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Ergodic theory and Weil measures for foliations

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Part I:

0. Introduction

The main theorem of this paper, Theorem 0.1 below, gives a relation between the transverse measure theory of a foliation and its secondary classes. While the technical antecedents of this work developed from the papers [37], [46], [44], our main result was inspired by a question posed in 1974–75 by Moussu-Pelletier and D. Sullivan: *Must a codimension-one C^2-foliation $\mathcal{F}$ with no leaves of exponential growth have zero Godbillon-Vey class*? This was settled after a progression of works [18], [53], [36], [37], [8], [7] each of which successively broadened the geometric hypothesis under which the vanishing of the Godbillon-Vey class $\mathrm{GV}(\mathcal{F})$ was known to hold. The culminating work by G. Duminy ([7], see also [4]) proved a much stronger and more natural result than the original question asked for. Duminy's theorem states: *If $\mathcal{F}$ is a codimension-one, C^2-foliation of a compact manifold and $\mathcal{F}$ has no resilient*

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leaves, then $GV(\mathcal{F})$ *is zero.* The proof of this theorem makes extensive use of the classification theory of codimension-one, C^2-foliations of Hector and Cantwell-Conlon (see references in [4]).

This paper begins with the study of ergodic theory of foliations, and especially the metric properties of measurable cocycles over discrete metric equivalence relations obtained from foliations. The results given in Sections 1 and 3 below are developed as needed for our main theory, but are preliminary to the full development of Pesin theory and Lyapunov exponent theory for foliations. These topics are discussed further in [25], [30]. In this paper, we use the ergodic theory techniques of Sections 1 and 3, and the methods of Weil measures from [17] to prove:

THEOREM 0.1. *Let* $\mathcal{F}$ *be a codimension-n,* C^2*-foliation on a smooth manifold M without boundary. Suppose that the measurable equivalence relation* $\mathcal{R}(\mathcal{F})$ *on M determined by the leaves of* $\mathcal{F}$ *is amenable. Then all residual secondary classes for* $\mathcal{F}$ *in degrees greater than* $(2n + 1)$ *must vanish.*

The hypothesis that $\mathcal{R}(\mathcal{F})$ is amenable is a purely measure-theoretic property of $\mathcal{F}$, and is invariant under measurable orbit equivalence of foliations, an extremely weak type of equivalence (cf. [39]). Characterizations of amenability and examples of amenable foliations are discussed in Section 1.

The question of Moussu-Pelletier and Sullivan has a direct extension to higher codimensions, as the notion of the growth type of a leaf is independent of the codimension (see §1, also [46]). A foliation, for which the set of leaves with positive exponential growth type is a set of measure zero, must be amenable, so that by Theorem 0.1 all of the residual secondary classes in degrees greater than $(2n + 1)$ must vanish. Moreover, Theorem 1 of [22] implies that all of the secondary classes in degrees $2n + 1$ vanish when almost every leaf of $\mathcal{F}$ has subexponential growth; so we obtain an affirmative solution to the Moussu-Pelletier-Sullivan question for arbitrary codimension:

COROLLARY 0.2. *Let* $\mathcal{F}$ *be a codimension-n,* C^2*-foliation on a smooth manifold M for which almost every leaf of* $\mathcal{F}$ *has subexponential growth. Then all residual secondary classes for* $\mathcal{F}$ *vanish.*

This corollary in particular implies that all of the "Godbillon-Vey" type classes (cf. §2) vanish for $\mathcal{F}$ of subexponential type. Note that some hypothesis on $\mathcal{F}$ in addition to amenability is required in order to conclude the secondary classes in degree $2n + 1$ must vanish. The Roussarie examples [13] and Thurston's examples [53] are all amenable foliations, but have non-zero Godbillon-Vey classes. A very interesting open problem is to find the precise

ergodic hypothesis on $\mathcal{F}$ between amenability and subexponential growth which is sufficient to force the Godbillon-Vey classes to vanish.

The measurable hypothesis in Theorem 0.1 can be reformulated in terms of von Neumann algebras. Associated to a C^1-foliation $\mathcal{F}$ is a von Neumann algebra $\mathcal{M}(\mathcal{F})$, represented as bounded operators on the Hilbert space $L^2(\mathcal{G})$ of measurable fields of functions which are L^2 along leaves of $\mathcal{F}$ (cf. [11]). We say $\mathcal{M}(\mathcal{F})$ is *approximately finite* if it is the weak closure of an increasing sequence of finite dimensional subalgebras. By Zimmer [57] (see also §5, [39]), $\mathcal{R}(\mathcal{F})$ is amenable if and only if $\mathcal{M}(\mathcal{F})$ is approximately finite; so we conclude:

COROLLARY 0.3. *Let $\mathcal{F}$ be a codimension-n, C^2-foliation whose von Neumann algebra $\mathcal{M}(\mathcal{F})$ is approximately finite. Then all residual secondary classes for $\mathcal{F}$ in degrees greater than $2n + 1$ must vanish.*

A particular case where $\mathcal{M}(\mathcal{F})$ is approximately finite occurs when $\mathcal{M}(\mathcal{F})$ has Murray-von Neumann type I (cf. §1.9). This corresponds to the foliation $\mathcal{F}$ admitting a measurable cross-section, or equivalently a measurable Epstein hierarchy [9]. Theorem 0.1 and Theorem 3.11 below combine to yield:

COROLLARY 0.4. *Let $\mathcal{F}$ be a C^2-foliation of a manifold M without boundary. If $\mathcal{M}(\mathcal{F})$ has type* I, *then all residual secondary classes for $\mathcal{F}$ vanish.*

These corollaries are stated explicitly to emphasize that some of the deepest problems in foliation theory arise when trying to relate properties of the von Neumann algebra $\mathcal{M}(\mathcal{F})$ with differential-geometric properties of $\mathcal{F}$. It would be extremely interesting to have direct proofs of Corollaries 0.3 and 0.4, i.e., to deduce the vanishing of characteristic classes directly from properties of von Neumann algebras, bypassing the use of ergodic properties of $\mathcal{R}(\mathcal{F})$.

We next discuss the result in ergodic theory which is crucial to the proof of Theorem 0.1, and then discuss how it is applied. We begin with a theorem of Zimmer: *Suppose ϕ: $\mathcal{R} \to G$ is a cocycle over an amenable ergodic equivalence relation $\mathcal{R}$ and G is a real algebraic group. Then ϕ is cohomologous to a cocycle ψ with values in an amenable subgroup $H \subset G$.* (See §1.7 below for a further discussion.) We consider the case where the orbits of $\mathcal{R}$ are equipped with a discrete good metric d, and G has a left-invariant norm. Then we ask: If ϕ is a tempered cocycle, is it possible to choose ψ tempered? Using Moore's classification of maximal amenable subgroups of $\mathrm{GL}(m, \mathbf{R})$, we prove the following result for foliations:

THEOREM 0.5. *Let $\mathcal{F}$ be a C^1-foliation of a Riemannian manifold M. Assume the equivalence relation $\mathcal{R}(\mathcal{F})$ on M is ergodic and amenable. Then for any tempered cocycle ϕ: $\mathcal{R}(\mathcal{F}) \to \mathrm{GL}(m, \mathbf{R})$, there are a maximal amenable*

subgroup H of $\mathrm{GL}(m,\mathbf{R})$ *and a tempered cocycle* ψ: $\mathscr{R}(\mathscr{F}) \to H$ *cohomologous to* ϕ. *Moreover, if the growth rates of the leaves of* $\mathscr{F}$ *are at most exponential of type* a *and the growth of* ϕ *is at most exponential of type* b, *then for all* $\varepsilon > 0$, *we can choose* $\psi = \psi_\varepsilon$ *to have growth of type at most* $\{(4m-3)a + (8m-6)b + \varepsilon\}$.

This is actually a special case of the more general result Theorem 3.2 which applies to discrete metric equivalence relations. For non-ergodic foliations, the conclusion of Theorem 0.5 must be modified to allow the subgroup H to vary on ergodic components. A precise statement is given in Corollary 3.3.ii) below. One other special case of Theorem 3.2 is of particular importance:

COROLLARY 0.6. *Let* Γ *be a finitely generated group with subexponential growth,* (X, μ) *a standard Borel measure space and* $\Gamma \times X \to X$ *a measurable, ergodic action of* Γ *on* X. *Let* ϕ: $\Gamma \times X \to \mathrm{GL}(m,\mathbf{R})$ *be a cocycle with exponential type* a. *Then there is a maximal amenable subgroup* H *of* $\mathrm{GL}(m,\mathbf{R})$ *and for all* $\varepsilon > 0$ *a cocycle* $\psi = \psi_\varepsilon$: $\Gamma \times X \to H$ *cohomologous to* ϕ *with exponential type* $(\mathrm{a} + \varepsilon)$.

This corollary has been used by the second author to establish noncommutative analogs of the Birkhoff ergodic theorem for finitely generated nilpotent groups, and also to deduce the existence of Lyapunov exponents for measure-preserving actions of such groups. These results allow one to extend the Pesin theory from actions of the integers to actions of this more general class of groups [30].

The relevance of Theorem 0.5 for the proof of Theorem 0.1 is seen when we apply it to the normal linear holonomy cocycle $D\gamma$ of an amenable foliation $\mathscr{F}$. The cohomology between $D\gamma$ and a tempered cocycle ψ gives a *measurable* (though leafwise smooth) framing for the normal bundle to $\mathscr{F}$, and in this normal framing, the linear holonomy ψ takes values in an appropriate maximal amenable subgroup of $\mathrm{GL}(n,\mathbf{R})$, where n is the codimension of $\mathscr{F}$. Moreover, the linear holonomy matrix ψ in this framing is uniformly bounded when evaluated on points a fixed distance apart in the leaf metric on M.

The Weil algebra formulation of the secondary classes, due to Kamber and Tondeur [28], shows that if $\mathscr{F}$ has a *smooth* normal framing for which the linear holonomy takes values in an amenable subgroup, then the vanishing of Theorem 0.1 follows from a Lie algebra calculation made in [21]. This is explained in Remark 5.12 below. In order to deal with measurable framings, we need two new techniques. The first is the Weil measure reformulation of the secondary invariants introduced in [7] and [17], which divides the task of calculating residual secondary classes into two steps: The Chern forms for the normal bundle

to $\mathcal{F}$ are first calculated, or assumed given. Then a linear functional, the appropriate Weil measure, is applied to this form to obtain a secondary invariant. These ideas are discussed in [17].

The second technique is to show the Weil measures can be calculated using any tempered cocycle cohomologous to $D\gamma$. This follows in three steps: The semisimplicial construction of the Weil homomorphism is used to reduce the calculation of Weil measures to local foliation charts equipped with O_n-related smooth framings of the foliation normal bundle in Section 5.1. It is then shown in Section 6 that if these local framings are measurable, bounded and leafwise C^2, then they can again be used to calculate Weil measures. Finally, in Section 7 the geometry of the symmetric space $\mathrm{GL}(n,\mathbf{R})/O_n$ is used to show that the normal framing does not have to be bounded, as long as the cocycle $D\gamma$ expressed in this framing is tempered. This is exactly the data provided by Theorem 0.5, and the general measurable case then follows as for the smooth case of normal framings with amenable linear holonomy.

Section 3 contains a number of further vanishing theorems for the Weil measures, and hence for the appropriate residual secondary classes, which are consequences of the techniques developed in Sections 5, 6 and 7.

We conclude this introduction with a comment on the role of Proposition 3.9 of [17] (see Lemma 5.11 below) in the proof of Theorem 0.1. For an amenable subgroup $H \subset \mathrm{GL}(n,\mathbf{R})$, this proposition implies that the map on continuous cohomology

$$H_c^*(\mathrm{GL}(n,\mathbf{R}),\mathbf{R}) \to H_c^*(H,\mathbf{R}) \tag{0.1}$$

vanishes in degrees greater than one. This follows from the Van Est theorem and a Lie algebra calculation. For a C^∞-foliation $\mathcal{F}$ of codimension n, Haefliger has shown that the natural extension of (0.1) to the continuous cohomology of the category of C^∞-local diffeomorphisms is the secondary map

$$\Delta_*\colon H^*(WO_n) \cong H^*(W(\mathfrak{gl}_n, O_n)) \to H^*(M).$$

The hypothesis that H is amenable becomes the requirement that $D\gamma$ be smoothly conjugate to a cocycle with values in an amenable group, and the vanishing of Δ_* follows as in Section 5.2. However, this hypothesis on $D\gamma$ is extremely restrictive, and does not in general correspond to any measurable hypothesis on $\mathcal{F}$. The Lie group H is a topological category with one object, so there is essentially a unique notion of amenability. However, for a foliation with $\mathcal{G}$ the groupoid associated to $\mathcal{F}$ and linear holonomy $D\gamma\colon \mathcal{G} \to \mathrm{GL}(n,\mathbf{R})$, $\mathcal{G}$ has both a *measurable* and a *smooth* structure on its object space. Thus, there are notions of smooth and measurable amenability. The point of this paper is to show that for the much weaker notion of measurable amenability for $\mathcal{F}$, one still

has the vanishing of the map Δ_* on appropriate classes. This vanishing should also be compared to the vanishing of primary classes in bounded cohomology for manifolds with amenable fundamental groups due to Gromov. This relation will be made quite precise in a subsequent paper.

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The main results of this paper have been announced in [26].

1. Ergodic theory of metric equivalence relations

In this section we discuss the ergodic theory of foliations and group actions necessary for the remainder of this paper. The fundamental object will be a metric equivalence relation on a measure space, which arises as a common discrete model for the following three geometric situations:

(F) a smooth foliation $\mathscr{F}$ on a Riemannian manifold without boundary, with the Riemannian metric determining a metric class on each leaf of $\mathscr{F}$;

(A) a locally-free measurable non-singular action of a connected Lie group G on a standard measure space, which involves two structures:

(a) the measure classes on "local transversals",

(b) the Riemannian metric classes on leaves or orbits.

Recall that by a metric class we mean an equivalence class of metrics which are quasi-isometric; i.e., they differ by an arbitrary but uniform multiplicative constant.

(Γ) a measurable non-singular action of a countable group Γ on a standard measure space X, with a choice of generating set Γ_0 for Γ.

The passage from the data (F) or (A) to a metric equivalence relation requires the choice of a uniform section, discussed in Section 1.5. The passage from (Γ) to a metric equivalence relation is a special case of the discussion of Section 1.4.

The effect of replacing a foliation with its metric equivalence relation is to construct a discrete, measurable model for it, where the geometry of the leaves is reduced to metric properties for the discrete equivalence relation on a uniform transversal. The notions of equivalence we use always involve measurable maps, so that Section 1 can be viewed on the study of foliations up to measurable equivalences, with leaves being sent to leaves. This naturally suggests studying

the broader category of measurable foliations, which we do not undertake here. One definition of measurable foliations has been given by Sinai [52].

After the basic definitions of Section 1.1, we introduce the appropriate notions of equivalence, growth rates of orbits, uniform sections, cocycle theory, exponents for cocycles and ergodic properties of cocycles over amenable equivalence relations. These topics are standard for the case of a flow on a measure space, but are seen here to also have natural extensions to metric equivalence relations.

1.1. *Metric equivalence relations.* Let $(X, \mathcal{B}, \mu)$ be a standard measure space (Lebesgue space). Here, X is a set, $\mathcal{B}$ is a Borel σ-algebra of subsets at X and μ is a σ-additive probability measure defined on $\mathcal{B}$. It is often the case that only the equivalence class of the measure μ is important, and we will then abuse notation by omitting σ-algebras and measures from our notation.

A discrete measured equivalence relation on $(X, \mathcal{B}, \mu)$ is a pair $(\mathcal{F}, \nu)$, where $\mathcal{F} \in X \times X$ is a Borel subset, ν is a measure on $\mathcal{F}$, and these satisfy (cf. [39]):

(1.1.1) $(x, x) \in \mathcal{F}$ for all $x \in X$,

(1.1.2) $(x, y) \in \mathcal{F}$ implies $(y, x) \in \mathcal{F}$,

(1.1.3) (x, y) and $(y, z) \in \mathcal{F}$ implies $(x, z) \in \mathcal{F}$,

(1.1.4) for every $x \in X$, the set $\mathcal{F}_x = \{y \in X: (x, y) \in \mathcal{F}\}$ is at most countable,

(1.1.5) the projection of ν from $\mathcal{F}$ to either coordinate X in $X \times X$ yields a measure equivalent to μ.

A continuous measured equivalence relation on $(X, \mathcal{B}, \mu)$ is a pair $(\mathcal{F}, \nu)$ satisfying (1.1.1), (1.1.2), (1.1.3) and (1.1.5), but for which (1.1.4) fails.

Two equivalence relations $(\mathcal{F}, \nu)$ on X and $(\mathcal{F}', \nu')$ on X' are orbit equivalent if there exists an isomorphism $P: X \to X'$ of measure spaces which carries equivalence classes of $\mathcal{F}$ onto those of $\mathcal{F}'$, and the measure ν onto a measure equivalent to ν'.

A subset $A \subset X$ is called *saturated* if $x \in A$ implies $\mathcal{F}_x \subset A$. We will denote by $\mathcal{B}(\mathcal{F}) \subset \mathcal{B}$ the σ-algebra of all measurable saturated subsets of X. A measurable saturated set A is called *ergodic* if it cannot be decomposed into a disjoint union of two measurable saturated sets of positive measure. We say that the equivalence relation $\mathcal{F}$ is *ergodic* if X itself is an ergodic saturated set.

A *metric* on $\mathscr{F}$ is a ν-measurable non-negative real-valued function d on $\mathscr{F}$ such that

$$\begin{aligned} d(x,y) &> 0 \quad \text{iff } x \neq y, \\ d(x,y) &= d(y,x), \\ d(x,y) + d(y,z) &\geq d(x,z). \end{aligned} \tag{1.2}$$

For any fixed $x \in X$ the metric d defines a *norm* $|\cdot|_x = d(x,\cdot)$ on $\mathscr{F}_x$. Condition (1.2) implies that

$$\begin{aligned} |y|_x &> 0 \quad \text{for } y \in \mathscr{F}_x \setminus \{x\}, \\ |x|_y &= |y|_x, \\ |y|_x + |z|_y &\geq |z|_x. \end{aligned} \tag{1.3}$$

Conversely, any measurable family of norms $|\cdot|_x$ defined on $\mathscr{F}_x$ and satisfying (1.3) determines a metric on $\mathscr{F}$. We will call the metric d on a discrete measured equivalence relation $\mathscr{F}$ *good* if for every $x \in X$ and $T > 0$ the T-ball around x on $\mathscr{F}_x$, $B(x,T,d) = \{y \in \mathscr{F}_x\colon\ d(x,y) \leq T\}$, contains finitely many elements.

The concept of a good metric will play the central role in what follows. A *metric equivalence relation* will mean a *discrete* measured equivalence relation provided with a good metric.

1.2. *Quasi-isometry and Kakutani equivalence.* The metric equivalence relations $(X, \mathscr{F}, d)$ and $(X', \mathscr{F}', d')$ are *quasi-isometric* if there exists an orbit equivalence P: $(X, \mathscr{F}) \to (X', \mathscr{F}')$ and constants A, B such that for every $(x,y) \in \mathscr{F}$

$$A \cdot d(x,y) < d'(Px, Py) < B \cdot d(x,y). \tag{1.4}$$

For most of our purposes quasi-isometric equivalence relations may be treated as indistinguishable. In other words we will be interested not in the metric d itself, but in the *metric class*, i.e. the class of uniformly equivalent metrics. However some of the invariants discussed below, e.g. the exponential growth rate, do depend on the metric itself.

For many purposes the notion of quasi-isometry is too strong. This becomes clear if one tries to describe the properties common for all uniform sections of a smooth foliation (cf. §1.5 below). An appropriate notion of equivalence for that purpose is *Kakutani equivalence* which we will describe now. For simplicity we will consider only ergodic equivalence relations.

The metric equivalence relation $(X', \mathscr{F}', d')$ is a *factor* of $(X, \mathscr{F}, d)$ if there exists a non-singular (but probably non-invertible) measurable map P of X

onto X' which maps equivalence classes of $\mathcal{F}$ onto equivalence classes of $\mathcal{F}'$ and such that for some constants A, B, C and every $(x, y) \in \mathcal{F}$

$$A \cdot d(x, y) - C \leq d'(Px, Py) \leq B \cdot d(x, y) + C.$$

Two ergodic metric equivalence relations are *Kakutani equivalent* if they have a common factor. Kakutani equivalence for ergodic measure-preserving transformations and cross-sections to flows was introduced in [27]; later it served as a basis of extensive development in ergodic theory [59], [40]. For foliations, Haefliger has studied a corresponding smooth version of this equivalence (cf. §1.1, [15]). For generalizations to groups actions see [60], [31]. For the von Neumann algebras obtained from a measurable equivalence relation [11], Kakutani equivalent relations yield Morita equivalent algebras.

1.3. *Growth rate.* The exponential growth rate of a good metric d on $\mathcal{F}$ at a point x is defined by

$$g(\mathcal{F}, d, x) = \limsup_{T \to \infty} \left(\frac{\log \operatorname{Card} B(x, T, d)}{T} \right).$$

It follows from the triangle inequality (1.2) that for $(x, y) \in \mathcal{F}$,

$$B(y, T - d(x, y), d) \subset B(x, T, d) \subset B(y, T + d(x, y), d)$$

so that $g(\mathcal{F}, d, x) = g(\mathcal{F}, d, y)$. This implies that the growth rate is constant on equivalence classes and since it is a measurable function, it is constant on ergodic saturated sets. We will say that the metric equivalence relation $(\mathcal{F}, \nu, d)$, or sometimes just $\mathcal{F}$ for short, has *exponential type* a if $g(\mathcal{F}, d, x) \leq a$ almost everywhere.

Let us notice that although the exponential growth rate is not preserved under a quasi-isometry, its property of being zero, a positive number or infinity is preserved. Moreover, this property is invariant under Kakutani equivalence so that we can speak of metric equivalence relations of subexponential, exponential and superexponential type.

The growth types of the leaves of a foliation were first systematically studied by Plante (cf. [46]), who related subexponential type to the existence of invariant measures and cohomology classes. Our interest is in how the growth type influences the canonical forms of tempered cocycles over an equivalence relation.

It is worth noticing that the *polynomial growth rate*

$$p(\mathcal{F}, d, x) = \limsup_{t \to \infty} \left\{ \frac{\log \operatorname{Card} B(x, T, d)}{\log T} \right\}$$

is invariant under quasi-isometry and under Kakutani equivalence.

There are several useful properties which a good metric d may possess:

d is *discrete* if there exists a constant $c > 0$, so that for $(x, y) \in \mathcal{F}$, $x \neq y$, $d(x, y) > c$.

d is *tempered* if for every T the cardinality of the set $B(x, T, d)$ is uniformly bounded for $x \in X$.

d is *uniformly connected* if there exists $T_0 > 0$ such that for any $(x, y) \in \mathcal{F}$ one can find a finite set of points $x = x_0, x_1, \ldots, x_N = y \in \mathcal{F}_x$ such that

$$d(x_i, x_{i+1}) < T_0 \quad \text{for } i = 0, \ldots, N-1.$$

The metric induced on any uniform section of a smooth foliation of a compact manifold by a Riemannian metric on the leaves is good, discrete, tempered and uniformly connected. Other examples of such metrics come from finitely generated pseudo-groups (cf. next section).

We will end this section with an interesting open question, which is the analogue via Mackey virtual group construction [34] for a metric equivalence relation of the proposition in Section 2 of Manning [35].

Problem 1.1. *Let $(X, \mathcal{F}, d)$ be an ergodic metric equivalence relation such that d is good, tempered and uniformly connected. Suppose that $\mathcal{F}$ has a finite transversal invariant measure $\mu \cdot$. Does*

$$\frac{\log \operatorname{Card} B(x, T, d)}{T}$$

converge to a limit as $T \to \infty$ for almost every $x \in \operatorname{Support}(\mu)$?

1.4. *Equivalence relations and discrete pseudo-groups.* Let us assume that the equivalence relation $\mathcal{F}$ is generated by a discrete pseudo-group $\Gamma = (A_n, \varphi_n)$ for $n = 1, 2, \ldots$ where $A_n \subset X$ is a measurable set and $\varphi_n\colon A_n \to X$ is an injective non-singular transformation so that the left inverse

$$\varphi_n^{-1}\colon \varphi_n(A_n) \to A_n$$

is defined. The equivalence relation $\mathcal{F}$ is defined as follows: $(x, y) \in \mathcal{F}$ if there exist $n_1, \ldots, n_k$ and $\varepsilon_1, \ldots, \varepsilon_k = \pm 1$ such that

$$\varphi = \varphi_{n_k}^{\varepsilon_k} \varphi_{n_{k-1}}^{\varepsilon_{k-1}} \cdots \varphi_{n_1}^{\varepsilon_1} \tag{1.5}$$

is defined at x and $\varphi(x) = y$. The *word metric* d_Γ on $\mathcal{F}$ is defined by

$$d_\Gamma(x, y) = \min\{k | n_1, \ldots, n_k \text{ satisfying (1.5) exists}\}.$$

The metric d_Γ is always discrete and uniformly connected. It is good if every $x \in X$ belongs only to finitely many of the sets A_n and $\varphi_n(A_n)$, and tempered if the number of A_n's and $\varphi_n(A_n)$'s covering any point is uniformly bounded.

One can define other metrics on $\mathscr{F}$ by introducing positive weight functions t_n on A_n and minimizing the sum of this function (φ) along any chain (1.5). If the functions t_n are uniformly bounded from above and bounded away from zero, this more general metric is quasi-isometric to the metric d_Γ.

It is interesting to notice that any metric equivalence relation with a uniformly connected metric can be naturally generated by a pseudo-group. Namely, let us fix any $T \geq T_0$ and order measurably all elements of the set $B(x, T, d) = \{x_1, \ldots, x_{k(x)}\}$.

Let

$$A_n = \{x | \text{Card } B(x, T, d) \geq n\} \qquad \text{and}$$
$$\varphi_n x = x_n.$$

For the word metric d_Γ we have $d_\Gamma(x, y) \geq T_0^{-1} d(x, y)$ (but the above estimate may not be true).

A slightly more general construction works for any good metric.

1.5. *Uniform sections.* A metric equivalence relation is a discrete object which is supposed to play the role of a "section" to a certain continuous object. The picture we have in mind is similar to the relationship between a flow and the Poincaré map defined on a section. The general continuous object in question can be called "foliated measure space." The precise definition is somewhat involved (cf. [52] and [39]); for our current purposes it is enough to state that a foliated measure space is a common generalization of the situations (F) and (A) above.

For a smooth codimension-n foliation the measurable structure on an n-dimensional transversal $\mathscr{N}$ is given by the n-dimensional smooth measures on $\mathscr{N}$. It can be naturally derived from the smooth measure class on the ambient manifold M.

The metric structures on the leaves are provided by fixing a smooth Riemannian metric on M and defining the distance on a leaf via the induced Riemannian metric. The compactness of M guarantees the equivalence of metrics obtained from different Riemannian metrics on the ambient manifold.

Let us consider the group action case in some detail. Let $T = \{T_g | g \in G\}$ be an action of the Lie group G on the measure space X. First, by fixing a Riemannian metric σ on the Lie algebra $\mathfrak{g}$ of G we define the Riemannian metric on the orbit Gx of $x \in X$. Let us assume that G acts on X from the left. Then the Riemannian metric obtained on the orbits will correspond to the *right* invariant Riemannian metric on G so that in general it is not preserved by the action of G. Obviously, different Riemannian metrics on the Lie algebra generate equivalent metrics on orbits. Let $B_G(x, r)$ be the ball in the orbit of x,

centered at x of radius r with respect to the metric defined above. A set $\Gamma \subset X$ is called an *r-section* for the G-action T if the following conditions (S1)–(S3) are satisfied:

(S1) For $x, y \in \Gamma$, $x \neq y$ implies $B_G(x, r) \cap B_G(y, r) = 0$,

(S2) $\mu(\bigcup_{x \in \Gamma} B_G(x, r)) > 0$.

(S3) The partition of $\Gamma_r \overset{\text{def}}{=} \bigcup_{x \in \Gamma} B_G(x, r)$ into the sets $B_G(x, r)$ is measurable.

The measure μ on Γ_r is naturally projected onto Γ thus defining a measure μ_Γ whose equivalence class does not depend on r, or on the choice of the metric on G. Moreover, if G is unimodular and the action T is measure-preserving then the measure μ_Γ itself is canonically defined.

Let M be a complete metric space. Given $0 < r < R$, a set $\Sigma \subset M$ is called an (r, R)-*net* if

(N1) $\operatorname{dist}(x_1, x_2) \geq r$ for $x_1, x_2 \in \Sigma$ and $x_1 \neq x_2$.

(N2) For every $x \in M$ there exists $y \in \Sigma$ such that $\operatorname{dist}(x, y) < R$.

The set Σ is called a *net* if it is an (r, R)-net for some r, R. The notion of net depends only on the equivalence class of the metric on M. Let us consider now the cases (F) and (A) simultaneously, having in mind that the concept of uniform section defined below can be extended to general foliated measure spaces.

A set $\Gamma \subset X$ is called a *uniform section* if:

(i) The intersection of Γ with any leaf or orbit is a net with respect to the Riemannian metric class.

(ii) For (F), Γ is a finite union of smooth local transversals.

For (A), Γ is an r-section for some $r > 0$.

The existence of a uniform section for the case (F) of foliations is obvious; for case (A) it can be proved by taking an r-section and considering its maximal (up to a set of measure zero) extension (cf. [10]). For the case (Γ), the equivalence relation

$$\mathscr{R} = \{(x, \gamma x) | \gamma \in \Gamma\} \subset X \times X$$

is already discrete, but we can view the diagonal inclusion $X \subset \mathscr{R} \subset X \times X$ as a uniform section.

Let $\Gamma \subset X$ be a uniform section. It possesses a measure class which makes it a standard measure space. The continuous equivalence relation of X into leaves or orbits defines a discrete equivalence relation on Γ. The restriction of a Riemannian metric to each equivalence class defines a good, discrete, tempered and uniformly connected metric on the equivalence relation. The metric equivalence relations corresponding to different uniform sections for the same foliation or action are Kakutani equivalent. In the foliation case we also obtain a

finitely generated groupoid generating the equivalence relation by considering the holonomy maps on the leaves which correspond to paths of bounded length.

1.6. *Cocycles over metric equivalence relations.* Let G be a Polish group. A G-*cocycle* over an equivalence relation $(X, \mathcal{F})$ is a measurable map ϕ: $\mathcal{F} \to G$ satisfying the cocycle equation: if $x \in X$, $y, z \in \mathcal{F}_x$, then

$$\phi(x, y) \cdot \phi(y, z) = \phi(x, z). \tag{1.6}$$

Two cocycles ϕ and ψ are *cohomologous* if there exists a measurable map f: $X \to G$ such that for $(x, y) \in \mathcal{F}$

$$\psi(x, y) = f(x)^{-1} \cdot \phi(x, y) \cdot f(y).$$

We will call f the *transfer function* and will denote the cohomology by $\phi_f\psi$ or sometimes simply by $\varphi \sim \psi$.

We now will consider G-cocycles over a metric equivalence relation $(X, \mathcal{F}, d)$. Let us fix a left-invariant metric ρ on the group G, or a left invariant pseudo-metric ρ such that the set

$$G_0 = \{g \in G | \rho(\mathrm{id}, g) = 0\}$$

is compact, and denote for $g \in G$, $|g| = \rho(\mathrm{id}, g)$.

In the following, we will have need of several different notions of norm and pseudonorm in matrix groups and in order to avoid confusion we will have to distinguish notations carefully. First, for any matrix group $G \subset \mathrm{GL}(N, \mathbf{R})$ there is a natural pseudonorm

$$|A|_M = \max(\log\|A\|, \log\|A^{-1}\|) \tag{1.7}$$

where $\|A\|$ is the usual matrix norm

$$\|A\| = \sup_{0 \neq v \in \mathbf{R}^N} \frac{\|Av\|}{\|v\|}. \tag{1.8}$$

The pseudonorm $|\ |_M$ determines a left-invariant pseudo-metric ρ_M, where

$$\rho_M(A, B) = |A^{-1} \cdot B|_M, \tag{1.9}$$

which is quasi-isometric to a left-invariant metric ρ on G in the sense of (1.4); i.e., there are constants a, b, c so that for all $A, B \in G$,

$$a \cdot \rho(A, B) - c \leq \rho_M(A, B) \leq b \cdot \rho(A, B) + c.$$

In general, the metric ρ is not generated by a Riemannian metric, though if $G = \mathrm{GL}(N, \mathbf{R})$ then ρ can be chosen that way. In the calculations which appear in the proofs of Theorems 3.1 and 3.2, as well as in Sections 6 and 7, we will use the norm $\|\ \|$, the pseudonorm $|\ |_M$, and the quantity

$$\|A\|^+ = \max(\|A\|, \|A^{-1}\|) = \exp|A|_M. \tag{1.10}$$

The G-cocycle φ over $(X, \mathscr{F}, d)$ is called *tempered* if there exists a continuous function c: $\mathbf{R}^+ \to \mathbf{R}^+$ such that

$$|\varphi(x, y)| < c(d(x, y)). \tag{1.11}$$

The cocycle φ has *exponential type* b if for almost every $x \in X$,

$$\limsup_{T \to \infty} \left\{ \frac{1}{T} \max_{y \in B(x, T, d)} \cdot |\varphi(x, y)|_M \right\} \leq b, \tag{1.12}$$

and b is the least real number such that (1.12) holds. We say φ has *exponential type* if for some real $b < \infty$, it has exponential type b. A cocycle φ has *moderate growth*, or is of *subexponential type*, if it has exponential type 0.

The following two lemmas are immediate corollaries of the definitions.

LEMMA 1.1. *If a metric d is uniformly connected, then every tempered cocycle has exponential growth.*

LEMMA 1.2. *If a cocycle φ has exponential type b, then for every $\varepsilon > 0$ there exists a measurable positive function C_ε on X such that*

$$|\varphi(x, y)| \leq C_\varepsilon(x) \exp\{(b + \varepsilon) \cdot |y|_x\}$$

for all $y \in \mathscr{F}_x$.

If the equivalence relation $\mathscr{F}$ is generated by the action $T = \{T_\gamma\}_{\gamma \in \Gamma}$ of a discrete finitely generated group Γ and the metric on $\mathscr{F}$ is generated by the word metric on the orbits (cf. subsection 1.4) then any cocycle φ can be interpreted as a function on $\Gamma \times X$, and (1.6) becomes

$$\varphi(\gamma_2 \cdot \gamma_1, x) = \varphi(\gamma_2, T_{\gamma_1} x) \cdot \varphi(\gamma_1, x).$$

In this case, the cocycle φ is tempered if and only if for every $\gamma \in \Gamma$ (or for every γ from a generating set in Γ) the function $\varphi(\gamma, x)$ on X takes values in a compact subset of G.

Cocycles arising from continuous geometric constructions are usually tempered; for example, the normal linear holonomy cocycle to a foliation, when calculated with respect to an orthonormal measurable framing of the normal bundle equipped with a smooth Riemannian metric, will be tempered with exponential type. Non-tempered cocycles can be obtained by using an orbit equivalence P: $(X, \mathscr{F}, d) \to (X', \mathscr{F}', d')$ where P is not a quasi-isometry (cf. [31], Chapter 11).

1.7. *Amenability.* Let K be a compact convex space and φ be a cocycle over the discrete equivalence relation $(X, \mathscr{F})$ with values in the group Aut K of affine automorphisms of K. The equivalence relation $\mathscr{F}$ is called *amenable* if for

any such cocycle φ one can find an invariant section, i.e., a measurable map $h\colon X \to K$ such that for $(x, y) \in \mathcal{F}$,

$$\varphi(x, y) \cdot h(y) = h(x)$$

(see [55]). Obviously, amenability is invariant under orbit equivalence. It is also inherited by the restriction of the equivalence relation to any measurable subset. The notion of amenability has an obvious extension to continuous equivalence relations, for example, those obtained from cases (F) and (A). Every such relation admits a section [48], and a continuous equivalence relation is amenable if and only if the discrete equivalence relation on a section is amenable.

If the equivalence relation $\mathcal{F}$ is defined by an action of an amenable Lie group Γ on $(X, \mathcal{B}, \mu)$ then $\mathcal{F}$ is amenable. Conversely, if $\mathcal{F}$ is defined by an almost everywhere free action of Γ, μ is invariant under the action, and $\mathcal{F}$ is amenable, then Γ is amenable. If μ is not invariant, this need not be true [55].

Every foliation defined by the action of a solvable Lie group is amenable, though it may have leaves of exponential growth type $b > 0$. Amenability for foliations is discussed further by Brooks in [3], and for general equivalence relations in [6].

There is an obvious similarity between the notion of amenability for an equivalence relation (in particular for a relation generated by a free group action) and the more common notion of amenable group. The amenability of a discrete group is equivalent to a combinatorial growth condition (Følner condition). A similar condition for metric equivalence relations turns out to be sufficient but not necessary for amenability. Here is the condition. For $x \in X$ we will call an increasing sequence of finite subsets $F_n \subset \mathcal{F}_x$ a *Følner sequence if*

$$\bigcup_{n=1}^{\infty} F_n = \mathcal{F}_x, \qquad \text{and for every } T > 0,$$

$$\lim_{n\to\infty} \frac{\operatorname{Card} \bigcup_{x\in F_n} B(x, T, d)}{\operatorname{Card} F_n} = 1.$$

It is easy to see that the last property depends only on the equivalence class of metric d.

PROPOSITION 1.3 (cf. [6]). *If a Følner sequence exists for almost every $x \in X$, then $(X, \mathcal{F}, d)$ is an amenable equivalence relation.*

For foliations defined by connected Lie group actions, the fact that this condition is not necessary for amenability is connected with incompatibility of the left and right invariant metrics. Here are two geometric propositions which imply the existence of a Følner sequence.

PROPOSITION 1.4 (cf. [46]). *If a metric equivalence relation* $(X, \mathscr{F}, \nu)$ *has exponential growth type* 0, *then for almost every* $x \in X$, *there exists a Følner sequence on* $\mathscr{F}_x$.

Proof. Let $g(\mathscr{F}, d, x) = 0$. Fix $T > 0$ and consider the balls $B(x, nT, d)$ for $n = 1, 2, \ldots$. Let

$$a(x, n) = \frac{\text{Card } B(x, (n+1)T, d)}{\text{Card } B(x, nT, d)}.$$

Obviously $a(x, n) \geq 1$ and Card $B(x, nT, d) = \prod_{k=0}^{n-1} a(x, k)$. Thus, there exists a sequence $n_k \to \infty$ such that $a(x, n_k) \to 1$ so that $B(x, n_k T, d)$ may serve as a Følner sequence for the given T. Applying this argument for $T = 1, 2, \ldots$ and using the diagonal process we finish the proof. □

A partial converse to Proposition 1.3 has been proven by Carrière and Ghys (cf. Théorème 4, [5]).

PROPOSITION 1.5. *Let* $\Gamma = (A_n, \varphi_n)$ *be a discrete pseudo-group acting essentially freely on* $(X, \mathscr{B}, \mu)$, *with* μ *an invariant probability measure. If the equivalence relation* $\mathscr{F}$ (*cf.* §1.4) *is amenable, then a Følner sequence exists for* μ*-almost every* $x \in X$.

We conclude this subsection with a discussion of the above concepts for an important subcase of (Γ). Suppose that Y is a compact manifold without boundary, $y_0 \in Y$ is a basepoint, $\Gamma = \pi_1(Y, y_0)$ is the fundamental group and a representation $A\colon \Gamma \to \text{Diff}^{(k)}X$ is given, where X is a closed n-manifold. Then A induces a left C^k-action $\overline{A}\colon \Gamma \times X \to X$, and Γ acts naturally on the universal cover $\tilde{Y}$; so the quotient manifold $M = M_A \equiv \Gamma \setminus (\tilde{Y} \times X)$ is defined. Moreover, M carries a C^k-foliation, $\mathscr{F}_A$, which is the quotient of the product foliation on $\tilde{Y} \times X$ by the leaves $\{\tilde{Y} \times \{x\} \mid x \in X\}$. The leaf $\mathscr{F}_x$ of $\mathscr{F}_A$ through $x \in X$ is the covering space of Y associated to the isotropy group of A at x. Note that the inclusion of the fiber $\{y_0\} \times X \subset M$ is a uniform section to the continuous equivalence relation $(M, \mathscr{F}_A)$, and the discrete model is $\overline{A}\colon \Gamma \times X \to X$.

The pair $(M, \mathscr{F}_A)$ is called the flat $\text{Diff}^{(k)}$ X-foliation associated to the representation A. The importance of these examples is that there is an extensive homotopy machinery available to construct such foliations, with given secondary class data (cf. [24]). Also, if Γ has subexponential growth, then $\mathscr{F}$ and $\overline{A}$ have exponential type 0. If Γ is amenable, then $\mathscr{F}$ and $\overline{A}$ will be amenable. However if $\mathscr{F}$ does not admit an invariant transverse measure, then $\mathscr{F}$ can be amenable with Γ nonamenable [55].

1.8. *Cocycles over amenable equivalence relations.* Zimmer has proved that there is a strong restriction on a cocycle over an amenable equivalence relation: its Mackey range must be amenable. Combining this with the virtual group theory of Mackey (cf. [33], [34], [47]), we obtain the following version of Zimmer's theorem, which is one of the starting points for the results of this paper.

THEOREM 1.6 (Zimmer [56]). *Let* $(X, \mathcal{F})$ *be an ergodic amenable discrete equivalence relation, and* G *a real algebraic group. Then for every cocycle* φ: $\mathcal{F} \to G$, *there is an amenable subgroup* $H_\varphi \subset G$ *and a cocycle* ψ: $\mathcal{F} \to H_\varphi$ *with* φ *cohomologous to* ψ *in* G.

Proof. This theorem is implicit in Zimmer's work; we discuss here only the steps needed to deduce it from the two results cited below. The cocycle defines the skew-product equivalence relation $\mathcal{F}^\varphi$ on the product $X \times G$ by the rule: the equivalence class of (x, g) consists of all elements $\{(y, \varphi(y, x) \cdot g | y \in \mathcal{F}_x\}$. The σ-algebra $\mathcal{B}(\mathcal{F}^\varphi)$ determines the space of ergodic components of $\mathcal{F}^\varphi$, which we denote by S_φ.

The right action of G on $X \times G$ projects onto a right action of G on S_φ, defining a factor-action called the *Mackey range* of φ (cf. [47], [55]). Since $(X, \mathcal{F})$ is ergodic, G acts ergodically on S_φ. By Proposition 3.5 of [56], G acts amenably on S_φ. Then G real algebraic implies there is a closed amenable subgroup $H \subset G$ and a G-equivariant map f: $S_\varphi \to H \backslash G$, [55]. It is then standard to construct, using f, a cocycle ψ: $\mathcal{F} \to H$ so that ψ is cohomologous to ϕ (cf. [47]). □

Let $(X, \mathcal{F})$ be an amenable discrete equivalence relation, and let Z be the space of ergodic components of X under $\mathcal{F}$. Let γ: $X \to Z$ be the natural quotient map, so that each $X_z = \lambda^{-1}(z) \subset X$ is ergodic under the action of $\mathcal{F}$. Given a cocycle ϕ: $\mathcal{F} \to G$ into a real algebraic group G, we apply Theorem 1.6 to the restriction of ϕ to each set X_z to obtain an amenable closed subgroup $H_z \subset G$ and a cocycle $\psi_z \sim \phi | X_z$ which takes values in H_z. Via a selection argument, we can conclude:

COROLLARY 1.7. *Let* $(X, \mathcal{F})$ *be an amenable discrete equivalence relation and* ϕ: $\mathcal{F} \to G$ *a cocycle with values in a real algebraic group. For* λ: $X \to Z$ *an ergodic decomposition space of* $(X, \mathcal{F})$, *there is a measurable field of closed amenable subgroups* $\{(z) \times H_z | z \in Z\} \subset Z \times G$ *and a cocycle* ψ: $\mathcal{F} \to G$ *such that* $\psi \sim \phi$, *and* ψ *restricted to* $\mathcal{F} | X_z$ *takes values in* H_z.

In the special case $G = \mathrm{GL}(n, \mathbf{R})$, there are precisely 2^n conjugacy classes of maximal amenable subgroups ([38]; cf. also subsection 4.2 below). Choose one

representative from each class and denote these by $\{H_i | i \leq i \leq 2^n\}$. For $(X, \mathcal{F})$ as in Corollary 1.7 and a cocycle ϕ: $\mathcal{F} \to \mathrm{GL}(n, \mathbf{R})$, for each $z \in Z$ there is an integer $n(z)$ such that H_z is conjugate to a closed subgroup of $H_{n(z)}$. The function n: $Z \to N$ is measurable; so there is a partition of X into at most 2^n measurable saturated sets,

$$X = \bigcup_{i=1}^{2^n} X_i$$

with $n \circ \lambda$ constant on X_i. Again using a selection lemma, we can incorporate this conjugation into the cocycle ψ and obtain a sharper form of Corollary 1.7:

COROLLARY 1.8. *Let $(X, \mathcal{F})$ be an amenable discrete equivalence relation and ϕ: $\mathcal{F} \to \mathrm{GL}(n, \mathbf{R})$ a cocycle. Then there is a cocycle ψ: $\mathcal{F} \to \mathrm{GL}(n, \mathbf{R})$ such that $\psi \sim \phi$, and a decomposition $X = \bigcup_{i=1}^{2^n} X_i$ such that ψ: $\mathcal{F} | X_i \to H_i$.*

1.9. *Murray-von Neumann classification.* The classification of ergodic measurable equivalence relations by their orbit equivalence types falls naturally into three categories, according to their Murray-von Neumann types (cf. [39]). We describe here these types as they apply to a foliation $\mathcal{F}$ on a manifold M.

Let M_{I} be the largest measurable saturated subset of M on which $\mathcal{F}$ is dissipative; that is, there is a measurable transversal $T \subset M_{\mathrm{I}}$ so that almost every leaf of $\mathcal{F} | M_{\mathrm{I}}$ intersects T precisely once, and the quotient measure space $M_{\mathrm{I}}/\mathcal{F}$ is a standard, non-atomic Borel space. The complement of M_{I} is partitioned into M_{II} and M_{III}, where M_{II} is the largest saturated measurable subset of $M - M_{\mathrm{I}}$ so that $\mathcal{F} | M_{\mathrm{II}}$ admits an *absolutely continuous* σ-finite transverse invariant measure μ with almost every leaf of $\mathcal{F} | M_{\mathrm{II}}$ being μ-essential. Thus, the only absolutely continuous transverse invariant measure for $\mathcal{F} | M_{\mathrm{III}}$ is the zero measure, and the quotient $M_{\mathrm{III}}/\mathcal{F}$ is a completely singular measure space.

A subset $B \in \mathcal{B}(F)$ inherits a decomposition into disjoint measurable saturated sets,

$$B = B_{\mathrm{I}} \cup B_{\mathrm{II}} \cup B_{\mathrm{III}}$$

with the above properties. We say $\mathcal{F} | B_{\mathrm{I}}$ (respectively, $\mathcal{F} | B_{\mathrm{II}}$ or $\mathcal{F} | B_{\mathrm{III}}$) has type I (respectively, II or III). There are various further subdivisions of types I, II and III which are of great importance for foliations, but are not used in this paper.

For $B \in \mathcal{B}(\mathcal{F})$ ergodic with positive Lebesgue measure, either $B = B_{\mathrm{II}}$ a.e. or $B = B_{\mathrm{III}}$ a.e., in which case we say B has type II or III, respectively. If $\mathcal{F}$ is ergodic on M, then we say $\mathcal{F}$ has type II or III, accordingly.

A foliation $\mathcal{F}$ is an $\mathrm{SL}(n, \mathbf{R})$-foliation if there is a non-vanishing closed n-form ω on M whose kernel defines the distribution $T\mathcal{F}$. In this case, $M = M_{\mathrm{I}} \cup M_{\mathrm{II}}$.

A leaf $L \subset M$ is *proper* if L is locally closed in the relative topology induced from M. For example, a compact leaf is proper and a dense leaf is not proper. There is the following geometric context which yields type I foliations:

PROPOSITION 1.9 [49], [58]. *For $B \in \mathscr{B}(\mathscr{F})$, suppose almost every leaf $L \subset B$ is proper. Then $\mathscr{F}|B$ has type* I. *If almost every leaf $L \subset M$ is proper, then $\mathscr{F}$ has type* I.

2. Characteristic classes for foliations

The secondary classes, and the related μ-classes and dual homotopy invariants of foliations have provided the primary means for studying the quantitative theory of foliations, and especially for analyzing the topological type of the foliation classifying spaces of Haefliger (cf. [1], [2], [14], [20], [24]). In this section, we briefly define the secondary classes and Weil measures of a codimension-n, C^2-foliation, $\mathscr{F}$, on an m-dimensional manifold M without boundary. If $\mathscr{F}$ has, in addition, a finite invariant transverse measure, μ, then the μ-classes of the pair $(\mathscr{F}, \mu)$ are also defined. More extensive treatments are found in the literature [1], [12], [16], [20], [28].

2.1. *Secondary classes.* Denote by $I(\mathrm{GL}(n, \mathbf{R}))$ the graded ring of adjoint invariant polynomials on the Lie algebra $\mathfrak{gl}_n$ of the real general linear group $\mathrm{GL}(n, \mathbf{R})$. As a ring, $I(\mathrm{GL}(n, \mathbf{R})) \cong R[c_1, \ldots, c_n]$ is a polynomial algebra on n generators, the Chern polynomials $\{c_1, \ldots, c_n\}$ where c_i is the polynomial of degree i (although c_i is assigned graded degree $2i$) defined by the equation

$$\det(\lambda \cdot I_n - 1/2\pi X) = \sum_{i=1}^{n} \lambda^{n-1} c_i(X) \tag{2.1}$$

for $X \in \mathfrak{gl}_n$ and I_n the identity in $\mathrm{GL}(n, \mathbf{R})$. Denote by $I(> n)$ the ideal in $I(\mathrm{GL}(n, \mathbf{R}))$ generated by the monomials of graded degree greater than $2n$. Define the truncated polynomial ring to be the quotient $I(\mathrm{GL}(n, \mathbf{R}))_n = I(\mathrm{GL}(n, \mathbf{R}))/I(> n)$.

Let $A^*(M)$ denote the algebra of differential forms on M, and let $\mathscr{I}$ be the ideal of differential forms which vanish when restricted to the leaves of $\mathscr{F}$. Observe that $\mathscr{I}^{n+1} = 0$. Let Q be the normal bundle to $\mathscr{F}$. Choose a basic connection θ^b on $Q \to M$, and let Ω^b denote its $\mathfrak{gl}_n$-valued curvature form. The i^{th}-Chern form of $\mathscr{F}$ is then $c_i(\Omega^b) \in A^{2i}(M)$. This is a closed form which belongs to $\mathscr{I}^i$.

Choose a Riemannian metric h on Q with metric connection θ^h and curvature Ω^h. Note that Ω^h is skew-symmetric; hence

$$c_i(\Omega^h) \equiv 0 \qquad \text{for } i \text{ odd}. \tag{2.2}$$

For each $t \in R$, form the connection $\theta^t = t \cdot \theta^b + (1-t)\theta^h$ on Q, and let its curvature form be Ω^t. For each c_i set

$$\Delta_{c_i}(\theta^b, \theta^h) = i \cdot \int_0^1 c_i(\theta^b - \theta^h, \Omega^t, \ldots, \Omega^t) \cdot dt$$

where we view c_i as an i-linear functional on $\mathfrak{gl}_n$ using polarization [16]. The form $\Delta_{c_i}(\theta^b, \theta^h) \in A^{2i-1}(M)$ satisfies the equation

$$d\Delta_{c_i}(\theta^b, \theta^h) = c_i(\Omega^b) - c_i(\Omega^h).$$

In particular, for i odd from (2.2) we have

$$d\Delta_{c_i}(\theta^b, \theta^h) = c_i(\Omega^b).$$

Let $\Lambda(y_1, y_3, \ldots, y_{n'})$ be the exterior algebra on generators $\{y_1, y_3, \ldots, y_{n'}\}$ where degree $y_i = 2i - 1$ and n' is the greatest odd integer less than $n + 1$. Define the graded differential algebra WO_n to be the graded tensor product

$$WO_n = \Lambda(y_1, y_3, \ldots, y_{n'}) \otimes I(\mathrm{GL}(n, \mathbf{R}))_n$$

with differential defined by $d(y_i \otimes 1) = 1 \otimes c_i$, $d(1 \otimes c_i) = 0$. The choice of θ^b and θ^h determines a graded differential algebra homomorphism

$$\begin{aligned} \Delta = \Delta(\theta^b, \theta^h)&: WO_n \to A^*(M), \\ \Delta(y_i \otimes 1) &= \Delta_{c_i}(\theta^b, \theta^h), \\ \Delta(1 \otimes c_i) &= c_i(\Omega^b). \end{aligned}$$

The induced map on cohomology, Δ_*: $H^*(WO_n) \to H^*(M)$, is independent of the choice of θ^b and θ^h, and the elements of $H^*(M)$ which belong to the image of Δ_* are called the secondary classes for $\mathcal{F}$.

We adopt the standard notation: $I = (i_1, \ldots, i_s)$ denotes an index set with $1 \le i_1 < \cdots < i_s \le n'$ with all i_j odd; $J = (j_1, \ldots, j_n)$ where each $j_k \ge 0$; $|J| = j_1 + 2j_2 \cdots + nj_n$;

$$y_I c_J = y_{i_1} \wedge \cdots \wedge y_{i_s} \otimes c_1^{j_1} \cdots c_n^{j_n}.$$

A basis of $H^*(WO_n)$ in degrees greater than $2n$, due to J. Vey [12], is given by the cocycles

$$\{y_I c_J \mid |J| \le n;\ i_1 + |J| > n;\ \text{for } k \text{ odd if } j_k > 0 \text{ then } i_1 \le k\}.$$

In degrees less than $2n + 1$, $H^*(WO_n)$ is the ring of Pontrjagin polynomials for Q. The classes in degrees greater than $2n$ in $H^*(WO_n)$ are divided into several

groups:

$y_I c_J$ is *residual* if $|J| = n$;
$y_1 c_J$ for $|J| = n$ are the *Godbillon-Vey* classes;
$y_I c_J$ with $i_1 = 1$ and $|J| = n$ are the *generalized Godbillon-Vey* classes;
$y_I c_J$ with degree $y_I > 1$ and $|J| = n$ are the *higher degree* residual classes;
$y_I c_J$ with $i_1 + |J| > n + 1$ are the *rigid* classes;
$y_I c_J$ with $i_1 + |J| = n + 1$ are the *variable* classes.

Notice that the generalized Godbillon-Vey classes constitute the variable residual classes. The names residual, variable and rigid reflect the behavior of these classes in various examples.

2.2. *Weil measures.* The Weil measures of [17] are a common generalization of the μ-classes of [20] and the Godbillon measure introduced by Duminy [7], [4]. The basic idea is that the transgression forms $\Delta_{c_i}(\theta^b, \theta^h)$, introduced in the definition of the secondary classes, are themselves intrinsically important data, and the Weil measures are "universal" functionals derived from these forms. Moreover, we will show in Section 5 that the transgression forms $\Delta_{c_i}(\theta^b, \theta^h)$ can be calculated from the normal linear holonomy cocycle; these forms are the "finest" de Rham invariants on M which one can derive from this cocycle. Thus, one expects the values of the Weil measures to be closely related to the dynamics of the foliation, and this is what we show in this paper. A key technical point is that the Weil measures depend continuously on the choices of connections θ^b and θ^h, so it is feasible to study their behavior in the limit, as the connection data limit to a "measurable connection". In contrast, the Chern forms $c_J(\theta^b)$ depend upon θ^b *and* its first derivatives. This effectively prevented the analysis, prior to the introduction of the Weil measures, of the secondary classes vis-a-vis dynamics, except under very controlled geometric conditions (cf. [18], [19], [36], [37]).

Let $A(M, \mathcal{F})$ denote the n-th power of the ideal $\mathcal{I}$. If Q is orientable, so that M admits a non-vanishing n-form ω whose kernel is the tangent bundle to $\mathcal{F}$, then $A(M, \mathcal{F}) = \omega \wedge A(M)$. A typical form $\varphi \in A^k(M, \mathcal{F})$, for $k \geq n$, can be written as $\varphi = \omega \wedge \hat{\varphi}$, where $\hat{\varphi}$ is a $(k - n)$-form. If Q is not orientable, then this just holds locally on M. The ideal $A(M, \mathcal{F})$ is closed under the exterior derivative, and we set

$$H^*(M, \mathcal{F}) = H^*(A(M, \mathcal{F}), d).$$

We will also need the ideal of *compactly supported* forms,

$$A_c(M, \mathcal{F}) = A(M, \mathcal{F}) \wedge A_c(M)$$

where $A_c(M)$ are the compactly supported forms on M. If Q is orientable, a

typical form $\varphi \in A_c^k(M, \mathscr{F})$ can be written $\phi = \omega \wedge \hat{\varphi}$, where $\hat{\varphi}$ is a compactly supported $(k - n)$-form. We again set

$$H_c^*(M, \mathscr{F}) = H^*(A_c(M, \mathscr{F}), d).$$

Note that for M compact, $H_c^*(M, \mathscr{F}) = H^*(M, \mathscr{F})$.

For notational convenience, the relative Lie algebra cohomology of $\mathfrak{gl}_n$ is identified with the exterior algebra of WO_n:

$$H^*(\mathfrak{gl}_n, O_n) \equiv \Lambda(y_1, y_3, \ldots, y_{n'}) \otimes 1.$$

Now assume M is oriented. Given $y \in H^p(\mathfrak{gl}_n, O_n)$, $\varphi \in A_c^{m-p}(M, \mathscr{F})$ and a measurable subset $B \subset M$, set

$$\chi_B(y)[\varphi] = \int_B \Delta(y) \wedge \varphi. \tag{2.3}$$

For $B \in \mathscr{B}(\mathscr{F})$, the integral (2.3) depends only on the cohomology class of φ in $A_c(M, \mathscr{F})$ and does not depend on the choice of θ^b or θ^h, [17].

THEOREM 2.1 (Heitsch-Hurder [17]). *Let $\mathscr{F}$ be a C^2-foliation of an oriented manifold M without boundary.*

(a) *For each $y \in H^p(\mathfrak{gl}_n, O_n)$ and $B \in \mathscr{B}(\mathscr{F})$ there is a well-defined continuous linear map $\chi_B(y)$: $H_c^{m-p}(M, \mathscr{F}) \to R$.*

(b) *For each $y \in H^p(\mathfrak{gl}_n, O_n)$, the correspondence $B \to \chi_B(y)$ defines a countably additive measure on $\mathscr{B}(\mathscr{F})$ with values in the continuous dual $H_c^{m-p}(M, \mathscr{F})^*$.*

We call $\chi(y)$ the *Weil measure* of $\mathscr{F}$ corresponding to y. The *Godbillon measure* (cf. [7]) is defined as $g = \chi(-2\pi \cdot y_1)$, with values in $H_c^{m-1}(M, \mathscr{F})^*$.

For each residual class $y_I c_J \in H^p(WO_n)$ and closed form $\varphi \in A_c^q(M)$, Proposition 1.4 of [17] implies there is a well-defined class $[\Delta(y_I c_J) \wedge \varphi] \in H_c^{p+q}(M, \mathscr{F})$. If now $y_I = \pm y_{I'} \cdot y_{I''}$, then for $q = m - p$,

$$\chi_M(y_{I'})[\Delta(y_{I''} c_J) \wedge \varphi] = \pm \langle \Delta_*(y_I c_J) \cup [\varphi], [M] \rangle.$$

By Poincaré duality, the secondary class $\Delta_*(y_I c_J)$ is thus completely determined by the measure $\chi(y_{I'})$ on M, and we obtain:

COROLLARY 2.2. *If $\chi_M(y_{I'}) = 0$ then all residual secondary classes $\Delta_*(y_I c_J) = 0$, where $y_{I'}$ is a factor of y_I.*

Given a residual $y_I c_J \in H^p(WO_n)$ and $B \in \mathscr{B}(\mathscr{F})$, we can define the restriction $\Delta_*(y_I c_J)|B \in H^p(M)$ via Poincaré duality and the rule, for $[\varphi] \in$

$H_c^{m-p}(M)$,

$$\langle \Delta_*(y_I c_J) | B \cup [\varphi], [M] \rangle = \chi_B(y_I)[\Delta(c_J) \wedge \varphi].$$

The countable additivity of $\chi(y_I)$ then implies:

PROPOSITION 2.3. *Let $M = \bigcup_{i=1}^{\infty} B_i$ be a countable partition of M, where each $B_i \in \mathcal{B}(\mathcal{F})$. Then for $y_I c_J$ residual,*

$$\Delta_*(y_I c_J) = \sum_{i=1}^{\infty} \Delta_*(y_I c_J)|B_i.$$

2.3. *μ-classes.* Let the manifold M be oriented and without boundary. Assume $\mathcal{F}$ admits an invariant transverse measure μ which is finite on compact transversals. The μ-classes are invariants of the pair $(\mathcal{F}, \mu)$ given by a map

$$\chi_\mu\colon H^*(\mathfrak{gl}_n, O_n) \to H^{n+*}(M).$$

For $y \in H^p(\mathfrak{gl}_n, O_n)$, to construct $\chi_\mu(y)$ consider first the Ruelle-Sullivan [51] current c_μ: $A_c^{m-n}(M) \to \mathbf{R}$ associated to μ. Define a closed current $\chi_\mu(y)^*$: $A_c^{m-n-p}(M) \to \mathbf{R}$ by the rule:

$$\chi_\mu(y)^*[\varphi] = c_\mu(\Delta(y) \wedge \varphi).$$

Poincaré duality implies that $\chi_\mu(y)^*$ determines a class $\chi_\mu(y) \in H^{n+p}(M)$. Complete details and applications are given in [20].

If the support of μ consists of compact leaves of $\mathcal{F}$, then $\chi_\mu(y)$ is the μ-average of the leaf classes corresponding to y for the leaves in the support of μ (Proposition 3.4, [20]). For μ absolutely continuous, $\chi_\mu(y_I)$ is in a more general sense the average of the leaf classes for the leaves in the support of μ. Intuitively, $\chi_\mu(y)$ reflects the transverse mixing of $\mathcal{F}$, and the main problem is to determine precisely what aspects of the dynamics of $(\mathcal{F}, \mu)$ are being detected. This is answered in part by Corollary 3.10 below.

3. Statement of results

This section contains precise technical formulations of the main results of this paper. The proofs of Theorems 3.1 and 3.2 will be given in Section 4, and the proofs of the remaining theorems given in Sections 5–8.

3.1. *Cocycles over metric equivalence relations.* The results formulated in this subsection form a part of a more general program of "ε-classification" of matrix-valued cocycles over amenable group actions and amenable equivalence relations. The roots of this approach can be found in the multiplicative ergodic theorem by Oseledec [42] and in the Pesin construction of a Lyapunov (adapted) metric ([43], Theorem 1.5.1) which plays a pivotal role in smooth ergodic theory

[43], [44], [61]. For different approaches to tempering processes in ergodic theory see [31], [32], [41], [50].

Let $H \subset \mathrm{GL}(n, \mathbf{R})$ be a linear Lie group. We will say that H has the *cone property* if the set

$$\{A^* \cdot A : A \in H\}$$

is a cone; i.e., for any $A, B \in H$ and $\alpha, \beta > 0$ the equation $\alpha A^* \cdot A + \beta B^* \cdot B = X^* \cdot X$ has a solution $X \in H$. Among the standard forms of semisimple groups the following ones have the cone property: $\mathrm{GL}(n, \mathbf{R})$, $\mathrm{GL}(n, \mathbf{C})$, $\mathrm{SO}^*(4n)$, one of the real forms of E_6 and $\mathrm{SO}(n, 1)$. On the other hand the group of diagonal matrices also has this property.

THEOREM 3.1. *Let $(X, \mathcal{F}, d)$ be an ergodic metric equivalence relation with exponential type* a, *$H \subset \mathrm{GL}(n, \mathbf{R})$ be a linear group with the cone property and ϕ be an H-cocycle over $(X, \mathcal{F}, d)$ of exponential type* b *with respect to the norm $|\ |_M$. Then for every $\varepsilon > 0$ there exists a tempered H-valued cocycle ψ such that $\phi_f \sim \psi$, where $f: X \to H$ and for $(x, y) \in \mathcal{F}$,*

$$|\psi(x, y)|_M < (a + 2b + \varepsilon)|x|_y. \tag{3.1}$$

Moreover, the transfer function f satisfies the inequality

$$|f(x)|_M \leq C(y) + (a + 3b + \varepsilon)|x|_y. \tag{3.2}$$

If H does not have the cone property, the tempering of H-valued cocycles in general remains an open question. Various partial results will be discussed in a separate paper. However there is one important case where the tempering is always possible, namely when H is a maximal amenable subgroup of $\mathrm{GL}(n, \mathbf{R})$.

THEOREM 3.2. *Let $(X, \mathcal{F}, d)$ be an ergodic metric equivalence relation with exponential growth rate a, $H \subset \mathrm{GL}(n, \mathbf{R})$ be a maximal amenable subgroup and ϕ be an H-valued cocycle over $\mathcal{F}$ of exponential type* b *with respect to the norm $|\ |_M$. Then for every $\varepsilon > 0$ there exists a tempered H-valued cocycle $\psi \sim \phi$ such that for $(x, y) \in \mathcal{F}$,*

$$|\psi(x, y)|_M \leq \{(4n - 3)a + (8n - 6)b + \varepsilon\} \cdot |x|_y. \tag{3.3}$$

Moreover, this can always be achieved with a diagonal cohomology.

From Theorem 3.2 and Corollary 1.8 we obtain a tempered version of Theorem 1.6:

COROLLARY 3.3(i) *Let ϕ be a $\mathrm{GL}(n, \mathbf{R})$-cocycle with exponential growth over an amenable ergodic metric equivalence relation $(X, \mathcal{F}, d)$. Then ϕ is*

cohomologous to a tempered cocycle ψ with exponential growth which takes values in an amenable subgroup $H \subset \mathrm{GL}(n, \mathbf{R})$.

(ii) *If the metric equivalence relation is not ergodic then there exists a decomposition of X into at most 2^n sets, $X_i \in \mathscr{B}(\mathscr{F})$, $i = 1, \ldots, N$ such that the restriction of ϕ to each set X_i is cohomologous to a tempered cocycle with values in an amenable subgroup $H_i \subset \mathrm{GL}(n, \mathbf{R})$.*

The next corollary will not be used directly in this paper but it represents the basic step in developing the ε-classification of cocycles over the actions of nilpotent groups [30].

COROLLARY 3.4. *Let $\mathscr{F}$ be defined by the action on X of a discrete group Γ with subexponential growth and let ϕ be a $\mathrm{GL}(n, \mathbf{R})$-valued cocycle of moderate growth. Then for every $\varepsilon > 0$ there exists a cocycle $\psi_\varepsilon \sim \phi$ where ψ_ε takes values in an amenable subgroup of $\mathrm{GL}(n, \mathbf{R})$ and for all $\gamma \in \Gamma$*

$$|\psi_\varepsilon(\gamma, x)|_M < \varepsilon|\gamma|.$$

3.2. *Characteristic classes and ergodic theory.* We discuss now the relation between the characteristic classes of a foliation and its ergodic theory. The most general results are in terms of the Weil measures. Recall that $\mathscr{B}(\mathscr{F})$ is the Σ-algebra of measurable $\mathscr{F}$-saturated subsets of the orientable manifold M.

THEOREM 3.5. *Given $B \in \mathscr{B}(\mathscr{F})$, assume that the restriction $\mathscr{F}|B$ is amenable. Then for $y \in H^l(\mathfrak{gl}_n, O_n)$ with $l > 1$, the Weil measure $\chi_B(y) = 0$.*

COROLLARY 3.6. *Let $\mathscr{F}$ be an amenable foliation. For all $y \in H^l(\mathfrak{gl}_n, O_n)$ with $l > 1$, the measure $\chi(y)$ is zero on $\mathscr{B}(\mathscr{F})$.*

COROLLARY 3.7. *Let $\mathscr{F}$ be an amenable foliation. Then all residual secondary classes $\Delta_*(y_I c_J) \in H^p(M)$ are zero for $p > 2n + 1$.*

Corollary 3.7 shows that the higher degree residual classes of $\mathscr{F}$, which are differential topological invariants, yield obstructions to $\mathscr{F}$ possessing the property of amenability which depends only on the transverse measure theory of $\mathscr{F}$. We cite two special geometric contexts to which Corollary 3.7 applies. First, recall that $\mathscr{F}$ is amenable if almost every leaf L of $\mathscr{F}$ has non-exponential growth; i.e., for every $x \in L$, $\lim_{T \to \infty} \inf (1/T) \log \mathrm{vol}\, B_L(x, T) = 0$, where $B_L(x, T)$ is the T-ball on L around x. Thus, we deduce:

COROLLARY 3.8. *If almost every leaf of $\mathscr{F}$ has non-exponential growth, then all residual secondary classes of $\mathscr{F}$ vanish in degrees greater than $2n + 1$.*

By the remarks of subsection (1.7), we also have:

COROLLARY 3.9. *Suppose $\mathcal{F}$ is defined by the C^2-action of an amenable Lie group on M. Then all residual secondary classes of $\mathcal{F}$ vanish in degrees greater than $2n + 1$.*

For a foliation of a locally homogeneous manifold defined by the cosets of a minimal parabolic subgroup, Bott and Haefliger proved (cf. [45]) that all of the secondary classes in degrees greater than $2n + 1$ are zero. As a minimal parabolic subgroup is solvable, hence amenable, Corollary 3.9 generalizes this result to arbitrary foliations defined by the action of a solvable Lie group. For the case of $\mathcal{F}$ defined by the action of the abelian group $\mathbf{R}^{m-n}$, this also generalizes M. Herman's vanishing theorem for foliations of the 3-torus by planes [18].

Let us next discuss the situation where $\mathcal{F}$ admits an absolutely continuous invariant transverse measure (a.c.i.t.) μ. Let $d\bar{x}$ be a non-vanishing smooth transverse measure to $\mathcal{F}$. Define

$$S(\mu) = \left\{ x \in M \middle| \frac{d\mu}{dx}(x) > 0 \right\},$$

so that $S(\mu)$ is saturated and $S(\mu) \in \mathcal{B}(\mathcal{F})$. If μ is finite on compact transversals to $\mathcal{F}$, then the μ-classes of $(\mathcal{F}, \mu)$ are defined as in subsection 2.3, and Theorem 3.5 yields:

COROLLARY 3.10. *Let μ be an* a.c.i.t. *measure for $\mathcal{F}$ which is finite on compact transversals. If $\mathcal{F}\,|S(\mu)$ is amenable, then the characteristic map χ_μ: $H^*(\mathfrak{gl}_n, O_n) \to H^{*+n}(M)$ of $(\mathcal{F}, \mu)$ is zero in positive degrees.*

When $\mathcal{F}$ admits an a.c.i.t. measure μ, not necessarily finite on compact transversals, the value of the Godbillon measure is restricted by:

THEOREM 3.11. *Assume $\mathcal{F}$ has an* a.c.i.t. *measure μ. For $B \in \mathcal{B}(\mathcal{F})$ with $B \subset S(\mu)$, the Godbillon measure $g(B) = 0$, and all Weil measures $\chi_B(y_1 y_I) = 0$ for $y_I \in H^*(\mathfrak{sl}_n, O_n)$.*

COROLLARY 3.12. *If $\mathcal{F}$ admits an* a.c.i.t. *measure μ with $S(\mu)$ equal to M up to a set of measure zero, then all Godbillon-Vey and generalized Godbillon-Vey classes of $\mathcal{F}$ are zero.*

COROLLARY 3.13. *Let $B \in \mathcal{B}(\mathcal{F})$ and suppose $\mathcal{F}\,|B$ has type I. Then all Weil measures $\chi_B(y_I)$ are zero, and for all of the residual secondary classes, their restrictions $\Delta_*(y_I c_J)|B = 0$.*

Proof. $\mathcal{F}\,|B$ type I implies $\mathcal{F}\,|B$ is amenable, so $\chi_B(y_I) = 0$ for all y_I with degree $y_I > 1$. Since $B/\mathcal{F}$ is a standard measure space, the measure on $B/\mathcal{F}$ induces an a.c.i.t. measure μ on B with $S(\mu) = B$ a.e., so that $\chi_B(y_1) = 0$ also. □

COROLLARY 3.14. *If $\mathcal{F}$ has type* I, *then all residual secondary classes for $\mathcal{F}$ are zero.*

This corollary yields a new relation between the geometry of a foliation and its secondary classes. In the paper [19], the first author proved that all residual classes are zero for a foliation with all leaves compact, using the Epstein filtration for such a foliation. In fact, it is well-known that a type I foliation admits a filtration with properties similar to the Epstein filtration, and this can be used to give an alternate proof of Corollary 3.14.

The last theorem relates the characteristic classes of $\mathcal{F}$ with distality of the linear holonomy of $\mathcal{F}$. We will say that the foliation $\mathcal{F}$ is *measurably linear distal* on $B \in \mathcal{B}(\mathcal{F})$ if the normal cocycle $D\gamma$ for $\mathcal{F}$ is cohomologous to a cocycle ϕ whose restriction to B takes values in a distal subgroup of $\mathrm{GL}(n, \mathbf{R})$. Recall that a subgroup $H \subset \mathrm{GL}(n, \mathbf{R})$ is *distal* if for each $h \in H$, the eigenvalues of h all have modulus one. The above definition extends to foliations a concept introduced by Fürstenberg for diffeomorphisms.

THEOREM 3.16. *Let $B \in \mathcal{B}(\mathcal{F})$ and suppose $\mathcal{F}$ is measurably linear distal on B. Then all Weil measures $\chi_B(y_I)$ are zero. If $\mathcal{F}$ is measurably linear distal on M, then all Weil measures on $\mathcal{B}(\mathcal{F})$ are zero, and all residual secondary classes for $\mathcal{F}$ vanish.*

4. Tempering procedures: Proofs of Theorems 3.1 and 3.2

4.1. *Proof of Theorem* 3.1. A $\mathrm{GL}(n, \mathbf{R})$-cocycle ϕ over the equivalence relation $\mathcal{F}$ determines an extension $\mathcal{F}_\phi$ of $\mathcal{F}$ to $X \times \mathbf{R}^n$; that is, $\mathcal{F}_\phi$ is the measurable equivalence relation on $X \times \mathbf{R}^n$ where $(x, v) \sim (y, w)$ for $(y, x) \in \mathcal{F}$ and $w = \phi(y, x)v$. If ϕ is a smooth cocycle over a manifold X, then $\mathcal{F}_\phi$ is a discrete form of a foliated vector bundle over $\mathcal{F}$ in the sense of Kamber and Tondeur [28]. The cohomology between cocycles can be viewed as a measurable fiberwise linear coordinate change. The idea of the proof of Theorem 3.1 is to introduce a new inner product $(\cdot, \cdot)_x$ on $X \times \mathbf{R}^n$, where the inner product on each fiber $\{x\} \times \mathbf{R}^n$ is defined by convolution with an exponentially decaying kernel. We then introduce a cohomology (i.e., a fiberwise coordinate change) which brings the new inner-product into the standard one. The cone property will guarantee that this coordinate change can be chosen within the group H.

For all of Section 4 we work with a fixed, but arbitrarily chosen, ε satisfying $0 < \varepsilon < 1$.

Let $(X, \mathcal{F}, d)$ be a metric equivalence relation with exponential type a, and $\phi\colon \mathcal{F} \to H$ a tempered cocycle with exponential type b. Suppose $H \subset \mathrm{GL}(n, \mathbf{R})$ and H has the cone property. Recall that $\|v\|$ is the usual Euclidean norm of a vector $v \in \mathbf{R}^n$.

LEMMA 4.1. *For all $x \in X$ and $v, w \in \mathbf{R}^n$, the sum*

$$(4.1)\quad (v, w)_x = \sum_{y \in \mathcal{F}_x} (\phi(y, x) \cdot v, \phi(y, x) \cdot w) \exp(-\{a + 2b + \varepsilon\} \cdot |y|_x)$$

is convergent and defines a measurable inner product on $X \times \mathbf{R}^n$.

Proof. It is enough to prove the lemma for the fiberwise norm $|v|_x = (v, v)_x$. By Lemma 1.2, there is a measurable function $C_1(x)$ so that

$$(4.2)\qquad \|\phi(y, x)\|^+ < C_1(x) \cdot \exp(\{b + \varepsilon/10\} \cdot |y|_x)$$

where $\| \quad \|^+$ is as in (1.10). Let us represent

$$\mathcal{F}_x = \bigcup_{n=0}^{\infty} \{B(x, n, d) \setminus B(x, n-1, d)\}.$$

Since the exponential growth rate $g(\mathcal{F}, d, x) \le a$, we again invoke Lemma 1.2 and find a measurable function $C_2(x)$ so that

$$\mathrm{Card}\{B(x, n, d)\} \le C_2(x) \cdot \exp\{(a + \varepsilon/10) \cdot n\}$$

and thus using this and (4.2), we have

$$\begin{aligned} |v|_x^2 &\le C_1(x)^2 \cdot \|v\|^2 \cdot \sum_{y \in \mathcal{F}_x} \exp(-\{a + 8\varepsilon/10\} \cdot |y|_x) \\ &\le C_1(x)^2 \cdot \|v\|^2 \cdot \sum_{n=0}^{\infty} \exp(-\{a + 8\varepsilon/10\} \cdot n) \\ &\quad \cdot (\mathrm{Card}\{B(x, n, d) \setminus B(x, n-1, d)\}) \\ &\le C_1(x)^2 C_2(x) \cdot \sum_{n=0}^{\infty} \exp(-\{7\varepsilon/10\} \cdot n) \cdot \|v\|^2 < \infty. \end{aligned}$$

The inner products $(\cdot, \cdot)_x$ are thus bounded above by a measurable function, and being the limit of a family of continuous inner products on $X \times \mathbf{R}^n$, are therefore measurable.

Since $|v|_x \ge \|v\|$ for all $x \in X$, the inner products $(\cdot, \cdot)_x$ are non-degenerate.

Let us now show that the norms $|\cdot|_x$ of Lemma 4.1 do not change very fast under the action of ϕ.

LEMMA 4.2. *For $y \in \mathscr{F}_x$ and $v \in \mathbf{R}^n$,*

$$|\phi(y, x)v|_y^2 \leq \exp(\{a + 2b + \varepsilon\} \cdot |y|_x) \cdot |v|_x^2. \tag{4.3}$$

Proof.

$$\begin{aligned}
|\phi(y, x)v|_y^2 &= \sum_{z \in \mathscr{F}_y} \|\phi(z, y) \cdot \phi(y, x)v\|^2 \cdot \exp(-\{a + 2b + \varepsilon\} \cdot |z|_y) \\
&= \sum_{z \in \mathscr{F}_y} \|\phi(z, x) \cdot v\|^2 \cdot \exp(-\{a + 2b + \varepsilon\} \cdot |z|_y) \\
&\leq \sum_{z \in \mathscr{F}_y} \|\phi(z, x) \cdot v\|^2 \cdot \exp(-\{a + 2b + \varepsilon\} \cdot \{|z|_x - |y|_x\}) \\
&= \exp(\{a + 2b + \varepsilon\} \cdot |y|_x) \cdot |v|_x^2. \qquad \square
\end{aligned}$$

The inner product $(\cdot, \cdot)_x$ is the limit of finite linear combinations of products of the form $(Av, Aw) = (v, A^*Aw)$ where $A \in H$. Using the cone property of H, we can deduce that any such combinations can again be expressed in a similar form. The assumption H is closed guarantees that the limit has a similar representation. It is also clear that such a representation can be chosen to depend on x in a measurable way. Thus, we can choose a measurable map f: $X \to H$ such that

$$(v, w)_x = (f^{-1}(x)v, f^{-1}(x)w) \qquad \text{for all } x \in X, \tag{4.4}$$

and define for $(y, x) \in \mathscr{F}$,

$$\psi(y, x) = f(y)^{-1} \cdot \phi(y, x) \cdot f(x).$$

It follows from (4.1), (4.3) and (4.4) that

$$\|\psi(y, x)v\|^2 \leq \exp(\{a + 2b + \varepsilon\} \cdot |y|_x) \cdot \|v\|^2. \tag{4.5}$$

This inequality applied to $\psi(y, x)$ *and* $\psi(x, y) = \psi(y, x)^{-1}$ implies conclusion (3.1) of Theorem 3.1.

To obtain conclusion (3.2), we simply notice that

$$f(y) = \phi(y, x) \cdot f(x) \cdot \psi(x, y)$$

and use the inequalities (4.2) and (3.1). $\square$

The method of proof of Lemma 4.1 yields somewhat more than indicated, and for subexponential systems there is a curious additional conclusion. Let $(X, \mathscr{F}, d)$ have exponential type a, and ϕ: $\mathscr{F} \to \mathrm{GL}(n, \mathbf{C})$ have exponential type b. We say a measurable field of Hermitian inner products $H = \langle \cdot, \cdot \rangle_x$ has

exponential type c if there is a measurable function $C_3(x)$ on X so that

$$\langle v, w\rangle_y \leq C_3(x) \cdot \exp(c \cdot |y|_x) \cdot \langle v, w\rangle_x, (y, x) \in \mathcal{F}. \tag{4.6}$$

Let $\mathcal{H}(\mathcal{F}, c)$ denote the real cone of such metrics on $X \times \mathbf{C}^n$.

Let $\mathcal{K}(\mathcal{F}, r)$ denote the space of measurable complex functions on $\mathcal{F}$ such that for $k \in \mathcal{K}(\mathcal{F}, r)$ and $\varepsilon > 0$, there is a measurable function $C_4(x)$ on X so that for all $(y, x) \in \mathcal{F}$,

$$|k(y, x)| \leq C_4(x) \cdot \exp(-\{r + \varepsilon\} \cdot |y|_x). \tag{4.7}$$

The same proof as in Lemma 4.1 then yields:

LEMMA 4.3. *For all $r \geq a + 2b + c$, then, there is a bilinear pairing, where ϕ: $\mathcal{F} \to \mathrm{GL}(n, \mathbf{C})$ has exponential type $\leq$ a,*

$$\hat{\phi}: \mathcal{K}(\mathcal{F}, r) \times \mathcal{H}(\mathcal{F}, c) \to \mathcal{H}(\mathcal{F}, r),$$

where for given k and $\langle \cdot, \cdot \rangle$ we define

$$\langle v, w\rangle'_x = \sum_{y \in \mathcal{F}_x} \langle \phi(y, x)v, \phi(y, x)w\rangle k(y, x). \tag{4.8}$$

The formula (4.8) makes it clear that the tempering procedure of Lemma 4.1 is obtained from a generalized Fourier transform, where the kernel k in (4.8) represents a combination of a weight used to define an appropriate L^2-space, for $x \in X$, $L^2(\mathcal{F}_x, |k(y, x)| \cdot d\nu(y, x))$ and ϕ is a connection on the fibers identifying $\{x\} \times \mathbf{C}^n$ with $\{y\} \times \mathbf{C}^n$. An element $H \in \mathcal{H}(\mathcal{F}, c)$ can be interpreted as a measurable map H: $X \to \mathrm{GL}(n, \mathbf{C})/U_n$ with a growth condition, and the cone structure on $\mathrm{GL}(n, \mathbf{C})/U_n$ is used to define the integral of such maps. In the proof of Theorem 3.1, the kernel $k(y, x) = \exp(-\{a + 2b + \varepsilon\} \cdot |y|_x)$ is a real positive function; so the convolution process yields the constant term in this generalized Fourier expansion. If $a = b = 0$, then Lemma 4.2 shows that as $\varepsilon \to 0$, the averaged metric does approximate a constant inner product with respect to ϕ. For $\mathcal{F}$ defined by a group action $\Gamma \times X \to X$, these analogies can be further pursued.

A second remark is that for $a = b = c = 0$, convolution defines a multiplication

$$\mathcal{K}(\mathcal{F}, 0) \times \mathcal{K}(\mathcal{F}, 0) \to \mathcal{K}(\mathcal{F}, 0)$$

by setting, for k; $k' \in \mathcal{K}(\mathcal{F}, 0)$

$$k''(z, x) = \sum_{y \in \mathcal{F}_x} k(z, y) \cdot k(y, x). \tag{4.9}$$

If we set $k^*(x, y) = \overline{k(y, z)}$, then $\mathcal{K}(\mathcal{F}, 0)$ becomes a *-algebra which is dense in the group-measure space von Neumann algebra obtained from $(X, \mathcal{F}, \nu)$.

LEMMA 4.4. *For* $(X, \mathcal{F}, d)$ *of subexponential growth and* $\phi\colon \mathcal{F} \to \mathrm{GL}(n, \mathbf{C})$ *a cocycle with moderate growth there is a natural action* $\hat{\phi}\colon \mathcal{K}(\mathcal{F}, 0) \times \mathcal{H}(\mathcal{F}, 0) \to \mathcal{H}(\mathcal{F}, 0)$ *of the *-algebra* $\mathcal{K}(\mathcal{F}, 0)$, *given by* (4.8).

As the space $\mathcal{H}(\mathcal{F}, 0)$ is a contractible cone, the action of $\mathcal{K}(\mathcal{F}, 0)$ on it should yield cohomological invariants of a subexponential metric equivalence relation.

4.2. *Proof of Theorem* 3.2. Let us first recall the classification up to conjugacy of maximal amenable subgroups of $\mathrm{GL}(n, \mathbf{R})$ in [38]. Let $(n_1, \ldots, n_r)$ be an ordered partition of n into the sum of positive integers: $n = n_1 + \cdots + n_r$. For every such partition we construct the group of all block matrices of the form

$$\begin{bmatrix} A_1 & * & * & * \\ 0 & \cdot & & * \\ 0 & \cdot & & * \\ 0 & 0 & 0 & A_r \end{bmatrix} \tag{4.10}$$

where each diagonal block A_i of size $n_i \times n_i$ consists of the scalar multiples of the $n_i \times n_i$-orthogonal matrices, so that $A_i = \mathbf{R}^* \cdot O(n_i)$. The elements above the diagonal blocks are arbitrary, and those below are zero. Every maximal amenable subgroup of $\mathrm{GL}(n, \mathbf{R})$ is conjugate to one of these groups.

Let $(X, \mathcal{F}, d)$ be a metric equivalence relation of exponential type a, and $\phi\colon \mathcal{F} \to H$ a cocycle with exponential type b, and H a maximal amenable subgroup of $\mathrm{GL}(n, \mathbf{R})$. We can assume that H has the canonical form (4.10) for appropriate $(n_1, \ldots, n_r)$.

The cocycle ϕ decomposes into blocks

$$\begin{bmatrix} \phi_1(y, x) & \phi_{12}(y, x) & \cdots & \phi_{1r}(y, x) \\ 0 & \phi_2(y, x) & \cdots & \phi_{2r}(y, x) \\ \cdot & & & \\ \cdot & & & \\ 0 & 0 & \cdots & \phi_r(y, x) \end{bmatrix} \tag{4.11}$$

where $\phi_i(y, x) \in \mathrm{GL}(n_i, \mathbf{R})$ and $\phi_{ij} \in \mathrm{End}(\mathbf{R}^{n_j}, \mathbf{R}^{n_i})$. Note that each diagonal block ϕ_i is again a cocycle with values in the group A_i. Consequently,

$$\|\phi_i(y, x)\|^{n_i} = \det \phi_i(y, x). \tag{4.12}$$

Define a scalar-valued cocycle λ_i over $\mathcal{F}$ by

$$\lambda_i(y, x) = \{\det \phi_i(y, x)\}^{1/n_i}, \ (y, x) \in \mathcal{F}.$$

The hypothesis that ϕ has exponential type b and the estimate $\|\phi_i(y, x)\| \leq \|\phi(y, x)\|$ imply that λ_i has exponential type $\leq$ b. By Theorem 3.1 applied to

λ_i, there is a measurable function $T_i(x)$ on X so that

$$\mu(y, x) = T_i(y)^{-1} \cdot \lambda_i(y, x) \cdot T_i(x) \tag{4.13}$$

is a $(2b + a + \varepsilon)$-tempered cocycle, and T_i has exponential type $(a + 3b + \varepsilon)$ along orbits. Let $T(x)$ denote the $n \times n$-diagonal matrix whose first n_1-diagonal entries are $T_1(x)$, the next n_2-diagonal entries are $T_2(x)$, and so forth with the last n_r-diagonal entries equal to $T_r(x)$. Define

$$\theta(y, x) = T(y)^{-1} \cdot \phi(y, x) \cdot T(x), \tag{4.14}$$

and write θ in the block form, for $(y, x) \in \mathscr{F}$,

$$\theta(y, x) = \begin{bmatrix} \theta_1(y, x) & \theta_{12}(y, x) & \cdots & \theta_{1r}(y, x) \\ 0 & \theta_2(y, x) & \cdots & \theta_{2r}(y, x) \\ \cdot & & & \\ \cdot & & & \\ 0 & 0 & \cdots & \theta_r(y, x) \end{bmatrix}.$$

Note that each θ_i: $\mathscr{F} \to A_i$ is a cocycle.

LEMMA 4.5. *For $1 \leq i \leq r$, θ_i is $(2b + a + \varepsilon)$-tempered.*

Proof. $\theta_i(y, x) = T_i(y)^{-1} \cdot \phi_i(y, x) \cdot T_i(x)$, where $\phi_i(y, x)$ is $\lambda_i(y, x)$ times an orthogonal $n_i \times n_i$-matrix. Thus, $\|\theta_i(y, x)\| = \mu_i(y, x)$ which is $(2b + a + \varepsilon)$-tempered by the choice of T_i. □

LEMMA 4.6. *For all $1 \leq j < i \leq r$, there is a measurable function $C(x)$ on X such that*

$$\|\theta_{ij}(y, x)\| \leq C(x) \cdot \exp(\{a + 4b + \varepsilon\} \cdot |y|_x). \tag{4.15}$$

Proof.

$$\|\theta_{ij}(y, x)\| \leq \|\theta(y, x)\| \leq \|T(y)\|^+ \cdot \|\phi(y, x)\|^+ \cdot \|T(x)\|^+$$

and (4.15) now follows from this, the growth hypothesis on ϕ and the asymptotic estimates on the T_i. □

COROLLARY 4.7. *The cocycle θ has exponential type $\leq (a + 4b + \varepsilon)$.* □

By Corollary 4.7 and the method of proof of Lemma 4.1, we can now define

$$\lambda(x) = \varepsilon^{-1} \sum_{y \in \mathscr{F}_x} \|\theta(y, x)\|^+ \cdot \exp(-\{2a + 4b + 2\varepsilon\} \cdot |y|_x), \tag{4.16}$$

and we immediately obtain:

LEMMA 4.8. $\|\lambda(y)\|^+ \leq \|\lambda(x)\|^+ \cdot \exp(\{2a + 4b + 2\varepsilon\} \cdot |y|_x)$ *for* $(y, x) \in \mathscr{F}$. □

Let $\Lambda(x)$ be the diagonal matrix whose first n_1-diagonal entries are equal to 1, the next n_2-diagonal entries are equal to $\lambda(x)^{-2}$, the next n_3-diagonal entries are $\lambda(x)^{-4}$, and so forth, with the last n_r-diagonal entries equal to $\lambda(x)^{2-2r}$. Then define

$$\psi(y,x) = \Lambda(y)^{-1} \cdot \theta(y,x) \cdot \Lambda(x). \tag{4.17}$$
$$= (T(y) \cdot \Lambda(y))^{-1} \cdot \phi(y,x) \cdot (T(x) \cdot \Lambda(x)).$$

Since $T(x) \cdot \Lambda(x)$ is a diagonal matrix in H, ψ is again H-valued. We will show that ψ is tempered, the first assertion of Theorem 3.2, and then establish (3.3). Write ψ in block form

$$\psi(y,x) = \begin{bmatrix} \psi_1(y,x) & \psi_{12}(y,x) & \cdots & \psi_{1r}(y,x) \\ 0 & \psi_2(y,x) & & \psi_{2r}(y,x) \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \psi_r(y,x) \end{bmatrix}.$$

It will suffice to show that the entries $\psi_i(y,x)$ and $\psi_{ij}(y,x)$ are tempered, and obtain a uniform exponential estimate. We begin with the diagonal blocks.

LEMMA 4.9. *For each* $1 \le i \le r$,

$$\|\psi_i(y,x)\|^+ \le \exp(\{(4i-3)a + (8i-6)b + (4i-3)\varepsilon\} \cdot |y|_x). \tag{4.18}$$

Proof. From (4.17) we have

$$\psi_i(y,x) = \left(\lambda(y)^{2-2i}\right)^{-1} \cdot \theta_i(y,x) \cdot \lambda(x)^{2-2i},$$

so that

$$\|\psi_i(y,x)\| \le \left(\|\lambda(y) \cdot \lambda(x)^{-1}\|^+\right)^{2i-2} \cdot \|\theta_i(y,x)\|^+$$

and (4.18) follows from Lemmas 4.5 and 4.8. □

For the off-diagonal terms it suffices to estimate the matrix norms. Note that

$$\psi_{ij}(y,x) = \lambda(y)^{2i-2} \cdot \theta_{ij}(y,x) \cdot \lambda(x)^{2-2j}$$

so that

$$\|\psi_{ij}(y,x)\| \le \|\theta_{ij}(y,x)\| \cdot \lambda(x)^{2(i-j)} \cdot \left(\lambda(y) \cdot \lambda(x)^{-1}\right)^{2i-2} \tag{4.19}$$
$$\le \lambda(x)^{-2} \cdot \|\theta(y,x)\| \cdot \exp(\{2i-2\} \cdot \{2a + 4b + 2\varepsilon\} \cdot |y|_x)$$

by Lemma 4.8 and since $\lambda(x)^{-1} < \varepsilon \le 1$.

LEMMA 4.10. *For* $(y, x) \in \mathscr{F}$,

$$\lambda(x)^{-2} \cdot \|\theta(y, x)\| \leq \varepsilon^2 \cdot \exp(\{2a + 4b + 2\varepsilon\} \cdot |y|_x).$$

Proof. It suffices to show

$$\lambda(x) \cdot \|\theta(y, x)\|^{-1} \geq \varepsilon^{-1} \cdot \exp(\{2a + 4b + 2\varepsilon\} \cdot |y|_x),$$

which is immediate from (4.16). □

The proof of Theorem 3.2 can now be completed. By Lemma 4.9, the diagonal terms of ψ are uniformly tempered with exponent no greater than $(4n - 3)a + (8n - 6)b + (4n - 3)\varepsilon$, as $i \leq n$. By Lemma 4.10 and the estimate (4.19), the off-diagonal terms of ψ are uniformly tempered with exponent no greater than $(2n - 3)(2a + 4b + 2\varepsilon)$. It follows that ψ is uniformly tempered with exponent no greater than the former estimate. As $\varepsilon > 0$ was arbitrary, this yields (3.3). □

5. Proof of Theorem 0.1 in the smooth case

The purpose of this section is two-fold: First, we obtain a local formula for calculating the Weil measures, where the abstract definition of the forms $\bar{y}_i$ in Section 2 is replaced with an explicit Lie algebraic construction. This essentially reduces the calculation of Weil measures to the level of calculating leaf classes (cf. [19], [21]). Secondly, we use the local form of the Weil measures given in Corollary 5.7, along with a result from [21] to deduce Theorem 0.1 in the smooth case. The proof of Theorem 0.1 in full generality consists of removing the smoothness restrictions made in this section.

5.1. *A local formula.* Let $\mathscr{F}$ be a fixed codimension-n, C^2-foliation on an oriented m-dimensional manifold M without boundary. Let π: $P \to M$ denote the smooth principal $\mathrm{GL}(n, \mathbf{R})$-bundle of frames of the normal bundle $Q \to M$ to $\mathscr{F}$, where $\mathrm{GL}(n, \mathbf{R})$ acts on the right on P.

The relative truncated Weil algebra $W(\mathfrak{gl}_n, O_n)_n$ is the O_n-basic subalgebra of the full Weil algebra, the differential graded algebra (d.g.a.)

$$W(\mathfrak{gl}_n)_n = \Lambda \mathfrak{gl}_n^* \otimes S(\mathfrak{gl}_n^*)_n,$$

where $S(\mathfrak{gl}_n^*)$ is the commutative symmetric algebra on the dual to the Lie algebra $\mathfrak{gl}_n$, with each $x \in \mathfrak{gl}_n^*$ considered to have degree 2, and the differential is defined as in Chapter 4 of [28]. The subscript n denotes the quotient algebra obtained by truncating the free algebra $S(\mathfrak{gl}_n^*)$ in degrees above $2n$. For further details, see e.g., (Chapter 4, [28]).

The algebra $W(\mathfrak{gl}_n, O_n)_n$ contains a d.g. subalgebra isomorphic to WO_n, and by abuse of notation we identify WO_n with this subalgebra: The element

$1 \otimes c_i \in WO_n$ is identified with $c_i \in S^i(\mathfrak{gl}_n^*)$, and $y_i \otimes 1 \in WO_n$ is identified with $y_i \in \Lambda(\mathfrak{gl}_n^*) \otimes S(\mathfrak{gl}_n^*)_n$. It is a basic fact that this inclusion induces an isomorphism i_*: $H^*(WO_n) \cong H^*(W(\mathfrak{gl}_n, O_n)_n)$ (cf. Chapter 5, [28]).

A basic connection θ^b on the normal bundle Q determines a $\mathrm{GL}(n, \mathbf{R})$-equivariant map, also denoted

$$\theta^b\colon TP \to \mathfrak{gl}_n,$$

from which we obtain a d.g.a. map

$$k(\theta^b)\colon W(\mathfrak{gl}_n, O_n)_n \to A^*(P)^{O_n} \cong A^*(P/O_n)$$

on O_n-basic forms.

A smooth (respectively, measurable) metric h on Q corresponds to a smooth (respectively, measurable) section s_h: $M \to P/O_n$ of the quotient bundle whose fibers are the symmetric space $S(n) = \mathrm{GL}(n, \mathbf{R})/O_n$. Thus, a smooth metric on Q yields a d.g.a. map

$$ds_h\colon A^*(P/O_n) \to A^*(M).$$

PROPOSITION 5.1. *Given a basic connection θ^b and a metric h on Q, with metric connection θ^h, there is a commutative diagram*

$$\begin{array}{ccc} H^*(WO_n) & \xrightarrow{i_*} & H^*(W(\mathfrak{gl}_n, O_n)_n) \\ \Big\downarrow{\scriptstyle \Delta(\theta^b, \theta^h)} & & \Big\downarrow{\scriptstyle k(\theta^b)} \\ H^*(M) & \xleftarrow{ds_h} & H^*(P/O_n) \end{array} \tag{5.1}$$

where $\Delta(\theta^b, \theta^h)$ is defined in Section 2.

Proof. This is a special case of Theorem 5.95 and Remark 5.112 of [28]. □

COROLLARY 5.2. *For all $y_I \in H^*(\mathfrak{gl}_n, O_n)$, $\bar{y}_I = \Delta(\theta^b, \theta^h)(y_I \otimes 1) = ds_h \circ k(\theta^b)(y_I)$.*

This corollary enables one to analyze separately the contributions of θ^b and h to the forms $\bar{y}_I$ on M. For this paper, we restrict our attention to only one component of the form $\bar{y}_I$. Let τ: $W(\mathfrak{gl}_n, O_n)_n \to \Lambda(\mathfrak{gl}_n^*)$ denote the projection onto the first factor, and set $\tau_I = \tau(y_I)$. For degree $y_I > 1$, the element y_I is the sum of the purely exterior component τ_I with components containing "curvature terms" from $S(\mathfrak{gl}_n^*)_n$. These latter components are exceedingly difficult to analyze, and the success of the Weil measure approach is due to its dependence only on the images of the components τ_I, which are relatively easy to study.

LEMMA 5.3. *The form τ_I is O_n-basic on $\mathfrak{gl}_n$, closed with respect to the Hochshild differential on $\Lambda(\mathfrak{gl}_n^*)$ and the set $\{\tau_1, \tau_3, \ldots, \tau_{n'}\}$ is an exterior algebra basis for $H^*(\mathfrak{gl}_n, O_n)$.*

Proof. This is classical (cf. [62]). □

A foliation chart (U, Φ) for $\mathcal{F}$ consists of an open set $U \subset M$ and a diffeomorphism, for $I = (-1, 1)$, Φ: $U \to I^{m-n} \times I^n$ such that for each $x \in I^n$, the set $(L \cap U)_x = \Phi^{-1}(I^{m-n} \times \{x\})$ is a connected component of the leaves of $\mathcal{F}|U$. We call $(L \cap U)_x$ the *plaque* of $\mathcal{F}$ through x. We say (U, Φ) is *regular* if there is an open set $W \subset M$ with the closure $\overline{U} \subset W$, and a diffeomorphism $\overline{\Phi}$: $W \to (-2, 2)^m$ extending Φ. The foliated manifold M always admits a locally finite covering by regular foliation charts. For the rest of this paper we fix a choice of such covering,

$$\{(U_\alpha, \Phi_\alpha) | \alpha \in A\}.$$

For $U \subset M$ open, set $P|U = \pi^{-1}(U)$ with the restricted projection π: $P|U \to U$. A *local smooth* (respectively, measurable) *framing* for Q over U is a smooth (respectively, measurable) section s: $U \to P|U$. From s, we obtain an isomorphism of bundles over U, T_s: $U \times \mathrm{GL}(n, \mathbf{R}) \to P|U$. Given a metric s_h: $M \to P/O_n$, a *local orthonormal framing* is a local framing, orthonormal with respect to h. Equivalently, it is a section s: $U \to P|U$ so that the diagram commutes:

$$\begin{array}{ccc} P|U & \longrightarrow & P/O_n|U \\ & {\scriptstyle s}\nwarrow \quad \nearrow{\scriptstyle s_h|U} & \\ & U & \end{array} \qquad (5.2)$$

For a foliation chart (U, Φ), the bundle $P|U$ is trivial, so a local orthonormal framing of Q exists over U for any choice of metric h.

For $\alpha, \beta \in A$, we say the pair (α, β) is *admissible* if $U_{\alpha\beta} = U_\alpha \cap U_\beta$ is not empty. For each $\alpha \in A$, set $P_\alpha = \pi^{-1}(U_\alpha)$. The given metric s_h restricts to local sections $s_{h,\alpha}$: $U_\alpha \to P_\alpha/O_n$, and these lift to local orthonormal framings s_α: $U_\alpha \to P_\alpha$ which satisfy the following compatibility condition:

Definition 5.4. A collection $\{t_\alpha$: $U_\alpha \to P_\alpha | \alpha \in A\}$ of measurable local framings is said to be O_n-related if for each admissible pair (α, β), there is a measurable function $o_{\alpha\beta}$: $U_{\alpha\beta} \to O_n$ such that $t_\alpha(x) \cdot o_{\alpha\beta}(x) = t_\beta(x)$ for all $x \in U_{\alpha\beta}$.

LEMMA 5.5. *An O_n-related collection of measurable local framings $\{t_\alpha\colon U_\alpha \to P_\alpha \mid \alpha \in A\}$ determines a global section $t\colon M \to P/O_n$ such that each t_α is a lift of $t|U_\alpha$.*

Fixing a metric h on TM defines an orthonormal splitting $TM = T\mathcal{F} \oplus Q$, and an exterior algebra isomorphism $\Lambda T^*M \cong \Lambda T\mathcal{F}^* \wedge \Lambda Q^*$. The *leafwise exterior derivative on functions*, $d_{\mathcal{F}}$, is defined by means of this splitting as the component of the exterior derivative d which lies in $T\mathcal{F}^*$. Note that $d_{\mathcal{F}}f$ can be defined for any function f on M such that the restriction of f to leaves of $\mathcal{F}$ is C^1 as follows. The *differential* of a C^1 map $f\colon M \to N$ of manifolds will be denoted $Df\colon TM \to TN$. The *leafwise differential* of a function $f\colon M \to \mathbf{R}$ is defined by the composition

$$D_{\mathcal{F}}f\colon T\mathcal{F} \subset TM \xrightarrow{Df} T\mathbf{R} \cong \mathbf{R} \oplus \mathbf{R} \xrightarrow{\pi_2} \mathbf{R}.$$

Note this is independent of the metric, and $D_{\mathcal{F}}f$ extends to functions which are only leafwise C^1. Then $d_{\mathcal{F}}f \overset{\text{def}}{=} D_{\mathcal{F}}f$ for f leafwise smooth.

The main result of this subsection is then:

PROPOSITION 5.6. *Let (U, Φ) be a foliation chart and $s\colon U \to P|U$ a local orthonormal framing. Then on U we have*

$$\Delta(\theta^b, \theta^h)(y_I) = A_s^*(\tau_I) \bmod A(U, \mathcal{F})$$

where $A_s\colon T\mathcal{F}|U \to \mathfrak{gl}_n$ is defined by (5.5) below, and A_s^* is the induced map on exterior forms.

Proof. By Corollary 5.2, we have

$$\begin{aligned}\Delta(\theta^b, \theta^h)(y_I) &= ds_h\big(k(\theta^b)(y_I)\big) \\ &= ds_h\big(k(\theta^b)(\tau_I)\big) \bmod A(M, \mathcal{F}),\end{aligned}$$

the second equality a consequence of the identity $k(\theta^b)(\phi) \wedge \pi^*(A(M, \mathcal{F})) = 0$ for all $\phi \in S(\mathfrak{gl}_n^*)_n$, which follows from Theorem 4.23 of [28].

We calculate $\bar{\tau}_I = ds_h(k(\theta^b)(\tau_I))$ in local coordinates on U. It suffices to evaluate $\bar{\tau}_I$ on vectors tangent to $\mathcal{F}$; so we consider the restrictions $\bar{\tau}_I|L$ for $L \subset U$ a plaque. The connection θ^b is flat when restricted to $\pi\colon P|L \to L$ and L is contractible, so θ^b defines a product structure $P|L \cong I^{m-n} \times \mathrm{GL}(n, \mathbf{R})$. That is, there are coordinates (y, g) on $P|L$ with $y \in I^{m-n}$ and $g \in \mathrm{GL}(n, \mathbf{R})$ so that $\pi(y, g) = y$ in these coordinates, and θ^b is given by:

$$\theta^b\colon T(P|L) \to \mathfrak{gl}_n;\ \theta^b(\partial/\partial y, X) = X \tag{5.3}$$

where X is a left-invariant vector field on $\mathrm{GL}(n, \mathbf{R})$. The section $s|L\colon L \to P|L$ is given in coordinates as $s(y) = (y, g(y))$, where $g\colon L \to \mathrm{GL}(n, \mathbf{R})$ is smooth.

It follows that $\theta^b \circ D_{\mathscr{F}}$: $TL \to \mathfrak{gl}_n$ is given by

$$\theta^b \circ D_{\mathscr{F}} s(\partial/\partial y|_y) = \theta^b(\partial/\partial y|_y, D_{\mathscr{F}} g(\partial/\partial y)|_{g(y)}) = g(y)^{-1} \cdot D_{\mathscr{F}} g(\partial/\partial y)|_{g(y)}. \tag{5.4}$$

Define A_s on $T_y\mathscr{F}$ by the rule

$$A_s(\partial/\partial y|y) = g(y)^{-1} \cdot D_{\mathscr{F}} g(\partial/\partial y)|_{g(y)}, \tag{5.5}$$

or more simply, $A_s = g^{-1} \cdot D_{\mathscr{F}} g$. From (5.4) and (5.5) we obtain

$$\Delta(\theta^b, \theta^h)(y_I) = A_s^*(\tau_I)$$

as claimed. Note that τ_I is O_n-basic, so the form $A_s^*(\tau_I)$ on each plaque of $\mathscr{F}|U$ is independent of the choice of local Riemannian framing, s, compatible with the metric, h, on Q. □

Let $\{\lambda_\alpha | \alpha \in A\}$ be a partition-of-unity subordinate to the open cover $\{U_\alpha\}$ of M. For the given metric h on Q, choose local orthonormal framings $\{s_\alpha | \alpha \in A\}$ as above. Then Proposition 5.6 directly yields:

COROLLARY 5.7. *For $B \in \mathscr{B}(\mathscr{F})$ and $[\varphi] \in H_c^r(M, \mathscr{F})$, where degree $(y_I) + r = m$,*

$$\chi_B(y_I)[\varphi] = \sum_{\alpha \in A} \int_{U_\alpha \cap B} \lambda_\alpha \cdot A_{s_\alpha}^*(\tau_I) \wedge \varphi. \tag{5.6}$$

Remark 5.8. Observe that the integral in (5.6) is still defined whenever $\lambda_\alpha \cdot A_{s_\alpha}^*(\tau_I)$ is an integrable form on U_α for all $\alpha \in A$.

5.2. *Proof in the smooth case.*

Definition 5.9. A smooth foliation $\mathscr{F}$ has a *smooth amenable reduction* if there is a maximal amenable subgroup $H \subset \mathrm{GL}(n, \mathbf{R})$ and a collection $\{s_\alpha | \alpha \in A\}$ of smooth, O_n-related local framings such that the maps A_{s_α}: $T\mathscr{F}|U_\alpha \to \mathfrak{gl}_n$ take values in the Lie algebra $\mathfrak{h}$ of H. Equivalently, in the terminology of [28], P admits a foliated H-reduction. In this case, we can always assume that the $o_{\alpha\beta}$ take values in the maximal compact subgroup $K = O_n \cap H$ of H, where H is in the canonical form of Section 4.

THEOREM 5.10. *Suppose $\mathscr{F}$ is a C^2-foliation which admits a smooth amenable reduction. Then for y_I with $\deg(y_I) > 1$, the Weil measure $\chi(y_I)$ is zero.*

Proof. By Corollary 5.7, it suffices to show

$$\sum_{\alpha \in A} \int_{B \cap U_\alpha} \lambda_\alpha \cdot A_{s_\alpha}^*(\tau_I) \wedge \varphi = 0$$

for all $B \in \mathscr{B}(\mathscr{F})$, $\varphi \in H_c^r(M, \mathscr{F})$. The key point is provided by Proposition 3.9 of [21].

LEMMA 5.11. *Let $H \subset \mathrm{GL}(n, \mathbf{R})$ be a maximal amenable subgroup, $K \subset H$ its maximal compact subgroup and $\mathfrak{h}$ its Lie algebra. Then the restriction map*

$$H^p(\mathfrak{gl}_n, O_n) \to H^p(\mathfrak{h}, K)$$

is zero for $p > 1$. In particular, for τ_I of degree p, there is a $(p-1)$-form σ_I on $\mathfrak{h}$, which is K-basic and $\tau_I|\mathfrak{h} = d_{\mathfrak{h}}(\sigma_I)$.

By Lemma 5.11, for each $\alpha \in A$, there is a $(p-1)$-form $\sigma_{I,\alpha} = A_{s_\alpha}^*(\tau_I)$ on U_α so that $d_{\mathscr{F}}(\sigma_{I,\alpha}) = \tau_I|U_\alpha$. Moreover, s_α and s_β are K-related on $U_{\alpha\beta}$ and σ_I is K-basic, so that $\sigma_{I,\alpha} = \sigma_{I,\beta}$ on $U_{\alpha,\beta}$. Thus, there is a well-defined form σ_I on M whose restriction to each U_α gives $\sigma_{I,\alpha}$. Now consider

$$\begin{aligned}\chi_B(y_I)[\varphi] &= \sum_{\alpha \in A} \int_{B \cap U_\alpha} \lambda_\alpha \cdot A_{s_\alpha}^*(\tau_I) \wedge \varphi \\ &= \sum_{\alpha \in A} \int_{B \cap U_\alpha} \lambda_\alpha \cdot d_{\mathscr{F}}(\tau_I) \wedge \varphi = 0\end{aligned}$$

by Stokes' theorem for foliations (§2, [17]). □

Remark 5.12. Theorem 5.10 can be proved without introducing the local forms $A_{s_\alpha}^*(\tau_I)$, by using the functoriality of the Weil homomorphism. A reduction $P' \subset P$ where P' has structure group H determines a commutative diagram:

$$\begin{array}{ccc} H^*(W(\mathfrak{gl}_n, O_n)_n) & \xrightarrow{r_*} & H^*(W(\mathfrak{h}, K)_n) \\ \downarrow{\scriptstyle k(\theta^b)} & & \downarrow{\scriptstyle k(\theta^b)} \\ H^*(P) & \longrightarrow & H^*(P') \\ {\scriptstyle ds_h}\searrow & & \swarrow{\scriptstyle ds_h} \\ & H^*(M) & \end{array} \tag{5.7}$$

A calculation using Lemma 5.11 above shows that r_* annihilates the residual classes of degree $> 2n + 1$. This calculation is essentially the same as that used to prove Theorem 5.10 above. Note that the Weil algebra approach via diagram (5.7) breaks down completely for connections θ^b on Q which are only measurable transversally. It is in order to handle this generality that the Weil measures are introduced, and this will allow the techniques of Sections 6 and 7 to be developed.

6. Measurable Weil theory: Smoothing the normal cocycle

We introduce the metrics, norms and pseudo-norms on forms needed for this and the next section. Fix a Riemannian metric h on TM, and give the exterior bundle ΛTM the induced metric. The norm of $v \in \Lambda^p T_x M$ is denoted $|v|_x$, with $\|\varphi\|$ the corresponding sup-norm on p-forms. Give $\mathrm{GL}(n, \mathbf{R})$ a left-invariant Riemannian metric which is also invariant for the right O_n-action, and so that O_n has total volume 1. This metric projects to a $\mathrm{GL}(n, \mathbf{R})$-invariant metric on the symmetric space $S(n) = \mathrm{GL}(n, \mathbf{R})/O_n$. Let ρ and $\bar{\rho}$ denote the distance functions on $\mathrm{GL}(n, \mathbf{R})$ and $S(n)$ corresponding to these Riemannian metrics. The fibration π: $P/O_n \to M$ has typical fiber $S(n)$, and the metric $\bar{\rho}$ on $S(n)$ admits a natural smooth extension to a family of metrics $\{\bar{\rho}_x | x \in M\}$ on the fibers of π.

The metric h defines an embedding $Q \to TM$ of normal vectors perpendicular to $T\mathcal{F}$. Let $\bar{h}$ denote the induced metric on Q, which defines a section $\bar{s}_h$: $M \to P/O_n$. For any open set $U \subset M$ let $\bar{s} \in \Gamma(U, P/O_n)$ be a bounded measurable metric on $Q|U$. We define the essential sup norm of $\bar{s}$ to be

$$\|\bar{s}\|_U = \operatorname*{ess\,sup}_{x \in U} \bar{\rho}_x(\bar{s}(x), \bar{s}_h(x)) \tag{6.1}$$

and for $U = M$, set $\|\bar{s}\| = \|\bar{s}\|_M$. Note that (6.1) defines a pseudo-norm on the local framings s: $U \to P|U$ by first considering the O_n-reduction, $\bar{s} \in \Gamma(U, P/O_n)$, then setting

$$\|s\|_U \overset{\text{def}}{=} \|\bar{s}\|_U. \tag{6.2}$$

For a matrix-valued, bounded measurable p-form ω: $\Lambda^p TM \to \mathfrak{gl}_n$, define the norm of ω by

$$\|\omega\| = \operatorname*{ess\,sup}_{x \in M} \left\{ \sup_{0 \neq v \in \Lambda^p T_x M} \frac{\|\omega(v)\|}{|v|_x} \right\} \tag{6.3}$$

and the $\mathcal{F}$-pseudonorm of ω by

$$\|\omega\|_{\mathcal{F}} = \operatorname*{ess\,sup}_{x \in M} \left\{ \sup_{0 \neq v \in \Lambda^p T_x \mathcal{F}} \frac{\|\omega(v)\|}{|v|_x} \right\}. \tag{6.4}$$

For a form ω defined only on an open set $U \subset M$, both the norm and $\mathcal{F}$-pseudo norm of ω on U are again defined as in (6.3) and (6.4). We will abuse notation by using the same notation as in (6.3) and (6.4) for the norms restricted to U when the context makes the domain clear.

The purpose of this section is to prove:

THEOREM 6.1. *Let* $\{s_\alpha | \alpha \in A\}$ *be an* O_n*-related collection of local measurable framings for* Q *which satisfy for each* $\alpha \in A$:

$$s_\alpha | L \text{ is } C^2 \text{ for each plaque } L \subset U_\alpha, \tag{6.5}$$

and there is a constant K_α *with*

$$\|s_\alpha\| \leq K_\alpha, \tag{6.6}$$

$$\|A_{s_\alpha}\|_{\mathscr{F}} \leq K_\alpha. \tag{6.7}$$

Then the collection can be used to calculate the Weil measures of $\mathscr{F}$. *That is, given* $y_I \in H^p(\mathfrak{gl}_n, O_n)$, $[\varphi] \in H^{m-p}(M, \mathscr{F})$ *and* $B \in \mathscr{B}(\mathscr{F})$,

$$\chi_B(y_I)[\varphi] = \sum_{\alpha \in A} \int_{B \cap U_\alpha} \lambda_\alpha \cdot A^*_{s_\alpha}(\tau_I) \wedge \varphi. \tag{6.8}$$

Proof. Note the integral in (6.8) is well-defined, for each $\lambda_\alpha \cdot A^*_{s_\alpha}(\tau_I) \wedge \varphi$ is an essentially bounded measurable m-form on $U_\alpha \cap B$ by (6.7), (5.5) and the remark following Lemma 5.5.

The main point of the proof is to construct a sequence of smooth, O_n-related local framings

$$\{s_{\alpha,i} | \alpha \in A, i = 1, 2, \dots\}$$

such that for the projections $\bar{s}_{\alpha,i} = s_{\alpha,i} \bmod O_n$ to the symmetric space $S(n)$,

$$\bar{s}_{\alpha,i} \to \bar{s}_\alpha \text{ a.e. } U_\alpha, \tag{6.9}$$

$$A^*_{s_{\alpha,i}}(\tau_I) \to A^*_{s_\alpha}(\tau_I) \text{ a.e. } U_\alpha, \tag{6.10}$$

$$\|A_{s_{\alpha,i}}\|_{\mathscr{F}} \leq K'_\alpha, \text{ for all } i \geq 0$$

for some new constants K'_α. We call such a sequence a *tempered smoothing* of the measurable framings $\{s_\alpha | \alpha \in A\}$.

The symmetric space $S(n)$ is naturally identified with the cone of positive definite matrices in $\mathrm{GL}(n, \mathbf{R})$. This allows us to consider an $S(n)$-valued function as being matrix valued, and to perform integration of such functions. In particular, integration of $\bar{s}_\alpha$ with respect to a probability distribution on U_α yields a function with values in $S(n)$ again.

Choose a sequence of smooth kernels $\{k_i | i = 1, 2, \dots\}$ on $I^n = (-1, 1)^n$ which converge to the δ-function, and whose supports converge uniformly to the diagonal in $I^n \times I^n$. Extend each k_i to a function on $I^{m-n} \times I^n \times I^n$ which is constant on the first factor, and via the identification $U_\alpha \cong I^{m-n} \times I^n$, lift each k_i to a kernel $k_{\alpha,i}$ on U_α. Let $s_{\alpha,i}$ denote the convolution (in the transversal direction) of $\bar{s}_\alpha$ with $k_{\alpha,i}$. Note that $s_{\alpha,i}$ is a framing of $Q | U_\alpha$ via its identifica-

tion with an element of $\mathrm{GL}(n, \mathbf{R})$, and denote by $\bar{s}_{\alpha,i}$ the formal identification of $s_{\alpha,i}$ as an element of $S(n)$. By (6.5), the framings $\{s_{\alpha,i}\}$ are smooth on U_α, and the convergence in measure of $\{\bar{s}_{\alpha,i} | i = 1, 2, \dots\}$ to $\bar{s}_\alpha$ is immediate. The bound (6.6) implies that the inverses $\{s_{\alpha,i}^{-1}\}$ admit a bound K''_α, uniform in i. Moreover, modulo the adjoint action of O_n, the inverses $\{s_{\alpha,i}^{-1} | i = 1, 2, \dots\}$ also converge in measure to s_α^{-1}.

To prove that (6.10) holds, first observe that (6.6) implies $\bar{s}_\alpha$ has a uniform bound on U_α; so by (6.7), $D_{\mathscr{F}}(\bar{s}_\alpha)$ also has a uniform bound. The convolution of $\bar{s}_\alpha$ with $k_{\alpha,i}$ does not involve the leaf directions, so there is again a uniform bound on the differentials $\{D_{\mathscr{F}}(\bar{s}_{\alpha,i}) | i = 1, 2, \dots\}$. As the $\{s_{\alpha,i}^{-1}\}$ are uniformly bounded in i, we conclude that $\|A^*_{s_{\alpha,i}}\|_{\mathscr{F}} \leq K'_\alpha$ for some K'_α. Similarly, the differentials $D_{\mathscr{F}}(\bar{s}_{\alpha,i})$ converge in measure to $D_{\mathscr{F}}(\bar{s}_\alpha)$. Thus, modulo the adjoint action of O_n, $\{A_{s_{\alpha,i}} | i = 1, 2, \dots\}$ converges in measure to A_{s_α}. As the form τ_I is O_n-basic, we obtain

$$A^*_{s_{\alpha,i}}(\tau_I) \to A^*_{s_\alpha}(\tau_I) \text{ a.e. } U_\alpha \tag{6.11}$$

and (6.10) follows.

Now apply the dominated convergence theorem, Corollary 5.7 and (6.10) to conclude

$$\begin{aligned}
\chi_B(y_I)[\varphi] &= \lim_{i \to \infty} \sum_{\alpha \in A} \int_{U_\alpha \cap B} \lambda_\alpha \cdot A^*_{s_{\alpha,i}}(\tau_I) \wedge \varphi \\
&= \sum_{\alpha \in A} \int_{U_\alpha \cap B} \lambda_\alpha \cdot \lim_{i \to \infty} A^*_{s_{\alpha,i}}(\tau_I) \wedge \varphi \\
&= \sum_{\alpha \in A} \int_{U_\alpha \cap B} \lambda_\alpha \cdot A^*_{s_\alpha}(\tau_I) \wedge \varphi. \qquad \square
\end{aligned}$$

7. Measurable Weil theory: Tempering the normal coboundary

This section proves two essential results needed for the proof of Theorem 3.5. We begin by constructing the linear holonomy cocycle $D\gamma$ over the foliation pseudogroup Γ. It is an easy consequence of the semi-simplicial construction of the Chern-Weil homomorphism (cf. Chapter 8, [28]) that the Weil measures of $\mathscr{F}$ depend only on the smooth cohomology class of $D\gamma$ over Γ. We will show a much stronger result, that the Weil measures depend only on the measurable cohomology class of $D\gamma$ over the measurable equivalence relation $(X, \mathscr{F}, d)$ defined by the foliation. We prove in subsection 7.1 that the cocycle $D\gamma$ is almost everywhere equal to a cocycle lifted from $\mathscr{F}$. This is equivalent to the statement that almost every leaf of $\mathscr{F}$ has trivial linear holonomy group. The

ergodic theory of Sections 1 and 4 can then be applied to this cocycle over $\mathcal{F}$ to deduce that there are canonical framings for $Q|B$ when $\mathcal{F}|B$ is either amenable or admits an a.c.i.t. measure. These framings are used to calculate the Weil measures, as given in the main Theorem 7.3 below.

7.1. *The pseudo-group associated to* $\mathcal{F}$. Recall that $\{(U_\alpha, \Phi_\alpha)|\alpha \in A\}$ is the locally finite open cover of M by regular foliation charts chosen in Section 5. For each $\alpha \in A$, set $T_\alpha = I^n$ for $I = (-1, 1)$, and define $X = \bigcup_{\alpha \in A} T_\alpha$ to be the disjoint union of these discs. Then $\mathcal{N} = \xi(X)$ is a complete transversal to $\mathcal{F}$, where $\xi\colon X \to M$ is defined by $\xi(x) = \Phi_\alpha^{-1}(0, x)$ for $x \in T_\alpha$. Let $\Phi_\alpha^2\colon U_\alpha \to I^n$ be the projection of Φ_α onto the second factor. For (α, β) admissible, let $T_{\alpha\beta} = \Phi_\alpha^2(U_{\alpha\beta}) \subset T_\alpha$, and define

$$\begin{aligned} &\gamma_{\beta\alpha}\colon T_{\alpha\beta} \to T_{\beta\alpha}, \\ &\gamma_{\beta\alpha}(x) = \Phi_\beta^2 \circ \Phi_\alpha^{-1}(I^{m-n} \times \{x\}). \end{aligned} \tag{7.1}$$

The collection $\{\gamma_{\beta\alpha}|(\alpha, \beta)$ admissible$\}$ of local diffeomorphisms generates the holonomy pseudogroup $\mathcal{G}$ acting on X associated to the given covering of M. From $\mathcal{G}$ one can construct the holonomy groupoid $\Gamma_{\mathcal{F}}$ (cf. [14]), and the equivalence relation $\mathcal{F}$ on X consists of the orbits of the $\mathcal{G}$-action.

Fix a basis of $\mathbf{R}^n$ and use this to define a framing of TX via the identifications $T(T_\alpha) \cong I^n \times \mathbf{R}^n$. Given $\gamma \in \mathcal{G}$ and $x \in \text{Domain}(\gamma)$, denote by $D\gamma(x) \in \text{GL}(n, \mathbf{R})$ the derivative matrix of γ at x with respect to the framing of TX. Note that $D\gamma(x)$ acts on the left on $\mathbf{R}^n$. For (α, β) admissible, this defines a map $D\gamma_{\beta\alpha}\colon T_{\alpha\beta} \to \text{GL}(n, \mathbf{R})$. The chain rule $D\gamma_{\delta\beta} \circ D\gamma_{\beta\alpha} = D\gamma_{\delta\alpha}$ shows that $D\gamma\colon \mathcal{G} \to \text{GL}(n, \mathbf{R})$ is a map of pseudo-groups.

7.2. *The triviality of linear holonomy.*

PROPOSITION 7.1. *There exists a* $\text{GL}(n, \mathbf{R})$*-cocycle* ψ *over* $(X, \mathcal{F})$ *such that for all* $\gamma \in \mathcal{G}$ *and almost every* $x \in \text{Domain}(\gamma)$, $D\gamma(x) = \psi(\gamma(x), x)$. *In particular, the linear holonomy of almost every leaf of* $\mathcal{F}$ *is zero.*

Proof. It is enough to show that for all $\gamma \in \mathcal{G}$ and a.e. $x \in \text{Domain}(\gamma)$ with $\gamma(x) = x$, that $D\gamma(x) = \text{Identity}$. For then one can define $\psi(y, x)$ to be the map $\psi(y, x) = D\gamma(x)$ where $\gamma \in \mathcal{G}$ is chosen so that $\gamma(x) = y$.

Let $A_\gamma = \{x \in \text{Domain}(\gamma)|\gamma(x) = x\}$ and assume that A_γ has positive Lebesgue measure. Let $x \in A_\gamma$ be a point of Lebesgue density one. We can approach x from any direction in the tangent space T_xX by a sequence of points from A_γ, so that $D\gamma(x) = \text{Identity}$. Since almost every point in A_γ has Lebesgue density one and $\mathcal{G}$ is countable, this proves the proposition. □

By virtue of Proposition 7.1, we use the notation $D\gamma = \psi$: $\mathscr{F} \to \mathrm{GL}(n, \mathbf{R})$ and call this the *linear holonomy* cocycle for $\mathscr{F}$.

For $B \in \mathscr{B}(\mathscr{F})$, let $X|B = \xi^{-1}(B \cap \mathscr{N})$ be the saturated subset of X corresponding to B. Define $\mathscr{F}|B$ to be the equivalence relation on $X|B$ induced from $\mathscr{F}$, so that $(X|B, \mathscr{F}|B)$ is a subequivalence relation of $(X, \mathscr{F})$.

A measurable map g: $X \to \mathrm{GL}(n, \mathbf{R})$ defines a measurable framing of TX, also denoted by g, by letting $g(x)$ act on the standard frame of $T_xX \cong \mathbf{R}^n$ chosen above. The derivative matrix of $\gamma \in \mathscr{G}$ at x with respect to the new frames $g(x)$ and $g(\gamma x)$ is denoted by

$$\psi(y, x) = g(y)^{-1} \cdot D\gamma(y, x) \cdot g(x). \tag{7.2}$$

In Section 4, the Moore classification of the conjugacy classes of maximal amenable subgroups of $\mathrm{GL}(n, \mathbf{R})$ was described in terms of the partitions of n. Let us number all subgroups of the form (4.10) by $\{H_i | i = 1, 2, \ldots, 2^n\}$. Set also $H_0 = \mathrm{SL}(n, \mathbf{R})$. The maximal compact subgroup of H_i is $K_i = H_i \cap O_n$. We need the following conclusion from Corollary 3.3:

COROLLARY 7.2. *Let $B \in \mathscr{B}(\mathscr{F})$ and suppose that $(X|B, \mathscr{F}|B)$ is amenable. Then there exist a disjoint partition of B into $\{B_i \in \mathscr{B}(\mathscr{F}) | 1 \leq i \leq 2^n\}$, and a measurable framing g: $X|B \to GL(n, \mathbf{R})$ such that the linear holonomy cocycle ψ of $\mathscr{F}|B$ with respect to this framing is tempered with*

$$\psi\colon \mathscr{F}|B_i \to H_i. \tag{7.3}$$

7.3. *The fundamental theorem.* The following result will be combined in Section 8 with Corollary 7.2 to prove Theorem 3.5, and consequently Theorem 0.1.

THEOREM 7.3. *Let ψ: $\mathscr{F} \to \mathrm{GL}(n, \mathbf{R})$ be a tempered measurable cocycle cohomologous to $D\gamma$. Then for all $B \in \mathscr{B}(\mathscr{F})$, $y_I \in H^p(\mathfrak{gl}_n, O_n)$ and $[\varphi] \in H^{m-p}(M, \mathscr{F})$*

$$\chi_B(y_I)[\varphi] = \sum_{\alpha \in A} \int_{B \cap U_\alpha} \lambda_\alpha \cdot A_\alpha^*(\tau_I) \wedge \varphi$$

where A_α: $B \cap U_\alpha \to \mathfrak{gl}_n$ is obtained from the restriction ψ: $\mathscr{F}|B \to \mathrm{GL}(n, \mathbf{R})$. Moreover, if ψ on $\mathscr{F}|B$ takes values in one of the subgroups H_i, $0 \leq i \leq 2^n$, then A_α: $T\mathscr{F}|B \cap U_\alpha \to \mathfrak{h}_i$ where $\mathfrak{h}_i$ is the Lie algebra of H_i.

The proof of this theorem will occupy the rest of Section 7. We first show that the cocycle ψ can be approximated by uniformly tempered cocycles ψ^c, $c > 0$, obtained from $D\gamma$ via c-tempered coboundary. Then in subsection 7.6, we show the Weil measures can be calculated from the ψ^c; in order to show that

A_α has values in the appropriate Lie algebra, it is necessary to use special care in constructing framings of Q from ψ^c.

7.4. *Estimates on tempered cocycles.* Recall that ρ is the left-invariant metric on $\mathrm{GL}(n,\mathbf{R})$ chosen in Section 6 which is also right O_n-invariant. We identify $S(n) = \mathrm{GL}(n,\mathbf{R})/O_n$ with the cone of positive definite matrices, and let $\bar{\rho}$ denote the quotient metric on $S(n)$. Let $\mathfrak{m} \subset \mathfrak{gl}_n$ denote the subspace of symmetric matrices, exp: $\mathfrak{m} \to S(n)$ the matrix exponential map and log: $S(n) \to \mathfrak{m}$ its inverse. The value of the Riemannian metric determining ρ at the identity element $I_n \in \mathrm{GL}(n,\mathbf{R})$ determines an inner product $\rho_{\mathfrak{m}}$ on $\mathfrak{m}$, with associated norm $\| \ \|_\rho$. For $A \in \mathrm{GL}(n,\mathbf{R})$, we let $\bar{A}$ denote its coset in $S(n)$, and set

$$|A|_\rho = \rho(A, I_n),\ |\bar{A}|_\rho = \bar{\rho}(\bar{A}, \bar{I}_n).$$

These metrics and norms satisfy some basic identities and inequalities:

LEMMA 7.4. *For all* $A, B, C \in \mathrm{GL}(n,\mathbf{R})$,

$$\rho(A, B) \geq \bar{\rho}(\bar{A}, \bar{B}), \tag{7.4}$$

$$|\bar{A}|_\rho = |\overline{A^{-1}}|_\rho \tag{7.5}$$

$$\bar{\rho}(A \cdot \bar{B}, A \cdot \bar{C}) = \bar{\rho}(\bar{B}, \bar{C}) \tag{7.6}$$

$$|A \cdot \bar{B}|_\rho \leq \bar{\rho}(A \cdot \bar{B}, \bar{A}) + \bar{\rho}(\bar{A}, \bar{I}_n) = |\bar{A}|_\rho + |\bar{B}|_\rho \tag{7.7}$$

$$|\exp(a)|_\rho = \|a\|_\rho \qquad \text{for } a \in \mathfrak{m}. \tag{7.8}$$

Proof. (7.4) is obvious, (7.5) and (7.6) follow from left-invariance of ρ, (7.7) from the triangle inequality, and (7.8) from the fact that the matrix exponential agrees with the Riemannian exponential in $S(n)$. □

The geometry of $S(n)$ enters into this section, and thus the proof of Theorem 0.1, via (7.8) and the next lemma, which is a special case of a result valid for any Riemannian manifold with no focal points (cf. [29] or [35]).

LEMMA 7.5. *Let* $a(t)$ *and* $b(t)$ *be unit-speed geodesics in* $S(n)$. *Then for all* $0 \leq t \leq T$,

$$\bar{\rho}(a(t), b(t)) \leq \bar{\rho}(a(0), b(0)) + \bar{\rho}(a(T), b(T)). \tag{7.9}$$

When $a(0) = b(0)$, Lemma 7.5 implies that $\bar{\rho}(a(t), b(t))$ is a monotone increasing function. We apply this lemma to a special situation in which $a(t)$ and $b(t)$ are related via a cohomology:

LEMMA 7.6. *Let* $a(t)$ *and* $b(t)$ *be* 1*-parameter subgroups of* $\mathrm{GL}(n,\mathbf{R})$ *so that* $\overline{a(t)}$ *and* $\overline{b(t)}$ *are unit-speed geodesics in* $S(n)$. *Suppose there exist*

$g_0, g_1 \in \mathrm{GL}(n, \mathbf{R})$ *satisfying* $g_0 \cdot \overline{a(t_0)} = b(t_1) \cdot \bar{g}_1$ *for some* $t_0, t_1 > 0$. *Then for all* $0 \leq t \leq T = \min(t_0, t_1)$,

$$\bar{\rho}\big(g_0 \cdot \overline{a(t)}, \overline{b(t)}\big) \leq |\bar{g}_0|_\rho + |\bar{g}_1|_\rho + |t_1 - t_0|. \tag{7.10}$$

Proof. For $t \leq T \leq t_0$, Lemma 7.5 and (7.8) yield

$$\begin{aligned}
\bar{\rho}\big(g_0 \cdot \overline{a(t)}, \overline{b(t)}\big) &\leq |\bar{g}_0|_\rho + \bar{\rho}\big(g_0 \overline{a(t_0)}, \overline{b(t_0)}\big) \\
&= |\bar{g}_0|_\rho + \bar{\rho}\big(b(t_1) \cdot \bar{g}_1, \overline{b(t_0)}\big) \\
&\leq |\bar{g}_0|_\rho + \bar{\rho}\big(b(t_1) \cdot \bar{g}_1, \overline{b(t_1)}\big) + \bar{\rho}\big(\overline{b(t_1)}, \overline{b(t_0)}\big) \\
&= |\bar{g}_0|_\rho + |\bar{g}_1|_\rho + |t_1 - t_0|.
\end{aligned}$$

□

7.5. *Tempering normal coboundaries.* Let $(X, \mathcal{F}, d)$ be a metric equivalence relation and $\phi, \psi\colon \mathcal{F} \to \mathrm{GL}(n, \mathbf{R})$ be tempered cocycles related by a coboundary $g\colon X \to \mathrm{GL}(n, \mathbf{R})$:

$$\psi(y, x) = g(y)^{-1} \cdot \phi(y, x) \cdot g(x).$$

We construct below a sequence of bounded transfer functions $\{g^c \mid c > 0\}$ so that for $(y, x) \in \mathcal{F}$,

$$\psi^c(y, x) \overset{\text{def}}{=} g^c(y)^{-1} \cdot \phi(y, x) \cdot g^c(x), \tag{7.11}$$

the cocycles $\{\psi^c\}$ are uniformly tempered and converge in measure to ψ as $c \to \infty$.

For $\bar{g}\colon X \to \mathrm{S}(n) \subset \mathrm{GL}(n, \mathbf{R})$, define

$$l\colon X \to \mathfrak{m}\colon l(x) = \log\big(\overline{g(x)}\big),\ x \in X. \tag{7.12}$$

For each positive real number c, define the truncation l^c of l by setting

$$l^c(x) = \begin{cases} l(x) & \text{if } \|l(x)\|_\rho < c \\ \dfrac{c}{\|l(x)\|_\rho} \cdot l(x) & \text{if } \|l(x)\|_\rho \geq c. \end{cases} \tag{7.13}$$

Set $g^c = \exp\{l^c\}\colon X \to \mathrm{S}(n)$, and define ψ^c by (7.11). Note that for each $x \in X$, there is a $k(x) \in O_n$ so that

$$g(x) \cdot k(x) = \lim_{c \to \infty} g^c(x).$$

The function $k\colon X \to O_n$ is clearly measurable.

PROPOSITION 7.7. *With notation as above,*

$$\|g^c(x)\|^+ \leq \exp(K_0 \cdot c) \text{ a.e. } x \in X, \text{ fixed } K_0 > 0, \tag{7.14}$$

$$\lim_{c\to\infty} g^c(x) = \overline{g(x)}, \text{ for all } x \in X, \tag{7.15}$$

$$\lim_{c\to\infty} \psi^c(y,x) = k(y)^{-1} \cdot \psi(y,x) \cdot k(x), \text{ for all } (y,x) \in \mathscr{F}, \tag{7.16}$$

$$|\overline{\psi^c(y,x)}|_\rho \leq 3\{|\overline{\phi(y,x)}|_\rho + |\overline{\psi(y,x)}|_\rho\}, \text{ a.e. } x \in X. \tag{7.17}$$

Proof. (7.15) and (7.16) are immediate from the definitions. To see (7.14) observe that there is a constant K_0 so that $\|a\| \leq K_0 \cdot \|a\|_\rho$ for all $a \in \mathfrak{m}$; then using that $g^c(x)$ is symmetric we have

$$\begin{aligned}
\log\|g^c(x)\|^+ &= \text{largest eigenvalue of } \pm\log(g^c(x)) \\
&\leq K_0 \cdot \|\log(g^c(x))\|_\rho \\
&= K_0 \cdot |g^c(x)|_\rho \text{ by (7.8)} \\
&\leq K_0 \cdot c \text{ by (7.13).}
\end{aligned}$$

To establish (7.17), we use Lemma 7.6 and formula (7.11):

$$\begin{aligned}
|\psi^c(y,x)|_\rho &= \bar{\rho}(\phi(y,x) \cdot \overline{g^c(x)}, \overline{g^c(y)}) \\
&= \bar{\rho}(g_0 \cdot \overline{a(s(x))}, \overline{b(s(y))})
\end{aligned} \tag{7.18}$$

where $a(t) = \exp\{t/\|l(x)\|_\rho \cdot l(x)\}$, $b(t) = \exp\{t/\|l(y)\|_\rho \cdot l(y)\}$ and $s(x) = \min\{c, \|l(x)\|\}$. Then set $t_0 = \|l(x)\|_\rho$ and $t_1 = \|l(y)\|_\rho$ so that

$$a(t_0) = \overline{g(x)}, b(t_1) = \overline{g(y)},$$

and let $g_0 = \phi(y,x)$, $g_1 = \psi(y,x)$. We then apply Lemma 7.6 to (7.18), and consider the possible cases: For $c \geq \max\{t_0, t_1\}$, we have $\psi(y,x) = k(y) \cdot \psi^c(y,x) \cdot k(x)^{-1}$ so that (7.17) is immediate. If $c \leq \min\{t_0, t_1\}$ then (7.18) and Lemma 7.6 yield

$$|\overline{\psi^c(y,x)}|_\rho \leq |\overline{\phi(y,x)}|_\rho + |\overline{\psi(y,x)}|_\rho + |t_1 - t_0|. \tag{7.19}$$

If $t_1 \geq c > t_0$, then we obtain

$$\begin{aligned}
|\overline{\psi^c(y,x)}|_\rho &= \bar{\rho}(g_0 \cdot \overline{a(t_0)}, \overline{b(c)}) \\
&\leq \bar{\rho}(g_0 \cdot \overline{a(t_0)}, \overline{b(t^0)}) + \bar{\rho}(\overline{b(t_0)}, \overline{b(c)}) \\
&\leq |\overline{\phi(y,x)}|_\rho + |\overline{\psi(y,x)}|_\rho + |t_1 - t_0| + |c - t_0|.
\end{aligned}$$

The case $t_0 \geq c > t_1$ is similar, so it always holds that

$$|\overline{\psi^c(y,x)}|_\rho \leq |\overline{\phi(y,x)}|_\rho + |\overline{\psi(y,x)}|_\rho + 2|t_1 - t_0|. \tag{7.20}$$

It remains to estimate $|t_1 - t_0|$; so consider:

$$\begin{aligned} t_0 &= \|l(x)\|_\rho = |\overline{g(x)}|_\rho \text{ by (7.8)} \\ &= |\phi(x,y) \cdot g(y) \cdot \overline{\psi(y,x)}|_\rho \\ &\leq |\overline{\phi(x,y)}|_\rho + |\overline{g(y)}|_\rho + |\overline{\psi(y,x)}|_\rho \text{ by (7.7)}, \end{aligned}$$

so that $|t_0 - t_1| \leq |\overline{\phi(y,x)}|_\rho + |\overline{\psi(y,x)}|_\rho$. The proof of (7.17) is now complete. □

7.6. *Extending the framing g^c.* We now assume the hypotheses of Theorem 3.5 are satisfied for $B \in \mathscr{B}(\mathscr{F})$. Let $(X, \mathscr{F}, d)$ be the metric equivalence relation of subsection 7.1 and $D\gamma$: $\mathscr{F} \to \mathrm{GL}(n, \mathbf{R})$ the normal linear holonomy cocycle with respect to a smooth framing of TX. By Corollary 7.2, we can assume there is a disjoint decomposition $X|B = \bigcup_{i=0}^{2^n} X_i$ into measurable saturated sets and a measurable function g: $X \to \mathrm{GL}(n, \mathbf{R})$ so that for $\psi(y,x) = g(y)^{-1} \cdot D\gamma(y,x) \cdot g(x)^{-1}$, we have $\psi|X_i$ takes values in H_i. Recall that $H_0 = \mathrm{SL}(n, \mathbf{R})$, and H_i for $1 \leq i \leq 2^n$ are the maximal amenable subgroups of subsection 7.2, chosen so that $H_i \cap O_n = K_i$ is the maximal compact subgroup of H_i.

By the construction of subsection 7.5, there are uniformly tempered cocycles ψ^c, which converge in measure to ψ, and are cohomologous to $D\gamma$ via a tempered coboundary. We next use the functions $\{g^c\}$ on $X|B$ to define collections $\{F_\alpha^c | \alpha \in A, c > 0\}$ of O_n-related bounded measurable framings for the foliation. In this step, the key point is to make sure the limit framings are such that the resulting leafwise maps $A_{F_\alpha^c}(x)$ take values in $\mathfrak{h}_i$ for x in the saturation B_i of X_i. We begin with some notation. For $\alpha, \beta \in A$, set

$$\begin{aligned} g_\alpha^c &= g^c | X_\alpha, \\ \psi_{\beta\alpha}^c(x) &= g_\beta^c\big(\gamma_{\beta\alpha}(x)\big)^{-1} \cdot D\gamma\big(\gamma_{\beta\alpha}(x), x\big) \cdot g_\alpha^c(x), \\ X(\alpha, i, c) &= \big\{x \in T_\alpha \cap X_i \,\big|\, \psi_{\beta\alpha}^c(x) \in H_i \text{ for all } \beta \text{ with } x \in T_{\alpha\beta}\big\}, \\ U(\alpha, i, c) &= \big(\Phi_\alpha^2\big)^{-1}\big(X(\alpha, i, c)\big), \\ D\gamma_{\beta\alpha}(z) &= D\gamma\big(\Phi_\beta^2(z), \Phi_\alpha^2(z)\big),\ z \in U_{\alpha\beta}, \\ \psi_{\beta\alpha}(z) &= \psi\big(\Phi_\beta^2(z), \Phi_\alpha^2(z)\big),\ z \in U_{\alpha\beta}. \end{aligned}$$

We identify $Q|U_\alpha \cong (\Phi_\alpha^2)^*(T\mathbf{R}^n)$, so that the composition

$$f_\alpha^c = g_\alpha^c \circ \Phi_\alpha^2 \to \mathrm{GL}(n, \mathbf{R})$$

defines a measurable framing of $Q|U_\alpha$ which is parallel along plaques. For each $c > 0$, we modify the local framings $\{f_\alpha^c | \alpha \in A\}$ to obtain O_n-related local framings. On the good sets $U(\alpha, i, c)$, the convexity of the subsets H_i/K_i in $S(n)$ will be used to keep the modification inside H_i, while on the remainder of M we simply use the convexity of $S(n)$.

Consider first the case $i > 0$. The group H_i is a semi-direct product of K_i and a maximal connected solvable subgroup $R(i)$. Thus, for $z \in U(\alpha, \beta, i, c)$ there is a unique expression

$$\psi_{\beta\alpha}^c(z) = r_{\beta\alpha}^c(z) \cdot k_{\beta\alpha}^c(z)$$

with $r_{\beta\alpha}^c(z) \in R(i)$ and $k_{\beta\alpha}^c(z) \in K_i$.

LEMMA 7.8. *For (α, β) and (α, δ) admissible, and $z \in U(\alpha, i, c) \cap U_\beta \cap U_\delta$,*

$$\psi_{\beta\alpha}^c(z) \cdot r_{\alpha\delta}^c(z) = r_{\beta\delta}^c(z) \cdot k_{\beta\alpha}^c(z). \tag{7.21}$$

Proof. This follows from the cocycle identities

$$\psi_{\beta\alpha}^c \cdot \psi_{\alpha\delta}^c = \psi_{\beta\delta}^c \quad \text{and} \quad k_{\beta\alpha}^c \cdot k_{\alpha\delta}^c = k_{\beta\delta}^c,$$

the latter holding because $R(i)$ is normal in H_i, $i > 0$. □

The subgroup $R(i)$ is convex, so we can make a center of gravity construction for functions taking values in $R(i)$. In particular, for $z \in U(\alpha, i, c)$ we use the partition-of-unity $\{\lambda_\alpha\}$ chosen in subsection 5.1 to define

$$\begin{aligned} r_\alpha^c(z) &= \sum_{(\alpha, \delta)} \lambda_\delta(z) \cdot r_{\alpha\delta}^c(z) \in R(i), \\ s_\alpha^c(z) &= f_\alpha^c(z) \cdot r_\alpha^c(z). \end{aligned} \tag{7.22}$$

LEMMA 7.9. *For $z \in U(\alpha, i, c) \cap U(\beta, i, c)$, the framings $s_\alpha^c(z)$ and $s_\beta^c(z)$ are K_i-related.*

Proof. Use relation (7.21) and the definition of s_α^c to obtain

$$\begin{aligned} D\gamma_{\beta\alpha}(z) \cdot s_\alpha^c(z) &= f_\beta^c(z) \cdot \psi_{\beta\alpha}^c \cdot \sum_\delta \lambda_\delta(z) \cdot r_{\alpha\delta}^c(z) \\ &= f_\beta^c(z) \cdot \sum_\delta \lambda_\delta(z) \cdot r_{\beta\alpha}^c(z) \cdot k_{\beta\alpha}^c(z) \\ &= s_\beta^c(z) \cdot k_{\beta\alpha}^c(z). \end{aligned}$$

□

The functions f_α^c and $r_{\alpha\delta}^c$ are constant along the plaques in $U(\alpha, i, c)$, so the restriction of s_α^c to this good set has leafwise derivative given by

$$D_{\mathscr{F}} s_\alpha^c = f_\alpha^c \cdot \sum_\delta D_{\mathscr{F}} \lambda_\delta \cdot r_{\alpha\delta}^c \tag{7.23}$$

and thus

$$A_{s_\alpha^c} = \left[\sum_\delta \lambda_\delta \cdot r^c_{\alpha\delta}\right]^{-1} \cdot \left[\sum_\delta D_{\mathscr{F}}\lambda_\delta \cdot r^c_{\alpha\delta}\right]. \tag{7.24}$$

As $R(i)$ is a Lie subgroup of $\mathrm{GL}(n, \mathbf{R})$, (7.23) implies that $A_{s_\alpha^c}$ takes values in the Lie algebra $\mathfrak{r}(i)$ of $R(i)$ on the set $U(\alpha, i, c)$. Moreover, from (7.24) one concludes that the norm $\|A_{s_\alpha^c}\|_{\mathscr{F}}$ is majorized by a constant multiple of $[\max_{(\alpha,\delta)}\|\psi^c_{\alpha\delta}\|^+]^2$.

Next, consider the case $i = 0$. Let $SS(n) \subset S(n)$ denote the set of positive definite symmetric matrices with determinant one. There is a natural projection $S\colon S(n) \to SS(n)$, obtained by suitably scaling the matrix, and this gives $SS(n)$ a cone structure. On the set $U(\alpha, 0, c)$, where the cocycle $\psi^c_{\alpha\beta}(z)$ takes values in $\mathrm{SL}(n, \mathbf{R})$, we define

$$s^c_\alpha(z) = f^c_\alpha(z) \cdot S\left[\sum_\delta \lambda_\delta(z)\bar{\psi}_{\alpha\delta}(z)\right] \tag{7.25}$$

for $\bar{\psi}_{\alpha\delta}(z)$ the O_n-reduction of f. One checks easily that $\{s^c_\alpha | \alpha \in A\}$ gives an O_n-related framing of Q on the sets $\{U(\alpha, 0, c) | \alpha \in A\}$. The restrictions of s^c_α to the plaques in $U(\alpha, 0, c)$ are smooth, and $\psi_{\alpha\beta}(z) \in \mathrm{SL}(n, \mathbf{R})$ for $z \in U(\alpha, 0, c)$ implies

$$A_{s_\alpha^c}(z) = s^c_\alpha(z)^{-1} \cdot D_{\mathscr{F}}(s^c_\alpha)|_z \in \mathfrak{sl}_n.$$

It remains to extend the framing s^c_α, defined on the unions $U(\alpha, c) = \bigcup_{i=0}^{2^n} U(\alpha, i, c)$, to the whole of U_α. First define sets

$$U'(\alpha, c) = \left\{z \in U(\alpha, i) \middle| z \in U(\alpha, i, c) \cap U_\beta \text{ implies } z \in U(\beta, i, c)\right\},$$

$$U''(\alpha, c) = U(\alpha, c) \setminus U'(\alpha, c).$$

For each $\alpha \in A$, $\lim_{c\to\infty} U(\alpha, i, c) = U_\alpha \cap B_i$, so that $\lim_{c\to\infty} U'(\alpha, c) = U_\alpha \cap B$.

For each $z \in U'(\alpha, c)$, set

$$F^c_\alpha(z) = s^c_\alpha(z)\begin{cases} \text{via (7.22)} & \text{for } z \in U(\alpha, i, c) \\ \text{via (7.25)} & \text{for } z \in U(\alpha, 0, c) \end{cases}. \tag{7.26}$$

Extend s^c_α to all of U_α by setting $s^c_\alpha(z) = f^c_\alpha(z)$ for $z \in U_\alpha \setminus U'(\alpha, c)$. Let $\overline{s^c_\alpha}$ denote the O_n-reduction with values in $S(n)$; then using the convexity of $S(n)$ set, for $z \in U_\alpha \setminus U'(\alpha, c)$,

$$F^c_\alpha(z) = \sum_\delta \lambda_\delta(z) \cdot D\gamma_{\alpha\delta}(z) \cdot \overline{s^c_\delta(z)}.$$

Then $\{F^c_\alpha | \alpha \in A\}$ is an O_n-related measurable framing of Q, and we require two

estimates on the norms of the family. First, note that for each $z \in U_\alpha \cap B$, the limit framing as $c \to \infty$ exists,

$$F_\alpha(z) = \lim_{c \to \infty} s_\alpha^c(z), \tag{7.27}$$

and we define maps $A_{F_\alpha^c}, A_\alpha$: $T\mathcal{F}|U_\alpha \to \mathfrak{gl}_n$ by:

$$A_{F_\alpha^c}(z) = F_\alpha^c(z)^{-1} \cdot D_{\mathcal{F}} F_\alpha^c|_z,$$

$$A_\alpha(z) = A_{F_\alpha}(z) = F_\alpha(z)^{-1} \cdot D_{\mathcal{F}} F_\alpha|_z.$$

Clearly, $A_{F_\alpha^c} \to A_\alpha$, and it remains to estimate the pull-back maps $(A_{F_\alpha^c})^*$ on O_n-basic forms.

PROPOSITION 7.10. *There exists a constant C_2, depending only on $\|\psi\| + \|D\psi\|$ and the partition-on-unity $\{\lambda_\alpha\}$ so that for every O_n-basic p-form τ on $\mathfrak{gl}_n$,*

$$\left\|(A_{F_\alpha^c})^*(\tau)\right\|_{\mathcal{F}} \leq C_2 \|\tau\|_\rho.$$

Proof. The form $(A_{F_\alpha^c})^*(\tau)$ depends only on the reduced framing $\bar{F}_\alpha^c$: $U_\alpha \to S(n)$, for τ is O_n-basic implies that it defines a $\mathrm{GL}(n, \mathbf{R})$-invariant p-form $\tilde{\tau}$ on $S(n)$, and clearly $(A_{F_\alpha^c})^*(\tau) = (D_{\mathcal{F}} \bar{F}_\alpha^c)^*(\tilde{\tau})$. Thus, it will suffice to show there is a uniform bound on the norm of the derivative maps $D_{\mathcal{F}}(\bar{F}_\alpha^c)$: $TU_\alpha \to TS(n)$ with respect to the invariant norm on $TS(n)$. For $z \in U'(\alpha, c)$, we have

$$\begin{aligned}
\left\|D_{\mathcal{F}}\left(\bar{F}_\alpha^c\right)\right\|_\rho &= \left\|\sum_\delta D_{\mathcal{F}} \lambda_\delta \cdot \psi_{\alpha\delta}^c\right\|_\rho \\
&\leq \sum_\delta \|D_{\mathcal{F}} \lambda_\delta\| \cdot \max_{(\alpha, \delta)} \left|\psi^c\left(\gamma_{\beta\alpha}(x), x\right)\right|_\rho \\
&\leq C_2 \cdot (\|\psi\| + \|D\gamma\|)
\end{aligned}$$

by (7.17). The cases $z \in U''(\alpha, c)$ and $z \in U_\alpha \setminus U(\alpha, c)$ follow similarly. □

We can now prove Theorem 7.3. First, the O_n-oriented framing $\{F_\alpha^c\}$ is bounded, and by Proposition 7.10 the forms $(A_{F_\alpha^c})^*(\tau_I)$ are uniformly bounded by a constant independent of α, c and $z \in U_\alpha$. Thus, the proof of Theorem 6.1 shows that these forms can be used to calculate the Weil measure $\chi_\beta(y_I)$. It follows from the definition (7.26) and observation (7.27) that on $U_\alpha \cap B_i$,

$$\lim_{c \to \infty} \left(A_{F_\alpha^c}\right)^*(\tau_I) = (A_{\alpha, i})^*(\tau_I),$$

where $A_{\alpha, i} = A_\alpha|B_i$: $T\mathcal{F}|U_\alpha \cap B_i \to \mathfrak{h}_i$, $0 \leq i \leq 2^n$. Then by the dominated

convergence theorem we obtain

$$\chi_B(y_I)[\varphi] = \sum_{\alpha \in A} \int_{U_\alpha \cap B} \lambda_\alpha \cdot \left(A_{F_\alpha^c}\right)^*(\tau_I) \wedge \varphi$$

$$= \sum_{\alpha \in A} \int_{U_\alpha \cap B} \lim_{c \to \infty} \lambda_\alpha \cdot \left(A_{F_\alpha^c}\right)^*(\tau_I) \wedge \varphi$$

$$= \sum_{\alpha \in A} \int_{U_\alpha \cap B} \lambda_\alpha \cdot A_\alpha^*(\tau_I) \wedge \varphi. \qquad \square$$

8. Applications to vanishing theorems

In this section, we use the idea of the proof of Theorem 5.10 along with Corollary 7.2 and Theorem 7.3 to prove Theorems 3.5, 3.11 and 3.16.

Let us first prove Theorem 3.5. Given $B \in \mathscr{B}(\mathscr{F})$ with $\mathscr{F}|B$ amenable, by Corollary 7.2 there is a tempered cocycle ψ: $\mathscr{F}|X \cap B \to \mathrm{GL}(n, \mathbf{R})$ and a partition $X \cap B = \bigcup_{i=1}^{2^n} X_i$ so that ψ: $\mathscr{F}|X_i \to H_i$, for H_i maximal amenable. Let B_i denote the $\mathscr{F}$-saturation in M of X_i. Then by Theorem 7.3, for $y_I \in H^p(\mathfrak{gl}_n, O_n)$ we have

$$\chi_B(y_I)[\varphi] = \sum_{i=1}^{2^n} \sum_{\alpha \in A} \int_{U_\alpha \cap B_i} \lambda_\alpha \cdot A_{\alpha,i}^*(\tau_I) \wedge \varphi. \tag{8.1}$$

Now assume that degree $y_I = p > 1$. Then by Lemma 5.11, there is a K_i-basic $(p-1)$-form σ_I on $\mathfrak{h}_i$ with $d_{\mathfrak{h}_i}\sigma_I = \tau_I$. From Lemma 7.9 we conclude that

$$A_{\alpha,i}^*(\sigma_I) = A_{\beta,i}^*(\sigma_I)$$

on $U_\alpha \cap U_\beta$. Thus there is a well-defined bounded measurable $(p-1)$-form $A_i^*(\sigma_I)$ defined on all of B_i such that $d_{\mathscr{F}} A_i(\sigma_I)|U_\alpha = A_{\alpha,i}^*(\tau_I)$. This implies the integrands in (8.1) are leafwise-exact; so by the Stokes' Theorem for foliations [17] the integrals vanish. $\square$

The proof of Theorem 3.11 is similar, but easier. Given $B \in \mathscr{B}(\mathscr{F})$ with $B \subset \mathrm{support}(\mu)$, where μ is an absolutely continuous invariant transverse measure, it is immediate that $D\gamma \sim \psi$ on $X \cap B$ where ψ: $\mathscr{F}|X \cap B \to \mathrm{SL}(n, \mathbf{R})$ is tempered. We then apply Theorem 7.3 for the case $H_0 = \mathrm{SL}(n, \mathbf{R})$ to deduce

$$\chi_B(y_I)[\varphi] = \sum_{\alpha \in A} \int_{U_\alpha \cap B} \lambda_\alpha \cdot A_{\alpha,0}^*(\tau_I) \wedge \varphi. \tag{8.2}$$

Now if $y_I = y_1 \wedge y_{I'}$, then as $A_{\alpha,0}$: $T\mathscr{F}|U_\alpha \cap B \to \mathfrak{sl}_n$, we have $A_{\alpha,0}^*(\tau_I) \equiv 0$ so the integral in (8.2) vanishes. $\square$

Finally, to prove Theorem 3.16 we combine both of these approaches. For y_I of degree $p > 1$, the hypothesis of Theorem 3.16 implies $D\gamma \sim \psi$ on X, where there is a measurable partition $X = \bigcup_{i=1}^{2^n} X_i$ into saturated sets with ψ: $\mathscr{F} | X_i \to H_i$. Then by Theorem 7.3 and the argument for (8.1) above, $\chi_B(y_I) = 0$. For y_1, it is given that $D\gamma \sim \psi$: $X \to \mathrm{SL}(n, \mathbf{R})$; so $\chi_B(y_1) = 0$ for all $B \in \mathscr{B}(\mathscr{F})$.

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