FIVE MOST RESISTANT PROBLEMS IN DYNAMICS

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1. Coexistence of KAM circles and positive entropy in area preserving twist maps

The standard area preserving map f_λ of the cylinder $C = S^1 \times \mathbb{R}$:

$$f_\lambda(x, y) = (x + y, y + \lambda \sin 2\pi(x + y)).$$

Problem 1 *Is metric entropy $h_{area}(f_\lambda)$ positive*

(i) for small λ, or (ii) for any *λ (assuming y is periodic too)?*

"Yes" implies existence of ergodic components of positive measure. (Pesin, 1977.)

1954–1962	Kolmogorov, Moser. Existence of invariant curves for small λ and around elliptic points.
1960s	Sinai attempts to solve (i) and formulates first important ideas of smooth ergodic theory.

EXPECTED ANSWER: Coexistence possible ("yes" for (i)).

HEART OF THE DIFFICULTY: if true, estimates unimaginable.

RELATED EVIDENCE: Positive topological entropy, homoclinic points, Melnikov method

HOPE FOR PROBLEM (ii): Parameter exclusion techniques,.

ATTEMPTS: Lazutkin, Noble, Kosygin–Sinai.

POSSIBLE SHORTCUT: random perturbations.

2. Smooth realization of measure preserving transformations

Problem 2 *Given an ergodic measure preserving transformation T of a Lebesgue measure space X with measure μ, under what conditions does there exist a diffeomorphism f of a compact manifold M preserving a smooth volume v such that (f, v) is measurably isomorphic to (T, μ)? In particular, is there any T with finite entropy, $h_\mu(T) < \infty$, for which such an f does not exist?*

1965 Kushnirenko. Finiteness of entropy: $h_v(f) < \infty$.
True for any Borel f-invariant measure ν.

1970 Anosov–A.K. Non-standard smooth realizations,
e.g. translations on ∞-dimensional torus on the disc.

1977 Pesin. $\dim M = 2$, f weakly mixing implies f is Bernoulli.

EXPECTED ANSWER: there are universal obstructions.

HEART OF THE DIFFICULTY: no good candidates for invariants exist or even imagined.

RESTRICTION IN LOW DIMENSION: circle maps, flows on surfaces. Most interesting: (Almost) No smooth realization for a Diophantine rotation on the disc (Herman's last geometric theorem).

HOPE IN DIMENSION THREE: K implies Bernoulii (stable ergodicity theory).

POSSIBLE REALIZATION RESULTS.

Several characteristic problems which in our view can be approached by a version of the approximation by conjugation method or its modification with decreasing chances of success.

Find a smooth realization of:

A Gaussian dynamical system with simple (Kronecker) spectrum (just solved: A.K.-A.Windsor, 2007);

A dense G_δ set of minimal interval exchange transformations;

An adding machine;

The time-one map of the horocycle flow on the modular surface $SO(2)\backslash SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ (which is not compact, so the standard realization cannot be used).

3. Orbit growth, existence of periodic orbits and ergodicity of billiards in polygons

Consider the billiard motion inside a polygon $P \subset \mathbb{R}^2$. Let $S(T)$ be the number of orbits of length $\leq T$ which begin and end in vertices.

Problem 3 *(i) Find above and below estimates for $S(T)$.*

(ii) Is there a periodic billiard orbit for any P?

(iii) Find conditions for ergodicity of the billiard flow with respect to Liouville measure. In particular is the billiard ergodic for almost every P?

1985 Boshernitzan, A.K. based on Kerckhoff–Masur–Smillie.
Dense G_δ set of ergodic polygons.

1987 A.K. Subexponential estimate: $T^{-1} \log S(T) = 0$.

1990 Masur. For *rational* polygons $C_1 T^2 \leq S(T) \leq C_2 T^2$.

EXPECTED ANSWERS: (i) Orbit growth slower than $T^{2+\epsilon}$ (although not quadratic)

(ii) More likely than not periodic orbit exist; proved for triangles with angels $< 100°$ (R. Schwartz, 2005)

(iii) Typical billiards are ergodic.

HEART OF THE DIFFICULTY: Lack of structure (unlike the rational case)

For the periodic orbit problem the behavior is parabolic; no periodic orbits in other (more characteristic) parabolic systems.

4. Invariant measures for hyperbolic actions of higher rank abelian groups

Key examples:(i) $\times 2, \times 3$ is the action of $\mathbb{Z}_+^2$ on S^1 generated by

$$E_2: \ x \mapsto 2x \pmod 1 \text{ and } E_3: \ x \mapsto 3x, \pmod 1.$$

(ii) Let $M = SL(n, \mathbb{R})/\Gamma, \ n \geq 3, \Gamma$ a lattice in $SL(n, \mathbb{R}), \ D \subset SL(n, \mathbb{R})$ positive diagonals isomorphic to $\mathbb{R}^{n-1}$. The *Weyl chamber flow (WCF),* is the action of D on M by left translations.

Problem 4 *Find all ergodic invariant measures for the* $\times 2, \times 3$ *and the Weyl chamber flow.*

1967 Furstenberg asks the question for $\times 2, \times 3$.

1990 Rudolph. The only *positive entropy (P. E.)* ergodic measure for $\times 2, \times 3$ is Lebesgue.

2003 Einsiedler–A.K–E. Lindenstrauss. P. E. for WCF algebraic.

POSSIBLE APPROACHES: harmonic analysis, geometric.

HEART OF THE DIFFICULTY: For harmonic analysis approach: poor description of measures among distributions; there are many of those.

For geometric approach: lack of structure to use hyperbolicity in the zero entropy case.

HOPE FOR INTERMEDIATE RESULTS: Larger semi-groups, restriction on *slow entropy* of invariant measures.

5. Topological classification of Anosov diffeomorphisms and differentiable classification of Anosov actions of $\mathbb{Z}^k$ $k \geq 2$

Let N be a simply connected nilpotent group, $\Gamma \subset N$ a lattice in N.

Problem 5 *Is every Anosov diffeomorphism of a compact manifold M topologically conjugate to a finite factor of an automorphism of a nil-manifold N/Γ?*

1967 Smale (acknowledging A. Borel's contribution) constructs examples on nilmanifolds which are not tori and asks the question.

1970 Franks develops crucial machinery and solves special cases.

1974 Manning solves the problem if M is a finite factor of N/Γ.

HEART OF THE DIFFICULTY: Lack of understanding of topology for possible counterexamples.

RELATED PROBLEMS (Possibly more accessible):

(i) Is every Anosov action of $\mathbb{Z}^k,\ k \geq 2$ a compact manifold M without rank one factors *differentiably* conjugate to a finite factor of an action by affine maps of a nil-manifold N/Γ?

(ii) Is every Anosov diffeomorphism of a compact manifold M with smooth stable and unstable foliations differentiably conjugate to a finite factor of an automorphism of a nil-manifold N/Γ?

HOPE for (i) and (ii): find invariant geometric structures.