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RIGIDITY OF REAL-ANALYTIC ACTIONS OF $SL(n,\mathbb{Z})$ ON $\mathbb{T}^n$: A CASE OF REALIZATION OF ZIMMER PROGRAM

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Abstract. We prove that any real-analytic action of $SL(n,\mathbb{Z})$, $n \geq 3$ with standard homotopy data that preserves an ergodic measure μ whose support is not contained in a ball, is analytically conjugate on an open invariant set to the standard linear action on the complement to a finite union of periodic orbits.

1. **Introduction. Formulation of results.** Let G be a semisimple Lie group whose simple factors all have real rank greater than one, or an irreducible lattice in such a group. "Zimmer program" first formulated in 1986 [12] and modified in the 1990s to take into account examples described in [6], aims at proving that volume preserving actions of G by diffeomorphisms of compact manifold M are "essentially algebraic". This means that M splits into disjoint union of open G-invariant subsets $U_1, \dots, U_n$ and a nowhere dense closed subset F such that the restriction of the action to each of the open sets is smoothly conjugate to the restriction of a certain standard algebraic action (homogeneous or affine) to an open dense G-invariant subset.

Without attempting an overview of results in the direction of Zimmer program let us point out that so far they have been either negative (such as non-existence of actions in low dimension), or perturbative (local differentiable rigidity of algebraic actions), or subject to dynamical restrictions such as existence of Anosov elements.

In this note we present what is, to the best of our knowledge, the first positive result free of such restrictions. Its principal limitation is that we consider real-analytic, rather than differentiable actions with the pay-off that the conjugacy is also real-analytic. On the other hand, instead of preservation of volume we make a weaker assumption of existence of an invariant measure with a "homotopically large" support.

Let us consider the torus $\mathbb{T}^n$. The group $SL(n,\mathbb{Z})$ acts on $\mathbb{T}^n$ by automorphisms which are projections of the linear maps in $\mathbb{R}^n$. We will call this action of $SL(n,\mathbb{Z})$ on $\mathbb{T}^n$ *the standard action* and denote it ρ_0. We will call the corresponding action on

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$\mathbb{Z}^n$ *the standard homotopy data*. Since $SL(n,\mathbb{Z})$ is a lattice in the simple connected Lie group $SL(n,\mathbb{R})$ of real rank $n-1$ we assume $n \geq 3$. Let ρ be an action of a group $\Gamma \subset SL(n,\mathbb{Z})$ by diffeomorphisms of $\mathbb{T}^n$. Induced action ρ_* on $\mathbb{Z}^n = \pi_1(\mathbb{T}^n)$ is called *the homotopy data* of ρ. We will say that ρ has *standard homotopy data* if ρ_* is the restriction of the standard homotopy data to Γ.

This note is dedicated to the proof of the following result:

Theorem 1.1. *Let $\Gamma \subset SL(n,\mathbb{Z})$, $n \geq 3$ be a finite index subgroup. Let ρ be a C^ω (real-analytic) action of Γ on $\mathbb{T}^n$ with standard homotopy data, preserving an ergodic measure μ whose support is not contained in a ball. Then:*

1. *There is a finite index subgroup $\Gamma' \subset \Gamma$, a finite ρ_0-invariant set F and a bijective real-analytic map*
$$H : \mathbb{T}^n \setminus F \to D$$
where D is a dense subset of $\operatorname{supp}\mu$, *such that for every $\gamma \in \Gamma'$,*
$$H \circ \rho(\gamma) = \rho_0(\gamma) \circ H.$$
2. *The map H^{-1} can be extended to a continuous (not necessarily invertible) map $P : \mathbb{T}^n \to \mathbb{T}^n$ such that $\rho \circ P = \rho_0 \circ P$. Moreover, for any $x \in F$, pre-image $P^{-1}(x)$ is a connected set.*
3. *For $\Gamma = SL(n,\mathbb{R})$ one can take $\Gamma' = \Gamma = SL(n,\mathbb{R})$.*

Since Lebesgue measure λ is the only non-atomic invariant measure for ρ_0 we deduce that $\mu = H_*\lambda$ and we have the following corollary

Corollary 1.2. *The measure μ is given by a real-analytic density on an open dense subset of its support. Furthermore, it is the only ergodic ρ-invariant measure whose support is not contained in a ball.*

The map H is the inverse of the conjugacy between the action ρ on an open set and the standard action in the complement to finitely many periodic orbits.

Thus ρ is obtained by "blowing up" finitely many periodic orbits of ρ_0 and leaving the rest the same up to a real-analytic time change. Possibility of such non-trivial blow-ups on the torus is an open question. Constructions from [6] produce real-analytic actions with blowups on some manifolds other than the torus, e.g by gluing in a projective space $\mathbb{R}P(n-1)$ by a σ-process or glueing two such projective spaces and thus attaching a kind of handle to the torus. The same construction produces a real-analytic action on the torus with an open round hole that obviously cannot be extended to a real-analytic action inside the hole. C^0 extension is possible but whether it can be extended in a smooth way is an open question.

2. **Proof of Theorem 1.1.** We will assume throughout this section that Γ is a finite index subgroup of $SL(n,\mathbb{Z})$, $n \geq 3$.

A particular case of [10, Theorem, 6.10] asserts that a Γ action on $\mathbb{T}^n$ with standard homotopy data preserving a measure μ with full support is essentially semi-conjugate to the standard linear action. The following proposition is an improvement of that statement in two respects: (i) we put a weaker condition on μ and (ii) we assert that μ is absolutely continuous. To achieve that we rely on results from [8] about actions of Cartan (maximal rank semisimple abelian) subgroups of $SL(n,\mathbb{Z})$. Let Γ be a finite index subgroup of $SL(n,\mathbb{Z})$.

Proposition 2.1. *Let ρ be a $C^{1+\alpha}$, $\alpha > 0$ Γ action on $\mathbb{T}^n$ with standard homotopy data. Assume that ρ preserves an ergodic measure μ whose support is not contained in a ball. Then μ is absolutely continuous and there is a finite index subgroup Γ' of Γ and a continuous map $h : \mathbb{T}^n \to \mathbb{T}^n$ such that for every $\gamma \in \Gamma'$.*

$$h \circ \rho(\gamma) = \rho_0(\gamma) \circ h. \tag{2.1}$$

and

$$h_*\mu = \lambda. \tag{2.2}$$

Furthermore, μ is unique measure satisfying (2.2).

Proof. The scheme of the proof is as follows: First we construct a measurable map h defined μ almost everywhere and satisfying (2.1). Then we show that h extends to a continuous map defined on $\operatorname{supp} \mu$. Next we show that the image of this map is a complement to a finite set and (2.2) holds. Finally we show that h extends uniquely from $\operatorname{supp} \mu$ to the whole torus.

Once we get this we argue as follows. Let $\nu = h_*\mu$; the measure ν is invariant and ergodic with respect to the linear Γ' action, hence it is either Lebesgue or atomic. It will not be atomic (see bellow) hence ν is Lebesgue measure λ, this means that μ is a large measure as defined in [8], i.e $h_*\mu = \lambda$. Since for any Cartan subgroup $C \subset SL(n, \mathbb{R})$ large invariant measure is unique and absolutely continuous by [8], if we take a Cartan subgroup $C \subset \Gamma'$ (which can be done since Γ' is of finite index in $SL(n, \mathbb{Z})$), measure μ must coincide with this large measure and absolute continuity and uniqueness follow.

Step 1. We shall use the following consequence of Zimmer's cocycle super-rigidity (see the proof of the Main Theorem in [2]):

Lemma 2.2. *Under the assumptions of Proposition 2.1 there is a measurable map $\phi : \mathbb{T}^n \to \mathbb{R}^n$ defined μ a.e. such that if we put $h_0(x) = x + \phi(x)$ then (2.1) holds for h_0, i.e. $h_0 \circ \rho(\gamma) = \rho_0(\gamma) \circ h_0$ for every $\gamma \in \Gamma'$ for some finite index subgroup $\Gamma' \subset \Gamma$.*

Let A be a set of full μ measure where the equality (2.1) holds for h_0. Without loss of generality we may assume that A is invariant with respect to $\rho(\gamma)$ for every $\gamma \in \Gamma'$.

Step 2. Now we prove that ϕ extends to a continuous map $\operatorname{supp} \mu \to \mathbb{R}^n$.

Let $\gamma \in \Gamma'$ be a hyperbolic matrix so that $\rho_0(\gamma)$ is an Anosov linear map. Then there is a continuous map $\phi_\gamma : \mathbb{T}^n \to \mathbb{R}^n$ such that for $h_\gamma(x) = x + \phi_\gamma(x)$ one has $h_\gamma \circ \rho(\gamma) = \rho_0(\gamma) \circ h_\gamma$. Let $v(x) = \phi_\gamma - \phi$. $v \circ \rho(\gamma) = \gamma v$ on $\mathbb{T}^n$, i.e. there is an integer $c \in \mathbb{Z}^n$ such that $v \circ \rho(\gamma) = \gamma v + c$. This implies that $v(x)$ is constant μ a.e. Indeed, let us write $c = (I - \gamma)c'$ and put $v' = v - c'$, then

$$v' \circ \rho(\gamma) = v \circ \rho(\gamma) - c' = \gamma v + c - c' = \gamma v - \gamma c' = \gamma v'.$$

Hence it is enough to see that v' is constant μ-a.e. Let L_C be the set where $|v'(x)| < C$, taking C large L_C has measure as close to 1 as wanted. Call $f = \rho(\gamma)$ and take $x \in L_C$ such that x returns infinitely many times to L_C in the future and the past. This set has full μ measure in L_C by Poincaré recurrence. Then $v'(f^n(x)) = \gamma^n v'(x)$. Observe that $\mathbb{R}^n$ splits as $E^u_\gamma \oplus E^s_\gamma$. Let $v'(x)$ decompose as $v'^s(x) + v'^u(x)$. Then

$$C \geq |v'(f^n(x))| = |\gamma^n v'(x)| \geq K|\gamma^n v'^u(x)| \geq K\lambda^n |v'^u(x)|$$

for some $\lambda > 1$ which implies that $v'^u(x) = 0$. Reversing time we obtain $v'^s(x) = 0$.

Hence $\phi = \phi_\gamma - c'$ μ a.e. and ϕ extends continuously to the support of μ; we will still denote this extended map by ϕ. Let $h : \mathrm{supp}(\mu) \to \mathbb{T}^n$, $h(x) = x + \phi(x)$, then we get that $h \circ \rho(\gamma) = \rho_0(\gamma) \circ h$ for every $\gamma \in \Gamma'$ since $h = h_0$ on a set of full μ measure. So, let us forget about h_0 and work with h. Notice that for every hyperbolic $\gamma \in \Gamma'$ there is h_γ homotopic to the identity such that $h_\gamma \circ \rho(\gamma) = \rho_0(\gamma) \circ h_\gamma$ and arguing as above we get that $h = h_\gamma + c'$ for some $c' \in (I - \Gamma)^{-1}(\mathbb{Z}^n)$ on the support of μ. Observe also that changing h_γ with $h_\gamma + c'$ we get that $h_\gamma + c'$ also conjugates $\rho(\gamma)$ with γ. Hence we may assume already that h and h_γ coincide on $\mathrm{supp}\,\mu$ for every $\gamma \in \Gamma'$ hyperbolic. Since hyperbolic elements generate a finite index subgroup of Γ' we obtained desired map $h : \mathrm{supp}\,\mu \to \mathbb{T}^n$.

Step 3. Let $\nu = h_*\mu$ be the push-forward of the measure μ. Obviously ν is invariant and ergodic with respect to ρ_0 restricted to Γ'. The only ergodic invariant measures for the Γ' linear action ρ_0 are Lebesgue measure and measures supported on periodic orbits. Let us show that ν cannot have finite support. If this were the case, the support of μ would belong to the union of pre-images of finitely many points. Take an element $\gamma \in \Gamma'$ inside a Cartan subgroup. Then we have that $h = h_\gamma$ and hence the pre-images of h are inside the pre-images by h_γ. Now, the results in [8] gives us that the pre-image of every point by h_γ is inside a ball, indeed the pre-image of every point is the intersection of nested cubes. Hence the support of the measure is inside a finite disjoint union of cubes. Now, it is not hard to see that this finite disjoint union of cubes fit inside a ball, which implies that the support of μ is inside a ball, a contradiction.

Thus we get that ν is Lebesgue measure and hence for every Anosov element of ρ_0, μ is a large measure as defined in [8].

Step 4. Let us see that h can be extended to the rest of the torus conjugating the whole action. Let us make the following observations. Let R be the set of regular points which we may assume that are regular for all hyperbolic elements of the action. It has full μ measure. Then for every x in R and for every $\gamma \in \Gamma$ belonging to a Cartan subgroup we have that $W^s_\gamma(x) \subset \mathrm{supp}(\mu)$.

Let U be a connected component of the complement of $\mathrm{supp}(\mu)$. Let us see that for every $\gamma \in \Gamma$ an element in a Cartan subgroup C_γ, $h_\gamma(U)$ is a point and moreover this point does not depend on γ. Fix first γ and take a point $x \in U$. We know from [8] that x lies inside the intersection of a family of nested cubes C_n such that $\bigcap C_n = (h_\gamma)^{-1} h_\gamma(x)$ and $h_\gamma(C_n)$ is a family of cubes such that $\bigcap h_\gamma(C_n) = h_\gamma(x)$. The boundary of the cubes C_n are formed by pieces of stable manifolds of regular elements with respect to different elements of C_γ. Hence the boundary of C_n is in $\mathrm{supp}(\mu)$ and since U is connected and $U \cap \mathrm{supp}(\mu) = \emptyset$ we get that $U \subset C_n$. In particular $h_\gamma(U) = h_\gamma(x)$. So the whole U is collapsed by h_γ into a single point. Let us take now another Cartan element γ'. Then $h_{\gamma'}(U) = h_{\gamma'}(x)$ also. Now, we want to proof that $h_\gamma(x) = h_{\gamma'}(x)$. But the boundary of U is in the support of μ and on the support of μ we know that h_γ, $h_{\gamma'}$ and h coincide. Hence

$$h_\gamma(x) = h_\gamma(U) = h_\gamma(\partial U) = h(\partial U) = h_{\gamma'}(\partial U) = h_{\gamma'}(U) = h_{\gamma'}(x).$$

Since Cartan subgroups generate $SL(n, \mathbb{Z})$ this finishes the proof. □

The next step in the proof of Theorem 1.1 is finding a periodic orbit for the action ρ.

Proposition 2.3. *Let ρ be a Γ action preserving a large invariant measure μ as in Proposition 2.1. Then if Γ' is the finite index subgroup provided by Proposition*

2.1, there is a periodic point $p \in \operatorname{supp}\mu$ for the action ρ restricted to Γ' such that the derivative of ρ at p coincides with the standard linear action on $\mathbb{R}^n$.

Proof. Let us recall another result from [8].

Theorem 2.4. *Given an action of $\mathbb{Z}^{n-1}$ on $\mathbb{T}^n$ with Cartan homotopy data, there is a proper periodic point, that is, if h_η is a semiconjugacy with the linear action (work with a finite index subgroup if needed) then there is a periodic point p for the action inside the support of the large measure such that $(h_\eta)^{-1}(h_\eta(p)) = p$. Moreover the Lyapunov exponents for this point coincide with the Lyapunov exponents of the linear map.*

Recall that by the proof of Proposition 2.1 Step 2, we may assume that the semiconjugacy h coincides with h_η for every $\eta \in \Gamma'$. Hence, if p is a proper periodic point for a Cartan action as in Theorem 2.4 then $p = h^{-1}(h(p))$. Call $q = h(p)$ and observe that since q is periodic for a linear Anosov map it has to be a rational point, hence it is periodic for the whole linear $SL(n,\mathbb{Z})$ action. Finally, since $h \circ \rho(\gamma) = \rho_0(\gamma) \circ h$ for every $\gamma \in \Gamma'$ we have that p is periodic for the ρ action restricted to Γ'. So, we may take a finite index subgroup $\Gamma'' \subset \Gamma'$ and assume that p is fixed. We will assume that Γ' already equals Γ''. Since the derivative cocycle (homomorphism) at p is nontrivial by Theorem 2.4, Margulis super-rigidity Theorem implies that the derivative at p coincides with the linear one up to some conjugacy that we may assume to be trivial by taking an appropriate coordinate chart at p. ☐

The next step is a linearization of our action in a neighborhood of the periodic point p. This is the only place where we use real analyticity of the action. We shall use the local linearization theorem of G. Cairns and E. Ghys for real analytic actions, [1].

Theorem 2.5. *Let Γ be any irreducible lattice in a connected semi-simple Lie group with finite center, no non-trivial compact factor group and of rank greater than 1. Every C^ω-action of Γ on $(\mathbb{R}^m, 0)$ is linearizable.*

In particular, Theorem 2.5 applies to Γ', a finite index subgroup of $SL(n,\mathbb{Z})$. In this case the linear action is conjugate to the standard action. So this theorem gives us a real analytic map $H : U \to \mathbb{T}^n$ where $U \subset \mathbb{T}^n$ is a neighborhood of q such that $H \circ \rho_0(\gamma)(x) = \rho(\gamma) \circ H(x)$ for every $\gamma \in \Gamma'$ and for every x where the above equality make sense. $H(q) = p$ and we may assume without loss of generality that $D_qH = Id$.

To finish the proof of Theorem 1.1 it is sufficient to show that the semiconjugacy h coincides with H^{-1} and hence it is analytic and invertible in a neighborhood V of p. For, then the set of injectivity for h contains a ρ-invariant open set $\mathcal{V} = \bigcup_{\gamma\in\Gamma'} \rho(\gamma)V$. The set $h(\mathcal{V})$ is open and ρ_0-invariant, hence is complement is finite. But h is real analytic on $\mathcal{V}$ since on $\rho(\gamma)V$ it coincides with $\rho_0(\gamma) \circ H^{-1} \circ \rho(\gamma)$.

Let us see that H^{-1} coincides with h in a neighborhood of p. Let $\gamma \in \Gamma'$ be an element in a Cartan subgroup C_γ and assume its stable manifold for the linear element is one dimensional. Call $W^s_\gamma(p)$ the stable manifold of p for $\rho(\gamma)$ and E^s_γ the stable space for $\rho_0(\gamma)$. C_γ acts locally transitively on $W^s_\gamma(p)$ and on $q + E^s_\gamma$. By uniqueness of the invariant manifolds, we have that

$$H(q + E^s_\gamma \cap B_\varepsilon(q)) \subset W^s_\gamma(p).$$

We know from the results in [8] that $h = h_\gamma$ is smooth at $W^s_\gamma(p)$. Hence we have that H^{-1} coincides with h along $W^s_\gamma(p) \cap B_\delta(p)$, where δ is such that $H(B_\varepsilon(q)) \supset B_\delta(p)$.

Now, $A = \bigcup_\gamma q + E^s_\gamma \cap B_\varepsilon(q)$ where $\gamma \in \Gamma'$ ranges over the elements belonging to a Cartan subgroup and with one-dimensional stable manifold is dense in $B_\varepsilon(q)$. This is because the projective action on S^{n-1} is minimal and A corresponds with the orbit of a point in S^{n-1}, namely the orbit of the direction associated to the stable manifold of one of this γ's. Hence $H(A)$ is dense in $B_\delta(p)$. On the other hand h and H^{-1} coincide on $H(A)$ hence by continuity they coincide on $B_\delta(p)$ which finishes the proof of statements (1) and (2) of Theorem 1.1.

To prove statement (3) notice that $SL(n,\mathbb{Z})$ is generated by its maximal Cartan subgroups. For any such subgroup C the semi-conjugacy P conjugates restriction of ρ to C with an affine action ρ_C with standard homotopy data (see [4] for a detailed proof). Elements of such an action ρ_C are compositions of automorphisms and rational translations. In particular, ρ_C preserves the finite set Fix_C of fixed points of ρ_0 restricted to C. Since the intersection of those fixed point sets for different Cartan subgroups is the identity, this implies that the only affine action of $SL(n,\mathbb{Z})$ with standard homotopy data is ρ_0.

Thus for any Cartan subgroup C, the action ρ preserves the set $P^{-1}(Fix_C)$ and hence their intersection $P^{-1}(0)$. This implies that ρ_C coincides with ρ_0 for any Cartan subgroup C and statement (3) follows.

3. Remarks and open problems.

A. Katok, J. Lewis and R. Zimmer, in the sequence of papers [5, 6, 7] proved that smooth, measure preserving, Anosov, $SL(n,\mathbb{Z})$ actions on $\mathbb{T}^n$ are conjugated to the linear action (the measure is assumed to be fully supported but something like our assumption will work already in their case). This in particular implies local rigidity for linear actions.

A. Zeghib [11] announced and gave a sketch of another proof of C^1 local rigidity for real analytic perturbations of the linear $SL(n,\mathbb{Z})$ action on $\mathbb{T}^n$ using the local linearization of [1]. He also announced some global rigidity result for real analytic actions with standard homotopy data similar to ours (i.e. existence of a smooth conjugacy outside a finite set), but with the additional assumption of the existence of a periodic orbit in the support of a non-atomic invariant measure. After this announcement, a preprint appeared with some more details of the sketches. Zeghib recognizes that his result is a reduction of the rigidity problem to existence of a periodic orbit.

Let us point out that, while we in principle can use Zeghib's method in place of the local transitivity argument at the last stage of our proof, this last part is only one part of our arguments, the principal ones being Propositions 2.1 and 2.3 based on hard results from [8]. Furthermore, our method for that last part has a chance to work in smooth category while Zeghib's is completely dependent on the analyticity assumption.

With a more careful analysis of the structure of finite index subgroups of $SL(n,\mathbb{Z})$ one can very likely prove that the semiconjugacy P also serves as a semiconjugacy between the whole action ρ and an affine action $\tilde{\rho}_0$ with the standard homotopy data. Basically the question reduces to consideration of finite order elements of Γ that are not products of elements of Cartan subgroups.

A much more interesting question concerns restrictions of ρ_0 to infinite index subgroups whose linear representations are rigid, such as integer lattices in other higher rank simple Lie groups or irreducible representations of $SL(n,\mathbb{Z})$ into $SL(N,\mathbb{Z})$ for large N. While Cairns-Ghys Theorem 2.5 is available in those settings, the key ingredient, a weaker form of rigidity for maximal abelian subgroups like in [3, 8], is missing. Progress in this direction beyond the Cartan case has been achieved recently in [9] but the strong simplicity condition of that paper is not satisfied in the interesting cases mentioned above. Furthermore, in some cases, such as the representation of $SL(n,\mathbb{Z})$ into $SL(n^2-1,\mathbb{Z})$ given by the adjoint action on traceless matrices, there are no Anosov elements altogether and one should hope to tie elements of rigidity for partially hyperbolic abelian subgroups together to produce rigidity for the whole action.

A more immediate and probably more accessible issue is extension of our results to the differentiable case. Here the situation is reversed: rigidity for Cartan actions is proven but local linearization is not, and it may not even be true. Rigidity for Cartan actions provides extensive information about the semi-conjugacy P. Already for a Cartan subgroup, it is smooth in Whitney sense on sets whose measure is arbitrary close to full measure, see [9]. Those sets include grids of codimension one smooth submanifolds that divide the space into "boxes", most of them small. Superimposing those pictures for different Cartan subgroups provides such grids in a dense set of directions. However, certain elements of uniformity needed to conclude smoothness on an open set, are still lacking.

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