Formulas for the Derivative and Critical Points of Topological Entropy for Anosov and Geodesic Flows*

Anatole Katok[1,2]**, Gerhard Knieper**[3] **and Howard Weiss**[1,**]

[1] Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA
[2] Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA
[3] Mathematisches Institut, SFB 170, Universität Göttingen, Bunsenstraße 3–5,
W-3400 Göttingen, Federal Republic of Germany

Abstract. This paper represents part of a program to understand the behavior of topological entropy for Anosov and geodesic flows. In this paper, we have two goals. First we obtain some regularity results for C^1 perturbations. Second, and more importantly, we obtain explicit formulas for the derivative of topological entropy. These formulas allow us to characterize the critical points of topological entropy on the space of negatively curved metrics.

I. Formulation of Results

The topological entropy, h_{TOP}, measures the exponential growth rate of the number of orbit segments distinguishable with arbitrarily fine but finite precision. In general, it behaves irregularly with respect to perturbations. In the discrete time case, Misiurewicz [Mi1] constructed examples showing that $h_{TOP}: \text{Diff}^\infty(M^n) \to \mathbb{R}$ is not continuous for $n \geq 4$, as well as examples showing that h_{TOP} is not upper-semicontinuous in the C^k topology for $k < \infty$ in every dimension $n \geq 2$ [Mi2]. Here $\text{Diff}^\infty(M^n)$ denotes the space of C^∞ diffeomorphisms on a compact n-dimensional manifold equipped with the C^∞ topology. Yomdin [Y] and Newhouse [N] proved that $h_{TOP}: \text{Diff}^\infty(M^n) \to \mathbb{R}$ is upper semicontinuous. For $n = 2$, Katok [K3] proved lower semicontinuity. By combining these two results, one sees that $h_{TOP}: \text{Diff}^\infty(M^2) \to \mathbb{R}$ is continuous. This result also holds for C^∞ flows on three dimensional manifolds.

The structural stability of Anosov diffeomorphisms [A] implies that h_{TOP} is locally constant. For Anosov flows, the structural stability [A] involves a time

---

change and the topological entropy may change. However, the structural stability implies that h_{TOP} is continuous. Misiurewicz [Mi2] has shown that for general C^k flows on $M^n, k < \infty$ and $n \geq 3$, h_{TOP} need not be continuous.

This paper represents part of a program to understand the behavior of topological entropy for Anosov and geodesic flows. The principal regularity results were obtained in [KKPW]. Later, related results were obtained by Contreras [C]. In this paper, we have two goals. First we obtain some regularity results for C^1 perturbations (Theorems 1 and 2). Second and more importantly, we obtain explicit formulas for the derivative of topological entropy. These formulas allow us to characterize the critical points of topological entropy on the space of negatively curved metrics.

Our first result is an almost straightforward consequence of the results by Manning [M] and Freire and Mañé [FM].

Theorem 1. *Let* (M^n, g) *be a compact* n-*dimensional* C^2 *Riemannian manifold without conjugate points and let* g_λ, $-\varepsilon \leq \lambda \leq \varepsilon$, *be a* C^1 *perturbation through metrics without conjugate points. If* ϕ_λ *is the geodesic flow associated to* g_λ, *then*

$$\lambda \to h_T(g_\lambda) := h_{\text{TOP}}(\phi_\lambda)$$

is a Lipschitz function.

The next two theorems contain the principle results of this paper:

Theorem 2. *Let* ϕ^t *be a* C^1 *Anosov flow on a compact* n-*dimensional manifold* X *and let* ϕ_λ^t, $-\varepsilon \leq \lambda \leq \varepsilon$ *be a* C^1 *perturbation of* $\phi^t = \phi_0^t$. *Then* $h_T(\lambda) = h_{\text{TOP}}(\phi_\lambda)$ *is a* C^1 *function and the derivative is given by*

$$\frac{\partial}{\partial \lambda}\bigg|_{\lambda = \lambda_0} h_T(\lambda) = h_T(0) \cdot \int_X \frac{\partial}{\partial \lambda}\bigg|_{\lambda = \lambda_0} a_\lambda(p) d\mu_0,$$

where μ_0 *denotes the Margulis measure (the unique measure of maximal entropy for* ϕ^t [Ma2]*) with respect to the unperturbed Anosov flow and the function* $a_\lambda(p)$ *compares the infinitesimal generator of* ϕ_λ *to the infinitesimal generator of* ϕ. *One can think of* $a_\lambda(p)$ *as an infinitesimal time change.*

Theorem 3. *Let* (M, g) *be a compact* C^2 *Riemannian manifold of negative sectional curvature and let* g_λ, $-\varepsilon \leq \lambda \leq \varepsilon$, *be a* C^2 *perturbation of* $g = g_0$. *If* ϕ_λ *is the geodesic flow associated to* g_λ, *then* $h_T(g_\lambda) = h_{\text{TOP}}(\phi_\lambda)$ *is a* C^1 *function, and the derivative is given by*

$$\frac{\partial}{\partial \lambda}\bigg|_{\lambda = \lambda_0} h_T(g_\lambda) = -\frac{h_T(g_0)}{2} \cdot \int_{SM} \frac{\partial}{\partial \lambda}\bigg|_{\lambda = \lambda_0} g_\lambda(v, v) d\mu_{g_0},$$

where μ_{g_0} *denotes the Margulis measure of* g_0.

Remarks.

a) Since the geodesic flow on a negatively curved manifold is Anosov [A], Theorem 2 also applies in this case. The two formulas are different. Furthermore, the proof of Theorem 3 does not use structural stability.

b) In [KKPW] the authors prove, among other regularity results, that for a C^2

perturbation of a C^1 Anosov flow, h_{TOP} is C^1. However, they do not obtain a formula for the derivative.

c) Theorem 3 can be extended to Rank 1 manifolds of nonpositive curvature provided one can ensure the existence of a unique measure of maximal entropy. This was established by Ursula Hamenstadt (unpublished). Once the theorem is known for Rank 1 manifolds, it follows from the classification of compact manifolds of nonpositive curvature and the rigidity of higher rank manifolds (see Appendix 1 of [BGS] for a convenient summary) that the theorem is valid for arbitrary manifolds of nonpositive curvature.

The following result contains an interesting application of the formula from Theorem 3:

Theorem 4. *Let M be a compact surface and let $\mathfrak{R}(M)$ denote the submanifold of negatively curved C^2 metrics on M having area equal to 1. Then $h_T: \mathfrak{R}(M) \to \mathbb{R}$ has a critical point at $g_0 \in \mathfrak{R}(M)$ if and only if g_0 is a metric of constant negative curvature.*

We would like to point out several open questions related to our results.

Conjecture 1. Let $\mathfrak{R}(M)$ denote the submanifold of negatively curved C^2 metrics on a compact manifold M having volume equal to 1. Then $h_T: \mathfrak{R}(M) \to \mathbb{R}$ has a critical point at g_0 if and only if the Margulis measure and the Liouville measure of the geodesic flow for g_0 coincide.

The next conjecture would follow from Conjecture 1 and a well-known entropy rigidity conjecture of the first author [K2, BK].

Conjecture 2. Let $\mathfrak{R}(M)$ denote the submanifold of negatively curved C^2 metrics on a compact manifold M having volume equal to 1. Then $h_T: \mathfrak{R}(M) \to \mathbb{R}$ has critical point at g_0 if and only if g_0 is locally symmetric.

Our last conjecture would imply Conjecture 1 even if it is established only under the extra assumption that μ is the Margulis measure (cf. proof of Theorem 4 below).

Conjecture 3. Let (M, g) be a compact Riemannian manifold of negative sectional curvature. Suppose μ is a Borel probability measure on SM which is invariant under the geodesic flow, and projects to Lebesgue measure on M, i.e., μ and the Liouville measure coincide on $\pi^{-1}(B(M)) := \{\pi^{-1}(A) \mid A \in B(M)\}$, where $B(M)$ is the σ-algebra of Borel subsets on M. Then μ is the Liouville measure.

II. Topological Entropy for Geodesic Flows on Manifolds Without Conjugate Points

Let (M, g) be a compact Riemannian manifold and $\tilde{M}$ the universal covering of M. For every $p \in \tilde{M}$, Manning [M] showed that the limit

$$\lim_{r \to \infty} \frac{\log \mathrm{vol}_g B_g(p, r)}{r} = \Lambda_g$$

exists and is independent of p. Here $B_g(p, r)$ denotes the geodesic ball of radius $r > 0$ in the universal covering $\tilde{M}$ and vol_g is the volume associated to g. Margulis

[Ma2], for metrics of negative curvature, Manning [M], for metrics with no focal points, and Freire and Mañé [FM], for metrics without conjugate points, showed that A_g coincides with the topological entropy $h_T(g)$ of the geodesic flow in the compact quotient.

Lemma 1. *Let (M, g) be a compact n-dimensional Riemannian manifold and let $\{g_\lambda\}, -\varepsilon < \lambda < \varepsilon$ be a C^1 perturbation of $g = g_0$. Then there are Lipschitz functions $M(\lambda)$ and $m(\lambda)$, $M(0) = m(0) = 1$ such that:*

a) *For all $p, q \in M$, $m(\lambda) \leqq \dfrac{d_\lambda(p, q)}{d_0(p, q)} \leqq M(\lambda)$, where d_λ denotes the distance generated by g_λ.*

b) *For all measurable sets $S \subset M$*

$$m(\lambda)^n \leqq \frac{\mathrm{vol}_\lambda(S)}{\mathrm{vol}_0(S)} \leqq M(\lambda)^n,$$

where vol_λ denotes the volume generated by g_λ.

Proof. Define

$$M(\lambda) = \max_{v \in (SM)_0} \|v\|_\lambda, \quad m(\lambda) = \min_{v \in (SM)_0} \|v\|_\lambda,$$

where $(SM)_0$ is the unit tangent bundle of M with respect to g_0, and $\|v\|_\lambda$ is the norm of v in the g_λ metric. From this definition, the inequality in (a) is evident.

Comparing the eigenvalues of g_λ and g_0 one obtains:

$$m(\lambda)^n \leqq \sqrt{\frac{\det g_\lambda}{\det g_0}} \leqq M(\lambda)^n.$$

This immediately implies (b).

To see that $M(\lambda)$ is Lipschitz we observe that (1) $\|v\|_0 = 1$ for all $v \in (SM)_0$ and (2) $D = \sup_\lambda \sup_{v \in (SM)_0} \dfrac{\partial}{\partial \lambda} \|v\|_\lambda < \infty$. Since $\|v\|_\lambda$ is a C^1 function on $(SM)_0 \times (-\varepsilon, \varepsilon)$, (1) and (2) imply that $|M(\lambda) - M(0)| \leqq D \cdot \lambda$. The same argument applies to $m(\lambda)$.

Remark. The inequalities (a) and (b) remain valid if we replace M by a covering space equipped with the lifted metric.

Proof of Theorem 1. Let $\tilde{M}$ be the universal cover of M and fix $p_0 \in \tilde{M}$. Then part (a) of Lemma 1 implies:

$$B_{g_0}\left(p_0, \frac{r}{M(\lambda)}\right) \subseteq B_{g_\lambda}(p_0, r) \subseteq B_{g_0}\left(p_0, \frac{r}{m(\lambda)}\right).$$

Hence,

$$\mathrm{vol}_{g_0}\left(B_{g_0}\left(p_0, \frac{r}{M(\lambda)}\right)\right) \leqq \mathrm{vol}_{g_0}(B_{g_\lambda}(p_0, r)) \leqq \mathrm{vol}_{g_0}\left(B_{g_0}\left(p_0, \frac{r}{m(\lambda)}\right)\right).$$

Therefore,

$$\frac{1}{M(\lambda)} \cdot h_T(g_0) \leqq \lim_{r \to \infty} \frac{\log \mathrm{vol}_{g_0} B_{g_\lambda}(p_0, r)}{r} \leqq \frac{1}{m(\lambda)} \cdot h_T(g_0).$$

Part (b) of Lemma 1 implies:

$$\lim_{r \to \infty} \frac{\log \mathrm{vol}_{g_0} B_{g_\lambda}(p_0, r)}{r} = \lim_{r \to \infty} \frac{\log \mathrm{vol}_{g_\lambda} B_{g_\lambda}(p_0 r)}{r} = h_T(g_\lambda).$$

III. Topological Entropy for Anosov Flows

Let ϕ be an Anosov flow on a compact space X. Margulis [Ma1] and Bowen [B] have shown that the topological entropy h_T is obtained by:

$$h_T(\phi) = \lim_{t \to \infty} \frac{\log \# P(t)}{t},$$

where $P(t)$ denotes the set of periodic orbits of (prime) period $\leq t$. To prove Theorem 2, it is important to count only those closed orbits which are almost uniformly distributed with respect to the Margulis measure. Given $\varepsilon > 0$ and a continuous function f, we say that a closed orbit $\{\phi^s v\}_{s=0}^l$ of period l is (ε, f) uniformly distributed with respect to a measure μ if

$$\left| \frac{1}{l} \int_0^l f(\phi^t v) dt - \int_X f d\mu \right| < \varepsilon.$$

Proposition 2. Let ϕ be an Anosov flow, μ be an invariant ergodic measure for ϕ, and f a continuous function on X. For every $\varepsilon > 0$ let $P_\mu^{\varepsilon, f}(t)$ be the set of (ε, f) uniformly distributed closed orbits of period $\leq t$. Then:

$$\liminf_{t \to \infty} \frac{\log \# P_\mu^{\varepsilon, f}(t)}{t} \geq h_\mu(\phi),$$

where $h_\mu(\phi)$ is the measure theoretic entropy of ϕ with respect to μ.

Proof. The proof follows from the methods which the first author developed in [K1] and [K2].

Remarks.
a) The proposition remains true for any flow ϕ if μ is an invariant ergodic measure with non-zero Lyapunov exponents.
b) If μ is the Margulis measure of ϕ then:

$$\lim_{t \to \infty} \frac{\log \# P_\mu^{\varepsilon, f}(t)}{t} = h_T(\phi).$$

The next ingredient for the proof of Theorem 2 is a refined version of the structural stability of Anosov flows.

Proposition 3. (*Structural Stability for Anosov Flows*) Let Φ be a C^1 vector field on a compact manifold X which generates an Anosov flow ϕ^t. Define

$$C_\Phi^0(X) = \left\{ u : X \to X \mid u \text{ homeomorphism such that } \frac{d}{dt}\bigg|_{t=0} u \circ \phi^t =: D_\Phi u \text{ exists} \right\},$$

and let $C^0(X, \mathbb{R})$ be the space of continuous functions on X. Then there exists a C^1

neighborhood $U(\Phi)$ *of* Φ *and a* C^1 *map*

$$S: U(\Phi) \to C^0_\Phi(X) \times C^0(X, \mathbb{R})$$
$$\psi \to (u, \gamma)$$

which solves the structural stability equation

$$\Psi \circ u = \gamma \cdot D_\Phi u.$$

Corollary 4. (*Dependence on Parameters*) *Let* Φ *be as in Proposition 3 and let* Φ_λ *be a* C^1 *perturbation of* $\Phi = \Phi_0$. *Then there exists* $\varepsilon > 0$ *such that:*
a) *For each fixed* $p_0 \in X$,

$$\lambda \to u_\lambda(p_0), \quad \lambda \to \gamma_\lambda(p_0) \quad \text{are} \quad C^1 \quad \text{for} \quad \lambda \in (-\varepsilon, +\varepsilon).$$

b) *For each fixed* $\lambda_0 \in (-\varepsilon, \varepsilon)$,

$$p \to u_{\lambda_0}(p), \quad p \to \gamma_{\lambda_0}(p)$$

and

$$p \to u_\lambda(p), \quad p \to \left. \frac{\partial}{\partial \lambda} \right|_{\lambda = \lambda_0} \gamma_\lambda(p)$$

are continuous functions on X.

Proof of Corollary 4. Since S is C^1 we obtain: For each fixed p_0, $\lambda \to S(\phi_\lambda)(p_0)$ is a C^1 function, and for each fixed λ_0 and $j \in \{0, 1\}$, $p \to \left. \frac{\partial^j}{\partial \lambda^j} \right|_{\lambda = \lambda_0} S(\phi_\lambda)(p)$ is continuous on X.

We present a sketch of the proof of Proposition 3 in the appendix. Also see the appendix to [LMM].

The following formulas for the time change in structural stability are obtained by integrating the infinitesimal time change.

Lemma 5. *Let the vector field* Ψ *be* C^1 *close to the Anosov vector field* Φ *and let* $(u, \gamma) \in C^\alpha_\Phi(X) \times C^\alpha(X, \mathbb{R})$ *denote the solution to the structural stability equation*

$$\Psi \circ u = \gamma \cdot D_\Phi u.$$

If ϕ^t *and* ψ^s *are the flows generated by* Φ *and* Ψ, *then the time change equations*

$$\psi^{s(t,p)} u(p) = u \circ \phi^t(p),$$
$$\psi^s u(p) = u \circ \phi^{t(s,p)}(p)$$

have the solutions

$$s(t, p) = \int_0^t \frac{1}{\gamma(\phi^x(p))} dx,$$

$$t(s, u^{-1}(p)) = \int_0^s \gamma(u^{-1} \phi^x(p)) dx.$$

Proof. Simple calculation.

The following inequality is the key step in the proof of Theorem 2 (cf. the proof of Theorem 1 in [K4]).

Proposition 6. *Let ϕ be a C^1 Anosov flow and ψ be a perturbation C^1 close to ϕ. Then*

$$\frac{h_T(\phi)}{\int_X \frac{1}{\gamma} d\mu_\phi} \leq h_T(\psi) \leq h_T(\phi) \cdot \int_X \gamma \circ u^{-1} d\mu_\psi,$$

where μ_ϕ and μ_ψ are the Margulis measures with respect to ϕ and ψ, u is a conjugating homeomorphism, and γ is the corresponding infinitesimal time change.

Proof. Given $\varepsilon > 0$, let $P_\phi^{\varepsilon,\gamma^{-1}}(t)$ be the set of closed orbits of ϕ of period $\leq t$, which are $(\varepsilon, \gamma^{-1})$ uniformly distributed with respect to the Margulis measure μ_ϕ.

Let p be a periodic point for ϕ with period $l_\phi(p)$ whose orbit σ belongs to the set $P_\phi^{\varepsilon,\gamma^{-1}}(t)$. The period of the perturbed orbit $\psi^s(u(p))$ is given by $s(l_\phi(p), p)$. Because σ is $(\varepsilon, \gamma^{-1})$ uniformly distributed, it follows from Lemma 5 that:

$$\frac{s(l_\phi(p), p)}{l_\phi(p)} \leq \varepsilon + \int_X \frac{1}{\gamma} d\mu_\phi =: a(\varepsilon).$$

Using Proposition 2, we obtain:

$$h_T(\phi) = \lim_{t \to \infty} \frac{\log \# P_\phi^{\varepsilon,\gamma^{-1}}(t)}{t} \leq a(\varepsilon) \cdot \lim_{t \to \infty} \frac{\log \# P_\psi(t \cdot a(\varepsilon))}{t \cdot a(\varepsilon)} = a(\varepsilon) h_T(\psi).$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$h_T(\phi) \leq h_T(\psi) \cdot \int_X \frac{1}{\gamma} d\mu_\phi.$$

Similarly, using the second assertion of Lemma 5, we derive:

$$h_T(\psi) \leq h_T(\phi) \int_X \gamma \circ u^{-1} d\mu_\psi.$$

Proof of Theorem 2. Let ϕ_λ be a C^1 perturbation of $\phi = \phi_0$, $-\varepsilon \leq \lambda \leq \varepsilon$. Proposition 6 implies that:

$$h_T(\phi_0) \left(\frac{1}{\int_X \frac{1}{\gamma_\lambda} d\mu_0} - 1 \right) \leq h_T(\phi_\lambda) - h_T(\phi_0)$$

$$\leq h_T(\phi_0) \left(\int_X \gamma_\lambda \circ u_\lambda^{-1} d\mu_\lambda - 1 \right),$$

where $\gamma_\lambda, u_\lambda$ are as in Proposition 6 but now depend on the parameter λ, and μ_λ is the Margulis measure for ϕ_λ. Linearizing in λ at $\lambda = 0$ leads to:

$$\frac{1}{\gamma_\lambda(p)} = 1 - \lambda \cdot \frac{\partial}{\partial s}\bigg|_{s=0} \gamma_s(p) + o(\lambda)$$

and

$$\gamma_\lambda(u_\lambda^{-1}(p)) = 1 + \lambda \cdot \frac{\partial}{\partial s}\bigg|_{s=0} \gamma_s(u_\lambda^{-1}(p)) + o(\lambda).$$

Since $\left|\dfrac{\partial}{\partial\lambda}u_\lambda^{-1}(p)\right| \leqq$ constant for all $p \in X$ and $\dfrac{\partial}{\partial s}\Big|_{s=0} \gamma_s$ is uniformly continuous, we obtain:

$$\gamma_\lambda(u_\lambda^{-1}(p)) = 1 + \lambda \cdot \frac{\partial}{\partial s}\Big|_{s=0} \gamma_s(p) + \lambda \cdot \varepsilon(\lambda) + o(\lambda),$$

where $\varepsilon(\lambda) \to 0$ as $\lambda \to 0$. Substituting this in the equation above yields:

$$h_T(\phi_0)\left(\lambda \cdot \int_X \left(\frac{\partial}{\partial s}\Big|_{s=0} \gamma_s\right)d\mu_0 + o_1(\lambda)\right) \leqq h_T(\phi_\lambda) - h_T(\phi_0)$$

$$\leqq h_T(\phi_0)\left(\lambda \cdot \int_X \left(\frac{\partial}{\partial s}\Big|_{s=0} \gamma_s\right)d\mu_\lambda + o_2(\lambda)\right),$$

for suitable $o_1(\lambda)$ and $o_2(\lambda)$.

We remark that this inequality already implies Lipschitz continuity for $\lambda \to h_T(\phi_\lambda)$. The proof of Theorem 2 follows from the next lemma.

Lemma 7. (*Weak Continuity of the Margulis Measure*) *Let ϕ_n be a sequence of Anosov flows, $\phi_n \xrightarrow{\;c^1\;} \phi$ and μ_n, μ be the associated Margulis measures. Then μ_n converges to μ in the weak topology.*

Proof. Let μ^* be any weak limit of μ_n. We show that $\mu^* = \mu$. Since ϕ is expansive, $\mu \to h_\mu(\phi)$ is upper semicontinuous [W]. This, along with the variational principle, implies that:

$$h_T(\phi) \geqq h_{\mu^*}(\phi) \geqq \limsup_{n_k \to \infty} h_{\mu_{n_k}}(\phi_{n_k}).$$

The structural stability of Anosov flows immediately implies that topological entropy is continuous. Hence,

$$\limsup_{n_k \to \infty} h_{\mu_{n_k}}(\phi_{n_n}) = \limsup_{n_k \to \infty} h_T(\phi_{n_k}) = h_T(\phi).$$

This implies that $h_T(\phi) = h_{\mu^*}(\phi)$, and since the measure of maximal entropy is unique, $\mu^* = \mu$.

IV. Topological Entropy for Geodesic Flows on Manifolds of Negative Curvature

The following inequality is a counterpart of the one in Proposition 6. Its proof is even closer to the proof of Theorem 1 in [K4].

Proposition 8. *Let (M, g_0) be a compact manifold of negative curvature and g be another metric of negative curvature on M. Then:*

$$\frac{h_T(g_0)}{\int_{(SM)_{g_0}} \|v\|_g d\mu_{g_0}} \leqq h_T(g) \leqq h_T(g_0) \cdot \int_{(SM)_g} \|v\|_{g_0} d\mu_g,$$

where μ_{g_0} and μ_g are the Margulis measures associated to g_0 and g.

Proof. Again we use the fact that the topological entropy is given by the exponential

growth rate of the number of closed geodesics, i.e.

$$h_T(g) = \lim_{t \to \infty} \frac{\log \# P_g(t)}{t},$$

where $P_g(t)$ denotes the set of closed geodesics of (prime) period $\leq t$ in the metric g. Let $\Pi(M)$ be the set of free homotopy classes of closed curves in M. Given a metric g of negative curvature on M, we associate to each $\alpha \in \Pi(M)$ the uniquely determined closed geodesic $c_{\alpha,g} \in \alpha$. If $L_g : \Pi(M) \to \mathbb{R}$ is the length function with respect to the metric g, then $c_{\alpha,g}$ is characterized by

$$L_g(c_{\alpha,g}) = \inf_{c \in \alpha} L_g(c).$$

Choose $\varepsilon > 0$ arbitrary. As before let $P_g^{\varepsilon, \| \ \|_{g_0}}(t)$ be the set of all closed geodesics in the metric g of period $\leq t$ which are $(\varepsilon, \| \ \|_{g_0})$ uniformly distributed with respect to the Margulis measure μ_g. If $c_{\alpha,g} \in P_g^{\varepsilon, \| \ \|_{g_0}}(t)$ is parametrized by arclength, we obtain the following for the period of the corresponding closed geodesics c_{α,g_0},

$$L_{g_0}(c_{\alpha,g_0}) \leqq L_{g_0}(c_{\alpha,g}) = \int_0^{L_g(c_{\alpha,g})} \| \dot{c}_{\alpha,g}(s) \|_{g_0} \, ds$$

$$\leqq L_g(c_{\alpha,g}) \left(\varepsilon + \int_{(SM)_g} \| v \|_{g_0} d\mu_g \right) =: L_g(c_{\alpha,g}) \cdot a(\varepsilon).$$

Applying Proposition 2 yields:

$$h_T(g) = \lim_{t \to \infty} \frac{\log \# P_g^{\varepsilon, \| \ \|_{g_0}}(t)}{t}$$

$$\leqq a(\varepsilon) \lim_{t \to \infty} \frac{\log \# P_{g_0}(t \cdot a(\varepsilon))}{t \cdot a(\varepsilon)} = a(\varepsilon) \cdot h_T(g_0).$$

Since $\varepsilon > 0$ was arbitrary, it follows that:

$$h_T(g) \leqq h_T(g_0) \cdot \int_{(SM)_g} \| v \|_{g_0} d\mu_g.$$

Replacing g by g_0 we obtain:

$$h_T(g_0) \leqq h_T(g) \cdot \int_{(SM)_{g_0}} \| v \|_g d\mu_{g_0}.$$

Proof of Theorem 3. Let g_λ be a C^1 perturbation of $g = g_0$. Then Proposition 8 yields:

$$h_T(\phi_0) \left(\frac{1}{\int_{(SM)_{g_0}} \| v \|_{g_\lambda} d\mu_{g_0}} - 1 \right) \leqq h_T(g_\lambda) - h_T(g_0)$$

$$\leqq h_T(\phi_0) \left(\int_{(SM)_{g_\lambda}} \| v \|_{g_0} d\mu_{g_\lambda} - 1 \right).$$

Linearizing $\| v \|_{g_\lambda}$ in λ at $\lambda = 0$, we have:

$$\| v \|_{g_\lambda} = \| v \|_{g_0} + \lambda \cdot \frac{\partial}{\partial s} \bigg|_{s=0} \| v \|_{g_s} + o(\lambda, v).$$

Substituting this in the above inequality we obtain for suitable $o_1(\lambda)$, $o_2(\lambda)$:

$$h_T(g_0)\left(-\lambda \int_{(SM)_{g_0}} \frac{\partial}{\partial s}\bigg|_{s=0} \|v\|_{g_s} d\mu_0 + o_1(\lambda)\right) \leq h_T(g_\lambda) - h_T(g_0)$$

$$\leq h_T(g_0)\left(-\lambda \int_{(SM)_{g_\lambda}} \frac{\partial}{\partial s}\bigg|_{s=0} \|v\|_{g_s} d\mu_{g_\lambda} + o_2(\lambda)\right).$$

The weak continuity of the Margulis measure implies that:

$$\frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_0} h_T(g_\lambda) = -h_T(g_0)\cdot \int_{(SM)_{g_0}} \frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_0} \|v\|_{g_\lambda} d\mu_{g_0}$$

$$= \frac{-h_T(g_0)}{2}\cdot \int_{(SM)_{g_0}} \frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_0} g_\lambda(v,v) d\mu_{g_0}.$$

Proof of Theorem 4. Let (M,g) be a compact C^2 Riemannian manifold of negative curvature *of arbitrary dimension* and let l_{g_0} denote the Liouville measure with respect to g_0. A straightforward calculation [Be] shows:

$$\int_{(SM)_{g_0}} \frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_0} g_\lambda(v,v) dl_{g_0} = \frac{\omega(n-1)}{n} \frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_0} \mathrm{vol}_{g_\lambda}(M),$$

where $\omega(n-1)$ denotes the volume of the Euclidean $(n-1)$ sphere. The formula for the derivative of the entropy immediately shows that g_0 is a critical point in $\mathfrak{R}(M)$ if the Margulis measure and the Liouville measure of g_0 coincide. Now assume that g_0 is a critical point for $h_T:\mathfrak{R}(M)\to\mathbb{R}$. Let ξ be a symmetric $(0,2)$ tensor on M such that $g_\lambda = g_0 + \lambda\xi$ is a perturbation of g_0 and $g_\lambda\in\mathfrak{R}(M)$. Then:

$$\frac{\omega(n-1)}{n}\cdot\frac{\partial}{\partial \lambda}\bigg|_{\lambda=\lambda_0} \mathrm{vol}_{g_\lambda}(M) = \int_{(SM)_{g_0}} \xi(v,v) dl_{g_0} = 0.$$

The formula for the derivative of entropy implies that:

$$\int_{(SM)_{g_0}} \xi(v,v) d\mu_{g_0} = 0,$$

where μ_{g_0} is the Margulis measure of g_0. Define the functionals

$$L_{l_{g_0}}(f) = \int_{(SM)_{g_0}} f\, dl_{g_0} \quad \text{and} \quad L_{\mu_{g_0}}(f) = \int_{(SM)_{g_0}} f\, d\mu_{g_0},$$

where f is a smooth function. Let $\Omega_{0,2}$ be the space of symmetric $(0,2)$ tensors on M. If $\xi\in\Omega_{0,2}$, then $\xi(v,v)$ is a smooth function on SM_{g_0}. The discussion above shows that if g_0 is a critical point for h_T, then $\mathrm{Ker}\, L_{l_{g_0}|\Omega_{0,2}} \subseteq \mathrm{Ker}\, L_{\mu_{g_0}|\Omega_{0,2}}$.

Proof. Choose $r\in\Omega_{0,2}$ and assume $L_{l_{g_0}}(r) = a$. Define $q\in\Omega_{0,2}$ as $q = a\cdot g_0$. This implies: $L_{l_{g_0}}(r-q) = L_{l_{g_0}}(r) - a = 0$. By hypothesis we obtain:

$$0 = L_{\mu_{g_0}}(r-q) = L_{\mu_{g_0}}(r) - L_{\mu_{g_0}}(q) = L_{\mu_{g_0}}(r) - a.$$

Hence, $L_{\mu_{g_0}}(r) = L_{l_{g_0}}(r)$.

This lemma shows that $L_{l_{g_0}}(f) = L_{\mu_{g_0}}(f)$ for all functions $f:SM_{g_0}\to\mathbb{R}$ which arise from symmetric $(0,2)$ tensors on M. Let $\pi:(SM)_{g_0}\to M$ denote the canonical

projection. If we associate to each smooth function $f: M \to \mathbb{R}$ the symmetric $(0,2)$ tensor $f \cdot g_0$, then Lemma 8 implies that $L_{l_{g_0}}(f \circ \pi) = L_{\mu_{g_0}}(f \circ \pi)$ for all smooth functions $f: M \to \mathbb{R}$.

The measures l_{g_0} and μ_{g_0} are invariant under the geodesic flow ϕ^t and coincide on $\pi^{-1}(B(M)) := \{\pi^{-1}(A) \mid A \in B(M)\}$, where $B(M)$ is the σ-algebra of Borel subsets on M. We would like to say this implies that the measures coincide on the σ-algebra of Borel subsets of SM, and that g_0 is locally symmetric (cf. Conjectures 2 and 3 above). Unfortunately we can only prove this for surfaces.

Lemma 9. *Let (M, g_0) be a compact negatively curved surface and suppose that l_{g_0} and μ_{g_0} coincide on $\pi^{-1}(B(M))$. Then g_0 is a metric of constant negative curvature.*

Proof. The function $H^+(v)$, which assigns to each $v \in SM_{g_0}$ the normal curvature of the unstable horocycle orthogonal to v, satisfies the Riccati differential equation $\dot{u} + u^2 + K(\pi \circ \gamma_v(t)) = 0$, where γ_v is the geodesic with initial velocity v and K is the Gaussian curvature. If we integrate the Riccati equation with respect to μ_{g_0}, we obtain:

$$\int_{(SM)_{g_0}} (H^+)^2 d\mu_{g_0} = \int_{(SM)_{g_0}} -K(\pi \circ v) d\mu_{g_0}.$$

Since all of our metrics are normalized to have area 1, our hypothesis and the Gauss–Bonnet Formula imply:

$$\int_{(SM)_{g_0}} -K(\pi \circ v) d\mu_{g_0} = \int_{(SM)_{g_0}} -K(\pi \circ v) dl_{g_0} = -2\pi E,$$

where E is the Euler characteristic of M. The Ruelle entropy estimate [R] and Jensen's inequality imply:

$$h_{\text{top}}^2 \leq \left(\int_{(SM)_{g_0}} H^+ d\mu_{g_0} \right)^2 \leq \int_{(SM)_{g_0}} (H^+)^2 d\mu_{g_0}.$$

Putting everything together we obtain $h_{\text{top}}^2 \leq -2\pi E$. In Theorem 3.1 of [K2], the first author proves that $h_{\text{top}}^2 \geq -2\pi E$, and the inequality is strict unless g_0 is a metric of constant negative curvature. We conclude that g_0 is a metric of constant negative curvature.

V. Appendix. Proof of Proposition 3

Proof. Let Ψ be C^1 close to Φ. The idea is to use the Implicit Function Theorem to solve:

$$\Psi \circ u = \gamma \cdot D_\Phi u \quad \text{for} \quad u \in C_\Phi^\alpha \quad \text{and} \quad \gamma \in C^\alpha(X).$$

Define:

$$G: V^1(X) \times C_\Phi^0(X) \times C^0(X, \mathbb{R}) \to C^0(X, TX),$$

$$G(\Psi, u, \gamma) = \Psi \circ u - \gamma \cdot D_\Phi u,$$

where $V^1(X)$ denotes the space of C^1 vector fields on X and $C^0(X, Y)$ the space of C^0 functions from X to Y. The spaces introduced above are Banach manifolds

and one can easily prove that G is a C^1 map. It is clear that

$$G(\Phi, \mathrm{id}, 1) = 0.$$

Let $D_{2,3}G$ be the derivative with respect to the second and third variables. Then

$$D_{2,3}G(\Phi, \mathrm{id}, 1): V_\Phi^0(X) \times C^0(X, \mathbb{R}) \to V^0(X)$$

is given by:

$$D_{2,3}G(\Phi, \mathrm{id}, 1)(v, \gamma) = D\Phi(v) - Dv(\Phi) - \gamma \cdot \Phi$$

$$= [\Phi, v] - \gamma \cdot \Phi$$

$$= \frac{d}{dt}\bigg|_{t=0} D\phi^{-t} v \circ \phi^t - \gamma \cdot \Phi,$$

where $V_\Phi^0(X)$ denotes the subspace of the space of continuous vector fields $V^0(X)$ such that the directional derivative with respect to Φ exists. $(T_{\mathrm{id}}C_\Phi^0 = V_\Phi^0(X))$. Because the kernel of the Lie derivative

$$L_\Phi(v) = [\Phi, v] = \frac{d}{dt}\bigg|_{t=0} D\phi^{-t} v \circ \phi^t$$

contains the infinitesimal generator Φ, it follows that $D_{2,3}G$ is not invertible. To overcome this problem choose a section transverse to Φ. For instance define $\Phi^\perp = \Gamma(E^s) \oplus \Gamma(E^u)$ and $V_{\Phi,\perp}^0(X) = V_\Phi^0 \cap \Phi^\perp$. Then it is easy to show that $L_\Phi^\perp: V_{\Phi,\perp}^0(X) \to V_{\Phi,\perp}^0(X)$ has an inverse given by:

$$(L_\Phi^\perp)^{-1}(v_u) = -\int_0^\infty D\phi^{-s} v_u \phi^s ds, \quad v_u \in \Gamma(E^u),$$

$$(L_\Phi^\perp)^{-1}(v_s) = \int_0^\infty D\phi^x v_s \phi^{-x} dx, \quad v_s \in \Gamma(E^s).$$

This proves that $(v, \gamma) \to L_\Phi^\perp(v) - \gamma \cdot \Phi = D_{2,3}G(\Phi, \mathrm{id}, 1)(v, \gamma)$ is invertible. Now apply the Implicit Function Theorem to the equation $G = 0$ with C_Φ^0 replaced by $C_{\Phi,\perp}^0(X) = \exp(V_{\Phi,\perp}^0(X))$.

References

[A]   Anosov, D.: Geodesic Flows on closed Riemannian manifolds with negative curvature. Proc. Stek. Inst. **90** (1967)

[B]   Bowen, R.: Periodic orbits for hyperbolic flows. Am. J. Math. **94**, 1–30 (1972)

[Be]   Besse, A.: Einstein Manifolds. Berlin, Heidelberg, New York: Springer 1987

[BGS] Ballmann, W., Gromov, M., Schroeder, V.: Manifolds of non-positive curvature. Prog. Math., vol. **61**. Baset, Boston: Birkhäuser 1985

[BK]   Burns, K., Katok, A.: Manifolds with non-positive curvature. Ergod. Th. Dyn. Sys. **5**, 307–317 (1985)

[C]   Contreras, G.: Regularity of Topological and Metric Entropy of Hyperbolic Flows, preprint

[FM]   Freire, A., Mañé, R.: On the entropy of the geodesic flow in manifolds without conjugate points. Invent. Math. **69**, 375–392 (1982)

[K1]   Katok, A.: Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. Publ. Math. IHES **51**, 137–173 (1980)

[K2] Katok, A.: Entropy and closed geodesics. Ergod. Th. Dyn. Syst **2**, 339–367 (1982)

[K3] Katok, A.: Nonuniform hyperbolicity and structure of smooth dynamical systems. Proc. of Intl. Congress of Math. 1983, Warszawa, vol. **2**, 1245–1254

[K4] Katok, A.: Four applications of conformal equivalence to geometry and dynamics. Ergod. Th. Dyn. Sys. **8**, 139–152 (1988)

[KKPW] Katok, A., Knieper, G., Pollicott, M., Weiss, H.: Differentiability and analyticity of topological entropy for Anosov and geodesic flows. Invent. Math. **98**, 581–597 (1989)

[LMM] de la Llave, R., Marco, J., Moriyon, R.: Canonical perturbation theory of Anosov systems and regularity results for Livsic cohomology equation. Ann. Math. **123**, 537–611 (1986)

[M] Manning, A.: Topological entropy for geodesic flows. Ann. Math. **110**, 567–573 (1979)

[Ma1] Margulis, G.: Applications of ergodic theory to the investigation of manifolds of negative curvature. Funct. Anal. Appl., **3**, (4), 335–336 (1969)

[Ma2] Margulis, G.: Certain measures associated with U-flows on compact manifolds. Funct. Anal. Appl. **4**, (1), 55–67 (1969)

[Mi1] Misiurewicz, M.: On non-continuity of topological entropy. Bull. Acad. Polon. Sci., Ser. Sci. Math. Astro. Phys. **19**, (4), 319–320 (1971)

[Mi2] Misiurewicz, M.: Diffeomorphisms without any measure with maximal entropy. Bull. Acad. Polon. Sci., Ser. Sci. Math. Astro. Phys., **21**, (10), 903–910 (1973)

[N] Newhouse, S.: Continuity properties of entropy. Ergod. Th. Dyn. Sys. **8**, 283–300 (1988)

[R] Ruelle, D.: An inequality for the entropy of differentiable maps. Bol. Soc. Brasil. Math., **9**, 83–87 (1978)

[W] Walters, P.: An introduction to ergodic theory. Graduate Texts in Math. vol. **79**. Berlin, Heidelberg, New York: Springer 1982

[Y] Yomdin, Y.: Volume growth and entropy. Israel J. Math. **57**, 285–300 (1987)