\exp 2\pi i x.$

From this condition it follows (A. N. Kolmogorov [99]), that for numbers $\alpha$, not too well approximated by rational numbers, 4.2.1 has a solution whose properties are "not much worse" than those of f.

We shall not discuss concrete consequences of the indicated situation, but only examples when f is a "good" function and $\psi$ is given by the series 4.2.2 and is sufficiently "bad," since the corresponding results relate basically to the 1950s.

D. V. Anosov [12] proved that even for some analytic functions f there exist $\alpha$ for which 4.2.2 diverges but 4.2.1 has a solution which does not belong to $L^2$ or even $L^1$.

4.3. A. N. Kolmogorov remarked in [99] that for some analytic functions f and numbers $\alpha$, not normally well approximated by rational numbers, the flow $T_t^{\alpha,f}$ may have continuous spectrum (be weakly mixing) which corresponds to the unsolvability of the equation

4.3.1. $\exp 2\pi i\lambda f(x)=\frac{\varphi(x+\alpha)}{\varphi(x)}$

for any real $\lambda$ in the class of measurable functions $\varphi$. M. D. Shklover (cf. [48]) proved that such $\alpha$ exist for any analytic function f which is not a trigonometric polynomial. In the previously mentioned dissertation of A. B. Katok sufficient conditions are found for the continuity of the spectrum for analytic f in terms of the connection between the character of approximation of the number $\alpha$ by rationals and the character of the decrease of the Fourier coefficients of the function f. For functions with sufficiently regular behavior of the Fourier coefficients, these conditions are close to the necessary conditions following from the divergence of 4.2.2. Later progress in this question is due to A. M. Stepin, who found an effective criterion for continuity of the spectrum, applicable not only to analytic functions, but also to functions of finite smoothness, and also to functions with singularities.

4.4. A. B. Katok [83, 93], using the method of periodic approximation, proved that for functions $f\in C^3$ the flow $T_t^{\alpha,f}$ has simple singular spectrum and is not mixing. The result on the absence of mixing was strengthened significantly by A. V. Kochergin [103]: For any function of bounded variation, the special flow $T_t^{\alpha,f}$ is not mixing. Using this result and the remark made in Paragraph 4.1, A. V. Kochergin proved that any flow on the torus without fixed points and without closed trajectories, satisfying a Lipschitz condition, is not mixing with respect to its invariant measure. A. B. Katok, using his result on exchanges (cf. Paragraph 3.1.6, Chap. 4) and Kochergin's method, proved that a special flow over an exchange with a function of bounded variation also is not mixing.

Another generalization, due to A. V. Kochergin himself, is connected with certain flows on surfaces of genus $p>1$. In A. V. Kochergin's dissertation (Moscow University, 1974), flows $T_t^{\alpha,f}$ are considered where $\alpha$ is such that for some $\theta>0$ and an infinite sequence of irreducible fractions $p_n/q_n$

$$|\alpha-p_n/q_n|<\theta/q_n^2\ln q_n,\quad q_n\to\infty,$$

and

$$f(x)=f_0(x)+\sum_{i=1}^{k}\left(b_i\ln\frac{1}{\{x-a_i\}}+d_i\ln\frac{1}{\{a_i-x\}}\right),$$

where $f_0$ is a function of bounded variation, $b_i$, $d_i \geqslant 0$ and $\sum_{i=1}^{k} b_i = \sum_{i=1}^{k} d_i$, and such flows are proved to be nonmixing. Logarithmic singularities of the function f arise upon representation of the flow as a special flow on a surface with smooth invariant measure and with nondegenerate saddles (cf. Paragraph 3.2, Chap. 4).

The following question is still unanswered.

4.4.1. Let $T_t$ be a smooth flow on a surface of genus $p \geqslant 2$ with smooth positive invariant measure, all of whose fixed points are nondegenerate saddles. Can $T_t$ be mixing? The distinguished results of A. V. Kochergin and A. B. Katok give a negative answer to this question in all probability.

4.5. On the other hand, A. V. Kochergin proved that an ergodic flow on a surface, all of whose fixed points are typical degenerate saddles, arising not from logarithmic but from power singularities of the function f, is mixing. In [105] he introduced the class $\mathscr{F}(a,b)$ of functions, twice differentiable and convex downward on the interval $(a, b)$, whose second derivative as $x \to a$ grows faster than $(x-a)^{-2}$, and satisfying a certain regularity condition. The basic result of [105] is the following.

Let $T:[0,1] \to [0,1]$ be an ergodic exchange of segments, $\{x_1, \ldots, x_p\}$ be a set containing all points of discontinuity of T, $I = \{1, \ldots, p\}$, $f(x) \geqslant c > 0$ and

$$f(x) = f_0(x) + \sum_{i \in I} f_i(\{x - x_i\}) + \sum_{i \in I' \subset I} g_i(\{x_i - x\}),$$

where $f_i$, $g_i \in \mathscr{F}(0,1)$, $i \in I$, and the function $f_0$ has bounded second derivative. Then the special flow $T_t^f$ is mixing. From this theorem one deduces the existence of smooth mixing flows on any compact orientable surface except the sphere (cf. Paragraph 1.2.2, Chap. 5).

4.6. Finally, we consider this question: Can it happen that 4.2.1 is unsolvable and and 4.3.1 is solvable for some $\lambda$? This means either the disappearance of part of the discrete spectrum when the set of $\lambda$ for which 4.3.1 is solvable is a subgroup of the group $\{m\alpha + n\beta\}$, $m$, $n \in \mathbf{Z}$, or the appearance of a "nonnormal" discrete spectrum if 4.3.1 is solvable for some $\lambda$ not of the form indicated. For smooth f such effects have not been discovered.

4.6.1. The simplest example of the first type was constructed by A. B. Katok (cf. [110]; in this example, f is a "step," the spectrum of the flow $T_t^{\alpha,f}$ is mixed, while the discrete component consists of numbers which are multiples of $\alpha$). On the basis of this example, A. B. Krygin [110] constructed an example with the same properties with a continuous function f.

4.6.2. In [91] with the aid of his results on good approximations of automorphisms (Sec. 2, Chap. 5), A. B. Katok proved that for any irrational number $\alpha$ the flow $T_t^{\alpha,f}$ with a function $f \in L^1$ (and by virtue of Ornstein's theorem from Paragraph 3.2 also with a continuous function) can have any discrete spectrum except for one generated by one frequency, and also a mixed spectrum with any discrete component.

## CHAPTER 7

## TRANSFORMATIONS WITH QUASI-INVARIANT MEASURE

Let $(X, \mu)$ be a Lebesgue space and G be a locally compact measurable group of transformations of this space. The measure $\mu$ is called quasi-invariant with respect to the action of the group G if the transformations from G preserve the type of the measure $\mu$ (nonsingular). Groups G and G' of transformations with quasi-invariant measure are isomorphic if there exists a nonsingular transformation T such that $TGT^{-1} = G'$. From many points of view there is interest in the problem of constructing a complete system of invariants for a sufficiently broad class of transformation groups with quasi-invariant measure. The construction of von Neumann associating a group G with quasi-invariant measure with a weakly closed symmetric $M_G$ of operators on a separable Hilbert space (cf., e.g., [96]), establishes an important connection between the problem of classification and the theory of operator algebras. For an ergodic freely acting group G, the algebra $M_G$ is a factor (i.e., an algebra with one-dimensional center), and the decomposition of the group G into ergodic components corresponds to the decomposition of the algebra $M_G$ into a direct integral of factors. An ergodic group G of transformations with quasi-invariant measure we shall consider as type I, $II_1$, $II_\infty$, or III, if the factor $M_G$ belongs to type I, $II_1$, $II_\infty$, or III; this classification is connected with the question of the existence and properties of an invariant measure for the group G. The first example of a nonsingular transformation of type III was constructed by Ornstein [57]. Moore [543] and Hill gave a complete answer to the question of when a product-measure belongs to class I, $II_1$, $II_\infty$, or III. V. Ya. Golodets [59] proved that each countable group has an action of type III.

Transformation groups G and G' are called trajectory isomorphic if their decompositions into trajectories are isomorphic (with respect to a nonsingular transformation). If the ergodic groups G and G' are trajectory isomorphic, then the factors generated by them are isomorphic. The transformation group $[G]=\{T: Tx\in Gx$ for a.e. x$\}$ is called the complete group for G. A group is called approximate-finite if $M_G$ is an approximate-finite algebra. It is known that all ergodic approximate-finite groups of transformations of type $II_1$ are trajectory isomorphic. A. M. Stepin [172] proved that the cohomology groups $H^k(G)$ are trajectory invariants of the transformation group G.

A complete invariant for trajectory isomorphism for some groups of type III was found by Krieger [470, 471]. Namely, a countable group G of transformation of the space $(X, \mu)$ contains the measure $\mu$ if: 1) the group $\{S\in[G]:S\mu=\mu\}$ is ergodic; 2) for any $T\in G$ the Radon–Nikodym derivative $\frac{dT\mu}{d\mu}$ assumes a countable set of values. We write $\Delta(G, \mu)$ for the multiplicative subgroup of $\mathbf{R}^+$, generated by the set $\bigcup_{T\in G}\left\{a\in\mathbf{R}^+:\mu\left(\left\{x:\frac{dT\mu}{d\mu}(x)=a\right\}\right)>0\right\}$. Krieger proved that the closure of $\Delta(G, \mu)$ is a complete invariant of trajectory isomorphism for transformation groups containing the measure. In particular, all transformations T containing the measure $\mu$, for which $\Delta(T, \mu)$ is everywhere dense in $R^+$, are trajectory isomorphic. V. Ya. Golodets [58] proved that a countable approximate-finite group of transformations is trajectory isomorphic with an action of the group $\sum_1^\infty \mathbf{Z}_2$.

For countable transformation groups, Krieger, in [474], introduced an invariant r(G) of trajectory equivalence, analogous to the invariant of factors introduced by Araki and Woods [216] in connection with the algebraic classification of factors. Namely, the set r(G) consists of all real numbers $a$ satisfying the following: For any measurable set A of positive measure and any $\varepsilon>0$ there exists a measurable set $B\subset A$, $\mu(B)>0$ and a transformation $T\subset G$ such that

$$(1-\varepsilon)a<\frac{dT\mu}{d\mu}(x)<(1+\varepsilon)a \text{ for a.e. } x\in B.$$

The set r(G) is closed in $[0, \infty)$, $r(G)\cap(0, \infty)$ is a multiplicative group. Krieger proved that r(G) is a complete invariant of trajectory equivalence in the case when $r(G)\neq\{0,1\}$. We assume that the countable group $\Gamma$ acts as a group of shifts in the direct product $\prod_{g\in\Gamma}(X, \Omega)$, where $(X, \Omega)$ is a measure space. Let $\mu$ be a measure in $\prod_{g\in\Gamma}(X, \Omega)$, invariant with respect to

$\Gamma$, $\nu$ be the projection of the measure $\mu$ on $(X, \Omega)$ and T be a nonsingular transformation of the space $(X, \Omega, \nu)$. In [478] the invariant r(G) of the ergodic group G generated by the transformation $T_g$,

$$(T_e\omega)_g=\begin{cases}T\omega_e, & g=e,\\ \omega_g, & g\neq e,\end{cases}\quad T_g=\tau_g^{-1}T_e\tau_g,\ g\in\Gamma,$$

of the space $(\prod_{g\in G}(X, \Omega), \mu)$ is computed. The set $r(G)\neq\{0,1\}$ in this case and hence is a complete invariant of the group G.

A. M. Vershik [41, 43] proposed a new method of study of trajectory partitions based on their representation as the set-theoretic intersection $\bigcap_n \xi_n$ of a decreasing sequence of measurable partitions $\xi_n$. Partitions admitting such a representation are called tame. A. M. Vershik [47] proved that the property of the trajectory partition of a freely acting group G of being tame depends only on the algebraic type of the group G.

In [45] A. M. Vershik investigated the structure of tame partitions and their connection with operator algebras. Each tame partition P of the Lebesgue space $(X, \mu)$ can be represented in standard form $\bigcap_{n=1}^{\infty}\bigvee_{i=n}^{\infty}\eta_i$, where $\eta_i$ is a finite measurable partition and $\bigvee_{i=1}^{\infty}\eta_i=\varepsilon$. A tame partition P is called Bernoullian (Markovian) if there exists a standard representation $P=\bigcap_1^{\infty}\bigvee_n^{\infty}\eta_k$, where $\eta_k$ is a sequence of independent (a Markovian sequence of) finite partitions. If the tame partition P of the Lebesgue space $(X, \mu)$ has countable elements and $P=\bigcap\xi_n$, $\bigwedge\xi_n=\nu$, then for any $\varepsilon>0$ there exists a measure $\mu'$ equivalent to $\mu$ such that $\int\left|1-\frac{d\mu'}{d\mu}\right|d\mu\subset\varepsilon$ and P is a Markovian partition in $(X, \mu')$. For Bernoullian partitions, in [45] there is obtained an exhaustive criterion for isomorphism, which allows one to distinguish trajectory partitions also in the case $r(G)=\{0, 1\}$. With the aid of this criterion, a classification theorem is obtained for a class of factors of type III: If the trajectory partition of the ergodic group G is Bernoullian, then its metric type is an invariant of the factor $M_G$. Krieger [479] constructed an example of an ergodic nonsingular transformation whose trajectory partition is not Bernoullian.

Yu. I. Kifer and S. A. Pirogov [98] proved a theorem on the decomposition of quasi-invariant measures into ergodic components. The problem of classification of transformation groups up to trajectory equivalence is studied in [308]. On an approach connected with the concept of virtual groups, cf. [617, 508].

A series of papers is devoted to the problem of the existence of invariant measures for countable groups of nonsingular transformations.

Measurable sets A and B are called congruent with respect to the transformation group G if there exist decompositions $A=\bigcup_1^{\infty}A_n$, $B=\bigcup_1^{\infty}B_n$ and transformations $T_n\in G$ such that $T_nA_n = B_n$. The set A is incompressible with respect to G if it is not congruent to any subset $A'\subset A$, $\mu(A\setminus A')>0$. The classical result of Hopf is that the group G is of type $II_1$ if and only if X is incompressible with respect to G and the measure $\mu$ is continuous. A measurable set W is called weakly wandering if there exists a sequence $\{T_n\}$ of transformations from G such that $T_n(W)$ are pairwise disjoint. Hajian and Kakutani [380] proved that a cyclic group of transformations is of type $II_1$ if and only if there does not exist a weakly wandering set of positive measure. Hajian and Ito [376] generalized this result to arbitrary countable transformation groups. In [500] another proof of the theorem of Hajian–Kakutani is given.

Sufficient conditions for the existence of a finite invariant measure for a family $\{T\}$ of nonsingular transformations were obtained in [255] and [388]. Let M be the set of finite products of elements of $\{T\}$; if $\inf_{\tau\in M}\mu(\tau^{-1}A)>0$, $\mu(A)>0$, then there exists a finite measure m, equivalent with $\mu$ and invariant with respect to $\{T\}$. For conditions for the existence of a finite invariant measure for an indiviaul transformation cf. [499, 643].

In [308] partitions $\{X_f, X-X_f\}$ of the space $(X\ \mu)$, are considered, where $X_f$ is the maximal part on which there exists a finite G-invariant measure equivalent with $\mu$. Let P be the projector in $L_p(X, \mu)$ onto the subspace of G-invariant vectors. If $f\in L_p(X, \mu)$, $1<p<\infty$, $f>0$, then

$$\{x:(Pf)(x)>0\}=X_f.$$

Jones and Krengel [421] constructed for each invertible nonsingular transformation of a probability space $(X, \mu)$ a measurable partition $(X_f, X_1, \ldots)$, such that: a) $X_f$ and $\bigcup_{i\geq 1} X_i$ are invariant with respect to T; b) the restriction of T to $X_f$ has a finite T-invariant measure equivalent with $\mu$; c) $X_1$ is the image of each $X_i$, $i\geqslant 1$, with respect to some power of the transformation T.

If a nonsingular transformation T of the space $(X, \mu)$ has no finite invariant measure equivalent with $\mu$, then the collection of sets A, for which the partition $(A, X\setminus A)$ is generating, is everywhere dense in the $\sigma$-algebra of measurable sets (and even in any exhaustive subalgebra) [458]. Kuntz generalized this result to group actions.

Brunel [277] proved that a positive compression T in $L_1(X, \mu)$ has an equivalent finite invariant measure if all compressions of the form $\Sigma p_i T^i$, $p_i\geqslant 0$, $\Sigma p_i=1$ are conservative. In [315] criteria are obtained for the existence of finite invariant measures for positive operators in $L_1(X, \mu)$ satisfying the condition

$$\sup_{N\geqslant 1}\left\|\frac{1}{N}\sum_{0}^{N-1}T^n\right\|<\infty.$$

The question of the existence of $\sigma$-finite invariant measures for Markovian operators is the subject of [535, 277, 287]. In [535] sufficient conditions are obtained for the existence and uniqueness of fixed points for conservative Markovian operators in the cone of nonnegative measurable functions on a space with $\sigma$-finite measure. Ornstein and Sucheston [587] weakened the condition $\|T\|_{L_\infty}=1$ in Harris' theorem on the existence of a $\sigma$-finite invariant measure for a Markovian operator T to the requirement $\underline{\lim}\, T^n h<\infty$ for all $h\in L_\infty$, $h\geqslant 0$.

Hajian, Ito, and Kakutani in [380] proposed a new construction of a transformation not having a $\sigma$-finite invariant measure. Let $(X, \mu)$ be a Lebesgue space and $\mathfrak{A}_\infty$ be the set of ergodic nonsingular transformations of the space $(X, \mu)$, having $\sigma$-finite infinite invariant measure equivalent with $\mu$. In [380] it is proved that there exists a $T\in\mathfrak{A}_\infty$ and a nonsingular transformation Q, commuting with T, such that Q does not preserve the measure, invariant with respect to T and equivalent with $\mu$.

In [374] for the generalized Gauss endomorphism $T:x\to[f^{-1}(x)]$ an algorithm is found for reconstructing for f an invariant measure equivalent with Lebesgue measure.

Keane [430] proved for ergodic rotations of the circle the existence of a continuum of pairwise singular continuous quasi-invariant ergodic measures. Krieger [476] generalized this result to strictly ergodic homeomorphisms of compact metric spaces, provided the space does not consist of one trajectory. The decisive result in this direction is due to I. P. Kornfel'd [101] and Katznelson and Weiss [428]: a homeomorphism T of a compact metric space has an uncountable set of inequivalent continuous ergodic quasi-invariant measures if and only if T has a recurrent point.

Kubo [491, 492] studied the class of automorphisms and flows admitting a transversal flow with quasi-invariant measure. Ito constructed transversal flows for topological Markov chains with measure of maximal entropy. In [570] it is proved that the spectrum of such a flow is discrete for the transition matrix $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$. Kubo [491] and Krengel [455] obtained a generalization of the theorem of Ambrose on special representations of flows to the case of a one-parameter group (semigroup) of nonsingular transformations of a σ-finite space with measure. Kubo [493] constructed a special representation for the group $\mathbf{R}^n$ of nonsingular transformations. The question of special representations of general group action is open.

We shall now consider transformations with infinite invariant measure (of type $II_\infty$).

Krengel [456] introduced the concept of mixing for transformations of a topological space with σ-finite Borel invariant measure and proved that the mixing transformations form a set of the first category in the group of all transformations (cf. Sec. 4, Chap. 5).

Then Krengel and Sucheston [464, 466] proposed another definition of mixing transformations, coinciding with the usual one in the case of finite invariant measure. We define the sufficient σ-algebra $\mathfrak{R}$ of sequences of measurable sets $\{A_n\}$ as the intersection of the σ-algebras $\mathfrak{R}_k$ generated by the sets $A_n$, $n\geqslant k$. A transformation T of a σ-finite space with measure is called mixing if in each sequence $T^{k_s}A$, $\mu(A)<\infty$, $k_s<k_{s+1}$, one can find a subsequence with trivial sufficient σ-algebra; if this is so for each measurable set A, then T is called completely mixing.

A transformation T, having infinite invariant measure is mixing if and only if $\mu(T^nA\cap A)\to 0$ as $n\to\infty$ for each set A of finite measure. Strict endomorphisms are completely mixing, however Kolmogorov endomorphisms with infinite invariant measure (cf., e.g., [318]) do not have this property. Further, the property of completely mixing of a nonsingular invertible transformation T is equivalent with the existence of a finite invariant measure, with respect to which T is mixing.

A transformation T with infinite invariant measure μ is called weakly mixing if $\frac{1}{n}\sum\mu(T^{-k}A\cap B)\to 0$ as $n\to\infty$ for each pair of sets A and B of finite measure. In [637] it is proved that mixing transformations form a set of the first category with respect to the weak topology, and weakly mixing ones are nowhere dense in the uniform topology (cf. Sec. 4, Chap. 5).

An invariant of a transformation T of a space $(X,\mu)$ with infinite measure, connected with the asymptotic behavior of the sequences $\mu\left(\bigcup_0^n T^{-k}A\right)$, $n\geqslant 1$, is proposed in Kakutani's survey [423].

An interesting example of a rational map of the line, preserving Lebesgue measure, was considered by Adler and Weiss [198]. They proved the ergodicity of the transformation T: $x\to x-\frac{1}{x}$. The method of proof allows one to establish the complete ergodicity of the transformation T.

Entropy, mixing properties, and spectra of transformations with infinite invariant measure are considered in [445, 590, 339].

CHAPTER 8

# SOME APPLICATIONS OF ERGODIC THEORY

Applications of ergodic theory to other parts of mathematics are quite diverse. Moreover, ergodic problems arise in many problems of physical origin, e.g., in the general theory of relativity, in plasma physics, in problems relating to the study of the motion of charged particles in electromagnetic fields, Fermi statistical acceleration, etc. A survey of all similar applications would exceed in volume all that relates directly to ergodic theory. Hence, in compiling this section, it was necessary to restrict ourselves to somewhat narrower boundaries. We resolved to devote it only to applications of ergodic theory to statistical mechanics, since ergodic theory arose from statistical mechanics and the latter has always been a source of problems for ergodic theory.

Upon consideration of a series of problems of equilibrium and nonequilibrium statistical mechanics there arose a class of infinite-dimensional dynamical systems. In contrast to the partial differential equations, which may often also be considered as generating operators of the corresponding infinite-dimensional dynamical systems, the dynamical systems of statistical mechanics differ in that for them all powers are equally free. We proceed to the direct description of such systems. Assume function U(r) is a potential of a dual interaction. For now we shall assume only that $U(r)\in C^1, U(r)\equiv 0$ for $r>r_1$.

The configurations of our system are an infinite subset X of the space $\mathbf{R}^d$, $d\geqslant 1$, which has the property that $|X\cap O|<\infty$, here O is a compact subset of $\mathbf{R}^d$, the sign $|\cdot|$ denotes the cardinality of the set standing inside. Points $x\in X$ will be called particles. The collection of velocity vectors of all particles $x\in X$ is naturally considered as an $\mathbf{R}^d$-valued function $V=\{v(x), x\in X\}$, defined on X. The pair (x, v) is a point of the phase space of our infinite-dimensional system, which we shall denote by $\mathscr{M}$. We shall write Newton's equations of motion, assuming that the particles have mass 1, interact pairwise with strength depending on the distance, whose potential is U(r):

$$\frac{dx}{dt}=v(x),\quad \frac{dv}{dt}=-\sum_{y\neq x,\ y\in X} U'(|x-y|) \tag{1}$$

In view of our conditions of X the right-hand side is finite. Nevertheless, (1) does not split into finite-dimensional systems, since distant particles can be connected by means of interaction, and there arises the far from trivial problem of proving a theorem like an existence theorem for systems of type (1). The nontriviality is already evident just from the fact that for $U\equiv 0$, if the velocity v(x) grows sufficiently rapidly to infinity and is directed toward the origin, then in an arbitrarily small time in a neighborhood of zero there are infinitely many particles and we fall outside the limits of the space $\mathscr{M}$.

The problem of existence of solutions of (1) is related to the problem of existence of solutions of partial differential equations. In the theory of partial differential equations Cauchy-type problems are posed and solved in certain function spaces. For (1), instead of choosing a function space, it is necessary to construct a Borel set $\mathscr{M}'\subset\mathscr{M}$, on which one can define a one-parameter group of transformations $\{S_t\}$ in such a way that along trajectories of this group the equations of motion (1) are satisfied. It is clear that one can devise trivial examples of such subsets. But the problem consists of choosing the most massive possible subset.

We shall dwell on the case d = 1. In the space $\mathscr{M}$ one can introduce a three-parameter family of Gibbsian equilibrium distributions $\mu(\rho, \bar{v}, h)$, where the parameters are the first three additive integrals of motion: ρ is the density, $\bar{v}$ is the mean speed, h is the energy, taken over one degree of freedom. The measure $\mu(\rho, \bar{v}, h)$ is constructed analogously to the process described in detail in Chap. II on hyperbolic systems. We consider a segment $\Delta_L=[-L, L]$ and we fix coordinates and velocities of all particles outside it. In the phase space of particles inside $\Delta_L$, we consider a measure $\mu_L$, whose density is

$$\frac{1}{\Xi}\exp\left\{-\beta\left[\sum_{x\in\Delta_L}\frac{(v(x)-\bar{v})^2}{2}+\sum_{x',x''\in\Delta_L}U(|x'-x''|)+\sum_{\substack{x\in\Delta_L\\ y\notin\Delta_L}}U(|x-y|)+\mu N\right]\right\},$$

where N is the number of particles hitting $\Delta_L$; $\beta$, $\mu$, $\bar{v}$ are parameters; $\Xi$ is a normalizing factor. As $L\to\infty$ the measures $\mu_L$ converge in a natural sense to a limit, which is called the Gibbsian equilibrium distribution. The parameters $\beta$, $\mu$ are uniquely connected with h, $\rho$.

Returning to our problem, we shall call the subset $\mathcal{M}'\subset\mathcal{M}$ massive if $\mu(\mathcal{M}'|\rho, \bar{v}, h)=1$ for all $\rho$, $\bar{v}$, $h$. The first time the problem of the construction of massive subsets $\mathcal{M}'$ was considered was in the paper of Lanford [507]. In his paper it was assumed that U(r) is bounded above, and under this assumption a massive subset was constructed. The case when the potential has a hard kernel, i.e., $U(r)\equiv\infty$ for $r\leqslant r_1$, was considered in Ya. G. Sinai's paper [163]. He constructed a massive subset on which (1) has a solution, reducing to so-called cluster dynamics. The latter means that the evolution of the entire infinite ensemble of particles occurs in such a way that at each moment of time t all particles decompose into finite groups (clusters), where each cluster moves for some time independently from the others. Then the clusters decompose, run into one another, and the particles again split up into clusters, etc. The results of [163] were carried over to systems with long-range potential by A. N. Zemlyakov [77].

In Sinai's paper [166] the subset $\mathcal{M}'$ on which the solution of the system (1) giving the cluster dynamics is defined was constructed also in the multidimensional case for $d>1$. However, $\mu(\mathcal{M}'|\rho, \bar{v}, h)=1$ only for sufficiently small values of the density. From the physical point of view, this is completely natural. The fact is that for $d>1$ there may occur in the system phase transitions, manifesting themselves in that given values of the parameters $\rho$, h can correspond to several Gibbsian equilibrium distributions, i.e., the limit which figures in the definition of the Gibbs equilibrium distribution can depend on the choice of the successive boundary conditions. For the Gibbs equilibrium distribution corresponding to a fluid or solid thermodynamical phase, it is unnatural to require a cluster character of the dynamics, which is specific for gas phases; viz., this also explains that $\mu(\mathcal{M}'|\rho, \bar{v}, h)=1$ for sufficiently small $\rho$, when the Gibbs equilibrium distribution is unique and corresponds to a gas phase.

We consider the subset $\mathcal{M}'$, on which the solution of (1) is defined. The motions $S_t$ along solutions of this system, constructed in the papers mentioned, leave any Gibbs equilibrium distribution invariant, and hence the question arises of the ergodic properties of the corresponding dynamical system. Up to now only the simplest cases have been studied successfully: ideal gas, when $U(r)\equiv 0$, and gas of one-dimensional hard globules, i.e., $U(r)=\infty$ for $0\leqslant r\leqslant r_0$ and 0 in the remaining cases. Gallavotti proved that the dynamical system $\{S_t\}$ corresponding to an ideal gas is mixing, and in the paper of K. L. Volkovysskii and Ya. G. Sinai [53] it was proved that it is a K-system. In Sinai's paper [162] it was proved that the gas of one-dimensional hard globules is a K-system; and in the paper of Aiserman, Goldstein and Leibowitz [199], that it is a Bernoullian system. In the paper of de Pazzis [608] systems of one-dimensional hard globules on a half-line with elastic reflections from the boundary were considered and the K-property was proved for them.

From the point of view of statistical mechanics the question of the ergodic properties $\{S_t\}$ should be posed somewhat differently than in ordinary ergodic theory. The fact is that from the infinite number of degrees of freedom, different probability distributions in $\mathcal{M}$ natural from the point of view of statistical mechanics as a rule are mutually singular among themselves. The irreversibility of statistical mechanics means that for the evolution of these distributions, generated by motions along solutions of (1), they must in some sense or other converge to the Gibbs equilibrium distribution. The absence of a sufficiently complete existence theorem for (1) does not allow one to approach this problem at the present time. Nevertheless, one can try to clarify what kind of limits, in principle, are possible for the evolution of nonequilibrium distributions. The first such formulation of the problem was made in the paper of B. M. Gurevich, Ya. G. Sinai, and Yu. M. Sukhov [69]; some results were obtained there for the case d = 1. Recently, B. M. Gurevich and Yu. M. Sukhov [70] obtained considerably stronger results, to whose description we shall now proceed.

In the space $\mathcal{M}$ one can introduce a class of measures that are natural from the point of view of statistical mechanics, which we shall call Gibbs distributions (not necessarily equilibrium). The definition of a Gibbs distribution is a variant of the general definition of a limit Gibbs distribution due to R. L. Dobrushin. Let X be an infinite configuration of particles, Q be a compact set of the space $\mathbf{R}^d$, F be a function defined on the set of all finite subsets of particles (by particles we now mean points of the space $R^{2d}$, i.e., coordinates and velocities) which the following properties:

1) F = 0, if the subset contains more than $n_0$ particles ($n_0$ is a constant);

2) $F \to 0$, if the distance between points is less than $\infty$;

3) $F = \infty$, if the distance between points is less than $r_0$ (rigid core).

Definition. A probability distribution $\mu$ in $\mathcal{M}$ is called a Gibbs distribution (with potential F) if the conditional distribution on the space of particles in Q for fixed particles outside Q is given almost everywhere by a density with respect to the measure $\lambda$, having the form $\Xi^{-1}\exp(-\Sigma F(z))$, where the summation is over all finite subsets z, at least one of whose elements is contained in Q. By the measure $\lambda$ is meant the direct product of the Lebesgue measure in the velocity space and the Poisson measure with constant density on configurations in Q.

Gibbs distributions form a sufficiently broad and natural class of distributions in M. O. Kozlov obtained rather general conditions on distributions under which they turn out to be Gibbsian.

In statistical mechanics, the chain of Bogolyubov equations, which describe the evolution of a nonequilibrium distribution, is well known. This chain is derived formally: In the derivation one passes to a limit to an infinite number of particles, whose validity is not evident. Nevertheless, the extended point of view is that in natural situations the chain of equations of Bogolyubov correctly describes the process of evolution of a distribution. The basic result of Gurevich and Sukhov can now be formulated as follows:

If a Gibbs distribution is a stationary solution of a Bogolyubov chain of equations, then it is a Gibbs equilibrium distribution.

This theorem is a mathematical proof of the known assertion of Landau that the additive integrals of motion reduce to energy, impulse, and density.

An interesting example of an infinite-dimensional dynamical system was considered in a recent paper of Goldstein [361]. Let there be situated on the plane an infinite set of circles of the same radius with centers at integral points of the plane. To complete this set of circles one scatters a statistically homogeneously (i.e., according to a Poisson distribution) a infinite number of particles of identical mass, each of which moves independently from the other particles and is reflected from the fixed circles according to the law of billiards. In [361] ergodic properties of this system are studied in relation to a group of measure-preserving transformations isomorphic with the direct product of $\mathbf{R}^1$ and $\mathbf{Z}^2$.

## CHAPTER 9

## ERGODIC THEOREMS

The question of the asymptotic behavior of ergodic means $\frac{1}{T}\int_0^T U_t\,dt$ of one-parameter groups of linear operators $\{U_t\}$ is of interest from various points of view. The classical ergodic theorem of von Neumann and Birkhoff–Khinchin asserts that for a measure-preserving flow $\{T_t\}$ in the Lebesgue space $(X, \mu)$, $\mu(X)=1$, the function $\frac{1}{T}\int_0^T f(T_t x)\,dt$ converges as $t\to\infty$ in mean square or almost everywhere, if $f\in L_2(X, \mu)$, or, respectively: $f\in L_1(X, \mu)$. Generalizations of this theorem to the case of transformations of a space with infinite measure, and also to the case of semigroups of operators in abstract normed and function space were obtained by F. Riesz, V. V. Stepanov, E. Hopf, Yosida, Kakutani, Doob, Dunford and Schwartz, and Chacon and Ornstein. A detailed survey of these results and the corresponding bibliographical citations can be found in the book of Dunford and Schwartz, and also in the paper of Vershik and Yuzvinskii [48]. We shall restrict ourselves here to enumerating papers that have appeared since 1967.

## §1. Statistical Ergodic Theorems

Sine [674] proved that the average of iterations of a compression T in a Banach space converges strongly if and only if the fixed points of the operator T are separated by the fixed points of the adjoint operator T*. Aribund [217] obtained a generalization of the statistical ergodic theorem to the case of a group of linear operators acting on a locally convex linear topological space. A statistical ergodic theorem for other methods of summation is proved in [504].

Let T and S be bounded linear operators in a Banach space B. We consider the following assertions:

(i) $T^n$ converges weakly to S,

(ii) $\frac{1}{n}\sum_{s=1}^{n} T^{i_s}$ converges strongly to S for each strictly increasing sequence $\{i_s\}$.

From (ii) follows (i). The converse was proved by Blum and Hanson for operators T in $L_2(X, \mu)$, induced by an automorphism of the Lebesgue space $(X, \mu)$, $\mu(X)=1$. Then the equivalence of (i) and (ii) was proved for compressions in a Hilbert space or in the space $L_1(X, \mu)$, where $\mu$ is $\sigma$-finite. Akcoglu, Huneke, and Rost [206] constructed an example of a compression in a Banach space for which (i) and (ii) are not equivalent. Jones [418] established a connection between the existence of the strong limit $\lim \frac{1}{n}\sum_{s=1}^{n} T^{i_s}$ and weak mixing in the sense that for some sequence $\{n_k\}$ of density 1, $T^{n_k}x$ has a limit.

A sequence of probability measure $\mu_n$ on a locally compact commutative group G is called summing if for any unitary representation U of the group G $\int U_g h\,d\mu_n$ converges to a U-invariant vector. In [253] equivalent characteristics of summing sequences are found.

## §2. Individual Ergodic Theorems

Let T be an ergodic automorphism of a space X with measure $\mu$; we set $f^*(x)=\sup\frac{1}{n}\sum_{i=1}^{n} f(T^i x)$. In addition to the Birkhoff ergodic theorem, D. Ornstein [577] obtained the following result: 1) if $\mu(x)<\infty$, $f\geqslant 0$, then $f^*\in L_1(X, \mu)$ if and only if $\max(f\log f,\ 0)\in L_1(X, \mu)$; 2) if $\mu(X)=\infty$, $f\geqslant 0$, then the function f* is not integrable.

Results on the convergence of the mean $\frac{1}{n}\sum T^k f$ almost everywhere for compressions in the spaces $L_p$ are grouped around the theorems on Dunford–Schwartz and Chacon–Ornstein.

THEOREM (Dunford–Schwartz). Let $(X, \mu)$ be a space with a $\sigma$-finite measure and $T_t$ be a strongly continuous semigroup of compressions in $L_1(X, \mu)$ satisfying the condition

$$\|T_t f\|_\infty \leqslant \|f\|_\infty \quad \text{for all} \quad f\in L_1(X, \mu)\cap L_\infty(X, \mu).$$

Then for $f \in L_1(X, \mu)$ the limit $\lim_{N\to\infty} \frac{1}{N} \int_0^N T_t f dt$ exists almost everywhere.

An analogous result also holds for semigroup $\{T^n\}$ with discrete time. Brunel [278] proved an individual ergodic theorem for a finitely generated semigroup of compressions in $L_1$ which are also compressions in $L_\infty$.

THEOREM (Chacon–Ornstein). For a positive compression T in $L_1(X, \mu)$ the means $\frac{1}{n} \sum_{k=1}^{n} T^k f$, $f \in L_1(X, \mu)$, converge almost everywhere.

For a new proof of this theorem, cf. [345] (cf. also [307, 355]).

Akcoglu and Cunsolo [204] proved a theorem of Chacon–Ornstein type for one-parameter semigroups: If the semigroup $T_t$ of positive compression in $L_1(X, \mu)$ is strongly continuous, then for each $g \in L_1(X, \mu)$ and each $f \in (X, \mu)$

$$\lim_{\alpha\to\infty} \frac{\int_0^\alpha T_t f dt}{\int_0^\alpha T_t g dt}$$

exists almost everywhere on the set $\left\{x \in X : \int_0^\infty (T_t g)(x)\, dt > 0\right\}$.

Terrel [693] found the following remarkable generalization of the theorems of Dunford–Schwartz and Akcoglu–Cunsolo. A family $p_t$ of measurable nonnegative functions is called admissible with respect to the semigroup of operators $T_t$ if:

1) $p_t(x)$ is measurable in t for almost all x,

2) if $f \in L_1(X, \mu)$ and $|f| \leqslant \int_n^{n+1} p_t dt$ almost everywhere, then for any $s > 0$ $|T_s f| \leqslant \int_{n+s}^{n+s+1} p_t dt$ almost everywhere.

THEOREM. If the semigroup $\{T_t\}$ of compressions in $L_1(X, \mu)$ is strongly continuous and the family $p_t$ is admissible with respect to $\{T_t\}$, then for each $f \in L_1(X, \mu)$,

$$\lim_{\alpha\to\infty} \frac{\int_0^\alpha T_t f dt}{\int_0^\alpha p_t dt}$$

exists almost everywhere on the set $\left\{\int_0^\infty p_t dt > 0\right\}$.

In the paper of Meyer [534], results of Rost, which give a new approach to the proof of the maximal ergodic theorem and the Chacon–Ornstein ergodic theorem are expounded.

Of other papers on individual ergodic theorems we mention [438, 645, 502, 651].

Thus in [438] the convergence of the means $\frac{1}{n} \sum_{0}^{n} T^k f$ almost everywhere also in $L_1(X, \mu)$ for a Markovian operator T in $L_1(X, \mu)$ under the additional hypothesis of weak compactness of the sequence $\frac{1}{n}\sum_{0}^{n} T^k\varphi$ for some function $\varphi \in L_1(X, \mu)$, $\varphi > 0$, is proved.

We shall say that a linear operator T in $L_p(X, \mu)$ admits a majorizing estimate with contant k, if for each $f \in L_p(X, \mu)$

$$\int_X \sup_n \left|\frac{1}{n}\sum_{k=1}^{n} T^k f\right|^p d\mu \leqslant k^p \int |f|^p d\mu.$$

If a compression T in $L_2(X, \mu)$ has this property, then the sequence $\frac{1}{n}\sum_{1}^{n} T^k f$ converges almost everywhere for all $f \in L_2(X, \mu)$. E. Stein proved that positive unitary operators admit majorizing estimates. In [287] it is proved that a positive compression T in $L_p(X, \mu)$ admits a majorizing estimate which constant $\frac{p}{p-1}$, if there exists a sequence of positive functions $h_n \in L_p(X, \mu)$, such that $\|T^n h_n\| = \|h_n\|$. There are close results in [288].

Akcoglu and Sucheston [208] reduced the question of the existence of a majorizing estimate for compressions in a Hilbert space H to the corresponding question for unitary operators in H. In [532] it is proved that:

1) A convex linear combination of two commuting isometries in $L_p$, $p \neq 2$, admits a majorizing estimate with constant $\frac{p}{p-1}$;

2) a convex linear combination of any collection of commuting positive invertible isometries in $L_2$ admits a majorizing estimate with constant 2.

On the connection of ergodic theorems with integral representations of harmonic functions in the circle, cf. [207].

Kowada [451] gave a uniform estimate for the speed of convergence of the means $\frac{1}{s}\int_0^s f(T_t x)\,dt$ to $\int f(x)\,d\mu$ for analytic flows on the two-dimensional torus, having analytic invariant measure and a rotation number, poorly approximated by rational numbers.

D. A. Moskvin [128] found conditions under which almost all points of a curve of class $C^2$ on $T^2$ are uniformly distributed with respect to an algebraic endomorphism of the two-dimensional torus, which has no eigenvalues of modulus one.

In a series of distinct generalizations of individual ergodic theorems there is special interest in connection with the analysis of dynamical systems which have the property of exponential dispersal of trajectories (cf. Chap. 2), represented by multiplicative ergodic theorems. Let $T_t$ be a one-parameter group of automorphisms of the Lebesgue space $(X, \mu)$, $\mu(X) = 1$ and A(x, t) be a measurable matrix function on $X \times \mathbf{R}$, satisfying the relation

$$A(x;\, t+s) = A(x;\, t)\,A(T_t x;\, s)\,.$$

The problem consists of studying the asymptotic behavior of the matrix function A(x; t) as

Under certain restrictions on A(x; t), V. M. Millionshchikov [124] and V. I. Oseledets [131] established the existence for almost all x of the precise characteristic exponents (cf. [40]) of the function A(x; t) and studied their properties.

In some questions of ergodic theory, the behavior of the means $\frac{1}{\tau}\int_0^\tau T_t f\,dt$ as $\tau \to 0$ is essential. The local ergodic theorem of Wiener asserts that

$$\lim_{\tau\to 0}\frac{1}{\tau}\int_0^\tau f(T_t x)\,dt = f(x) \quad \text{a.e.,}$$

if $f \in L_1(X, \mu)$ and $T_t$ is a measurable flow in the space $(X, \mu)$, $\mu(X)=1$.

In [501] a generalization of this theorem of Wiener to the case of strongly continuous semigroups of positive operators in $L_1(X, \mu)$ is obtained. In [694] it is proved that for a strongly continuous semigroup of compressions $T_t$ in $L_1$ the local ergodic theorem is equivalent with the maximal ergodic inequality

$$\mu\left\{\overline{\lim_{\tau\to+0}}\frac{1}{\tau}\int_0^\tau T_t f\,dt > \delta\right\} \leqslant \frac{c}{\delta}\|f\|_1.$$

Terrel [693] proved the local ergodic theorem for an n-parameter strongly continuous semigroup of positive compressions in $L_1(X, \mu)$, and also for a semigroup of compressions in $L_1(X, \mu)$ which are compressions in $L_\infty(X, \mu)$. It is interesting to note that in the local case, the assertion analogous to the maximal ergodic lemma, in general, is untrue for n-parameter semigroups of positive compressions in $L_1$, $n>1$.

The application in ergodic theory of the method of summation of Abel leads to the question of the behavior as $\lambda\to 0$ of means of the form $\int_0^\infty e^{-\lambda t}T_t f\,dt$. For the corresponding theorem on convergence almost everywhere, cf. [325] and [650]. For a family of positive operators $T_\alpha$, where $\alpha$ runs through a semigroup $V\subset R^+$, Berk [248] considered the mean

$$\int_{V\cap[0,\beta]}(T_\alpha f)\,dv(\alpha);$$

here v is a nondecreasing, upper semicontinuous solution of the renewal equation $v(\alpha)=1+\int_0^\alpha v(\alpha-\beta)\,w(d\beta)$.

In [387] conditions are found on a matrix $(a_{nk})$, which are necessary and sufficient for the convergence in $L_2(X, \mu)$ of the sequence $S_n(x)=\sum_{k=0}^\infty a_{nk} f(T^k x)$. Sato [647] obtained a generalization of the Chacon ergodic theorem for weighted means.

Fong and Sucheston [343] proved the equivalence for a compression T in the space $L_1(X, \mu)$ of the following two properties:

1) $T^n f$ converges weakly in $L_1(X, \mu)$ for each $f$ $L_1(X, \mu)$;

2) if $f \in L_1(X, \mu)$, then the sequence $\sum_i a_{ni}T^i f$ converges in $L_p(X, \mu)$ for each matrix $(a_{ni})$ satisfying the uniform regularity conditions:

$$\sup_n \sum_i |a_{ni}| < \infty, \quad \lim_n \sum_i a_{ni} = 1, \quad \lim_n \max_i |a_{ni}| = 0.$$

An analogous result is true for compressions in $L_2$. Of other papers on ergodic theorems for weighted means we note [247] and [516].

In [279, 646, 304], pointwise convergence of means of subsequences is studied, i.e., expressions of the form $\frac{1}{n}\sum_{i=1}^n T^{k_i} f$, $f \in L_1(X, \mu)$. We note in connection with this that Krengel [460] constructed an increasing sequence of natural numbers $\{k_i\}$, such that for any aperiodic automorphism T there exists a characteristic function f, for which

$$\overline{\lim}\frac{1}{n}\sum_1^n f(T^{k_i}x)=1, \quad \underline{\lim}\frac{1}{n}\sum_1^n f(T^{k_i}x)=0$$

almost everywhere.

References [623, 442, 443] are devoted to probabilistic ergodic theorems.

In [446] a generalization of the individual ergodic theorem of information theory to the case of infinite invariant measure is obtained. B. S. Pitskel' [138] showed that the condition of finiteness of entropy of the partition in the Shannon–MacMillan–Breiman theorem cannot be weakened, requiring only convergence of the series $\sum\mu(C_\xi)|\log\mu(C_\xi)|^q$ for all $0<q<1$. R. M. Belinskaya [20] obtained a generalization of the individual ergodic theorem of information theory to the case of skew product.

## §3. Actions of General Groups

For actions of general groups of transformations, from various points of view there is interest in the question of the convergence almost everywhere of "time means." A series of results in this direction was obtained by A. A. Tempel'man [178, 179]. Analogous results are obtained in [291, 250]. Bewley [249] constructed an example of an action of a free group with two generators for which the individual ergodic theorem is false.

Let G be a locally compact group with left-invariant mean, $\{H_n\}$ be a averaging sequence of subsets of G, $\lambda$ be the left-invariant Haar measure on G. Under the assumption that there exists a constant K for which $\lambda(H_nH_n^{-1})\leqslant K\lambda(H_n)$, the individual ergodic theorem for actions of the group G is proved in [292] (cf. [692]).

A. M. Vershik [47] introduced the class of completely hyperconical groups and proved for actions of groups of this class the individual ergodic theorem. His proof is based on the theorem on convergence of martingales.

Of other papers on ergodic theorems for groups of operators, we note [688, 689, 437, 242].

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