Rigidity of symplectic Anosov diffeomorphisms on low dimensional tori

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Abstract. We show that any symplectic Anosov diffeomorphism of a four torus T^4 with sufficiently smooth stable and unstable foliations is smoothly conjugate to a linear hyperbolic automorphism of T^4.

1. *Introduction*

A diffeomorphism ϕ of a closed connected manifold M is called an *Anosov diffeomorphism* if the tangent bundle of M splits as a direct sum of two sub-bundles E^- and E^+ and the tangent map ϕ^* contracts the bundle E^- and expands the bundle E^-. More precisely, the latter property means that if we endow M with a Riemannian metric there exist constants $C > 0$ and $0 < \kappa < 1$ for which
(1) $\|\phi^{*n}\xi_s\| < C\kappa^n\|\xi_s\|$ for all $\xi_s \in E^-$ and positive integers n and
(2) $\|\phi^{*n}\xi_u\| < C\kappa^{-n}\|\xi_u\|$ for all $\xi_u \in E^+$ and negative integers n.
One can easily verify that the definition does not depend upon a particular choice of a metric and that the splitting $TM = E^- \oplus E^+$ is continuous (in fact, Hölder continuous). The bundle E^- is called the *stable or contracting bundle*; similarly, E^+ is called the *unstable or expanding bundle*.

In all known examples of Anosov diffeomorphisms, M is an infra-nilmanifold, i.e. a finite factor of $\Gamma\backslash N$ where N is a simply connected nilpotent Lie group and Γ a uniform lattice in N. Franks and Manning [**Fr, Ma**] proved that any Anosov diffeomorphism on an infra-nilmanifold M is topologically conjugate to an algebraic model, i.e. to a map that is induced by a hyperbolic automorphism of the Lie algebra of N. In general, the topological conjugacy is only Hölder continuous and need not be any smoother; to assure that, it is enough to make the eigenvalues of the linearization at corresponding periodic points different. On the other hand, we believe that the following fact is true:

CONJECTURE. *If the stable and unstable bundles of a C^k Anosov diffeomorphism ϕ on a compact manifold M are C^2 then M is an infra-nilmanifold and ϕ is $C^{\max(2,k)}$*

conjugate to an hyperbolic automorphism of M. (*See* [K] *for a similar conjecture for geodesic flows*).

In [Hu-Ka], Hurder and Katok proved, among other things, this conjecture for C^∞ Anosov diffeomorphisms in dimension two. Ghys has proved a similar result for geodesic flows on surfaces [Gh].

Recently Kanai [K] introduced a method which allowed him to show that the geodesic flow on a compact Riemannian manifold M with curvature strictly pinched between -1 and $-9/4$ is C^∞ conjugate to the geodesic flow on a manifold of constant negative curvature if (and only if) the geodesic flow on M has C^∞ stable and unstable foliations. In fact Kanai's method also provides an easier proof of Ghys's result (compare with Proposition 1 below). Feres and Katok have improved the pinching condition in Kanai's theorem allowing the curvature to range strictly between -1 and -4 [FeKa1] and removing the pinching assumption first for dim $M = 3$ [FeKa2], and finally for arbitrary odd dimension [Fe].

In this article we use Kanai's technique, together with a result from [FeKa2] to investigate the structure of symplectic Anosov diffeomorphisms in dimensions two and four. In the former case we improve on the smoothness assumption in [HuKa]:

THEOREM 1. *Let γ be an area-preserving Anosov C^1 diffeomorphism of the two-torus T^2. Assume that the stable and unstable foliations of γ are of class C^r for $r \geq 2$. Then γ is C^r conjugate to the action of a linear map.*

Remarks. (a) The previous theorem is optimal because from [HuKa] it is known that the stable and unstable foliations of a C^3 Anosov diffeomorphism are $C^{2-\varepsilon}$ for every $\varepsilon > 0$; more exactly they are $C^{1+x\log x}$. More importantly, our proof is self-contained in the sense that it does not use the deep Herman Linearization Theorem for diffeomorphisms of the circle.

(b) A simple variation on the proof of the above Theorem also yields a new and simpler proof of Ghys's result in [Gh].

The main result of this article is the following theorem:

THEOREM 2. *Let γ a C^1 be an Anosov diffeomorphism of the four-dimensional torus T^4 preserving a smooth symplectic form Ω on T^4. Assume that the stable and unstable foliations of γ are of class C^∞. Then γ is C^∞ conjugate to a linear automorphism of T^4.*

2. Preliminaries

We recall the definition of the Kanai connection. Let (M, Ω) be a smooth symplectic manifold, with M closed and connected, and let $\mathcal{F}^+$ and $\mathcal{F}^-$ be two transversal Lagrangian foliations of class C^r. Denote by E^+ and E^- the bundles of vectors tangent to the foliations $\mathcal{F}^+$ and $\mathcal{F}^-$, respectively. Kanai's connection is the unique C^{r-1} linear connection ∇ on the tangent bundle TM that satisfies the following properties:

(i) $\nabla\Omega = 0$;
(ii) the connection ∇ is torsionless;
(iii) $\nabla E^+ \subset E^+$ and $\nabla E^- \subset E^-$.

To see that there exists a connection that satisfies the properties above let us introduce the $(1, 1)$ tensor field ι defined by the involution of TM

$$\iota : (\xi, \eta) \in E^+ \oplus E^- \mapsto (\xi, -\eta) \in E^+ \oplus E^-. \tag{1}$$

The tensor field ι is as smooth as the foliations $\mathcal{F}^+$ and $\mathcal{F}^-$. If we consider the identity mapping of TM as a $(1, 1)$ tensor field I we see that $(I + \iota)/2$ and $(I - \iota)/2$ are the projections of $TM = E^+ \oplus E^-$ onto E^+ and E^-. Therefore condition (iii) is equivalent to

$$\nabla \iota = 0. \tag{iii'}$$

We define a non-degenerate symmetric tensor field g on M by setting $g(X, Y) = \Omega(X, \iota Y)$ for all vector fields X and Y. We claim that the Levi-Civita connection ∇ induced by g coincides with the Kanai connection. Indeed, as the Levi-Civita connection is torsionless, condition (ii) is satisfied. Since $\nabla g = 0$, if we show that the condition (iii') is also satisfied, that is $\nabla \iota = 0$, then the condition (i) will follow immediately and we have proved our claim. To prove (iii'), it suffices to show that

$$g(\nabla_X \iota Y - \iota \nabla_X Y, Z) = 0, \tag{iii''}$$

whenever X, Y, Z are commuting vector fields and each of them is a section of E^+ or E^-.

Note that if X, Y and Z are as in (iii''), then the six vector fields X, Y, Z, ιX, ιY and ιZ commute. From the definition of g and the formula for the Levi-Civita connection, we obtain

$$2g(\nabla_X \iota Y - \iota \nabla_X Y, Z) = 2g(\nabla_X \iota Y, Z) + 2g(\nabla_X Y, \iota Z)$$

$$= Xg(\iota Y, Z) + (\iota Y)g(X, Z) - Zg(X, \iota Y)$$

$$+ Xg(Y, \iota Z) + Yg(X, \iota Z) - (\iota Z)g(X, Y)$$

$$= (\iota Y)(\Omega(X, \iota Z)) + Y(\Omega(X, Z))$$

$$- Z(\Omega(X, Y)) - (\iota Z)(\Omega(X, \iota Y)). \tag{2}$$

Since the form Ω is closed one has

$$U\Omega(V, W) + V\Omega(W, U) + W\Omega(U, V) = 0, \tag{3}$$

whenever U, V and W commute. By applying (3) twice to (2), we obtain

$$2g(\nabla_X \iota Y - \iota \nabla_X Y, Z) = X\Omega(\iota Y, \iota Z) + Y\Omega(X, Z) - Z\Omega(X, Y) = 0$$

Thus (iii'') holds and ∇ is Kanai's connection.

Conversely, if ∇ denotes Kanai's connection then the symmetric form g is parallel and therefore ∇ is also the Levi-Civita connection for g. Thus we have shown that Kanai's connection exists and is unique.

Parallel transport along the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$ for Kanai's connection has a simple geometric description. In fact, consider a curve $\gamma(t)$ included in a leaf of $\mathcal{F}^-$ and let X_0 be a vector tangential to $\mathcal{F}^+$ at $\gamma(0)$. Let $F^+(t)$ be the leaf of $\mathcal{F}^+$ passing through the point $\gamma(t)$. The holonomy map along the leaves of the foliation

$\mathscr{F}^-$ induces a local diffeomorphism of a neighbourhood of $\gamma(0)$ in $F^+(0)$ with a neighbourhood of $\gamma(t)$ in $F^+(t)$. The derivative of the holonomy map sends X_0 to a vector X_t at $\gamma(t)$ tangent to the leaf $F^+(t)$. The curve $t \mapsto X_t$ is the parallel transport of X_0 along $\gamma(t)$. In this way we have described the parallel transport of E^+ along the leaves of $\mathscr{F}^-$. The parallel transport of E^- along a leaf of $\mathscr{F}^-$ can now be defined using the symplectic form: namely, a vector field $Y_t \subset E^-$ along a curve $\gamma(t)$ included in a leaf of $\mathscr{F}^-$ is parallel if the Ω product with any parallel vector field of E^+ is constant. One can see from this description that the curvature of Kanai's connection is the obstruction to the existence of local coordinates $(p_1, \ldots, p_n, q^1, \ldots, q^n)$ such that the surfaces $q^i = \mathrm{const}_i$, $i = 1, \ldots, n$, are locally leaves of $\mathscr{F}^+$, the surfaces $p_i = \mathrm{const}_i$, $i = 1, \ldots, n$, are locally leaves of $\mathscr{F}^-$ and $\Omega = dp_i \wedge dq^i$.

We want to consider only the case when Kanai's connection is invariant under its own parallelism. When this is not the case, we replace Kanai's connection with a new connection invariant by its parallelism and still satisfying the properties (i) and (ii). Let ∇ be either Kanai's or this new connection and let R and T be its curvature and torsion tensors, respectively. It is well known that ∇ is invariant under its parallelism if and only if $\nabla T = 0$ and $\nabla R = 0$. In this situation every point of M has a neighbourhood affinely diffeomorphic to a neighbourhood of a reductive homogeneous space G/A endowed with its canonical connection [No, Prop. 18.1] [KoNo, vol. 2, Prop. 2.4]. If the connection ∇ on M is complete (i.e. if every tangent vector is the initial velocity of a geodesic arc $t \in [0, 1] \mapsto \gamma(t)$) then the universal cover of M, which we denote by $\tilde{M}$, is affinely diffeomorphic to G/A [KoNo, vol. 1, Prop. 7.8]. Since the group of all affine transformations of $\tilde{M}$ preserving Ω and the foliations acts transitively on $\tilde{M}$ (cf. [KoNo, Cor. 7.9]), one can take this group as G; then the subgroup A is identified with the stabilizer of some point $p_0 \in \tilde{M}$. The fundamental group of M acts affinely on $\tilde{M}$ and therefore is represented faithfully as a discrete subgroup Π of G acting without fixed points on G/A.

The Lie algebras $\mathbf{g}$ of G and $\mathbf{a}$ of A are given in terms of the torsion and curvature tensors T and R. Let us chose a reference point $p_0 \in M$ and denote by $\mathbf{p}^+$ and $\mathbf{p}^-$ the tangent spaces $E_{p_0}^+$ and $E_{p_0}^-$. Let us also denote by $\mathbf{a}$ the linear subspace of all endomorphisms $\mathscr{A}$ of $T_{p_0}M = E_{p_0}^+ \oplus E_{p_0}^-$ preserving the splitting $E_{p_0}^+ \oplus E_{p_0}^-$ and satisfying the identity

$$R(\mathscr{A}\xi, \eta) + R(\xi, \mathscr{A}\eta) = \mathscr{A} \circ R(\xi, \eta) - R(\xi, \eta) \circ \mathscr{A} \quad \text{for all } \xi, \eta \in T_{p_0}M.$$

Then we have

$$\mathbf{g} = \mathbf{p}^+ \oplus \mathbf{p}^- \oplus \mathbf{a}$$

and the commutation relations are given by

$$[\xi, \eta] = -R(\xi, \eta) - T(\xi, \eta), \quad [\mathscr{A}, \xi] = \mathscr{A}\xi, \quad [\mathscr{A}, \mathscr{B}] = \mathscr{A}\mathscr{B} - \mathscr{B}\mathscr{A}, \qquad (4)$$

where $\xi, \eta \in T_{p_0}M$ and $\mathscr{A}, \mathscr{B} \in \mathbf{a}$.

It follows from the invariance of ∇ under its parallel transport that the Lie algebra above is independent of a choice of the reference point p_0.

Conversely, assume that **g** is a Lie algebra that splits linearly as $\mathbf{g} = \mathbf{a} \oplus \mathbf{p}^+ \oplus \mathbf{p}^-$ and satisfies the following conditions

$$[\mathbf{a}, \mathbf{a}] \subset \mathbf{a}, [\mathbf{a}, \mathbf{p}^+] \subset \mathbf{p}^+, [\mathbf{a}, \mathbf{p}^-] \subset \mathbf{p}^-, [\mathbf{p}^+, \mathbf{p}^+] = 0, [\mathbf{p}^-, \mathbf{p}^-] = 0; \qquad (5.1)$$

$$\text{if } \mathcal{A} \in \mathbf{a} \text{ and } [\mathcal{A}, \mathbf{p}^+] = [\mathcal{A}, \mathbf{p}^-] = 0 \text{ then } \mathcal{A} = 0; \qquad (5.2)$$

there exists $\mathcal{A}_0 \in \mathbf{a}$ with the property that $[\mathcal{A}_0, \mathbf{a}] = 0$ and $\mathrm{ad}\mathcal{A}_0$ has roots with negative real part on $\mathbf{p}^-$ and positive real part on $\mathbf{p}^+$. (5.3)

Let G be a connected Lie group with Lie algebra **g** and A the Lie subgroup generated by **a**. Then A is closed. The connection on the principal A bundle $G \to G/A$ which is defined by choosing as horizontal distribution the left translations under G of $\mathbf{p}^+ \oplus \mathbf{p}^-$ is invariant under its parallelism. Denote by P^+ and P^- the subgroups of G generated by $\mathbf{p}^+$ and $\mathbf{p}^-$. Then, since $P\pm$ are normalized by A, the orbit foliations of the right actions of P^+ and P^- on G project to two transversal foliations $\hat{\mathscr{F}}^+$ and $\hat{\mathscr{F}}^-$ of G/A. Furthermore the leaves of the foliations $\hat{\mathscr{F}}^\pm$ are totally geodesic submanifolds of G/A. For future reference we point out that *any leaf of $\hat{\mathscr{F}}^+$ intersects any leaf of $\hat{\mathscr{F}}^-$ at most in one point.* For a proof of this simple fact see Lemma 3.5 of [K].

Let G be a connected Lie group and A be a connected Lie subgroup of G and let **g** and **a** denote their Lie algebras. Assume that there exists abelian subalgebras $\mathbf{p}^+$ and $\mathbf{p}^-$ of **g** such that $g = \mathbf{a} + \mathbf{p}^+ + \mathbf{p}^-$ and the conditions (5.1)–(3) are satisfied. Then we shall call G/A endowed with its canonical connection a *bi-polarized homogeneous space.*

Proof of Theorem 1

PROPOSITION 1. *Let (M, Ω) be a smooth two-dimensional symplectic manifold, $\mathscr{F}^+$ and $\mathscr{F}^-$ be two transversal foliations of class C^2 and let ∇ denote the Kanai connection. Assume that Γ is a group of C^1 diffeomorphisms of M which preserves Ω and the foliations and whose action on M is ergodic with respect to the measure induced by Ω. Then ∇ is locally symmetric.*

Proof. The curvature tensor R of ∇ is a continuous tensor field on M. Since Γ is a group of diffeomorphisms of M preserving both the form Ω and the foliations $\mathscr{F}^\pm$, the uniqueness of ∇ implies that this connection is invariant by the action of Γ. At a given point $p \in M$ the expression

$$\frac{\Omega(R(\xi, \eta)\xi, \eta)}{\Omega(\xi, \eta)^2}$$

is independent of the choice of $\xi \in E_p^+$, $\eta \in E_p^-$ and hence defines a continuous function on M. This function is invariant under the action of Γ and therefore, by the ergodicity of Γ, it is everywhere equal to a constant k. The usual symmetries of R plus the identity

$$\Omega(R(\xi, \eta)\zeta, \theta) + \Omega(\zeta, R(\xi, \eta)\theta) = 0$$

allow us to conclude that

$$R = k\Omega \otimes \iota \qquad (6)$$

where ι is the C^2 tensor field defined in (1). It follows that R is differentiable and since $\nabla\Omega = 0$ and $\nabla\iota = 0$ we also have $\nabla R = 0$.   □

It is well known (and easy to see) that the only connected compact two-dimensional manifold admitting an Anosov diffeomorphism is the two-torus T^2. Thus, we now assume that Γ is the cyclic group generated by an area-preserving Anosov diffeomorphism γ of T^2. Then we have:

PROPOSITION 2. *Let γ be a symplectic Anosov diffeomorphism of the two-torus T^2 with stable and unstable foliations of class C^2. Then Kanai's connection ∇ is complete and flat.*

PROOF. By Proposition 1, ∇ is locally symmetric. We postpone the proof of its completeness till the next section, since it holds in greater generality. Assuming completeness, it follows that ∇ lifts to a connection $\tilde{\nabla}$ on $\mathbf{R}^2$ and that $\mathbf{R}^2$ is diffeomorphic to G/A. Here G is the Lie group of affine symplectic transformations of $(\mathbf{R}^2, \tilde{\nabla})$ which preseve Ω and the lift of the foliations $\mathscr{F}^\pm$ and A is the stabilizer in G of a point $p_0 \in \mathbf{R}^2$. Furthermore the fundamental group of T^2 acts by affine transformations on $(\mathbf{R}^2, \tilde{\nabla})$. Hence there is a discrete group $\Pi \subset G$ isomorphic to $\mathbf{Z}^2$ such that $\Pi\backslash G/A$ is diffeomorphic to T^2.

The Lie algebra **g** is determined by the curvature tensor R via the formulas (4). Since $R = k\Omega\otimes\iota (k \in \mathbf{R})$ we see that if ∇ is not flat, then **g** is isomorphic to $\mathbf{sl}_2(\mathbf{R})$, the Lie algebra of traceless 2×2 real matrices, and **a** can be taken to be the subalgebra of diagonal matrices. We shall show that this implies a contradiction and hence that ∇ is flat.

Let G^0 and A^0 denote the components of the identity of G and A. Since $G/A \approx \mathbf{R}^2$ is connected and simply connected we have $A^0 = A \cap G^0$. The adjoint action maps G^0 onto the group of inner automorphisms of $\mathbf{sl}_2(\mathbf{R})$, which we identify with $PSL_2(\mathbf{R})$, sending A^0 onto a Cartan subgroup C of $PSL_2(\mathbf{R})$. So we have a covering map

$$G/A \approx G^0/A^0 \to PSL_2(\mathbf{R})/C.$$

Since G is a finite cover of G^0, by possibly passing to a subgroup of finite index in Π, we can assume that the adjoint action maps Π onto a subgroup Π^* in $PSL_2(\mathbf{R})$. But two elements in $PSL_2(\mathbf{R})$ commute if and only if they belong to some one-parameter group. It follows that Π^* is included in a one-parameter group $(g_t) \subset PSL_2(\mathbf{R})$ and that we have a continuous map

$$\Pi\backslash G/A \to (g_t)\backslash PSL_2(\mathbf{R})/C.$$

However it can be easily seen that $(g_t)\backslash PSL_2(\mathbf{R})/C$ is never compact and this contradicts the assumption that $\Pi\backslash G/A$ is a torus.   □

Proof of Theorem 1. Now, let $\tilde{\gamma}$ be a lift of γ to $\mathbf{R}^2$ and p_0 be the fixed point of $\tilde{\gamma}$. Pick non-zero vectors $X^u_{p_0} \in E^u_{p_0}$ and $X^s_{p_0} \in E^s_{p_0}$. Since the connection is flat, we can transport $X^u_{p_0}$ and $X^s_{p_0}$ in parallel to any point in $\mathbf{R}^2$ hence defining two vector fields X^u and X^s on $\mathbf{R}^2$. The fields X^u and X^s are C^{r-1} because the connection is C^{r-1}. Since the connection is torsionless, X^u and X^s, as well as the flows that they generate, commute. In particular we see that X^u and X^s are invariant by the holonomy of

the C^r foliations $\mathscr{F}^+$ and F^- and hence are themselves C^r. For $(t_1, t_2) \in \mathbf{R}^2$, we set $\langle t_1, t_2 \rangle$ to be the point $\exp t_1 X^u \circ \exp t_2 X^s(p_0)$. It follows from the fact that $\tilde{\gamma}$ acts affinely on $\mathbf{R}^2$ that in the above coordinates γ has the form

$$\langle t_1, t_2 \rangle \mapsto \langle \lambda t_1, \lambda^{-1} t_2 \rangle.$$

Let π be an element of the fundamental group Π. Since π acts on $\mathbf{R}^2$ affinely and preserves the foliations W^u and W^s as well as the area, π is given in our coordinates by

$$\langle t_1, t_2 \rangle \mapsto \langle A(\pi)t_1 + B_1(\pi), A^{-1}(\pi)t_2 + B_2(\pi) \rangle.$$

If $A(\pi) \neq 1$, then π has a fixed point and this contradicts the fact that the fundamental group acts on $\mathbf{R}^2$ without fixed points. Hence π is a pure translation and Π leaves the vector fields X^u and X^s invariant. Hence X^u and X^s project to vector fields on T^2. This concludes the proof of Theorem 1. $\qquad\square$

3. Completeness

In this section we prove the following proposition which completes the proof of Theorem 1 and is also necessary to the proof of Theorem 2.

PROPOSITION 3. *Let* (M, ∇) *be a compact locally bi-polarized homogeneous space and* γ *be an affine Anosov diffeomorphism of* M. *Then the connection* ∇ *is complete.*

Proof. Let U be the subset of TM where the exponential map is defined. We claim that $U \supset \{X \in TM: \|X\| < \varepsilon\}$ for some $\varepsilon > 0$. Otherwise there is a sequence $X_n \in TM$ such that $\|X_n\| \to 0$ and $\exp X_n$ is not defined. The compactness of M allows us to choose a subsequence X_{n_i} so that $p := \lim_i \pi(X_{n_i})$ exists (here $\pi: TM \to M$ denotes the bundle map). Let $\varepsilon > 0$ be chosen so that the set $\{X \in TM_p: \|X\| < 2\varepsilon\}$ is included in the domain of $\exp_p$. Then there is a neighbourhood U of p such that for all q in U and all $X \in TM_q$ with $\|X\| < \varepsilon$, $\exp_q X$ is defined. This is an obvious contradiction and proves our claim. Since the Anosov diffeomorphism γ is affine the domain of the exponential map is invariant under γ_*. It follows that U contains both the stable and the unstable bundle.

Let $\tilde{M}$ be the universal cover of M. The unstable and stable foliations $\mathscr{F}^+$ and $\mathscr{F}^-$ lift to foliations $\tilde{\mathscr{F}}^+$ and $\tilde{\mathscr{F}}^-$. The leaves of all these foliations are embedded copies of $\mathbf{R}^n$ and, in particular, they are simply connected. We are assuming that M, and hence $\tilde{M}$, is locally modelled on a bipolarized homogeneous space G/A: any point of $\tilde{M}$ has a neighbourhood affinely diffeomorphic to a neighbourhood in G/A. Without loss of generality we can assume that G/A is simply connected. Consider the developing map $D: \tilde{M} \to G/A$. In order to establish the proposition we need to prove that D is a diffeomorphism of $\tilde{M}$ onto G/A. Since G/A is simply connected it suffices to show that D is a covering map.

First, we claim that D maps the leaves of the foliation $\mathscr{F}^+$ ($\mathscr{F}^-$) affinely and bijectively onto leaves of the foliation $\hat{\mathscr{F}}^+$ ($\hat{\mathscr{F}}^-$) of G/A. In fact, it is easily seen that local affine maps must send the local leaves of the foliation $\mathscr{F}^+$ affinely into leaves of the foliation $\hat{\mathscr{F}}^+$. Since the leaves of $\mathscr{F}^+$ are complete they are affinely mapped onto the leaves of $\hat{\mathscr{F}}^+$. If this mapping were not bijective there would exist

a closed geodesic loop in a leaf of $\hat{\mathscr{F}}^+$ and hence in every leaf of this foliation. This is impossible because by acting by the one-parameter group $\exp t\mathscr{A}_0$, where $\mathscr{A}_0$ was defined in (5.3), one can shrink to zero the length of the loop in the leaf through A. (The same argument is valid, mutatis mutandis, for the foliations $\mathscr{F}^-$ and $\hat{\mathscr{F}}^-$.)

For $p \in \tilde{M}$ let $L^+(p)$ be the leaf of $\mathscr{F}^\pm$ containing p; we define similarly $\hat{L}^\pm(\hat{p})$ for $\hat{p} \in G/A$. Set also $W(p) = \bigcup_{q \in L^+(p)} L^-(q)$ and $\hat{W}(\hat{p}) = \bigcup_{\hat{q}+\hat{L}^+(\hat{p})} \hat{L}^-(\hat{q})$; $W(p)$ and $\hat{W}(\hat{p})$ are of course open. Now, let L_1 and L_2 be two different leaves of $\mathscr{F}^+$ (resp. $\mathscr{F}^-$) that intersect a leaf L_3 of $\mathscr{F}^-$ (resp. $\mathscr{F}^+$); then, since any leaf of $\hat{\mathscr{F}}^+$ intersects a leaf of $\hat{\mathscr{F}}^-$ in at most a point (cf. § 2) we have $D(L_1) \neq D(L_2)$. This implies that D maps $W(p)$ injectively onto $\hat{W}(D(p))$ for all $p \in \tilde{M}$; it also follows that if $p_1 \neq p_2$ and $D(p_1) = D(p_2)$ then $W(p_1) \cap W(p_2) = \varnothing$. So we have showed that D covers evenly the open sets $\hat{W}(D(p))$, $p \in \tilde{M}$. In order to conclude that D is a covering we need to argue that D is surjective. Since $D(\tilde{M})$ is open it suffices to show that $D(\tilde{M})$ is closed because G/A is connected. Let $x \in G/A$ be a limit point of $D(\tilde{M})$. Let V be a neighbourhood of x with local product structure, i.e. such that $\hat{L}^+(y) \cap \hat{L}^-(y') \neq \varnothing$ whenever y, y' are in V. We denote by $[y, y']$ the only point in $\hat{L}^+(y) \cap \hat{L}^-(y')$. Pick $z = D(p) \in U$; then

$$D(L^+((p)) = \hat{L}^+(z) \Rightarrow [z, x] \in D(\tilde{M}) \Rightarrow \hat{L}^-([z, x]) \subset D(\tilde{M}) \Rightarrow x \in D(\tilde{M}).$$

Hence $D(\tilde{M})$ is closed and the claim follows.   □

4. *Diffeomorphisms of four-dimensional tori*
In this section we prove Theorem 2.

PROPOSITION 4. *Under the assumptions of Theorem 2, let ∇ be the Kanai connection associated with the stable and unstable foliation of γ. Then ∇ is locally symmetric.*

Proof. Let us denote by E^- and E^+ the bundles tangential to the stable and unstable foliations. Let

$$\omega(X_1, X_2, \ldots, X_5) := \Omega(\nabla R(X_1, X_2, X_3)X_4, X_5).$$

Then by a result of Feres and Katok ([**FeKa2**, Theorem 1, Lemma 3 and Lemma 6]), by changing their notation from flows to diffeomorphisms, we have that under the assumptions of Theorem 2 the following dichotomy holds: either

(1) ∇ is locally symmetric (i.e. $\omega = 0$ and $\nabla R = 0$), or
(2) there exist γ-equivariant C^∞ smooth splittings $E^- = E^{-1} \oplus E^{-2}$ and $E^+ = E^{+1} \oplus E^{+2}$ with the following properties:
 (2a) the distributions $E^{\pm i}$ extend the Oseledec decomposition of TM induced by γ and the Lyapunov exponent for almost every $v \in E^{\pm i}$ is $\pm i\lambda$, with $\lambda > 0$;
 (2b) the line bundles $E^{\pm 1}$ are invariant by the holonomy of the foliations $\mathscr{F}^\mp$;
 (2c) the codimension 1 bundles $E^- \oplus E^{+1}$ and $E^+ \oplus E^{-1}$ are integrable;
 (2d) at least one of the following is true:

$$\omega(E^{+1}, E^{-1}, E^{+1}, E^{-2}, E^{+1}) \neq 0$$

or

$$\omega(E^{-1}, E^{+1}, E^{-1}, E^{+2}, E^{-1}) \neq 0;$$

furthermore $\omega(E^{+1}1, E^{-1}, E^{+1}, E^{-2}, E^{+1}) \neq 0$ if and only if $E^{-1} \subset [E^{+1}, E^{-2}]$; similarly $\omega(E^{-1}, E^{+1}, E^{-1}, E^{+2}, E^{-1}) \neq 0$ if and only if $E^{+} \subset [E^{-1}, E^{+2}]$.

Thus in order to conclude the proof we prove that the second case of the dichotomy does not occur. Assume, in contradiction, that the second case occurs. First we notice that

if τ is a γ-invariant covariant r-tensor then

$$\tau(E^{i_1}, E^{i_2}, \ldots, E^{i_r}) \neq 0 \quad \text{only if } i_1 + i_2 + \cdots + i_r = 0. \tag{7}$$

There is an obvious reformulation of (7) which also applies to γ-invariant tensors of the mixed type. Now let $\pi_{\pm i}$, $i = 1, 2$, be the projection on $E^{\pm i}$ associated with the splitting $TM = E^{-1} \oplus E^{-2} \oplus E^{+1} \oplus E^{+2}$. We define a new connection by setting $\nabla' = \sum_{i=-2,-1,1,2} \pi_i \nabla \pi_i$. Denote by T' and R' the torsion and the curvature tensors of ∇'. We list some of the properties of ∇'.

(a) Since the projections $\pi_{\pm i}$ are equivalent by γ, the connection ∇' is γ-invariant; furthermore by definition we havce $\nabla' E^i \subset E^i$ for all $i = \pm 1, \pm 2$.

(b) $\nabla' \Omega = 0$. In fact, by (7) we have $\Omega(E^{+1}, E^{-2}) = \Omega(E^{+2}, E^{-1}) = 0$ and therefore $\Omega(X, Y) = \sum_{i=-2,-1,1,2} \Omega(\pi_i X, \pi_{-i} Y)$. It follows that

$$(\nabla'_X \Omega)(Y, Z) = \sum_{i=-2}^{2} [X(\Omega(\pi_i Y, \pi_{-i} Z)) + \Omega(\pi_i \nabla_X \pi_i Y, Z) + \Omega(Y, \pi_i \nabla_X \pi_i Z)]$$

$$= \sum_{i=-2}^{2} [X(\Omega(\pi_i Y, \pi_{-i} Z)) + \Omega(\pi_i \nabla_X \pi_i Y, \pi_{-i} Z)$$

$$+ \Omega(\pi_i Y, \nabla_X \pi_{-i} Z)]$$

$$= \sum_{i=-2}^{2} (\nabla_X \Omega)(\pi_i Y, \pi_{-i} Z) = 0.$$

(c) By (7) we see that $T'(E^i, E^j) \subset E^{i+j}$; hence, since the torsion is antisymmetric, the only non-zero terms in the torsion are possibly given by $T'(E^1, E^{-2})$ and $T'(E^{-1}, E^2)$. Indeed, if X^1 and X^{-2} are vector fields belonging to E^1 and E^{-2} we have

$$T'(X^1, X^{-2}) = \nabla'_{X^1} X^{-2} - \nabla'_{X^{-2}} X^1 - [X^1, X^{-2}]$$

$$= \pi_{-2} \nabla_{X^1} X^{-2} - \pi_1 \nabla_{X^{-2}} X^1 - \nabla_{X^1} X^{-2} + \nabla_{X^{-2}} X^1$$

$$= -\pi_{-1} \nabla_{X^1} X^{-2} + \pi_2 \nabla_{X^{-2}} X^1$$

$$= -\pi_{-1}(\nabla_{X^{-2}} X^1 + [X^1, X^{-2}]) + \pi_2(\nabla_{X^1} X^{-2} + [X^{-2}, X^1])$$

$$= -\pi_{-1}[X^1, X^{-2}] + \pi_2[X^{-2}, X^1] = -\pi_{-1}[X^1, X^{-2}]$$

(here we have used the fact that since the bundle $E^- \oplus E^{+1}$ is integrable we have $[X^{-2}, X^1] \subset E^- \oplus E^{+1}$). Similarly we have $T'(X^{-1}, X^2) = -\pi_1[X^{-1}, X^2]$.

LEMMA 5. *The connection ∇' is locally homogeneous* i.e. $\nabla' T' = 0$ and $\nabla' R' = 0$.

Proof of the lemma. In fact we have

$$(\nabla'_{E^i} T')(E^j, E^k) \subset \nabla'_{E^i}(T'(E^j, E^k)) + T'(\nabla'_{E^i} E^j, E^k) + T'(E^j, \nabla'_{E^i} E^k)$$

$$\subset \nabla'_{E^i} E^{j+k} + T'(E^j, E^k) \subset E^{j+k}. \tag{8}$$

On the other hand by (7) we have $(\nabla'_{E^i} T')(E^j, E^k) \subset E^{i+j+k}$. We conclude that $\nabla' T' = 0$. Similarly we obtain inclusions analogous to (8) for R' and again by (7) we conclude that $\nabla' R' = 0$. Hence (T^4, ∇') is locally homogeneous and the lemma is proved.   □

Pick a point $p_0 \in T^4$ and let **a** be the algebra of linear maps

$$\mathbf{a} \equiv \{\mathscr{A} : T_{p_0} T^4 \to T_{p_0} T^4 \mid \mathscr{A} E^i_{p_0} \subset E^i_{p_0}, \ \mathscr{A}^* \Omega = 0, \ \mathscr{A}^* T' = 0, \ \mathscr{A}^* R' = 0\}$$

(here we have implicitly extended **a** to the tensor algebra: then, for example

$$\mathscr{A}^* \Omega(v, w) = \Omega(\mathscr{A} v, w) + \Omega(v, \mathscr{A} w)$$

and

$$\mathscr{A}^* R'(v, w)z = R'(v, w)\mathscr{A} z + R'(\mathscr{A} v, w)z + R'(v, \mathscr{A} w)z - \mathscr{A}(R'(v, w)z)$$

$$\text{for all } v, w, z \in T_{p_0}).$$

Notice that for all $v, w \in T_{p_0} T^4$, $R(v, w)$ belongs to **a**.

Pick a basis in $T_{p_0} T^4$ of vectors $v_i \in E^i_{p_0}$ such that $\Omega(v_1, v_{-1}) = \Omega(v_2, v_{-2}) = 1$. Then $\mathscr{A} E^i_{p_0} \subset E^i_{p_0}$ implies that every $\mathscr{A} \in \mathbf{a}$ is diagonal in this basis, i.e. $\mathscr{A} v_i = \alpha_i v_i$, and $\mathscr{A}^* \Omega = 0$ implies that $\alpha_i = -\alpha_{-i}$. Since, by the conditions (2b) and (c) above, $T'(v_1, v_{-2})$ and $T'(v_{-1}, v_2)$ cannot vanish simultaneously, the equation $\mathscr{A}^* T' = 0$ implies that $\alpha_2 = 2\alpha_1$ (hence $\alpha_{-2} = 2\alpha_{-1} = -2\alpha_1$). Then **a** is one-dimensional and generated by $\mathscr{A}_0$ where $\mathscr{A}_0 v_i = i v_i$.

Now we are able to compute the curvature R'. Indeed notice that $\nabla' E^k \subset E^k$ implies that $R'(E^i, E^j)E^k \subset E^k$; on the other hand, since R' is a tensor, by (7) we have also $R'(E^i, E^j)E^k \subset E^{i+j+k}$. It follows that $R'(E^1, E^{-2}) = R'(E^2, E^{-1}) = 0$. Then the Bianchi identities

$$\sum_{(v,w,z)} R'(v, w)z = \sum_{(v,w,z)} T'(T'(v, w), z),$$

where $\sum_{(v,w,z)}$ denotes the sum over the cyclic permutations of v, w and z, imply that $R'(E^1, E^{-1})E^{\pm 2} = 0$. For all $v, w \in T_{p_0} T^4$, $R(v, w)$ is a multiple of $\mathscr{A}_0$ so we must also have $R'(E^1, E^{-1}) = 0$. Hence only $R'(v_2, v_{-2})$ is left to be determined. Using again the Bianchi identities we have $R'(v_2, v_{-2})v_1 = T'(T'(v_{-2}, v_1), v_2)$ and $R'(v_2, v_{-2})v_{-1} = T'(T'(v_{-1}, v_2), v_{-2})$.

Now we need to distinguish two cases:

Case 1. either $T'(v_1, v_{-2})$ or $T'(v_{-1}, v_2)$ vanishes (we have already mentioned that they cannot vanish simultaneously).

Let us assume that $T'(v_1, v_{-2}) \neq 0$. The other case is similar. Then $R'(v_2, v_{-2})v_{-1} = 0$; since $R'(v_2, v_{-2})$ is a multiple of $\mathscr{A}_0$ we have $R'(v_2, v_{-2}) = 0$ and therefore $R' = 0$.

Define a Lie algebra **g** as the linear span of $T_{p_0} T^4$ and $\mathscr{A}_0$ with the only non-trivial commutation relations given by

$$[v_1, v_{-2}] = -T'(v_1, v_{-2}) = \text{const} \times v_{-1} \quad \text{and} \quad [\mathscr{A}_0, v_i] = i v_i.$$

The Lie algebra **g** is a bipolarized homogeneous solvable Lie algebra. Hence T^4 is a locally bipolarized homogeneous space and by Proposition 3 it is complete. Hence T^4 can be represented as $\Pi \backslash G / A$, where G is a connected Lie group with Lie

algebra **g** and A a closed Lie subgroup with Lie algebra **a** and Π is a rank 4 abelian subgroup of G. We claim that G is simply connected and $A = \exp \mathbf{a}$. Let $\tilde{G}$ be the universal cover of G, $\pi : \tilde{G} \rightarrow G$ the covering map, $N = \ker \pi$ and $\tilde{A} = \pi^{-1}$. Since $\mathbf{R}^4 = \tilde{G}/\tilde{A}$ is simply connected, $\tilde{A}$ is connected and hence equal to $\exp \mathbf{a}$; in particular we have $N \subset \exp \mathbf{a}$. But we have Ad $N \mid T_{p_0} T^4 = 0$ and this together with the previous inclusion shows that $N = 0$ and G is simply connected.

The ordering $(v_2, v_{-1}, v_1, v_{-2}, \mathcal{A}_0)$ generates an increasing sequence of ideals of **g** of dimension $1, 2, \ldots, 5$. It follows that the exponential map is a diffeomorphism of **g** onto G [Di]. Hence Π is the exponential of the integer lattice of an abelian subgroup of rank 4 of **g**. Since **g** does not have a rank 4 abelian subalgebra we see that Case 1 cannot occur.

Case 2. In the second case $T'(v_1, v_{-2}) = \alpha v_{-1}$, $T'(v_{-1}, v_2) = -\beta v_1$ and $\alpha\beta \neq 0$. Then $R'(v_2, v_{-2})v_{-1} = T'(T'(v_{-1}, v_2), v_{-2}) = -\alpha\beta v_{-1}$ and therefore $R'(v_2, v_{-2}) = \alpha\beta\mathcal{A}_0$.

Let us define a Lie algebra **g** as the linear span of $T_{p_0} T^4$ and $\mathcal{A}_0$ with the commutation relations given by

$$[v_1, v_{-1}] = 0, \quad [v_2, v_{-2}] = -\alpha\beta\mathcal{A}_0, \quad [\mathcal{A}_0, v_i] = iv_i$$
$$[v_1, v_{-2}] = -T'(v_1, v_{-2}) = -\alpha v_{-1}$$
$$[v_2, v_{-1}] = -T'(v_2, v_{-1}) = -\beta v_1.$$

The Lie algebra **g** is a bipolarized homogeneous Lie algebra isomorphic to $\mathbf{sl}_2(\mathbf{R}) \ltimes \mathbf{R}^2$. Hence T^4 is a locally bipolarized homogeneous space and by Proposition 3 it is complete. It follows that T^4 can be represented as $\Pi\backslash G/A$, where G is a connected Lie group with Lie algebra $\mathbf{sl}_2(\mathbf{R}) \ltimes \mathbf{R}^2$, A a closed Lie subgroup of G with Lie algebra **a** and Π an abelian discrete group of G of rank 4. As before we can show that G is simply connected and hence equal to $\widetilde{SL_2(\mathbf{R})} \ltimes \mathbf{R}^2$. Also A is the exponential of the diagonal matrices in $\mathbf{sl}_2(\mathbf{R})$.

Let $\widetilde{SL_2(\mathbf{R})} \ltimes \mathbf{R}^2$ act on $\mathbf{R}^2$ by $((A, y), x) \in (\widetilde{SL_2(\mathbf{R})} \ltimes \mathbf{R}^2) \times \mathbf{R}^2 \mapsto \bar{A}x + y \in \mathbf{R}^2$, where $\bar{A}$ is the projection to $SL_2(\mathbf{R})$ of A. Let Π' be the image of Π in $SL_2(\mathbf{R}) \ltimes \mathbf{R}^2$. Then, since Π is abelian, either Π' fixes a unique $q \in \mathbf{R}^2$ or it fixes a direction on $\mathbf{R}^2$. In the first case Π' is conjugate in $SL_2(\mathbf{R}) \ltimes \mathbf{R}^2$ to a subgroup of $SL_2(\mathbf{R})$ and in the second case Π' is a conjugate to a subgroup of $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \ltimes \mathbf{R}^2$ (notice that this latter group is isomorphic to the three-dimensional Heisenberg group). In the first case, since γ normalizes Π, the projection of γ to $SL_2(\mathbf{R}) \ltimes \mathbf{R}^2$ also fixes q and we obtain that up to conjugacy Π' is included in $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$. In both cases the fact that Π is an abelian group of rank 4 brings us to a contradiction. We conclude that also Case 2 is impossible.

We have therefore concluded the proof of Proposition 4 by showing that the second case of the above dichotomy is impossible. $\quad\square$

From the fact that Kanai's connection ∇ is locally symmetric and the fact that the lift of the Anosov map γ to $\mathbf{R}^4$ has a fixed point it follows that (M, ∇) is a locally bipolarized symmetric space. Hence, by Proposition 3, Kanai's connection ∇ lifts to a complete connection $\tilde{\nabla}$ on $\mathbf{R}^4$. If we denote by G the Lie group of affine symplectic transformations of $(\mathbf{R}^4, \tilde{\nabla})$ which preserve Ω and the foliations and denote

by A the stabilizer in G of a point $p_0 \in \mathbf{R}^4$, we have that $\mathbf{R}^4$ is diffeomorphic to G/A. Furthermore, notice that both the deck transformations and lifts of γ act affinely on $\mathbf{R}^4$. Hence there is a discrete group $\Pi \subset G$ isomorphic to $\mathbf{Z}^4$ such that $\Pi\backslash G/A$ is diffeomorphic to T^4 and an element $\gamma' \in A$ such $\gamma'\Pi(\gamma')^{-1} = \Pi$. We want to show that ∇ is flat. First we prove the following proposition:

PROPOSITION 6. *Let* $\mathbf{r}$ *be the radical of* $\mathbf{g}$. *Then* $\mathbf{p}^+ + \mathbf{p}^- \subset \mathbf{r}$.

Proof. By the Levi decomposition there is a semisimple Lie algebra $\mathbf{s}$ such that $\mathbf{g} = \mathbf{s} + \mathbf{r}$. The Lie algebra $\mathbf{g} = \mathbf{p}^- + \mathbf{a} + \mathbf{p}^+$ is graded in the sense that if we set $\mathbf{g}_0 = \mathbf{a}$, $\mathbf{g}_{\pm 1} = \mathbf{p}^\pm$, we have $[\mathbf{g}_i, \mathbf{g}_j] = \mathbf{g}_{i+j}$. If we set $\mathbf{r}_i = \mathbf{r} \cap \mathbf{g}_i$ one can show that $\mathbf{r} = \mathbf{r}_{-1} + \mathbf{r}_0 + \mathbf{r}_{+1}$ [K, eq. (3.2)]. Hence, via the sequence $0 \to \mathbf{r} \to \mathbf{g} \to \mathbf{s} \to 0$ the semisimple part $\mathbf{s}$ inherits a grading $\mathbf{s}_{-1} + \mathbf{s}_0 + \mathbf{s}_{+1}$ from the grading of $\mathbf{g}$. Let G, S, R be the simply connected groups corresponding to $\mathbf{g}$, $\mathbf{s}$ and $\mathbf{r}$ and let A, $G^\pm$, S_0 and $S^\pm$ be the connected subgroups generated respectively by the sub-algebras $\mathbf{a} = \mathbf{g}_0$, $\mathbf{g}_0 + \mathbf{g}_{\pm 1}$, $\mathbf{s}_0$ and $\mathbf{s}_0 + \mathbf{s}_{\pm 1}$. By Lemma 3.5 of [K] both G^+ and S^+ are closed. The epimorphism $G \to S$ maps G^+ to S^+ and has fibre R. Hence it induces a fibration of the homogeneous spaces $G/G^+ \to S/S^+$ whose fibre is G^+R and hence is connected. Since G/G^+ can be identified with the space of unstable leaves (cf. § 3.2 of [K]) we have that G/G^+ is diffeomorphic to a stable leaf, that is diffeomorphic to $\mathbf{R}^2$. If $\mathbf{s}$ does not have $\mathrm{sl}_2(\mathbf{R})$ factors intersecting $\mathbf{p}$ nontrivially, we have that S/S^+ is compact by Lemma 3.3 of [K]. Then S/S^+ consists of a single point and therefore $\mathbf{s}_{-1} = 0$. Similarly one shows that $\mathbf{s}_{+1} = 0$. We conclude that $\mathbf{p}^+ + \mathbf{p}^- \subset \mathbf{r}$. If instead $\mathbf{s}$ does have a $\mathrm{sl}_2(\mathbf{R})$ factor $\mathbf{s}'$ which intersects $\mathbf{p}$ nontrivially, it can be seen that $\mathbf{s}_0'$ is a split Cartan and that the decomposition $\mathbf{s}_{-1}' + \mathbf{s}_0' + \mathbf{s}_{+1}'$ is the corresponding root decomposition of $\mathbf{s}'$. In particular we have $S^+S^- \neq S$. Let $L^+(p_0)$ and $L^-(p_0)$ be the unstable and stable leaves of p_0 in $\mathbf{R}^4$. The map $(p, q) \in L^+(p_0) \times L^-(p_0) \mapsto L^+(q) \cap L^-(p) \in \mathbf{R}^4$ is a diffeomorphism, thus $G^+G^- = G$. Thus $S^+S^- \neq S$ is impossible and we conclude that $\mathbf{p}^+ + \mathbf{p}^- \subset \mathbf{r}$.   □

It now follows that the smallest subalgebra of $\mathbf{g}$ containing $\mathbf{p}^+ + \mathbf{p}^-$ is solvable. We claim that in fact it must be abelian.

PROPOSITION 7. $\mathbf{p}^+ + \mathbf{p}^-$ *is an abelian subalgebra of* $\mathbf{g}$. *Hence* ∇ *is flat.*

Proof. Denote by $\mathbf{g}'$ the smallest subalgebra of $\mathbf{g}$ containing $\mathbf{p}^+ + \mathbf{p}^-$. It is clear that $\mathbf{g}' = \mathbf{p}^+ + R(\mathbf{p}^+, \mathbf{p}^-) + \mathbf{p}^-$. By Proposition 6, $R(\mathbf{p}^+, \mathbf{p}^-)$ belongs to the derived algebra of $\mathbf{r}$ and hence it is a nilpotent subalgebra. It follows that there are $\xi^+ \in \mathbf{p}^+$ and $\xi^- \in \mathbf{p}^-$ for which $R(\mathbf{p}^+, \mathbf{p}^-)\xi^+ = 0$. This implies that $R(\xi^+, \cdot) = 0$. Let $\eta^+ \in \mathbf{p}^+$ be linearly independent of ξ^+ such that $\Omega(\eta^\pm, \xi^\mp) = \pm 1$. If $R(\eta^+, \eta^-) = 0$ then ∇ is flat and we have concluded the proof. If $R(\eta^+, \eta^-) \neq 0$ then, since $R(\eta^+, \eta^-)$ is nilpotent, we have $R(\eta^+, \eta^-)\eta^\pm = c^\pm \xi^\pm$ for some $c^\pm \in \mathbf{R}$. Since the curvature operators $R(\cdot, \cdot)$ are infinitesimally symplectic, from $\Omega(R(\eta^+, \eta^-)\eta^+, \eta^-) + \Omega(\eta^+, R(\eta^+, \eta^-)\eta^-) = 0$ we obtain $c^+ = -c^-$. Renormalizing our vectors we can assume that $c^+ = -c^- = 1$ (or that $c^+ = -c^- = -1$, which corresponds to considering the opposite Lie algebra). Hence $\mathbf{g}'$ is the five-dimensional Lie algebra with generators $\eta^\pm$; $\xi^\pm$ and $\mathcal{R}$ whose

only non-trivial commutation relations are

$$[\eta^+, \eta^-] = \mathscr{R} \quad \text{and} \quad [\mathscr{R}, \eta^\pm] = \pm\xi^\pm.$$

It is not difficult to verify that the relations (4) imply that **g** is six dimensional and that the sixth generator is precisely the element $\mathscr{A}_0$ defined in (5.2) which satisfies the relations

$$[\mathscr{A}_0, \mathscr{R}] = 0 \quad [\mathscr{A}_0, \eta^\pm] = \pm\eta^\pm \quad [\mathscr{A}_0, \xi^\pm] = \pm\xi^\pm.$$

In particular we see that **g** is solvable. Consider the fundamental group Π of T^4; by possible passing to a subgroup of finite index we may and shall assume that Π is included in the component of the identity G^0 of G. We claim that Π is contained in the derived group of G. In fact, let G_1 be the analytic subgroup corresponding to the abelian ideal of G spanned by $\mathscr{R}$, ξ^+ and ξ^-. Then G_1 is closed and normal. The quotient group G^0/G_1 has a faithful matrix representation by matrices

$$\begin{pmatrix} a & b & & \\ 0 & 1 & & \\ & & a^{-1} & c \\ & & 0 & 1 \end{pmatrix}.$$

Here and later blank spaces in the matrices are assumed to be zeroes. We see that either Π is included in the derived group or its image in G^0/G_1 is contained in a one-parameter subgroup of G^0/G_1. Since Π is normalized by the Anosov map $\gamma \in A$ we conclude that in the latter case Π/G_1 is included in $\exp t\mathscr{A}_0/G_1$, $(t \in \mathbf{R})$. It follows that Π is contained in the analytic subgroup G_2 generated by $\mathscr{A}_0$, $\mathscr{R}$, ξ^+ and ξ^-. Considerations similar to the one we just made imply that either Π is included in the derived group or in the stabilizer A. The latter possibility is absurd. We have proved our claim that Π is included in the derived group. Since the derived group is nilpotent it follows that there is an abelian group of rank four in G. But an inspection of the commutation relations shows that this is again impossible. Hence we reach a contradiction starting from the assumption that the curvature tensor does not vanish and conclude the proof. $\qquad\qquad\square$

Now we have that the curvature is zero. We can therefore introduce coordinates $x = (x_1, x_2, x_3, x_4)$ in $\mathbf{R}^4$ where (x_1, x_2) coordinatize the stable leaf of the fixed point of $\tilde{\gamma}$ and (x_3, x_4) coordinatize the unstable leaf and $\Omega = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$. For simplicity, we shall call the plane (x_1, x_2) (resp. (x_3, x_4)) the stable (resp. unstable) space. In these coordinates the map $\tilde{\gamma}$ is a symplectic linear map given by $\begin{pmatrix} E & \\ & (E^T)^{-1} \end{pmatrix}$ where E is a contraction in the plane (x_1, x_2). The group $\Pi \approx \mathbf{Z}^4$ is also a group of affine symplectic transformations preserving the stable and unstable foliations; hence for each $m \in \mathbf{Z}^4$ we have a mapping

$$\pi_m : x \in \mathbf{R}^4 \mapsto A_m x + B_m \tag{9}$$

with $B_m \in \mathbf{R}^4$ and $A_m = \begin{pmatrix} C_m & \\ & (C_m^T)^{-1} \end{pmatrix}$. Of course we have

$$\pi_m \pi_n = \pi_n \pi_m = \pi_{m+n}. \tag{10}$$

LEMMA 8. *The group* Π *is a group of translations. In other words* $A_m =$ *Identity for all* $m \in \mathbf{Z}^4$.

Proof. If some A_m does not have 1 as an eigenvalue then we can solve the equation $x = A_m x + B_m$ and therefore we find that π_m has a fixed point. This is impossible because the fundamental group acts without fixed points. Hence all A_m have 1 as an eigenvalue. We can assume that the fixed space of some A_m restricted to the stable space (x_1, x_2) is one-dimensional, otherwise the lemma is true. Then there is a vector in the stable space and a vector in the unstable space that are fixed by all A_m. Let W be the two-dimensional subspace spanned by these two eigenvectors. The maps π_m are affine maps of $\mathbf{R}^4 / W$. If some A_m had an eigenvalue different from 1 we would find that some π_m has, and hence all π_m have, a fixed point in $\mathbf{R}^4 / W$. In other words there exist a two-plane W' parallel to W that is left invariant by all π_m. Since Π acts discretely on $\mathbf{R}^4$ there is a rank 2 subgroup of Π which fixes W' pointwise. Again this contradicts the fact that Π acts without fixed points.

Now we can assume that all A_m have 1 as their only eigenvalue. Hence in a suitable basis we have

$$
A_m = \begin{pmatrix} 1 & a_m & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & -a_m & 1 \end{pmatrix}.
$$

Now it follows that Π is a subgroup of the nilpotent group N given by the affine maps

$$
\begin{pmatrix} 1 & a & b & & & \\ 0 & 1 & c & & & \\ 0 & 0 & 1 & & & \\ & & & 1 & 0 & d \\ & & & -a & 1 & e \\ & & & 0 & 0 & 1 \end{pmatrix}.
$$

Since Π is discrete and abelian, it is included in an abelian rank 4 Lie subgroup of N. But there is only one such subgroup, namely the translations of $\mathbf{R}^4$.   □

This concludes the proof of Theorem 2.

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