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Bernoulli diffeomorphisms and group extensions of dynamical systems with non-zero characteristic exponents

By M. I. BRIN, J. FELDMAN, and A. KATOK

1. Every manifold of dimension greater than one carries a Bernoulli diffeomorphism

We begin by fixing some terminology:

Manifold— C^∞ compact connected manifold (possibly with boundary);

Smooth positive measure—probability measure on a manifold which can be represented in any local coordinate system by a positive C^∞ density;

Bernoulli diffeomorphism— C^∞ diffeomorphism of a manifold M which preserves a smooth positive measure μ and, considered as a measure-preserving transformation of the Lebesgue space (M, μ) , is isomorphic to a Bernoulli shift (analogously: *K-diffeomorphism*, *weakly mixing diffeomorphism*).

We will also use definitions and notation from [1], Section 1: In particular, the definition of the classes $\text{Diff}_\rho^\infty(\mathbf{D}^n)$, those diffeomorphisms of the n -dimensional disc \mathbf{D}^n which are sufficiently flat near the boundary. The flatness of the m th derivatives is controlled by a non-negative function ρ_m which is positive inside \mathbf{D}^n . It will be convenient to denote the sequence (ρ_0, ρ_1, \dots) by ρ .

The main result is this:

1.1. THEOREM. *Every manifold of dimension greater than one carries a Bernoulli diffeomorphism.*

The proof has four main components, (i)–(iv). The first and last are taken from A. Katok [1] which deals with the particular case of Theorem 1.1 for two-dimensional manifolds; the second is carried out in subsequent sections of this paper; the third is due to D. Rudolph [2]. We describe them briefly.

(i) Construction of a Bernoulli diffeomorphism f_0 on \mathbf{D}^2 ([1], Theorem A).

For any fixed sequence $\rho = (\rho_0, \rho_1, \dots)$, such a diffeomorphism can in fact be constructed in $\text{Diff}_\rho^\infty(\mathbf{D}^2)$.

(ii) Construction of a C^∞ map $h: \mathbf{D}^2 \rightarrow \mathbf{T}^{n-2}$ (the $(n-2)$ -torus) which vanishes in a neighborhood of $\partial\mathbf{D}^2$ and such that the \mathbf{T}^{n-2} -extension \hat{f}_0 of f_0 corresponding to h is weakly mixing. We show two methods of doing this, obtained independently by two of the authors.

One method, due to Feldman, is similar to methods of Livšič and Sinai [3], [4]. It is not difficult, but is quite special, and leans heavily on the specific properties of the example in [1]. The construction is carried out in Sections 2 and 3. (This author wishes to acknowledge valuable conversations with Mary Rees, in connection with Section 2.)

The other method, due to Brin, involves proving and then applying a more general theorem of independent interest, which extends earlier work [8] of the same author. This theorem will be proved in Sections 4, 5, and 6. Its statement is the following:

COROLLARY 4.5. *Given a K -diffeomorphism $f: N \rightarrow N$ which has non-zero Lyapunov exponents, a compact Lie group G , and a non-empty open set $V \subset N$, there exists a C^∞ map $h: N \rightarrow G$ such that for $x \in N \sim V$, $h(x)$ is the identity element of G , and the G -extension $\hat{f}: N \times G \rightarrow N \times G$ defined by*

$$\hat{f}(x, g) = (fx, h(x)g)$$

is a K -diffeomorphism.

(iii) A deep theorem of D. Rudolph [2]: If a compact group extension of a Bernoulli shift is weakly mixing then it is metrically isomorphic to a Bernoulli shift.

(iv) The Reduction Theorem ([1], Proposition 1.2, and the proof of Theorem B): If for every admissible sequence of functions there exists a Bernoulli diffeomorphism $f \in \text{Diff}_\rho^\infty(\mathbf{D}^n)$ preserving the Lebesgue measure on \mathbf{D}^n then there exists a Bernoulli diffeomorphism on every n -dimensional manifold M .

Here is how the four components are used to give Theorem 1.1.

We start with the Bernoulli diffeomorphism f_0 of \mathbf{D}^2 described in (i). This diffeomorphism has non-zero Lyapunov exponents ([1], Proposition 2.2). Then we use either of the methods described in (ii) to construct a weakly mixing diffeomorphism

$$\begin{aligned} \hat{f}_0: \mathbf{D}^2 \times \mathbf{T}^{n-2} &\rightarrow \mathbf{D}^2 \times \mathbf{T}^{n-2}, \\ \hat{f}_0(x, g) &= (f_0x, h(x) \cdot g), \end{aligned}$$

where $h: \mathbf{D}^2 \rightarrow \mathbf{T}^{n-2}$ is a C^∞ function equal to the identity in a neighborhood of the boundary.

Now we show that if \hat{f}_0 is a Bernoulli diffeomorphism in $\text{Diff}_{\rho^0}(\mathbf{D}^2 \times \mathbf{T}^{n-2})$, with ρ^0 sufficiently rapidly decreasing, we may transfer it to \mathbf{D}^n in any preassigned $\text{Diff}_\rho(\mathbf{D}^n)$.

First, we show how to map $\mathbf{D}^2 \times \mathbf{T}$ onto \mathbf{D}^3 by a continuous map which is a C^∞ diffeomorphism on the interior, and which sends normalized Lebesgue product measure on $\mathbf{D}^2 \times \mathbf{T}$ to normalized Lebesgue measure on \mathbf{D}^3 . Represent \mathbf{D}^2 as pairs $(x_1, x_2): x_1^2 + x_2^2 \leq 1$; \mathbf{T} as points $z = (z_1, z_2)$ on the unit circle; and \mathbf{D}^3 as triples $(y_1, y_2, y_3): y_1^2 + y_2^2 + y_3^2 \leq 1$. Our map will send (x_1, x_2, z) to

$$\left(\eta(x_1), z_1 \sqrt{1 - \eta(x_1)^2} \sqrt{\frac{x_2}{2\sqrt{1 - x_1^2}} + \frac{1}{2}}, z_2 \sqrt{1 - \eta(x_1)^2} \sqrt{\frac{x_2}{2\sqrt{1 - x_1^2}} + \frac{1}{2}} \right)$$

where η is a certain increasing function from $[-1, 1]$ onto $[-1, 1]$. For fixed x_1 the map sends the cylinder $\{(x_1, x_2, z): x_2^2 \leq 1 - x_1^2\}$ onto the disc $\{(\eta(x_1), y_2, y_3): y_2^2 + y_3^2 \leq 1 - \eta(x_1)^2\}$, in a \mathbf{T} -invariant manner which sends normalized product measure on the cylinder to normalized plane Lebesgue measure on the disc. Now η is that unique function which causes the map to give the proper image measure at the 3-dimensional level (this is equivalent to the differential equation $d\eta/dx_1 = (8/3)(\sqrt{1 - x_1^2}/(1 - \eta^2))$, from which the regularity of η and hence of the map become clear).

Now suppose we are given any j , $0 \leq j < n - 2$, and a preassigned measure $\rho^{j+1} = (\rho_0^{j+1}, \rho_1^{j+1}, \dots)$ of flatness at the boundary of $\mathbf{D}^{j+3} \times \mathbf{T}^{n-j-3}$. Topologically, $\mathbf{D}^{j+3} \times \mathbf{T}^{n-j-3} = \mathbf{D}^j \times (\mathbf{D}^3 \times \mathbf{T}^{n-j-3})$, which may be obtained by applying the map described in the previous paragraph to the $\mathbf{D}^2 \times \mathbf{T}$ factor in

$$\mathbf{D}^j \times (\mathbf{D}^2 \times \mathbf{T}) \times \mathbf{T}^{n-j-3} = (\mathbf{D}^j \times \mathbf{D}^2) \times \mathbf{T}^{n-j-2} = \mathbf{D}^{j+2} \times \mathbf{T}^{n-j-2},$$

The homeomorphisms: $\mathbf{D}^{j+3} \rightarrow \mathbf{D}^j \times \mathbf{D}^3$ and $\mathbf{D}^j \times \mathbf{D}^2 \rightarrow \mathbf{D}^{j+2}$ may be obtained via homeomorphisms which are C^∞ diffeomorphisms on the interior and which make product Lebesgue measure correspond to Lebesgue measure; we leave this point to the reader.

Now, there is some degree of flatness ρ^j at the boundary of $\mathbf{D}^{j+2} \times \mathbf{T}^{n-j-2}$ such that if \hat{f}_j is in $\text{Diff}_{\rho^j}(\mathbf{D}^{j+2} \times \mathbf{T}^{n-j-2})$, then the corresponding \hat{f}_{j+1} on $\mathbf{D}^{j+3} \times \mathbf{T}^{n-j-3}$ will be on $\text{Diff}_{\rho^{j+1}}(\mathbf{D}^{j+3} \times \mathbf{T}^{n-j-3})$. Thus, starting from \hat{f}_0 in a sufficiently flat $\text{Diff}_{\rho^0}(\mathbf{D}^2 \times \mathbf{T}^{n-2})$, we get \hat{f}_{n-2} in $\text{Diff}_{\rho^{n-2}}(\mathbf{D}^n)$, where ρ^{n-2} is our preassigned ρ . Since all identifications were on sets of measure 0, \hat{f}_0 and \hat{f}_{n-2} are isomorphic as measure-preserving transformations; in particular, if \hat{f}_0 is a Bernoulli diffeomorphism, so is \hat{f}_{n-2} .

1.2. *Remark.* For the diffeomorphism \hat{f} constructed in this paper, $n - 2$

of the Lyapunov exponents are zero.

1.3. *Problem.* Show that any manifold of dimension greater than 1 carries a Bernoulli diffeomorphism none of whose Lyapunov exponents are zero (cf. [6]).

The simplest manifolds for which the answer is unknown are \mathbf{D}^n , n odd and S^n , $n \geq 3$.

2. Hyperbolicity of the Katok example

We examine the example of [1] briefly. Let g_0 be a hyperbolic automorphism of the 2-torus which leaves fixed the four points $x_1 = (0, 0)$, $x_2 = (1/2, 0)$, $x_3 = (0, 1/2)$, $x_4 = (1/2, 1/2)$. The significance of the x_i is that they are fixed points of the map $x \mapsto -x$ on the 2-torus. Then there is a commutative diagram

$$\begin{array}{ccccccccc} \mathbf{T}^2 & \xrightarrow{\phi_0} & \mathbf{T}^2 & \xrightarrow{\phi_1} & \mathbf{T}^2 & \xrightarrow{\phi_2} & \mathbf{S}^2 & \xrightarrow{\phi_3} & \mathbf{D}^2 \\ \downarrow g_0 & & \downarrow g_1 & & \downarrow g_2 & & \downarrow g_3 & & \downarrow f_0 \\ \mathbf{T}^2 & \xrightarrow{\phi_0} & \mathbf{T}^2 & \xrightarrow{\phi_1} & \mathbf{T}^2 & \xrightarrow{\phi_2} & \mathbf{S}^2 & \xrightarrow{\phi_3} & \mathbf{D}^2 \end{array},$$

and g_1 is obtained by “slowing down” g_0 near the x_i . It leaves invariant a smooth probability measure $d\nu = \rho d\lambda$, where λ is Lebesgue measure and ρ is positive and C^∞ except for infinities at the x_i . The conjugating function ϕ_0 of the diagram is only known to be a homeomorphism. However, ϕ_1 is a homeomorphism which is in fact a diffeomorphism except at the x_i , is also the identity outside a neighborhood of $\{x_1, x_2, x_3, x_4\}$, and carries $d\nu$ into Lebesgue measure.

Following [1], we choose coordinates (ξ_u, ξ_s) in the tangent space T_x to \mathbf{T}^2 at x , so that $Dg_0(x)$ has the form $(\xi_u, \xi_s) \mapsto (\alpha\xi_u, \alpha^{-1}\xi_s)$ where $\alpha > 1$. Thus $\{\xi_u = 0\}$ is the stable subspace of g_0 , and $\{\xi_s = 0\}$ the unstable one. (Be warned that this is not necessarily an orthogonal coordinate system for the Riemann metric.) Set

$$K_x^+ = \{(\xi_u, \xi_s): |\xi_u| \geq |\xi_s|\}, \quad K_x^- = \{(\xi_u, \xi_s): |\xi_u| \leq |\xi_s|\}.$$

In [1] it is shown that $Dg_1(x)(K_x^+) \subset K_{g_1x}^+$, while $Dg_1^{-1}(x)(K_x^-) \subset K_{g_1^{-1}x}^-$.

Denote by $\mathfrak{V}_u, \mathfrak{V}_s$ the images under ϕ_0 of the systems of stable curves and unstable curves of g_0 ; these are then systems of stable and unstable curves for g_1 . They no longer contract exponentially, but still contract uniformly (since ϕ_0 is uniformly continuous). Our purpose in this section is to show that this contraction is nevertheless exponential in a certain limited sense.

It is further shown in [1] that the curves of \mathfrak{V}_u and \mathfrak{V}_s are smooth, and meet transversally. Thus each T_x has a splitting into $E_x^u \oplus E_x^s$. It is also the case that $E_x^u \subset K_x^+$ and $E_x^s \subset K_x^-$; thus we may consistently choose a "positive" unit unstable vector $v_u(x)$ and unit stable vector $v_s(x)$; consistency means that $Dg_1(x)v_s(x)$ is a positive multiple of $v_s(g_1x)$, and similarly $Dg_1^{-1}(x)v_u(x)$ of $v_u(g_1^{-1}x)$.

2.1. LEMMA. *Given any neighborhood U of $\{x_1, x_2, x_3, x_4\}$ there exists $\alpha_0 > 0$ such that for all x outside of U we have*

$$\begin{aligned} \|v_u(x) - \partial K_x^+\| &> \alpha_0, \\ \|v_s(x) - \partial K_x^+\| &> \alpha_0. \end{aligned}$$

Proof. This is implicit in [1]. There, in (4.3) and (4.4) it is pointed out that there is an inequality of this sort at each x ; but furthermore the form of (4.2) shows that the right hand side stays bounded away from 0 outside any neighborhood of $\{x_1, x_2, x_3, x_4\}$. \square

Let $l_r(x, k) = \#\{j: 0 \leq j \leq k, g_j^i x \in U\}$.

2.2. LEMMA. *For any neighborhood U of $\{x_1, x_2, x_3, x_4\}$, there is some $c_0 > 0$ and $\sigma_0 < 1$ such that if $x \in U$, $g_1^k x \in U$ and $l = l_r(x, k)$, then the opening of the cone $Dg_1^k(K_x^+)$ has angle $< c_0 \sigma_0^l \times$ angle of opening of the cone K_x^+ .*

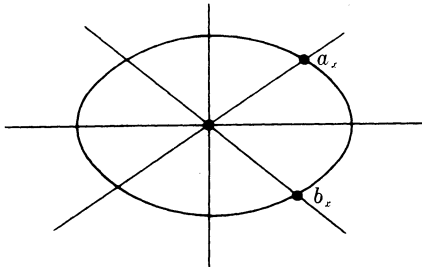
This lemma follows from (4.7) and (4.9) in [1].

2.3. PROPOSITION. *Given any neighborhood U of $\{x_1, \dots, x_4\}$, there exist $c > 0$ and $\sigma < 1$ such that if x and $g^k x$ are not in U , and $l = l_r(x, k)$, then $\|Dg_1^k(x)v_s(x)\| \leq c\sigma^l$.*

Proof. This is essentially a repetition of the proof of Proposition 2.2 from [1]. Let

$$\begin{aligned} Dg_1^k v_s(x) &= S_k(x)v_s(x) \\ Dg_1^k v_u(x) &= U_k(x)v_u(x). \end{aligned}$$

Let $a_x = A_u(x)v_u(x) + A_s(x)v_s(x)$ be the point where ∂K_x^+ meets the set $\xi_s^2 + \xi_u^2 = 1$ with $A_u(x), A_s(x)$ positive and let $b_x = B_u(x)v_u(x) + B_s(x)v_s(x)$ be



the point where ∂K_x^+ meets the set $\xi_1^2 + \xi_2^2 = 1$ with $B_u(x)$ positive and $B_s(x)$ negative. Then $|A_u(x)|$, $|A_s(x)|$, $|B_u(x)|$ and $|B_s(x)|$ are all bounded away from 0 and 1, because of Lemma 2.1. So Lemma 2.2 says the angle between $Dg_1^k a_x$, $Dg_1^k b_x$ is less than $c_0 \sigma_0^l$ times the angle between a_x and b_x . But

$$Dg_1^k a_x = A_u(x)S_k(x)v_u(x) + A_s(x)U_k(x)v_s(x)$$

and

$$Dg_1^k b_x = B_u(x)S_k(x)v_u(x) + B_s(x)U_k(x)v_s(x).$$

A little algebra then reveals that

$$\frac{U_k(x)}{S_k(x)} \leq \text{const.} \times \sigma_0^l.$$

But g_1^k preserves a measure whose density is continuous outside U . Thus, $U_k(x)S_k(x)$ is bounded above and below by positive constants. Therefore $U_k(x)^2 \leq \text{const.} \times \sigma_0^l$, so $U_k(x) \leq c\sigma^l$. \square

2.4. COROLLARY. *Given any neighborhood V of $\{x_1, x_2, x_3, x_4\}$, there exist $c' > 0$, $\sigma < 1$, and $\delta > 0$ such that whenever x, y lie on the same stable curve of g_1 with $x, g_1^k x \in V$ and $d(x, y) < \delta$, then $d(g_1^l x, g_1^l y) < c'\sigma^l d(x, y)$, where $l = l_V(x, k)$.*

Proof. It is shown in [1] that for any small open ball O disjoint from $\{x_1, x_2, x_3, x_4\}$, the map which assigns to x in O the connected component of stable curve through x in O is continuous from O into the space of curves in the C^1 topology. Then there is some δ_0 such that if y is any point on the connected segment of stable curve through x in the δ_0 -ball around x , then the segment of stable curve from x to y has arc length $< 2d(x, y)$.

Now choose a neighborhood U of $\{x_1, x_2, x_3, x_4\}$ with $\bar{U} \subset V$; say $d(U, V^c) < \text{some positive } \varepsilon_1$. There is some $\delta_1 < \delta_0$ such that, if x, y are on the same stable curve, and $d(x, y) < \delta_1$, then $d(g_1^j x, g_1^j y) < \varepsilon$, for all $j \geq 0$. Thus, if $x \in V$ and $g_1^k x \in V$ then $y \in U$ and $g_1^k y \in U$.

If y lies on the component of the stable curve through x in the δ_1 -neighborhood of x , and if x and $g_1^k x$ are outside V , then we may use the above remark and the previous proposition for all points on the stable segment from x to y to get this chain of inequalities: $d(g_1^k x, g_1^k y) < \text{arc length of stable curve from } g_1^k x \text{ to } g_1^k y < c\sigma^{l_V(x, u)} \times \text{arc length of stable curve from } x \text{ to } y < 2c\sigma^{l_V(x, u)} d(x, y)$. Since $l_V(x, k) \geq l_V(x, k)$, we conclude: $d(g_1^k x, g_1^k y) < 2c\sigma^l d(x, y)$. \square

3. Construction of the skewing function

3.1. LEMMA. *Let f be an ergodic transformation on the probability space (X, μ) . Let h be a measurable function: $X \rightarrow \mathbf{T}^m$, and write*

$h(x) = (h_1(x), \dots, h_m(x))$. Then the skew product transformation $(x, z) \mapsto (fx, zh(x))$ is weakly mixing if and only if there is no m -tuple of integers $(i_1, \dots, i_m) \neq (0, \dots, 0)$ for which the function $h_1^{i_1} \dots h_m^{i_m}: X \rightarrow \mathbf{T}$ has a form $\beta(b(fx)/b(x))$ where β is a constant and $b: X \rightarrow \mathbf{T}$ a measurable function.

Proof. If $h_1^{i_1}(x) \dots h_m^{i_m}(x) = \beta(b(fx)/b(x))$ for almost every x , then set $\psi(x, z) = b^{-1}(x)z_1^{i_1} \dots z_m^{i_m}$, where $z = (z_1, \dots, z_m) \in \mathbf{T}^m$. A calculation reveals that $\psi(fx, zh(x)) = \beta\psi(x, z)$ for all z and almost every x . Clearly ψ is not almost everywhere constant, so the skew product transformation cannot be weakly mixing.

Conversely: if $\psi(fx, zh(x)) = \beta\psi(x, z)$ for almost every (x, z) , then we expand ψ as a Fourier series in

$$z: \psi(x, z) = \sum_{i_1, \dots, i_m} a_{i_1, \dots, i_m}(x) z_1^{i_1} \dots z_m^{i_m}.$$

The equation on ψ gives

$$a_{i_1 \dots i_m}(fx) h_1^{i_1}(x) \dots h_m^{i_m}(x) = \beta a_{i_1 \dots i_m}(x) \quad \text{for a.e. } x.$$

By ergodicity of f , the modulus of each $a_{i_1 \dots i_m}$ is almost everywhere constant. If ψ is not a.e. constant, then there is some $(i_1, \dots, i_m) \neq (0, \dots, 0)$ for which $|a_{i_1 \dots i_m}(x)| = c > 0$ a.e. Set $b(x) = c^{-1} a_{i_1 \dots i_m}(x)$. Then b takes values in \mathbf{T} , and satisfies the equation of the lemma. \square

Now we return to the diffeomorphism $g_1: \mathbf{T}^2 \rightarrow \mathbf{T}^2$.

3.2. LEMMA. *If H is a Lipschitz continuous function: $\mathbf{T}^2 \rightarrow \mathbf{T}$, and $H = 1$ in a neighborhood of $\{x_1, \dots, x_i\}$, and $H(x) = \beta(b(g_1x)/b(x))$ almost everywhere for some measurable $b: \mathbf{T}^2 \rightarrow \mathbf{T}^1$, then there is a set A of measure 1 in \mathbf{T}^2 and for each $\varepsilon > 0$, some $\delta_0 > 0$, such that, if $d(x, y) < \delta_0$, x, y are in A , and x and y both lie on the same stable or unstable curve of g_1 , then $|b(x) - b(y)| < \varepsilon$.*

Proof. There is a g_1 -invariant set S of measure 1 such that $H(x) = \beta(b(g_1x)/b(x))$ for all $x \in S$. Choose a compact set K of measure $> 9/10$ on which b is uniformly continuous. If $x \in S$ and $k > 0$, we may write

$$b(x) = b(g_1^k x) / H(x) \dots H(g_1^{k-1} x).$$

Thus, if $x, y \in S$, we have

$$\frac{b(x)}{b(y)} = \frac{b(g_1^k(x))}{b(g_1^k(y))} \frac{H(y)}{H(x)} \dots \frac{H(g_1^{k-1}y)}{H(g_1^{k-1}x)}.$$

By ergodicity, there is a set F of measure 1 such that if $x \in F$ then $g_1^k x$ lies in K for a set of k of density $> 9/10$. Thus, if $x, y \in F$, then there exists an arbitrarily large k such that both $g_1^k x$ and $g_1^k y$ lie in K . Hereafter we will always assume that x and y lie in $S \cap F$ and on the same stable curve.

Let us consider the factor $(H(y) \cdots H(g_1^{k-1}y))/(H(x) \cdots H(g_1^{k-1}x))$, and denote by V some neighborhood of $\{x_1, \dots, x_k\}$ with $\bar{V} \subset U$. Then $\varepsilon_0 = d(\bar{V}, T^2 \setminus U) > 0$. Choose δ_0 so small that if $d(x, y) > \delta_0$ then $d(g_1^j x, g_1^j y) < \min(\varepsilon_0, \delta)$ for $j = 0, 1, \dots, k$, where δ is as in Corollary 2.4. Given x, y with $d(x, y) < \delta_0$, let $0 < j_1 < \dots < j_l \leq k$ be the values of j between 0 and k for which $g_1^{j_i} x \notin V$. If j is not some j_i , so that $g_1^j x \in V$, then $g_1^j y \in U$, so that $H(g_1^j x) = H(g_1^j y) = 1$. Thus,

$$\frac{H(y) \cdots H(g_1^{k-1}y)}{H(x) \cdots H(g_1^{k-1}x)} = \prod_{i=1}^l \frac{H(g_1^{j_i}y)}{H(g_1^{j_i}x)}.$$

The product may, of course, be empty, i.e., $l = 0$, in which case it has the value 1. If it is nonempty, then for $1 < i \leq l$ we have

$$\begin{aligned} |H(g_1^{j_i}y) - H(g_1^{j_i}x)| &\leq Ld(g_1^{j_i}y, g_1^{j_i}x) \quad (L \text{ being the Lipschitz} \\ &\quad \text{constant for } H) \\ &\leq Lc'\sigma^{i-1}d(g_1^{j_1}x, g_1^{j_1}y). \end{aligned}$$

Now let us further restrict δ_0 so that if $d(x, y) < \delta_0$ then $d(g_1^j x, g_1^j y) < \delta_1$, where δ_1 is so small that

$$\prod_{i=1}^{\infty} (1 + Lc'\sigma^{i-1}\delta_1) < \varepsilon/2.$$

Then, since

$$\left| \frac{H(g_1^{j_i}y)}{H(g_1^{j_i}x)} - 1 \right| < Lc'\sigma^{i-1}\delta_1,$$

we have

$$\left| \prod_{j=0}^k \frac{H(g_1^j y)}{H(g_1^j x)} - 1 \right| < \varepsilon/2 \quad \text{for all } k \geq 0.$$

As for the factor $b(x)/b(y)$, we restrict δ_0 further: choose ε_1 so that if two points in K are within ε_1 of each other, then their b -values are within $\varepsilon/2$ of each other. Restrict δ_0 further so that if $d(x, y) < \delta_0$ then $d(g_1^j x, g_1^j y) < \varepsilon_1$ for all $j \geq 0$. Now choose k for which $g_1^k x$ and $g_1^k y$ are both in K . Then if $d(x, y) < \delta_0$ we have also

$$\left| \frac{b(g_1^k x)}{b(g_1^k y)} - 1 \right| < \varepsilon/2, \quad \text{and so} \quad \left| \frac{b(x)}{b(y)} - 1 \right| < \varepsilon.$$

Using g^{-1} we construct sets S' and F' . Then the set $A = S \cap F \cap S' \cap F'$ satisfies the assertion of the lemma.

3.3. COROLLARY. *Given H and b as in Lemma 2.3, there is a set of measure 1 on which b is uniformly continuous.*

Proof. This follows by use of absolute continuity of foliations, as in

[3]. More precisely: since the Lyapunov exponents of g_1 are nonzero almost everywhere ([1], Proposition 2.2), and since the stable and unstable foliations are globally defined, it follows as in [5] that each of the stable and unstable foliations is absolutely continuous. Thus, there is a set $B \subset A$, of measure 1, such that if $x, y \in B$ and x, y are reasonably close then there exist x_1, y_1 in A and on the same unstable curve, such that x and x_1 are on the same stable curve, and likewise y and y_1 . If x and y are close it easily follows that x and x_1, y and y_1 , and x_1 and y_1 are all close. Thus Lemma 3.2 may be applied to get the desired uniform continuity. \square

Now we shall exhibit functions h from \mathbf{D}^2 to \mathbf{T}^m which are C^∞ , take on the identity of \mathbf{T}^m near $\partial\mathbf{D}^2$, and give weakly mixing skew products over f_0 . Choose U in \mathbf{T}^2 small enough that there are at least two g_1 -periodic points x, y such that the orbits of $x, -x, y$, and $-y$ are disjoint from each other and from \bar{U} . Let $\phi = \phi_3 \circ \phi_2 \circ \phi_1$. Then the images of x and y under ϕ are f_0 -periodic points with orbits disjoint from each other and from $\phi(\bar{U})$. Let the periods be k and l respectively. Let h be a C^∞ function $= \mathbf{D}^2 \rightarrow \mathbf{T}^m$ which vanishes in $\phi(\bar{U})$. Suppose the skew product made with f_0 and h is not weak mixing. Then, by Lemma 3.1, there exist $(i_1, \dots, i_m) \neq (0, \dots, 0)$, $\beta \in \mathbf{T}$ and $b: \mathbf{D}^2 \rightarrow \mathbf{T}$ with $h_1^{i_1} \dots h_m^{i_m} = \beta(b \circ f_0/b)$ almost everywhere. Then

$$(h_1 \circ \phi)^{i_1} \dots (h_m \circ \phi)^{i_m} = \beta \frac{b \circ \phi \circ g_1}{b \circ \phi} \quad \text{a.e. on } \mathbf{T}^2.$$

Now, $H = (h_1 \circ \phi)^{i_1} \dots (h_m \circ \phi)^{i_m}$ is C^∞ , takes on 1 in U , and is almost everywhere equal to $\beta((b \circ \phi \circ g_1)/(b \circ \phi))$. Then by Corollary 3.3, $b \circ \phi$ is almost everywhere equal to a continuous function. Then the same is true for b , at least on the interior of \mathbf{D}^2 . In particular, it holds at $\phi(x), f_0\phi(x), \dots, f_0^{k-1}\phi(x)$ and $\phi(y), f_0\phi(y), \dots, f_0^{l-1}\phi(y)$. Thus, we have

$$(H(x) \dots H(g_1^{k-1}x))^l = \beta^{kl} = (H(y) \dots H(g_1^{l-1}y))^k.$$

But we are free to assign any values we want to h_1, \dots, h_m at the finitely many point of the f_0 -orbits of $\phi(x)$ and $\phi(y)$, and for almost all such assignments the above equation cannot hold for any $(i_1, \dots, i_m) \neq (0, \dots, 0)$. See [4].

4. Statement of results about group extensions

In this and subsequent sections we study in a more general way ergodic properties of group extensions of dynamical systems with non-zero characteristic exponents. The corresponding problems for group extensions of Anosov systems were studied in [7], [8]. Our main goal is to find out to what

extent ergodic properties of the base system are inherited by its group extensions. According to the results of Pesin [6], for any diffeomorphism preserving a smooth measure and having non-zero characteristic exponents, the measure of almost every ergodic component is positive, and the restriction of the diffeomorphism to such a component is isomorphic to the direct product of a finite cyclic permutation and a Bernoulli shift. Under some additional conditions the diffeomorphism is a Bernoulli shift. Our main result asserts that by taking a small perturbation of a given group extension one can “lift” the mentioned ergodic properties of the base diffeomorphism to its group extension.

All our results with appropriate changes in formulations remain valid for smooth flows.

Let h be a C^2 -diffeomorphism of a smooth compact Riemannian manifold $N^m = N$, preserving a smooth positive measure m_h . The measurable function

$$(**) \quad X(x, v) = \lim_{k \rightarrow \infty} \frac{1}{k} \ln \|dh^k v\|, \quad v \in T_x N$$

is called the characteristic exponent [9]. For almost every x in N the limit $(**)$ exists for every tangent vector v in $T_x N$, and for fixed x , only finitely many different values are taken on by $X(x, v)$. Let $X^-(x)$ be the greatest negative value and let $X^+(x)$ be the smallest positive value. Denote by L the h -invariant set of all points x such that:

- (i) x is a regular point ([9], [10]).
- (ii) $X(x, v) \neq 0$ for every v in $T_x N$.

The following theorem summarizes some of the results obtained by Ja. B. Pesin in [10].

4.1. THEOREM (Ja. B. Pesin). *Let $a(x) = (1/1000) \min(-X^-(x), X^+(x))$. There exist two dh -invariant measurable families of subspaces $E^s(x)$, $E^u(x) \subset T_x N$, $x \in L$, measurable functions $C(x)$, $K(x)$, $b(x) > 0$, and two h -invariant families of local stable and unstable C^2 -manifolds $V^s(x)$ and $V^u(x)$ satisfying the following conditions:*

- (a) $C(x)/C(hx) \geq e^{-a(x)}$, $K(x)/K(hx) \leq e^{a(x)}$, $b(x)/b(hx) \leq e^{a(x)}$.
- (b) $T_x N = E^s(x) \oplus E^u(x)$, and the angle between these subspaces is greater than $K(x)$.
- (c) For every positive integer n ,

$$\begin{aligned} \|dh^n v\| &\leq C(x) e^{(a(x) + X^-(x))n} \|v\|, \\ \|dh^{-n} v\| &\geq C^{-1}(x) e^{-(a(x) + X^-(x))n} \|v\|, \quad v \in E^s(x); \\ \|dh^n v\| &\geq C^{-1}(x) e^{(X^+(x) - a(x))n} \|v\|, \\ \|dh^{-n} v\| &\leq C(x) e^{-(X^+(x) - a(x))n} \|v\|, \quad v \in E^u(x). \end{aligned}$$

(d) *The local stable and unstable manifolds $V^s(x)$ and $V^u(x)$ are defined in the $b(x)$ -neighborhood of every point x in L ; their intersection is precisely x ; for every point y belonging to $V^s(x)$ and any positive integer n ,*

$$d(h^n x, h^n y) \leq C(x) e^{(X^-(x) + a(x))n} d(x, y);$$

if the distance $d(h^n x, h^n y)$ is always less than $b(h^n x)$ and less than $Ce^{(X^-(x) + a(x))n}$ for some constant C , then y belongs to $V^s(x)$; and the same is true for $V^u(x)$ and any negative n .

(e) *Let us set $L_r = \{x \in L: \min(-X^-(x), X^+(x)) \geq r^{-1}\}$, and*

$$L_r^s = \{x \in L_r: C(x), K^{-1}(x), b^{-1}(x) < s\};$$

then the families E^s, E^u, V^s, V^u are continuous on every set L_r^s .

(f) *There is a function $S = S(s, r)$ such that, if x and y belong to L_r^s and $V^s(x)$ intersects $V^u(y)$ at a point z , then z belongs to L_r^s .*

(g) *The foliations V^s and V^u are absolutely continuous (cf. [10], [11], [12]); moreover, the Jacobians of succession (or holonomy) maps and of conditional measures on stable and unstable leaves are bounded on each set L_r^s .*

4.2. Definition. We say that a diffeomorphism h preserving a smooth positive measure m_h is a diffeomorphism with non-zero characteristic exponents, if for almost every point x in N (with respect to m_h), $X(x, v) \neq 0$ for every tangent vector v in $T_x N$, i.e., $L = N$ almost everywhere.

Let M and N be smooth compact connected Riemannian manifolds (maybe with boundary), and let G be a compact connected Lie group.

4.3. Definition (cf. [8]). A diffeomorphism f of M is called a G -extension of a diffeomorphism h of N , if M is a smooth principal G -bundle over N [13] (with projection $p: M \rightarrow N$) and if the following conditions are satisfied:

- (i) $p \cdot f = h \cdot p$;
- (ii) $f \cdot R_g = R_g \cdot f$ ($R_g: M \rightarrow M$ is the right action of an element $g \in G$).

Group extensions have been studied from different points of view. The existence of group symmetries for these dynamical systems along the vertical direction (i.e., along the fibers $P^{-1}(x)$) makes it possible to study the structure of ergodic components, K -components and Bernoulli components, even in the abstract situation (cf., e.g., [14], [2]). In the case when h is an Anosov diffeomorphism these components are smooth compact submanifolds of M ([7], [8]). In this case every component turns out to be a principal subbundle of M . Moreover, since f commutes with the right action of G , the partitions of M into topological components P_i (i.e., the sets on which f is topologically transitive; in the case when h is Anosov, these sets really

constitute a partition), ergodic components P_e , or K -components P_K , are invariant with respect to every R_g . Therefore, all the components of the same type (topological, ergodic, or K) are diffeomorphic to one another, and the corresponding factor spaces M/P_i , M/P_e , M/P_K are homogeneous spaces of G . The stabilizers of those spaces measure the degree of integrability of the group extension, each of the stabilizers being the structure group of the corresponding subbundle. It is proved [7] that any group extension of an Anosov diffeomorphism h (preserving a smooth measure) can be perturbed in the space of group extensions of h in such a way that the perturbed diffeomorphism is ergodic and is a K -automorphism. In the present paper we generalize this perturbation theorem, and prove that the same is true if h is a diffeomorphism with non-zero characteristic exponents.

Let M and N be smooth compact Riemannian manifolds (perhaps with boundary). The following theorem is our main result concerning group extensions.

4.4. THEOREM. *Let $f: M \rightarrow M$ be a C^r - G -extension of $h: N \rightarrow N$, $r \geq 2$. Suppose h preserves a smooth positive measure and C_h is an ergodic component of h with positive measure, C_h is a disjoint union of C_h^i , $i = 1, \dots, k$, and the restriction $h^k|_{C_h^i}$ has non-zero characteristic exponents and is a K -automorphism. Then for every neighborhood U of f in the space of all the C^r - G -extensions of h there exists an $\bar{f} \in U$ such that the set $p^{-1}(C_h)$ is an ergodic component of \bar{f} , and the restriction $\bar{f}^k|_{p^{-1}(C_h^i)}$ is a K -automorphism for every i .*

4.5. COROLLARY. *Let $f: M \rightarrow M$ be a C^r - G -extension of $h: N \rightarrow N$, $r \geq 2$. Suppose h has non-zero characteristic exponents and is a K -automorphism. Then there exists a C^r -perturbation \bar{f} of f which is a K -automorphism. The perturbation can be concentrated in any neighborhood on N , i.e., for any neighborhood $V \subset N$ there exists an arbitrary small perturbation \bar{f} such that:*

- (i) \bar{f} is a K -automorphism;
- (ii) $\bar{f}(w) = f(w)$ if $p(w)$ does not belong to V .

4.6. Remark. By Pesin's results [6] every K -diffeomorphism with non-zero characteristic exponents is actually a Bernoulli diffeomorphism. Thus, combining Corollary 4.5 with Rudolph's result from [2], mentioned above (§ 1, (iii)), we can see that the diffeomorphism \bar{f} from Corollary 4.5 is a Bernoulli diffeomorphism.

4.7. COROLLARY. *Let $f: M \rightarrow M$ be a C^r - G -extension of $h: N \rightarrow N$, $r \geq 2$. Suppose h has non-zero characteristic exponents. Then there exists a C^r -perturbation \bar{f} of f such that for every ergodic component $C_e(\bar{f})$ of \bar{f} with*

positive measure, and for every K -component $C_K(\bar{f})$ with positive measure, there are components $C_e(h)$ and $C_K(h)$ for which:

- (i) $p^{-1}(C_e(h)) = C_e(\bar{f})$ almost everywhere;
- (ii) $p^{-1}(C_K(h)) = C_K(\bar{f})$ almost everywhere.

The general scheme of the proof of Theorem 4.4 and Corollaries 4.5 and 4.7 and techniques applied are similar to those of [8]. We begin by proving that the stable foliation W_h^s and the unstable foliation W_h^u of h can be "lifted" to the stable W_f^s and unstable W_f^u foliations of f respectively.

It is known (see [15]) that $\pi(f) \leq \nu(W_f^s) \wedge \nu(W_f^u)$, where $\pi(f)$ is the Pinsker partition for f . Our arguments will show that, if h has non-zero characteristic exponents on an ergodic component C_h (of positive measure), then f can be perturbed in such a way that the intersections of the Pinsker algebras for h and f with C_h and $p^{-1}(C_h)$, respectively, are isomorphic in the following sense: for any set A which consists almost everywhere of global stable and unstable manifolds of f , there exists a measurable subset B in N such that $p^{-1}(B) = A$ almost everywhere and B consists almost everywhere of stable and unstable manifolds of h . These arguments include both the study of Pinsker algebras and the perturbation itself.

5. Main lemmas

Throughout this section h is a C^2 -diffeomorphism of a compact Riemannian manifold N (maybe with boundary) preserving a smooth positive measure m_h , and $f: M \rightarrow M$ is a C^2 - G -extension of h preserving the measure m_f which is the product of m_h and the Haar measure.

Suppose that h has non-trivial characteristic exponents; i.e., there is a set L^0 of positive measure such that, if $x \in L^0$, then there exists a vector v in $T_x N$ such that $X(x, v) \neq 0$, and if x does not belong to L^0 , then $X(x, v) = 0$ for every v . Since h preserves a smooth measure, it is clear that for any point x in L^0 we can find two vectors v_1 and v_2 in $T_x N$ such that their characteristic exponents are non-zero and of different signs. That means (see [10]) that there exist global stable W_h^s and unstable W_h^u foliations,

$$W_h^s(x) = \bigcup_{n \geq 0} h^{-n}(V_h^s(h^n x)) \quad \text{and} \quad W_h^u(x) = \bigcup_{n \geq 0} h^n(V_h^u(h^{-n} x)).$$

Let L^1 be the set of points for which stable and unstable manifolds exist; $L^1 = L^0$ almost everywhere.

5.1. LEMMA. (a) *For every point $w \in p^{-1}(L^0)$ there are two tangent vectors with non-zero characteristic exponents of different signs.*

(b) *For every point w outside $p^{-1}(L^0)$ all the characteristic exponents are equal to 0.*

(c) For every point $w \in p^{-1}(L^0)$ there exist a global stable manifold $W_f^s(w)$ and a global unstable manifold $W_f^u(w)$.

(d) $p(W_f^s(w)) = W_f^s(p(w))$; $p(W_h^u(w)) = W_h^u(p(w))$.

Proof. Let w belong to $p^{-1}(L^0)$, $p(w) = x$. The differential dp maps $T_w M$ onto $T_x N$, and $\|u\| \geq \|dpu\|$ for every vector $u \in T_w M$. Since there are vectors v_1 and v_2 in $T_x N$ with non-zero characteristic exponents of different signs, it is clear that there exist two vectors u_1 and u_2 in $T_w M$ with non-zero characteristic exponents of different signs. This proves (a).

Now let w be outside $p^{-1}(L^0)$. Then for every tangent vector u in $T_w M$ the characteristic exponent $X(x, dpu)$ is equal to 0. The mapping dp does not increase the norm, and $dpdf = dhdp$; besides the characteristic exponent is 0 for every vector u_0 tangent to $p^{-1}(x)$. Therefore, $X(w, u) = 0$ and (b) is proved.

It follows from (a) and (b) (see [10], [12] and Theorem 2.1) that there exist local stable manifolds $V_f^s(w)$ and local unstable manifolds $V_f^u(w)$ for almost every point w in $p^{-1}(L^0)$ (actually it is obvious that these manifolds exist for every point w in $p^{-1}(L^1)$). Since $pf = hp$, we have $p(V_f^s(w)) \subset V_h^s(p(w))$ and $p(V_f^u(w)) \subset V_h^u(p(w))$. Let $x = p(w)$ belong to L^1 and let x_1 belong to $W_h^s(x)$. Since the distance between the fibers $p^{-1}(h^n x)$ and $p^{-1}(h^n x_1)$ decreases (exponentially), and since f acts isometrically in every vertical fiber, there exists a single point w_1 in $p^{-1}(x_1)$ such that the distance $d(f^n w, f^n w_1)$ tends to 0 (exponentially) as n tends to infinity. Thus, (c) and (d) are proved. \square

Suppose now that h has an ergodic component C_h with positive measure and such that $X(x, v) \neq 0$ for every point x in C_h and every vector v in $T_x N$. We shall use now the results of Theorem 4.1. Without loss of generality we can assume that local stable and unstable manifolds pass through every point in C_h . We need the following lemma to prove that for every element A of the Pinsker algebra of f there exists an element B of the Pinsker algebra of h such that $A = p^{-1}(B)$ almost everywhere.

5.2. LEMMA. *Let $A \subset M$ be a set of positive measure which consists almost everywhere of entire global stable and unstable leaves. Suppose w_0 is a density point of A (i.e., if we consider a ball $B(w_0, r)$ of radius r with the center at w_0 , then the relative measure of A in this ball tends to 1 as r tends to 0). Suppose also that $w_1 \in W_f^s(w_0)$ and $w_2 \in W_f^u(w_0)$ and that all the three points $x_0 = p(w_0)$, $x_1 = p(w_1)$, and $x_2 = p(w_2)$ are density points of some L^s (see Theorem 4.1). Then both w_1 and w_2 are density points of A .*

Proof. It is sufficient to prove only the first half of the statement, i.e.,

the half which concerns w_1 . Let us note that $h^n(L_r^s) \subset L_r^{s(n)}$ (where $s(n) = s(0)e^{n\alpha(x_0)}$), and the size of the stable manifold decreases not faster than $e^{-n\alpha(x_0)}$ along the trajectory. Besides, f and h are smooth, and hence they map density points of sets to density points of their images. Therefore, if we apply f^n and h^n for n large enough, the images $h^n x_0$ and $h^n x_1$ will belong to the same local stable manifold. Thus, without loss of generality we can assume that x_1 belongs to $V_h^s(x_0)$ and w_1 belongs to $V_f^s(w_0)$. Moreover, we can assume that the distance between the points w_0 and w_1 measured along the leaf $V_f^s(w_0)$ is much smaller than $\inf\{b(x): x \in L_r^s\}$, the minimal diameter of the local stable manifolds in L_r^s .

The vertical fibers $p^{-1}(x)$ form a smooth foliation W^0 which is absolutely continuous because of its smoothness (see [12]) and is transversal to the foliations V_f^s and V_f^u . It follows from Lemma 2.1 that the pairs of foliations W^0, V_f^s and W^0, V_f^u are integrable (see [9]), and their integral manifolds form two foliations W^{0s} and W^{0u} . It follows from Theorem 2.1 that all the aforementioned foliations are absolutely continuous, with bounded Jacobians of succession mappings on the set L_r^s (see 4.1 (g)), and that all of them are transversal, the angles between them being bounded away from 0 on the set L_r^s (see 4.1 (b)). We use here the same notation for both $L_r^s(f)$ and $L_r^s(h)$ because $L_r^s(f) = p^{-1}(L_r^s(h))$.

By assumption there exists a set A_1 such that:

- (i) $A_1 = A$ almost everywhere;
- (ii) A_1 consists of entire unstable leaves;
- (iii) A_1 consists almost everywhere of entire stable leaves.

Let us denote:

$B^s(w, q)$ —the ball in $V_f^s(w)$ of radius q with center at w ;

$B^u(w, q)$ —the ball in $V_f^u(w)$ of radius q with center at w ;

$B^0(w, q)$ —the ball in $V^0(w)$ of radius q with center at w ;

$B^{0s}(w, q) = \bigcup_{\bar{w} \in B^0(w, q)} B^s(\bar{w}, q)$, $w \in L_r^s$;

$U(w, q) = \bigcup_{\bar{w} \in B^{0s}(w, q) \cap L_r^s} B^u(\bar{w}, q)$.

The foliations are transversal; therefore, to prove the lemma it is sufficient to show that the relative measure of A_1 in $U(w_1, q)$ tends to 1 as q tends to 0.

Now, w_0 is a density point of both A_1 and L_r^s , and the foliations V_f^s and W^{0s} are absolutely continuous. Thus, for q_1 small enough, the relative measure of those points in $B^{0s}(w_0, q_1)$ which lie in a leaf $V_f^s(w)$ belonging to A_1 is arbitrarily close to 1 (it tends to 1 as q_1 tends to 0). Hence, arbitrarily (independently of q_1) close to w_0 we can find a point w_1 such that:

(i) The intersection $A_1 \cap W^{os}(w_3)$ consists mod 0 of entire stable leaves;
(ii) The relative measure of those points in $B^0(w_3, q_1)$, which lie in a stable leaf belonging almost everywhere to A_1 , tends to 1 as q_1 tends to 0. Thus, arbitrarily close to w_1 there exists a leaf $W^{os}(w_3)$ whose intersection with A_1 consists almost everywhere of stable leaves, the relative measure of A_1 on this leaf tending to 1 as q_1 tends to 0. Now, since A_1 consists of entire unstable leaves, the relative measure of A_1 in some $U(w_1, q_2)$ is close to 1. \square

5.3. LEMMA. *Let $A \subset M$ be a set of positive measure which consists almost everywhere of entire global stable and unstable leaves. Suppose w_0 is a density point of A and (w_0, w_1, \dots, w_k) is a sequence of points such that:*

- (i) $p(w_i)$ is a density point of some L^* ;
- (ii) every two neighboring points w_i, w_{i+1} belong to the same stable or unstable leaf.

Then w_k is a density point of A .

This lemma immediately follows from Lemma 3.2.

The following four lemmas show that every group extension f can be perturbed in such a way that for the perturbed system every set A of positive measure, which consists almost everywhere of entire global stable and unstable leaves, coincides almost everywhere with $p^{-1}(p(A))$. The most interesting may be the case when $f = h \times I$, where I is the identity mapping of the group G .

The perturbations of f considered below will have the form $f \circ L_z$, where $z: U \rightarrow G$ is a smooth function defined in a small neighborhood U and L_g is the left action of $g \in G$ (this action can be defined only locally, if M is not homeomorphic to the direct product of N and G). If we want the perturbation to be C^r -small, the function z should be close enough to e in the C^r -topology.

5.4. LEMMA. *Let $\bar{f} = f \circ L_z$. Suppose that w_1 belongs to $W_f^s(w_0)$ and w_2 belongs to $W_f^u(w_0)$, $p(w_i) = x_i$. Then*

$$\begin{aligned} w_f^s(w_0) \cap p^{-1}(x_1) &= \lim_{n \rightarrow \infty} \bar{f}^{-n}(W_f^s(\bar{f}^n w_0) \cap p^{-1}(h^n x_1)), \\ w_f^u(w_0) \cap p^{-1}(x_2) &= \lim_{n \rightarrow \infty} \bar{f}^n(W_f^u(\bar{f}^{-n} w_0) \cap p^{-1}(h^{-n} x_2)). \end{aligned}$$

This lemma is obvious, because every stable leaf is uniquely determined by its infinite future, and every unstable leaf is uniquely determined by its infinite past (see [7]).

Suppose $M = N \times G$; then the vertical difference between two points w_1 and w_2 belonging to the same stable (or unstable) leaf can be measured

by an element of G . In this case f can be represented in the following form: $f(x, g) = (hx, L_{z(x)}g)$, where $z: N \rightarrow G$ is a smooth function and L denotes the left action of G on itself. Now, let $p(w_i) = x_i$; then the difference $\Delta_s(x_1, x_2)$ between the vertical coordinates of the points w_1 and w_2 can be calculated by the following formula (see Lemma 3.4):

$$\Delta_s(x_1, x_2) = \lim_{n \rightarrow \infty} \left(\prod_{i=1}^n z^{-1}(h^i x_2) \right) \circ \left(\prod_{i=1}^n z^{-1}(h^i x_1) \right)^{-1}.$$

The limit exists because the distance between the images $h^n x_1$ and $h^n x_2$ tends to 0 exponentially. It is obvious that the same is true when the bundle is not trivial (but in this case it is necessary to consider small neighborhoods over which the bundle is locally trivial).

Let us consider a perturbation $\bar{f} = f \circ L_z$, where z is a smooth function which differs from e in a finite number of small neighborhoods U_i , $i = 1, 2, \dots, k$. Let (x_1, x_2, \dots, x_k) be a sequence of points such that every two neighboring points x_i, x_{i+1} belong to the same stable or unstable leaf. For every point $w_1 \in p^{-1}(x_1)$ and for every perturbation \bar{f} there exists a single sequence (w_1, w_2, \dots, w_k) such that $p(w_i) = x_i$ and every neighboring point belongs to the same leaf for \bar{f} (see Lemma 3.1); the point w_k is uniquely determined by the function z . To estimate the vertical difference $\Delta(z)$ between $w_k(e)$ (e corresponds to $z \equiv e$) and $w_k(z)$ it is necessary to consider k infinite sequences of images $h^n x_i$. Each time $h^n x_i$ hits some U_j we get a non-zero contribution to $\Delta(z)$. Let us note that, while studying a stable leaf, it is necessary to apply h^n with positive n , and while studying an unstable leaf, it is necessary to apply h^n with negative n . Therefore, since \bar{f} acts isometrically along the vertical fiber $p^{-1}(x)$, the contribution of every image $h^n x_i$ to the difference $\Delta(z)$ decreases exponentially as n tends to infinity. That means that, if we consider a function z which equals e everywhere except a small neighborhood U of $h(x_k)$ (we assume that x_k and x_{k-1} belong to the same stable leaf; $z(x) = g_{t(x)}$, where $t(x)$ is positive inside U and 0 outside U , and g_t is a one parameter subgroup) and if the points x_i are not periodic and are not images of one another, then for U small enough the distance between the real difference $\Delta(z)$ and $g_{t(hx_k)}$ does not exceed $1/10$ of the distance between $g_{t(hx_k)}$ and e . It is obvious that $\Delta(z)$ depends continuously on z .

5.5. LEMMA. *Let (x_1, x_2, \dots, x_k) be a sequence of points in N such that:*

- (i) *Every two neighboring points belong to the same stable or unstable leaf;*
- (ii) *If $h^n x_i = x_j$, then $i = j$ and $n = 0$.*

Then for every positive ε there exists a positive δ such that for any g_0

belonging to a δ -neighborhood of e there is a perturbation function z satisfying the following conditions:

- (i) $\Delta(z) = g_0$;
- (ii) The C^r -distance between f and \bar{f} is less than ε .

Proof. For every g belonging to the ball $B = \{g \in G: d(g, e) \leq a\}$, consider its one-parameter subgroup $g_t (g_0 = e, g_1 = g)$. Let $t(x)$ be a function which is positive inside U , is 0 outside U , and has a maximum equal to 1 at the point $h(x_k)$ (we assume here that x_{k-1} and x_k belong to the same stable leaf). If a is small enough, the perturbation to the function $z(x) = g_{t(x)}$ will satisfy the second requirement for every $g \in B$. The neighborhood U can be chosen so small that the distance between $\Delta(z)$ and g does not exceed $a/10$. Thus, we have a continuous mapping $g \rightarrow \Delta(z)$ of the ball B into G such that for every point g belonging to the boundary the distance between its image and itself does not exceed $1/10$ of the radius. Therefore, the image of the ball B covers a smaller ball. \square

5.6. PERTURBATION LEMMA. *Let $(x_1^i, x_2^i, \dots, x_{k(i)}^i)$, $i = 1, 2, \dots, l$, be a finite number of sequences in N such that:*

- (i) *Every two neighboring points x_j^i and x_{j+1}^i belong to the same stable or unstable leaf;*
- (ii) *If $h^n x_j^i = x_{j_0}^{i_0}$, then $i = i_0$, $j = j_0$, and $n = 0$.*

Then for every positive ε there exists a positive δ such that for any l elements g_i belonging to a δ -neighborhood of e there is a perturbation function z satisfying the following conditions:

- (i) $\Delta^i(z) = g_i$;
- (ii) The C^r -distance between f and \bar{f} is less than ε .

Proof. Due to the second assumption all the sequences are “independent.” Therefore, if the perturbation is concentrated in a finite number of neighborhoods which are sufficiently small, then the mapping $z \rightarrow \Delta^1(z), \dots, \Delta^l(z)$ is close to the direct product of the mappings $z \rightarrow \Delta^i(z)$, $i = 1, 2, \dots, l$. The argument similar to the one used in the proof of the previous lemma shows that, if a is small enough, then for every point belonging to the boundary $\partial(B \times B \times \dots \times B)$ its image lies near the point itself. Hence, the image of the product $B \times B \times B \times \dots \times B$ covers the product of smaller balls. \square

Henceforth we will be interested in sequences of points in N for which their beginnings coincide with ends.

5.7. LEMMA. *Let all the assumptions of Lemma 5.6 be valid, but let*

$x_1^i = x_{k(i)}^i = x_0$. Then for every positive ε there exists a positive δ such that for any l elements g_i belonging to a δ -neighborhood of e there is a perturbation function z satisfying the following conditions:

- (i) $\Delta^i(z) = g_i$;
- (ii) The C^r -distance between f and \bar{f} is less than ε .

Proof. Let us note that the perturbation does not have to be concentrated near the image (or pre-image) of the point x_k (see Lemma 5.5). Thus, the perturbation can be transferred to the image (or pre-image) of any other point x_i . Therefore, the arguments of Lemmas 5.5 and 5.6 go through. \square

6. Proof of Theorem 4.4 and Corollaries 4.5 and 4.7

Let all the assumptions of Theorem 4.4 be valid. It is known (see [5]) that every element of the Pinsker partition of any diffeomorphism f consists almost everywhere of entire stable and unstable leaves. Let A be a set of positive measure which consists almost everywhere of entire stable and unstable leaves. Suppose the set $A_i = A \cap C_h^i$ has positive measure. Then (see Theorem 2.1) there is a point w_0 in A_i such that:

- (i) w_0 is a density point of A_i ;
- (ii) $p(w_0) = x_0$ is a density point of L_r^s for some s and r ;
- (iii) x_0 is a density point of both $V_h^s(x_0) \cap L_r^s$ and $V_h^u(x_0) \cap L_r^s$.

There exists a small neighborhood U of the point x_0 such that:

- (i) The relative measure of L_r^s in U is at least 0.99;
- (ii) $\text{diam}(U)$ is less than $(S(s, r))^{-1}$ (see Theorem 2.1 (g));
- (iii) Let $S = S(s, r)$; then $[x_1, x_2] = V_h^s(x_1) \cap V_h^u(x_2) \in U$ if $x_1, x_2 \in L_r^s \cap U$.

Therefore, for any l we can find l sequences $x^j = (x_0, x_1^j, x_2^j, x_3^j)$, $j = 1, 2, \dots, l$, such that:

- (i) x_k^j is a density point of L_r^s ;
- (ii) $x_1^j = V_h^s(x_0) \cap V_h^u(x_2^j)$, $x_3^j = V_h^u(x_0) \cap V_h^s(x_2^j)$.

According to Lemma 5.1, for any point w belonging to $p^{-1}(x_0)$, every such sequence can be "lifted" to the corresponding sequence of length 5 in M . That is, there exist l sequences $(w, w_1^j, w_2^j, w_3^j, w_4^j)$ such that:

- (i) $w_1^j = V_f^s(w) \cap V_f^u(w_2^j)$, $w_3^j = V_f^u(w_4^j) \cap V_f^s(w_2^j)$;
- (ii) $p(w_4^j) = x_0$.

Let us identify w_0 with e . Then for every sequence x^j there is an element $g^j \in G$ such that $w_4^j(w) = L_{g^j} w$ (since $w_0 = e$, the left action of G is defined in $p^{-1}(x_0)$). Indeed, the mapping $w \rightarrow w_4^j(w)$ commutes with the right action of G (since f commutes); hence, this mapping is a left translation.

Let $D(x)$ denote the set of density points of A_i which belong to the fiber $p^{-1}(x)$. Lemma 5.3 states that the set $D(x_0)$ is invariant with respect to the

left action of the subgroup G_l generated by the elements g^1, g^2, \dots, g^l . It is obvious that for any neighborhood in a Lie group one can find a finite number of elements belonging to this neighborhood, such that the subgroup generated by them is dense. Therefore, according to Lemma 5.7 there exists a perturbation \bar{f} of f such that the corresponding elements g^1, \dots, g^l generate a dense subgroup G_l . It is also clear that for almost any point x in U which belongs to L_r^s the corresponding subgroup is conjugate to G_l . Actually, the left action of G in a vertical fiber $p^{-1}(x)$ is determined only up to the choice of an element corresponding to e ; so, all the left actions are conjugate. Let us note that the set $D(x)$ is measurable for almost every point x in U . Thus, it is clear now that $D(x) = p^{-1}(x)$ for almost every point x in $U \cap L_r^s = U_r^s$. Thus we have proved that $A_i \supset p^{-1}(U_r^s)$ almost everywhere.

Define

$$B = \{x \in N: p^{-1}(x) \subset A_i(\text{mod } 0)\}.$$

It is obvious that B is measurable and consists almost everywhere of entire global unstable leaves $V_h^u(x)$. Furthermore its measure is positive, hence (see [6], [15]), $B = C_h^i$ almost everywhere. It follows that $A_i = p^{-1}(C_h^i)$ almost everywhere.

Notice now that the perturbation obtained possesses the following property: for almost every point w in $P^{-1}(U_r^s)$, the assumption “ w is a density point for a set A' consisting almost everywhere of stable and unstable leaves” implies that A' contains $p^{-1}(C_h^i)$ almost everywhere. But for every measurable set A' almost every point w in M (and in particular $p^{-1}(C_h^i)$) is either a density point of A' or a density point of its complement. Therefore, the perturbation \bar{f} which was constructed, possesses the property that every set of positive measure consisting almost everywhere of stable and unstable leaves coincides almost everywhere with $p^{-1}(C_h^i)$ for some i . Thus, $p^{-1}(C_h^i)$ is a K -component, and the restriction $\bar{f}^k|_{P^{-1}(C_h^i)}$ is a K -automorphism. This completes the proof of Theorem 4.4.

6.1. Remark. It follows from the concluding arguments in the proof of Theorem 4.4 that the perturbation can be concentrated in an arbitrarily small neighborhood V of any density point of the ergodic component C_h ; i.e., $\bar{f}(w) = f(w)$ if $p(w)$ is outside V . If the assumptions of Corollary 4.5 are valid, then h is a K -automorphism, and the perturbation can be concentrated in any neighborhood of every point.

6.2. Remark. If h is a diffeomorphism with non-zero characteristic exponents, then the number of ergodic components of positive measure is at most countable (see [6]). Let us note that Lemma 5.6 can be applied to a

sequence of elements g_i , $i = 1, 2, \dots, l$, such that $g_i = e$, for $i = 1, 2, \dots, n$. Therefore, by induction one can get a converging sequence of perturbations f_k such that:

(i) For every f_k the statement of Corollary 5.7 is true for the first k ergodic components of h ;

(ii) The differences $\Delta_j^i(z_k)$, which correspond to the j -th ergodic component and to the k -th perturbation, do not change for $k > j$.

The second property guarantees the ergodicity of the limit diffeomorphism on the pre-image $p^{-1}(C_k)$ for every ergodic component C_k of h . This proves Corollary 4.7.

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