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NEW EXAMPLES IN SMOOTH ERGODIC THEORY. ERGODIC DIFFEOMORPHISMS

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Contents

Introduction

1. In this paper we use a uniform construction method in order to obtain examples of measure-preserving ergodic C^∞-diffeomorphisms defined on certain smooth manifolds which have various, sometimes unexpected metric properties. The manifolds to be considered will be compact, connected, smooth manifolds (with or without boundary) possessing a nontrivial smooth free group of circular rotations,[1] or, as we shall call it, a periodic current. The invariant measure in our examples will have a positive smooth density. The metrical properties of the diffeomorphisms which we shall construct may have the following properties depending on the choice of the parameters in the construction:

a) a discrete spectrum with an arbitrary given (finite or infinite) number of basic (i.e. linearly independent over the ring $\mathbf{Z}$ of integers) frequencies;

b) a simple continuous singular spectrum in the absence of the mixing property;

c) properties a) and b) combined.

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[1] Nontrivial in the sense that each point of the manifold is displaced by at least one element of the group.

These examples obviously show that [2] there is but a rather weak connection between the topological properties of a manifold and the ergodic properties of the diffeomorphisms defined on it.

2. It is well known (cf. [6]) that an arbitrary ergodic automorphism of a Lebesgue space with discrete spectrum generated by h basic eigenfrequencies is metrically isomorphic to some shift transformation of the h-dimensional torus. The problem of the realization of such metric automorphisms by diffeomorphisms of smooth manifolds with smooth invariant measure is considerably more difficult. It is easy to prove that the manifold must necessarily be an h-dimensional torus if, in addition, the eigenfunctions are assumed to be smooth. The diffeomorphism is in this case smoothly equivalent to a shift operator. On the other hand, in the case of the two-dimensional torus A. N. Kolmogorov [11] has constructed a real-analytic diffeomorphism with an analytic invariant measure which is metrically isomorphic to the group shift and which possesses discontinuous eigenfunctions. (This implies that these diffeomorphisms can be not even topologically conjugate to a shift.) Until now no smooth realization of automorphisms with discrete spectrum having h basic frequencies have been known on any manifolds other than the h-dimensional torus. There exists a conjecture (cf. [10]) to the effect that on a real-analytic m-dimensional manifold an ergodic, analytic diffeomorphism with analytic invariant measure may have a discrete spectrum with only m basic frequencies. This problem is still open even though our results also show that the C^∞ version of it is not true.

3. Even in the simple case of the two-dimensional circle

$$\mathbf{D}^2 = \{(x, y) \colon x^2 + y^2 \leqq 1\}$$

the problem is still open whether there exists an ergodic diffeomorphism conjugate to Lebesgue measure. However, for different special functions ρ the topological properties of those mappings of the circle which are expressible in polar coordinates in the form

$$(r, \phi) \to (\rho(r, \phi), \phi + \leftarrow) \tag{0.1}$$

were analyzed a long time ago. An equivalent problem was first studied by Poincaré ([19], Chapter 19). In connection with the investigation of the neighborhood of a periodic orbit of a three-dimensional dynamic system, L. G. Šnirel'man [20] and A. S. Besicovitch [2] have constructed examples of topologically transitive[3] mappings of the form (0.1). Their examples were

[2] We refer to the case of two or more dimensions.

[3] Topological transitivity means that there exists an everywhere dense orbit (which must then be a continuum; cf. [15]). It is interesting that Šnirel'man and Besicovitch at first calculated as if all points of the circle, except the center and the points of its boundary, had everywhere dense orbits. The error was corrected in [3] and [5].

not smooth, but smoothness can be attained without essentially altering the construction (cf. [21]). However, it is easy to prove that a transformation of the form (0.1) of the circle cannot have an ergodic invariant measure equivalent to Lebesgue measure. A careful analysis of the examples given by Šnirel′man and Besicovitch has also been one of the starting points for the present paper.

4. The diffeomorphisms which we will exhibit form a nowhere dense set in the space of all diffeomorphisms of a given manifold which preserve a given measure, provided with the topology of C^∞ or C^n for some $n \geqq 1$. This is related to the fact that our diffeomorphisms belong to the closure of the set of periodic diffeomorphisms, which is nowhere dense in that space, and even to the set of those diffeomorphisms among them which are shifts with respect to the orbits of periodic flows. It is evident that even the set of all ergodic diffeomorphisms is not everywhere dense in the space described (in the two-dimensional case this follows from results by J. Moser [13], [14] under a sufficient smoothness assumption).

In the class of homeomorphisms the situation is different. In 1941 it was shown by Oxtoby and Ulam [17] that for any triangulated manifold[4] and for an arbitrary measure satisfying natural conditions, the ergodic homeomorphisms which preserve that measure form a set of the second category (an everywhere dense G_δ set) in the space of all (measure-preserving) homeomorphisms under the corresponding topology.[5] From the purely metrical point of view Halmos [7] (cf. [6]) showed that the situation is analogous: In the space of all automorphisms of a Lebesgue space provided with the weak topology, the ergodic automorphisms form a set of the second category. At the basis of both results there lies the same phenomenon: The periodic automorphisms (in the metric case) and the homeomorphisms which are periodic outside a set of sufficiently small measure (in the topological case) are everywhere dense in the corresponding basis. This is in good agreement with the fact that also in the smooth case the metric situation on the closure of the set of periodic diffeomorphisms resembles the topological case and the purely metrical case in many respects.

5. The methods of this paper are suitable for establishing also a number of additional results. Thus on manifolds with periodic flows A. B. Katok has constructed nonergodic diffeomorphisms which have an arbitrary finite or denumerably infinite number of ergodic components, where for each of them any one of the subcases a), b) or c) of subsection 1 is possible and, furthermore, the following modifications may occur:

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[4] This is also true with respect to polyhedra of a somewhat more general type.

[5] An application of the uniform approximation technique allows one to derive sharper results from the work of Oxtoby and Ulam (cf. [9]).

a) As ergodic components one may choose open sets (with complementary sets of measure zero).

b) Each ergodic component intersects with any open set in a set of positive measure; hence, in particular, almost all orbits are everywhere dense.

As we have already stated, all these results are concerned with a special class of manifolds. Recently E. A. Sirodov [21] has proved that on many manifolds of dimension three or larger there exists a topologically transitive smooth flow (and consequently a diffeomorphism). Using some modifications of the methods of our paper, D. V. Anosov has proved that on an arbitrary manifold of dimension three or larger there exists an ergodic smooth flow and consequently a diffeomorphism (cf. [1]). (This was preceded by a more complicated construction of analogous flows with four or more dimensions due to A. B. Katok.) For the sake of completeness we also wish to mention that recently A. A. Blohin has constructed examples of smooth ergodic currents on all closed surfaces excepting spheres, projective planes and Klein bottle, on which no such flows exist. Similar examples were also investigated by J. Milnor.

6. In the first two sections we present here for convenience auxiliary material which is undoubtedly well known. §3 is the basic part of the paper. There we introduce an inductive process on an arbitrary manifold with periodic flow which enables us to construct, for a given smooth measure μ, a sequence of periodic μ-preserving diffeomorphisms which converges to an ergodic diffeomorphism under the C^∞ topology. This construction possesses a significant nonuniqueness property. In §§4 to 6 it is shown that, under a suitable choice of the parameter in the construction, we may obtain a diffeomorphism with given metrical properties. Furthermore we can construct a diffeomorphism which is metrically isomorphic to a circular rotation. (For the construction of such a diffeomorphism, subsections 5, 6 and 7 of §3 are not needed except for the almost trivial verification of the inductive assumption at the end of subsection 5.) In §7 it is shown that in the closure of the set of diffeomorphisms belonging to a periodic flow, the ergodic diffeomorphisms form a set of the second category. The basic results of this paper have been announced in [1].

§1. Background material about measures on smooth manifolds

1. A measure μ on an n-dimensional manifold M^m of class C^∞ is called a measure of class C^∞ if on any coordinate neighborhood it induces an infinitely differentiable density relative to the local coordinates. For brevity such a measure will be called positive if this density does not vanish identically on any coordinate neighborhood. If μ_1 and μ_2 are two positive measures of class C^∞ on M^m then there exists a positive function $\rho(x)$ of class C^∞ such that $\mu_1 = \rho(x)\mu_2$. Consequently all positive measures have the same class of null sets.

THEOREM 1.1. *Let M^m be an m-dimensional open connected manifold of class C^∞, and let μ_1 and μ_2 be positive measures of class C^∞ on M^m which are identical outside a compact set $N \subset M^m$, for which $\mu_1(M^m) = \mu_2(M^m)$. Then there exists a diffeomorphism S: $M^m \to M^m$ which is the identical mapping outside a compact set $N_1 \subset M^m$ such that $S^*\mu_1 = \mu_2$, using the notation*

$$(S^*\mu_1)(A) = \mu_1(S^{-1}(A)).$$

This theorem is formally different from Theorem 1 of Moser [16] inasmuch as we do not assume the manifold to be closed, but the theorem follows from the same Lemmas 1 and 2 of the paper [16].

By a "manifold" we shall henceforth always mean a connected, compact manifold of class C^∞, closed or with boundary, unless something else is specified.

THEOREM 1.2. *Let μ_1 and μ_2 be two normalized positive measures of class C^∞ on the manifold M^m. Then there exists a diffeomorphism S: $M^m \to M^m$ of class C^∞ such that $S^*\mu_1 = \mu_2$.*

For closed manifolds this assertion reduces to the Theorem 1 from the paper [16] which we have already mentioned. For manifolds with boundary it follows from Theorem 1.1 if the measures μ_1 and μ_2 coincide on some neighborhood of the boundary. Therefore it suffices to prove the existence of a diffeomorphism $\hat{S}$ such that the measure $\hat{S}^*\mu_1$ coincides with μ_2 on some neighborhood of the boundary.

In order to prove this we consider nonoverlapping tube-shaped neighborhoods B_i of the connected components A_i $(i = 1, \cdots, n)$ of the boundary. Then B_i is diffeomorphic to the direct product $A_i \times [0, 10]$. Hence we can introduce coordinates (y, t) on B_i, where $y \in A_i$, $0 \leqq t \leqq 10$. We consider some fixed positive measure λ of class C^∞ on A_i, and we let

$$d\mu_1(y, t) = \rho_1(y, t)\,dt\,d\lambda(y), \quad d\mu_2(y, t) = \rho_2(y, t)\,dt\,d\lambda(y).$$

Then we introduce the function $\alpha(y, t)$ by the relation

$$\int_0^t \rho_1(y, s)\,ds = \int_0^{\alpha(y,t)} \rho_2(y, s)\,ds.$$

The function $\alpha(y, t)$ is defined for any y and for all sufficiently small positive t. Let $\delta \in (0, 3)$ be such that for $0 \leqq t \leqq \delta$ the function $\alpha(y, t)$ is defined and satisfies the condition $|\alpha| \leqq 1$. We construct functions ϕ, $\psi \in C^\infty[0, 10]$ with the following properties:

The function ϕ is nonincreasing, $\phi(t) = 1$ for $t \in [0, 2\delta/3]$ and $\phi(t) = 0$ for $t \geqq \delta$, with $\phi'(t) > -4/\delta$ for all t.

The function ψ is nondecreasing for $t \in [0, \delta]$, with $\psi(t) = 0$ for $t \in [0, \delta/3]$ and $\psi(t) = 4t/\delta$ for $t \in [2\delta/3, \delta]$, with $\psi'(t) > -1$ for $t \geqq \delta$ and $\psi(t) = 0$ for $t \geqq 8$.

It is easy to verify that the function

$$F(y,t)=\varphi(t)\,\alpha(y,t)+(1-\varphi(t))\,t+\psi(t)$$

coincides with $\alpha(y,t)$ for $t\in[0,\delta/3]$ and with t for $t\geqq 8$; furthermore, it satisfies $F_t'(x,t)>0$. Therefore the mapping $\hat{S}_i\colon B_i\to B_i$ defined by the formula

$$\hat{S}_i(y,t)=(y,F(y,t)),$$

is the identity mapping for $t\geqq 8$; it may be extended to all points M^m outside B_i in a unique manner, and for $X\subset A_i\times[0,\delta/3]$ we have $\mu_1(X)=\mu_2(\hat{S}_iX)$. Hence the diffeomorphism $\hat{S}=\coprod_i\hat{S}_i$ has the property that $\hat{S}^*\mu_1=\mu_2$ in some neighborhood of the boundary.

2. **Theorem** 1.3. *Let O^M be an open, connected manifold of class C^∞, let F_i and G_i $(i=1,\cdots,k)$ be two systems of open subsets from O^m whose closures $\overline{F_i}$ and $\overline{G_i}$ are C^∞-diffeomorphic to the m-dimensional sphere $\mathbf{D}^m$, with $F_i\cap\overline{F_j}=\emptyset$ and $G_i\cap G_j=\emptyset$ for $i\neq j$. Then there exists a C^∞-diffeomorphism S^1: $O^m\to O^m$ which is equal to the identity mapping outside some compact set $N_1\subset O^m$ and which maps F_i into G_i.*

This fact is well known and easily derived, for example, from [18].

In this paper we always denote by $\mathrm{Diff}^\infty(M^m,\mu)$ the space of those diffeomorphisms of class C^∞ of the manifold M^m which preserve a given positive finite measure μ of class C^∞, provided with the natural topology.

Lemma 1.1. *Under the conditions of Theorem 1.3 let μ be a given measure of class C^∞ on the manifold O^m such that $\mu(F_i)=\mu(G_i)$, $i=1,\cdots,k$. Then for any $\epsilon>0$ there exists a diffeomorphism $S\in\mathrm{Diff}^\infty(O^m,\mu)$ which coincides with the identity mapping outside a given compact set $N\subset O^m$ and which satisfies the inequality*[6] $\mu(SF_i\triangle G_i)<\epsilon, i=1,\cdots,k$.

Proof. We construct a diffeomorphism S^1 by means of Theorem 1.3. Let $(S^1)^*\mu=\rho(x)\mu=\mu_1$. By the assumptions of the lemma we have

$$\mu_1(G_i)=\int_{G_i}\rho(x)\,d\mu=\mu(G_i),\quad i=1,\ \dots,\ k.$$

We choose open sets G_i^1, G_i^2, G_i^3 whose closures are diffeomorphic to $\mathbf{D}^m$ and which furthermore satisfy

$$\overline{G_i^1}\subset G_i^2\subset\overline{G_i^2}\subset G_i^3\subset\overline{G_i^3}\subset G_i,$$

$$\mu(G_i^1)=\mu(G_i)-\delta,\ \mu(G_i^2)<\mu(G_i)-\frac{2\delta}{3},\ \mu_1(G_i^3)>\mu_1(G_i)-\frac{\delta}{3},$$

where the number $\delta>0$ is small enough so that

[6] As A. B. Krygin has shown, this lemma is in reality true with $SF_i=G_i$. But we shall not need this version.

$$\mu_1(G_i^1) > \mu(G_i) - \frac{\varepsilon}{2}. \tag{1.1}$$

We construct on G_i a positive function $\rho_i(x)$ of class C^∞ which satisfies

$$\rho_i(x) = 1 \quad \text{for } x \in G_i^2, \quad \rho_i(x) = \rho(x) \quad \text{for } x \in G_i \setminus G_i^3,$$
$$\int_{G_i} \rho_i(x)\, d\mu = \mu(G_i).$$

(This is possible since $\mu(G_i^2) + \mu_1(G_i \setminus G_i^3) > \mu(G_i) - \delta/3$.)

Now we let $\mu^{(i)} = \rho_i(x)\mu$. By Theorem 1.1 there exists a diffeomorphism $S^{(i)}: G_i \to G_i$ which is the identity mapping along the boundary of the sphere G_i and which satisfies the equation $S^{(i)*}\mu_1 = \mu^{(i)}$. Extending $S^{(i)}$ as the identity mapping on the entire manifold O^m, we obtain $\hat{S} = \coprod_i S^{(i)}$. Now we consider the manifold $\hat{O}^m = O^m \setminus \bigcap_i G_i^1$. On $\hat{O}^m$ we have the two measures μ and $\hat{\mu} = \hat{S}^*\mu_1$. They coincide within the compact set $N_1 \cap (O^m \setminus \bigcup_i G_i^2)$ where N_1 stands for the same set as in Theorem 1.3. Furthermore, $\mu(\hat{O}^m) = \hat{\mu}(\hat{O}^m)$. Applying once more Theorem 1.1, we obtain a diffeomorphism $\widetilde{S}: \hat{O}^m \to \hat{O}^m$, equal to the identity mapping outside some compact set N_2, for which $\widetilde{S}^*\hat{\mu} = \mu$. We extend $\widetilde{S}$ as the identity mapping to all points of O^m, and we let $S = \widetilde{S}\hat{S}S^1$. Clearly $S^*\mu = \mu$. The diffeomorphism S is the identity outside some compact set, since this is true for each of the diffeomorphisms S^1, $\hat{S}$ and $\widetilde{S}$.

Now we define $F_i^1 = (S^1)^{-1}G_i^1$. Obviously $SF_i^1 \subset G_i$, and therefore by (1.1) we have

$$\mu(F_i^1) = \mu_1(G_i^1) > \mu(G_i) - \frac{\varepsilon}{2},$$

i.e. $\mu(SF_i \triangle G_i) < \epsilon$. This proves the lemma.

§2. Periodic flows

A flow $\{S_t\}: M^m \to M^m$ is called periodic if S_τ is the identity mapping for some $\tau > 0$. In the sequel we shall always assume that on the manifold M^m there exists a nontrivial periodic flow of class C^∞ (i.e. generated by a vector field of class C^∞), and this flow will be denoted by $\{S^t\}$. For $x \in M^m$ we define

$$t(x) = \inf\{\tau : \tau > 0,\ S_\tau x = x\}.$$

Without loss of generality we may assume that $\max_x t(x) = 1$.

The following assertion follows immediately from a theorem by Bochner on the smooth operation of a compact group (cf. [19], §5.2, Theorem 1).

Proposition 2.1. *The closed set $\Omega_1 = \{x \mid t(x) < 1\}$ has measure zero and its complementary set is connected.*

Indeed, by Bochner's theorem, in the neighborhood of a fixed point, expressed in some coordinates, a group acts as a linear transformation. Therefore, the set of fixed points of our flow is a submanifold of codimension ≥ 2.

The points x with $t(x) = 1/q$ $(q = 3, 4, \cdots)$ are fixed points under the cyclic group of mappings generated by $S_{1/q}$. Consequently they form locally a submanifold of codimension $\geqq 2$. Only the points with $t(x) = \frac{1}{2}$ have to be considered separately. They may form a submanifold M^{m-1} of codimension one which is a local decomposition of M^m. But $S_{1/2}$ transforms the points on one side of M^{-1} into points on the other side. Consequently these points are joined with each other by orbits of the flow $\{S_t\}$.

REMARK. Each periodic flow possesses a positive invariant measure of class C^∞ on M^m. Let $S_t^*\mu = \rho_t(x)\mu$, and define

$$\hat{\mu} = \left(\int_0^1 \rho_t(x)\, dt\right)\mu_1.$$

Then the measure $\hat{\mu}$ is positive, it belongs to the class C^∞ and is invariant with respect to $\{S_t\}$.

PROPOSITION 2.2. *Let μ be a positive measure of class C^∞ on M^m. If there exists on M^m a periodic flow $\{S_t\}$, then there exists also on M^m a periodic flow with the same period which preserves the measure μ.*

PROOF. We construct a positive measure $\hat{\mu}$ of class C^∞ which is invariant with respect to $\{S_t\}$. We may assume that $\mu(M^m) = \hat{\mu}(M^m)$. By Theorem 1.2 there exists a diffeomorphism $S: M^m \to M^m$ satisfying $S^*\hat{\mu} = \mu$. Hence it follows that the measure μ is invariant with respect to the periodic flow $\{S^{-1}S_tS\}$.

We introduce some simple examples of periodic flows:

1) A one-parameter group of rotations of the unit sphere, or spherical surface, in $\mathbf{R}^{n+1}$ about a fixed $(n-1)$-dimensional linear subspace.

2) If on a Riemannian manifold there exists an oriented field of two-dimensional tangential planes, then this induces on a single tangential fibering a periodic flow of rotations about the orthogonal complement to that plane. For example, this is true on a tangential fibering of an oriented surface.

3) The existence of the periodic flow on a manifold M^m allows us, in an obvious manner, to construct a periodic flow on the direct product $M^m \times Y$, where Y is an arbitrary manifold of class C^∞.

Now we consider a manifold M^m with a periodic flow $\{S_t\}$. We need the following assertion on the existence of a special kind of mutually related fundamental domains for the mappings $S_{1/q}$ for the different values of q:

PROPOSITION 2.3. *There exists a system of sets $\Delta_q \subset M^m$ $(q = 1, 2, \cdots)$ with the following properties:*

2.3.1. *The set Δ_q is contained in the closure of the set of its interior points.*

2.3.2. *The boundary of Δ_q is a null set.*

2.3.3. *The set of interior points of Δ_q is open.*

2.3.4. $\bigcup_{k=0}^{q-1} S_{k/q}\Delta_q = M^m$.

2.3.5. $\bigcup_{k=1}^{q-1}(S_{k/q}\Delta_q \cap \Delta_q) \subset \Omega_1$ (*where* Ω_1 *is the exceptional set from Proposition* 2.1).

2.3.6. *The intersection of* Δ_q *with any orbit is connected.*

2.3.7. $\Delta_q = \bigcup_{l=0}^{k-1} S_{l/kq}\Delta_{qk}$ *for any positive integer* k.

PROOF. Let Ω_2 be the boundary of the manifold M^m. Then the set

$$\Omega = \Omega_1 \cup \Omega_2 \tag{2.1}$$

is obviously closed. The open manifold $M^m \backslash \Omega$ may be decomposed into equivalence classes each of which consists of the points belonging to the same orbit of the flow. On the factor-space N^{m-1} so obtained we introduce in a natural way the structure of an open manifold of class C^∞. The set $M^m \backslash \Omega$ is a fibering with the basis N^{m-1} and the circumference S^1 as fiber. Let π be the natural projection of $M^m \backslash \Omega$ onto N^{m-1}. Then the formula $\hat{\mu}(A) = \mu(\pi^{-1}(A))$ defines a positive measure of class C^∞ on N^{m-1}. In N^{m-1} there exists a closed subset E of measure zero such that the closed submanifold $N^{m-1} \backslash E$ is connected and on it the fibering is a direct product. We construct a smooth decomposition Δ_0 of this direct product. To the decomposition Δ_0 we add the entire set Ω_1 and one point belonging to $\pi^{-1}(E) \cup (\Omega_2 \backslash \Omega_1)$ from each orbit of the flow $\{S_t\}$, such that these points belong to the closure of Δ_0. We denote the set so obtained by $\hat{\Delta}$, and we let $\Delta_q = \bigcup_{0 \le t < 1/q} S_t \hat{\Delta}$. The verification of properties 2.3.1–2.3.7 does not present any difficulty.

§3. The basic construction

1. Suppose on the manifold M^m there is given a (nontrivial) periodic flow which preserves a given positive normalized measure μ of class C^∞. We shall augment the exposition by figures related to a simple case, namely the two-dimensional disk D^2 with Lebesgue measure and with the flow $\{S_t\}$ which is expressible in the form $S_t(r,\phi) = (r, \phi + 2\pi t)$ in polar coordinates. As sets Δ_q we select the sectors $\Delta_q = \{(r,\phi)\colon\ 0 \le \phi \le 2\pi/q\}$ which clearly satisfy the conditions 2.3.1–2.3.7.

2. Returning to the general case, we introduce the following notation. We let $S_{k/q}\Delta_q = \Delta_{k,q}$. The sets $\Delta_{k,q}$, $k = 0, \cdots, q-1$, form a decomposition mod 0 of the manifold M^m. We denote this decomposition by η_q. The automorphism S_α, where $\alpha = p/q$ (with integer p), maps the decomposition η_q into itself; if p and q are relatively prime then the factor-automorphism S_α/η_q permutes the $\Delta_{k,q}$ cyclicly. We observe that the interior $\operatorname{Int}\Delta_{k,q}$ is an open manifold for which

$$S_{\frac{k}{q}}(\operatorname{Int}\Delta_{0,q}) = \operatorname{Int}\Delta_{k,q}.$$

On M^m we introduce an arbitrary Riemannian metric of class C^∞ which is invariant under $\{S_t\}$. All distances which will be mentioned in the sequel are taken in this fixed metric. (In the case of the disk D^2 we consider the

Euclidean metric.) By a standard method we introduce in the space $\mathrm{Diff}^\infty(M^m)$ of all diffeomorphisms M^m of class C^∞ a sequence of metrics $\rho_r(T_1, T_2)$ measuring the proximity of the r-flows of T_1 and T_2 as well as T_1^{-1} and T_2^{-1}.

3. We construct inductively a sequence T_n of diffeomorphisms of the manifold M^m satisfying the following conditions which involve arbitrarily small numbers $\epsilon_n > 0$, positive integers k_n, l_n and positive integers $a_n(i) \leqq q_n$, defined for $i = 0, \cdots, k_n - 1$, where q_n is expressed inductively in terms of k_s and l_s with $s < n$.

First Step. Let $T_n = B_n^{-1} S_{\alpha_n} B_n$, where $B_n = A_n \cdots A_1$, and each A_k is an element of $\mathrm{Diff}^\infty(M^m, \mu)$ which is the identity map within some neighborhood of the set Ω (cf. (2.1)). Here we define the numbers $\alpha_n = p_n/q_n$, where p_n and q_n are relatively prime positive integers. Finally, we let

$$\alpha_{n+1} = \alpha_n + \beta_n, \quad \text{where } \beta_n = \frac{1}{k_n l_n q_n^2}, \tag{3.1}$$

$$S_{\alpha_n} A_{n+1} = A_{n+1} S_{\alpha_n}.$$

Thus

$$p_{n+1} = k_n l_n q_n p_n + 1, \quad q_{n+1} = k_n l_n q_n^2, \tag{3.2}$$

$$T_{n+1} = B_n^{-1} S_{\alpha_n} A_{n+1}^{-1} S_{\beta_n} A_{n+1} B_n.$$

No conditions will be imposed directly on k_n and l_n but for the Second, Third, and Fourth Steps it will be necessary to choose these parameters sufficiently large (see subsections 4 and 5 below); furthermore, in §§5 and 6 additional stipulations will be made on the size of k_n and l_n.

Second Step. Let $\xi_{n+1} = B_{n+1}^{-1} \eta_{q_{n+1}}$. There exists a set $E_{n+1} \subset M^m$ such that $\mu(E_{n+1}) > 1 - \epsilon_n$, where the diameter of the intersection between elements of the decomposition ξ_{n+1} with the set E_{n+1} is less than $1/2^{n+1}$ (which implies that $\xi_{n+1} \to \epsilon$ as n tends to infinity provided, of course, that $\lim_{n\to\infty} \epsilon_n = 0$).

Third Step. $\rho_{[1/\epsilon_n]}(T_{n+1}, T_n) < \epsilon$ (which implies that the sequence T_n converges under the topology of the space $\mathrm{Diff}^\infty(M^m, \mu)$ if $\sum_{k=1}^\infty \epsilon_k < \infty$).

Fourth Step. We let

$$R^{(n)} = \bigcup_{i=0}^{k_n - 1} \Delta_{a_n(i) k_n + i,\, k_n q_n}, \tag{3.3}$$

so that

$$\mu(A_{n+1}^{-1} R^{(n)} \triangle \Delta_{0, q_n}) < \epsilon_n. \tag{3.4}$$

It should be noted that the arbitrariness in our construction, which will be used in §§4 to 6 in order to obtain diffeomorphisms $T = \lim T_n$ with different metric properties, arises from the possibility of choosing the parameters $a_n(i)$.

REMARK 3.1. We introduce the decomposition[7)]

$$\eta_{n,n+1} = \{S_{\frac{k}{q_n}} R^{(n)},\ k = 0, 1, \ldots, q_n - 1\},$$

$$\xi_{n,n+1} = B_{n+1}^{-1}\eta_{n,n+1},\ \xi'_n = B_{n+1}^{-1}\eta_{k_n q_n}.$$

It follows from the Second Step and (3.4) that $\xi_{n,n+1} \to \epsilon$ as n tends to infinity. But since $\xi_{n,n+1} < \xi'_n$, this implies that $\xi'_n \to \epsilon$.

4. The number $\alpha_0 = p_0/q_0$ may be chosen arbitrarily. Suppose the numbers $\alpha_0, \cdots, \alpha_n$ and the diffeomorphisms $A_1, \cdots, A_n$ have already been defined in such a way that for each $k < n$ the conditions listed under Steps 1–4 are satisfied (with n replaced by k).

Since the mapping B_n^{-1} is uniformly continuous, there exists a number $\gamma_{n+1} > 0$ such that the diameter of the set U is less than $1/2^{n+1}$ if the diameter of the set $B_n(U)$ is less than γ_{n+1}.

Let k_n be a fixed, sufficiently large number (to be determined later) and consider the decomposition η_{k_n, q_n}. For any choice of $a_n(i)$ the set $R^{(n)}$ contains with any one point all the orbits of periodic diffeomorphisms S_{α_n} (or S_{1/q_n}, which amounts to the same) outside Ω_1.

It is our aim to construct a diffeomorphism

$$A_{n+1} \in \mathrm{Diff}^\infty(M^m, \mu),$$

which is the identity mapping within some neighborhood of the set Ω and which possesses the properties (3.1), (3.4) and (3.5).

For a suitable positive integer h_n and for every s, $0 \leqq s < h_n k_n$, there exists a set

$$R_s^{(n)} \subset \Delta_{a_n\left(\left[\frac{s}{h_n}\right]\right) k_n h_n + s,\ h_n k_n q_n},$$

(3.5) such that

$$\mu(R_s^{(n)}) > \frac{1-\varepsilon_n}{k_n h_n q_n} \quad \text{and} \quad \operatorname{diam} A_{n+1}^{-1} R_s^{(n)} < \gamma_{n+1}.$$

The existence of such a diffeomorphism will be proved in two steps. First we consider the special case where $R^{(n)} = \Delta_{0,q_n}$ (i.e. where $a_n(i) = 0$ for all $i = 0, \cdots, k_n - 1$). In this case it is possible to choose $h_n = 1$ in (3.5) as we have done. We select a number k_n of mutually disjoint open sets $F_i \subset \operatorname{Int} \Delta_{0,q_n}$, $i = 0, \cdots, k_n - 1$, with the following properties:

---

7) The notation $\xi_{n,n+1}$ and $\eta_{n,n+1}$ is explained by the fact that in subsection 6 of this section we will introduce decompositions $\xi_{n,m}$ and $\eta_{n,m}$ with $m > n$ of which the decompositions $\xi_{n,n+1}$ and $\eta_{n,n+1}$ are special cases.

the numbers $\mu(F_i)$ are equal for all $i = 0, \ldots, k_n - 1$;

(3.6) $$\mu(F_i) > \frac{1 - \varepsilon_n/2}{k_n q_n};$$

$\overline{F}_i$ is diffeomorphic to $\mathbf{D}^m$;

diam $F_i < \gamma_{n+1}$.

On the other hand we select within each of the sets Δ_{r,k_nq_n}, $r = 0, \cdots, k_n - 1$, an open set G_i whose closure is diffeomorphic to $\mathbf{D}^m$ and which satisfies

$$\mu(G_i) = \mu(F_i).$$

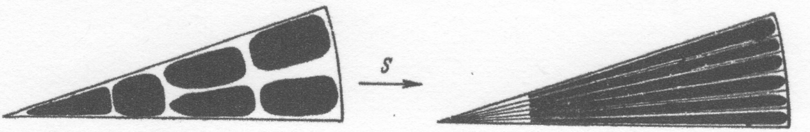

FIGURE 1

We apply Lemma 1.1 to the open set $O^m = \operatorname{Int} \Delta_{0,q_n}$ and the family of sets F_i, G_i, choosing $\epsilon = \varepsilon_n/2k_nq_n$. We denote the set $SF_i \cap G_i$ by $R_i^{(n)}$. Since S is the identity mapping outside the compact set $N \subset O^m$, we may extend it as the identity on the entire set Δ_{0,q_n} and subsequently we may define it on the entire set M^m by the formula

(3.7) $$Sx = S_{\frac{r}{q_n}} S S_{-\frac{r}{q_n}} x \text{ for } x \in \Delta_{r,q_n}, \quad r = 1, \ldots, q_n - 1.$$

Such an extension guarantees that S and S_{α_n} commute. Letting $S = A_{n+1}$ we obtain a mapping which satisfies the conditions (3.1), (3.4) and (3.5).

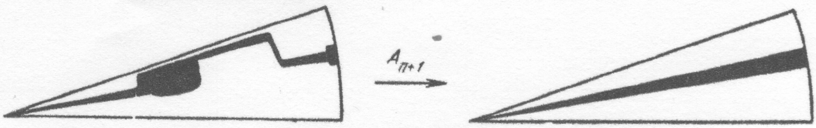

FIGURE 2

5. We now turn to the general case where not all $a_n(i)$ in (3.3) vanish. Let $b_r = q_n\mu(R^{(n)} \cap \Delta_{r,q_n})$, $r = 0, \cdots, q_n - 1$; then $b_r = k_{n,r}/k_n$, where

(3.8) $$k_{n,r} = \# \{ i : 0 \leqslant i < k_n, \; a_n(i) = r \}.$$

As in §2, we consider the factor space N^{M-1} of the open manifold $M^M \setminus \Omega$ (see (2.1)). As a result we obtain an identification of the orbits of the flow $\{S_t\}$ and the measure $\hat{\mu}$ on them. We choose open sets $c_0, \cdots, c_{q_n-1} \subset N^{m-1}$ with mutually nonoverlapping closures. diffeomorphic to Int $\mathbf{D}^m$ and such that $b_r \leqq \hat{\mu}(c_r) \geqq b_r - \epsilon_n/4k_nq_n$. We consider on $\bigcup_{r=0}^{q_n-1} c_r$ the function which is equal to r/q_n if $x \in c^r$, $r = 0, \cdots, q_n - 1$. This function may be extended to the entire set N^{m-1} in such a way that the resulting function f is of class C^∞ and vanishes outside a compact set $N \subset N^{m-1}$. We introduce on M^m the function

$$f^*(x) = \begin{cases} f(\pi(x)), & \text{if} \quad x \notin \Omega, \\ 0, & \text{if} \quad x \in \Omega. \end{cases}$$

Clearly this function belongs to the class C^∞. We define the mapping $\hat{S} \in \operatorname{Diff}^\infty(M^m, \mu)$ by $\hat{S}x = S_{f^*(x)}x$. Thus $\hat{S}$ is the identity mapping along Ω. We note that

$$\left| \mu(\hat{S}\Delta_{0,q_n} \cap \Delta_{r,q_n}) - \frac{b_r}{q_n} \right| < \frac{\varepsilon_n}{4k_nq_n}. \tag{3.9}$$

Now let us consider the set $\Gamma_r = \Delta_{0,q_n} \cap \pi^{-1}c_r$. Obviously

$$\left| \mu(\Gamma_r) - \frac{b_r}{q_n} \right| < \frac{\varepsilon_n}{4k_nq_n^2},$$

so that for $b_r \neq 0$ (in which case $b_r \geqq 1/k_n$) one has

$$\mu(\Gamma_r) > \frac{1 - \varepsilon_n/4}{k_nq_n} k_{n,r}.$$

If the integer $h_n > 0$ is sufficiently large, then outside each of the sets Γ_r we may choose $h_nk_{n,r}$ (see (3.8)) nonoverlapping open sets F_j^r ($j = 0, \cdots, h_nk_{n,r} - 1$) in such a way that

the values $\mu(F_j^r)$ are equal for all j and r;

$$\mu(F_j^r) > \frac{1 - \varepsilon_n/4}{h_nk_nq_n};$$

$\overline{F_j^r}$ is diffeomorphic to $\mathbf{D}^m$;

$$\operatorname{diam} F_j^r < \gamma_{n+1}.$$

We consider the decomposition $\eta_{h_nk_nq_n} = \{\Delta_{s,h_nk_nq_n}\}$. In each of the sets $\Delta_{s,h_nk_nq_n}$ with $s = 0, \cdots, h_nk_n - 1$ we choose an open set G_s with $\mu(G_s) = \mu(F_j^r)$ whose closure is diffeomorphic to $\mathbf{D}^m$.

Let

$$0 \leqslant i_r(0) < i_r(1) < \ldots < i_r(k_{n,r} - 1) < k_n$$

be those indices i for which $a_n(i) = r$. We associate with each set F_j^r the set G_s with the index

$$s = s(r, j) = h_ni_r\left(\left[\frac{i}{h_n}\right]\right) + j - h_n\left[\frac{j}{h_n}\right].$$

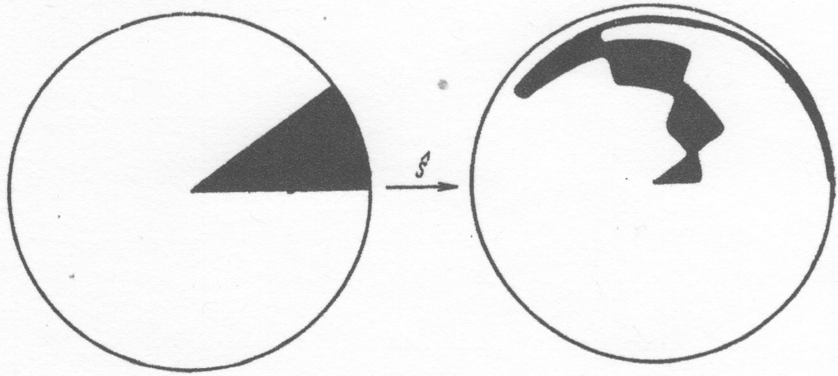

FIGURE 3

Remark on the sketch. For the disk D^2 the factor space N^{m-1} is identical with an interval. In this case the set c_r is also an interval. This case is illustrated by Figure 3.

We apply Lemma 1.1 to the open manifold $\operatorname{Int}\Delta_{0,q_n}$ with the two systems of sets F_j^r and $G_{s(r,j)}$ and with $\epsilon_n/4h_nk_nq_n$ instead of ϵ.

The mapping S so obtained can be extended as the identity on the entire set Δ_{0,q_n}, and subsequently it can be extended to the entire set M^m by formula (3.7).

We will show that we can make $A_{n+1} = S\hat{S}$ and that the conditions (3.1), (3.4) and (3.5) are satisfied.

Condition (3.1) holds for A_{n+1} since it is satisfied with respect to the sets $\hat{S}$ and S.

Let us verify condition (3.4). Since A_{n+1} preserves the measure μ, it suffices to show that $\mu(A_{n+1}\Delta_{0,q_n}\triangle R^{(n)'}) < \epsilon_n$. It follows from our definitions that

$$R^{(n)} = \bigcup_{r=0}^{q_n-1} \bigcup_{j=0}^{h_n k_{n,r}-1} S_{\frac{r}{q_n}} \Delta_{s(r,j),h_n k_n q_n},$$

$$\mu\Big(\Delta_{0,q_n} \setminus \bigcup_{r=0}^{q_n-1} \bigcup_{j=0}^{h_n k_{n,r}-1} F_j^r\Big) < \frac{\varepsilon_n}{4q_n},$$

and also

$$\mu\left(\Delta_{s(r,j),h_n k_n q_n} \setminus G_{s(r,j)}\right) < \frac{\varepsilon_n}{4h_n k_n q_n}$$

Therefore

$$\mu(A_{n+1}\Delta_{0,q_n} \triangle R^{(n)}) \leqslant \frac{\varepsilon_n}{2q_n}$$

$$(3.10) \qquad + \mu\Big((A_{n+1} \bigcup_{r=0}^{q_n-1} \bigcup_{j=0}^{h_n k_{n,r}-1} F_j^r) \triangle (\bigcup_{r=0}^{q_n-1} \bigcup_{j=0}^{h_n k_{n,r}-1} S_{\frac{r}{q_n}} \Delta_{s(r,j),h_n k_n q_n})\Big).$$

On the other hand, $F_j^r \subset \Gamma_r$, and on Γ_r the mapping $\hat{S}$ equals S_{r/q_n}; thus

$$A_{n+1}F_j^r = S\hat{S}F_j^r = S_{\frac{r}{q_n}} SF_j^r,$$

Now the inequality $\mu(SF_j^r \Delta G_{s(r,j)}) < \epsilon_n/4h_n k_n q_n$ implies that $\mu(\cdots) < \epsilon_n/4q_n$ on the right-hand side of (3.10). Consequently the left-hand side is less than $\epsilon_n/q_n < \epsilon_n$.

Finally, condition (3.5) is satisfied if there exist numbers r and j for which $s = s(r,j)$, and as the set $R_s^{(n)}$ we may choose $SF_j^r \cap G_{s(r,j)}$.

Thus the construction of the diffeomorphism A_{n+1} is completed, and it is evident that for any arbitrary multiple l_n of h_n formula (3.2) yields a diffeomorphism T_{n+1} which satisfies the conditions of the First and the Fourth Step.

If we now choose as l_n a sufficiently large multiple of h_n, then it follows from the condition of the Third Step that

$$\rho_{\left[\frac{1}{\varepsilon_n}\right]}(T_{n+1}, T_n) < \varepsilon_n$$

since in (3.2) T_{n+1} converges to T_n under the C^∞ topology as β_n tends to zero.

It remains to verify the conditions of the Second Step. Since $\operatorname{diam} A_{n+1}^{-1}R_s^{(n)} < \gamma_{n+1}$, we have $\operatorname{diam} B_{n+1}^{-1}R_s^{(n)} < 1/2^{n-1}$ (see the beginning of subsection 4). We take as E_{n+1} the set

$$E_{n+1} = \bigcup_{k=0}^{q_n-1} S_{\frac{k}{q_n}} \bigcup_{s=0}^{h_n k_n - 1} R_s^{(n)}.$$

It follows from (3.5) that $\mu(E_{n+1}) > 1 - \epsilon_n$. The intersection of an arbitrary element of the decomposition ξ_{n+1} with E_{n+1} is contained in some set of the form $S_{k/q_n}R_s^{(n)}$ (since an element of the decomposition η_{q_n+1} is contained in some $\Delta_{m,h_n q_n}$), and since S_{k/q_n} is an isometry, the diameter of this intersection does not exceed $1/2^{n+1}$.

6. The sequence $\xi_n = B_n^{-1}\eta_{q_n}$ of decomposition is, generally speaking, not monotonic if $R^{(n)} \neq \Delta_{0,q_n}$. We show how from this sequence a monotonic one may be obtained.

We associate with the element Δ_{k,q_n} of the decomposition η_{q_n} the element $S_{k/q_n}R^{(n)}$ of the decomposition $\eta_{n,n+1}$ (see Remark 3.1). This correspondence defines a mapping

$$C_n : M^m / \eta_{q_n} \to M^m / \eta_{n,n+1}.$$

For a set E consisting of elements of the decomposition η_{q_n} we consider as image E' under the mapping C_n the projection of this set into M^m/η_{q_n}, and we denote by C_nE the complete inverse image of the set E' under the canonical projection $M^m \to M^m/\eta_{n,n+1}$. Since the elements of the decomposition $\eta_{n,m+1}$ consist of elements of the decomposition $\eta_{q_{m+1}}$, we may map the sets $C_{n-1} \cdots C_m\Delta_{k,q_m}$ for $m < n$. We denote by $\eta_{m,n}$ the decomposition $\{C_{n-1} \cdots C_m\Delta_{k,q_m}\}$. It is not difficult to see that $\eta_{m+1,n} > \eta_{m,n}$. Now we define $\xi_{m,n} = B_n^{-1}\eta_{m,n}$.

Lemma 3.1. *If $\sum q_n\epsilon_n < \infty$,*[8] *then the sequence $\xi_{m,n}$ for fixed m converges to a decomposition $\xi_{m,\infty}$ as n tends to infinity. The sequence $\xi_{m,\infty}$ has the following properties:*

3.1.1. $\xi_{m+1,\infty} > \xi_{m,\infty}$, $m = 1, 2, \cdots$.

3.1.2. $T_m\xi_{m,\infty} = \xi_{m,\infty}$, *where the factor-automorphism* $T_m/\xi_{m,\infty}$ *is a cyclic permutation of the elements of the decomposition* $\xi_{m,\infty}$.

3.1.3. $\xi_{m,\infty} \to \epsilon$ *for* $m \to \infty$.

Proof. We define a correspondence P_n^m between the elements of the decompositions ξ_m and $\xi_{m,n}$. Let $c \in \xi_m$ be the element $c = B_m^{-1}\Delta_{k,q_m}$. We define

$$P_n^m c = B_n^{-1}\ C_{n-1} \ldots C_m\Delta_{k,q_m}.$$

For a set E consisting of elements c_i of the decomposition ξ_m we let $P_n^m E = \bigcup P_n^m c_i$. It is easy to see that $P_n^m = P_n^l P_l^m$ for $n > l > m$. We will give an upper bound for

$$\sum_{c \in \xi_m} \mu(P_n^m c \triangle P_{n+1}^m c).$$

The element $d = C_{n-1} \cdots C_m\Delta_{k,q_m}$ consists of q_n/q_m sets Δ_{l,q_n}. Similarly, the element $d' = A_{n+1}^{-1}C_nC_{n-1} \cdots C_m\Delta_{k,q_n}$ consists of the sets $A_{n+1}^{-1}S_{l/q_n}R^{(n)}$ with the same l. From (3.1) and (3.4) we have

$$\mu(d \triangle d') \leqslant \frac{q_n}{q_m}\mu(\Delta_{0,q_n} \triangle A_{n+1}^{-1}R^{(n)}) < \frac{q_n}{q_m}\varepsilon_n.$$

Since the mapping B_n^{-1} is measure preserving, the same inequality is valid for $\mu(P_n^m c \triangle P_{n+1}^m c)$ for any $c \in \xi_m$. Therefore

$$\sum_{c \in \xi_m} \mu(P_n^m c \triangle P_{n+1}^m c) < q_n\varepsilon_n. \tag{3.11}$$

From this we obtain for $n_2 > n_1 \geq m$ the inequality

[8] We choose the numbers ϵ_n after having determined the numbers q_n.

$$\sum_{c\in\xi_m} \mu(P^m_{n_1}c \triangle P^m_{n_2}c) < \sum_{k=n_1}^{n_2-1} q_k\varepsilon_k. \tag{3.12}$$

which shows the convergence of the sequence $\xi_{m,n}$ as n tends to infinity, and also the existence of the limit $\lim_{n\to\infty} P^m_n c$, which we denote by $P^m_\infty c$.

Property 3.1.1 is satisfied, i.e. $\xi_{m+1,n} > \xi_{m,n}$ for any n.

Property 3.1.2 is also satisfied, since the corresponding assertion holds true for the decomposition $\xi_{m,n}$ for arbitrary n.

Next we verify Property 3.1.3. From (3.12) we have

$$\sum_{c\in\xi_m} \mu(c \triangle P^m_\infty c) < \sum_{l=m}^{\infty} q_l\varepsilon_l = \varepsilon'_m. \tag{3.13}$$

Let A be an arbitrary measurable set. Since $\xi_m \to \epsilon$ for $m\to\infty$, there exist, in the Boolean σ-algebra $\mathfrak{B}(\xi_m)$, sets A_m for which $\mu(A\triangle A_m)\to 0$. Let $A_m = \bigcup_{i\in I_m} c^{(i)}$, where $c^{(i)}\in\xi_m$, and let $A'_m = \bigcup_{i\in I_m} P^m_\infty c^{(i)}$. Clearly $A'_m \in \mathfrak{B}(\xi_{m,\infty})$. On the other hand, (3.13) implies

$$\mu(A\triangle A'_m) \leqslant \mu(A\triangle A_m) + \varepsilon'_m,$$

and consequently $\mu(A\triangle A'_m)$ converges to zero as m tends to infinity. This proves the lemma.

REMARK 3.2. The mapping $P^\infty_n\colon M^m/\xi_n \to M^m/\xi_{n,\infty}$ is measure preserving and commutes with T_n.

7. Let $T = \lim_{n\to\infty} T_n$. By 3.1.2 and 3.1.3 the decompositions $\xi_{n,\infty}$ and the automorphisms T_n define a cyclic approximation of the automorphism T by periodic mappings (APM) with a certain speed of approximation in the sense of [8]. In order to estimate the speed of approximation we have to find a bound on the expression

$$\sum_{c\in\xi_{n,\infty}} \mu(Tc\triangle T_n c) \leqslant \sum_{k=n}^{\infty}\sum_{c\in\xi_{n,\infty}} \mu(T_{k+1}c\triangle T_k c) \leqslant \sum_{k=n}^{\infty}\sum_{c\in\xi_{k,\infty}} \mu(T_{k+1}c\triangle T_k c)$$

(here 3.1.1 is satisfied). In each term $c = P^k_\infty d$, where $d\in\xi_k$. We obtain

$$\sum_{d\in\xi_k} \mu(T_{k+1}P^k_\infty d\triangle T_k P^k_\infty d) \leqslant \sum_{d\in\xi_k} \mu(T_{k+1}P^k_\infty d\triangle T_{k+1}d)$$
$$+ \sum_{d\in\xi_k} \mu(T_k P^k_\infty d\triangle T_k d) + \sum_{d\in\xi_k} \mu(T_{k+1}d\triangle T_k d).$$

Since T_{k+1} and T_k are measure preserving, it follows from (3.13) that here the first two summations do not exceed ϵ'_k.

Now we give an estimate for each individual term in the last summation. There d has the form $d = B_k^{-1}\Delta_{i_{q_k}}$. Since (3.1), (3.2) and (3.4) hold, and since B_k and S_t are measure preserving, we have

$$\mu(T_{k+1}d \triangle T_k d) = \mu((B_k^{-1}T_{k+1}B_k\Delta_{l,q_k}) \triangle (B_k^{-1}T_kB_k\Delta_{l,q_k}))$$
$$= \mu((A_{k+1}^{-1}S_{\alpha_{k+1}}A_{k+1}\Delta_{l,q_k}) \triangle S_{\alpha_k}\Delta_{l,q_k}) = \mu((A_{k+1}^{-1}S_{\beta_k}A_{k+1}\Delta_{l,q_k}) \triangle \Delta_{l,q_k})$$
$$\leqslant \mu(S_{\beta_k}S_{\frac{l}{q_k}}R^{(k)} \triangle S_{\frac{l}{q_k}}R^{(k)}) + 2\varepsilon_k = \mu(S_{\beta_k}R^{(k)} \triangle R^{(k)}) + 2\varepsilon_k.$$

The set $R^{(n)}$ consists of k_n sets of the form Δ_{s,k_nq_n}. Clearly

$$\mu(\Delta_{s,k_kq_k} \triangle S_{\beta_k}\Delta_{s,k_kq_k}) = 2\beta_k,$$

i.e.

$$\mu(S_{\beta_k}R^{(k)} \triangle R^{(k)}) \leqslant 2k_k\beta_k = \frac{2}{l_kq_k^2}.$$

Finally,

$$\sum_{c\in\xi_{n,\infty}} \mu(Tc \triangle T_nc) < 2\sum_{k=n}^{\infty} \varepsilon_k' + 2\,\varepsilon_n' + \sum_{k=n}^{\infty} \frac{2}{l_kq_k} \leqslant 4\sum_{k=n}^{\infty} \varepsilon_k' + \sum_{k=n}^{\infty} \frac{2}{l_kq_k}. \tag{3.14}$$

From this inequality it is evident that if l_n increases sufficiently rapidly and if ϵ_n decreases sufficiently rapidly, then the summation $\sum_{c\in\xi_{n,\infty}}\mu(Tc \triangle T_nc)$ may be made less than $f(q_n)$, where $f(n)$ is a sequence of positive numbers which can be made to grow arbitrarily fast. In other words, we can construct the automorphism T in such a way that it possesses a cyclic APM with a speed $f(n)$ of approximation given in advance.

If, for example, $f(n) = o(1/n)$, then the diffeomorphism T is ergodic (cf. [8]) but it is not mixing, and the shift operator U_T in the space $L_2(M^m, \mu)$ has a simple singular spectrum.

§4. A diffeomorphism which is metrically isomorphic to a circular rotation

1. It turns out that in the most natural case $R^{(n)} = \Delta_{0,q_n}$ we may obtain considerably more extensive information on the metric structure of the limiting diffeomorphism T than what was given at the end of the preceding section.

THEOREM 4.1. *If in the mappings of §3 the set $R^{(n)}$ equals Δ_{0,q_n} for all n, then the limiting diffeomorphism T is metrically isomorphic to the circular rotation by the angle $2\pi\alpha = \lim_{n\to\infty} 2\pi\alpha_n$.*

The proof is based on the following abstract lemma.

LEMMA 4.1. *Let M_1 and M_2 be Lebesgue spaces and let $\xi_n^{(i)}$ $(i = 1, 2)$ be monotonic sequences of finite measurable decompositions of the spaces M_i, $\xi_n^{(i)} \nearrow \epsilon$; furthermore, let $T_n^{(i)}$ be automorphisms of the spaces M_i for which $T_n^{(i)}\xi_n^{(i)} = \xi_n^{(i)}$ and $\lim_{n\to\infty} T_n^{(i)} = T^{(i)}$, where the limit is taken in the weak topology of the space of automorphisms. Suppose there exist metric isomorphisms*

$$K_n : M_1/\xi_n^{(1)} \to M_2/\xi_n^{(2)},$$

for which

$$K_n^{-1}\, T_n^{(2)}/\xi_n^{(2)} K_n = T_n^{(1)}/\xi_n^{(1)}$$

and

(4.1)
$$K_n \xi_{n-1}^{(1)} = \xi_{n-1}^{(2)}$$

(in the last case, $\xi_{n-1}^{(i)}$ stands for the corresponding decomposition of the factor space $M_i/\xi_n^{(i)}$). Then the automorphisms T_1 and T_2 are metrically isomorphic.

The proof of the lemma is shorter than its wording. First we observe that (4.1) implies the equation $K_n \xi_k^{(1)} = \xi_k^{(2)}$ for any $k < n$. Hence there follows also the existence of an automorphism K: $M_1 \to M_2$ for which $K/\xi_n^{(1)} = K_n$. [Indeed, let $x \in M_1$, $x \in c_n(x)$, $c_n(x) \in \xi_n^{(1)}$. Then $\{x\} = \bigcap_{n=1}^{\infty} c_n(x)$ for almost all $x \in M_1$. We let $Kx = \bigcap_{n=1}^{\infty} K_n(c_n(x))$.] It is easy to see that $K^{-1} T_n^{(2)} K/\xi_n^{(1)} = T_n^{(1)}$. Therefore

$$T^{(1)} = \lim T_n^{(1)} = \lim K^{-1} T_n^{(2)} K = K^{-1} \lim T_n^{(2)} K = K^{-1} T^{(2)} K.$$

PROOF OF THEOREM 4.1. Under the conditions of Lemma 4.1, let $M_1 = M^m$ and $M_2 = \mathbf{S}^1$, and let $\xi_n^{(1)} = \xi_n$ and $\xi_n^{(2)}$ be decompositions of the circle $\mathbf{S}^{-1} = \{\zeta\colon \zeta \in \mathbf{C},\ |\zeta| = 1\}$ into the arcs

$$c_{n,k} = \left\{\zeta : \frac{2\pi k}{q_n} \leqslant \arg \zeta < \frac{2\pi (k+1)}{q_n}\right\},$$
$$k = 0, \ldots, q_n - 1;$$

furthermore let $T_n^{(1)} = T_n$, and let $T_n^{(2)}$ be the circular rotation by the angle $2\pi p_n/q_n$. By K_n we denote the mapping which maps the element $B_n^{-1}\Delta_{k,q_n} \in \xi_n^{(1)}$ onto the element $c_{n,k} \in \xi_n^{(2)}$. The condition (4.1) turns out to be a consequence of the equation $R^{(n)} = \Delta_{0,q_n}$. Thus Theorem 4.1 follows from Lemma 4.1.

REMARK 4.1. Theorem 4.1 may be reworded by stating that the shift operator U_T in the space $L_2(M^m, \mu)$ has a discrete spectrum with one independent eigenfrequency, i.e. in the space $L_2(M^m, \mu)$ there exists a basis consisting of eigenfunctions

$$f_n(x) = (f_1(x))^n \text{ and } U_T f_n(x) = \exp(2\pi i n \alpha)\, f_n(x).$$

These eigenfunctions must necessarily be discontinuous. It can be shown that in the case of continuous eigenfunctions the manifold M^m must be the circumference (but the functions $f_n(x)$ may, of course, be continuous at some points).

REMARK 4.2. Even if the metrical structure of the diffeomorphisms constructed above is simple, their topological structure may turn out to be quite complicated. These diffeomorphisms may have closed invariant sets of a complicated type, singular invariant measures which are positive on any open set and relative to which the metrical properties of T are completely different from those relative to μ, etc.

§5. A weakly mixing diffeomorphism

1. In this section we show that by a suitable choice of the set $R^{(n)}$ the limiting automorphism T can in general be made not to have eigenfunctions. For this proof we need some facts from ergodic theory which are summarized in the following theorem.

THEOREM 5.1. *Let T be an automorphism of a Lebesgue space (M, μ). Then the following assertions are equivalent.*

5.1.1. *The shift operator U_T does not have eigenfunctions other than constants.*

5.1.2. *For an arbitrary pair of measurable sets $F, G \subset M$ there exists an increasing sequence n_k of positive integers with density one such that*

$$\mu(T^{n_k}F \cap G) \to \mu(F)\cdot\mu(G) \quad \text{for} \quad k \to \infty.$$

5.1.3. *There exists a sequence of positive integers $n_k \to \infty$ such that for any pair of sets $F, G \subset M$*

$$\mu(T^{n_k}F \cap G) \to \mu(F)\cdot\mu(G) \quad \text{for} \quad k \to \infty.$$

5.1.4. *There exists a sequence of finite measurable decompositions $\xi_k \to \epsilon$ and a sequence of positive integers $n_k \to \infty$ such that*

$$\sum_{c_1, c_2 \in \xi_k} |\mu(T^{n_k}c_1 \cap c_2) - \mu(c_1)\mu(c_2)| \to 0$$

as k tends to infinity.

The equivalence of the assertions 5.1.1 and 5.1.2 is the subject of the well-known "Weak Mixing Theorem", a proof of which may be found in [6], Russian pp. 58–60. The equivalence of 5.1.2 and 5.1.3 is a simple fact; a proof may be found, for example, in [22].

We now give a proof of the equivalence of the assertions 5.1.3 and 5.1.4 which is also very simple.

Suppose 5.1.3 is satisfied. We consider a fixed sequence of measurable decompositions $\xi_n \to \epsilon$. Let $c_1 \cdots, c_{q_n}$ be the elements of the decomposition ξ_n. In view of 5.1.3, for each pair (i, j), $1 \leq i, j \leq q_n$ there exists a number $k_{i,j,n}$ such that

$$|\mu(T^{n_k}c_i \cap c_j) - \mu(c_i)\mu(c_j)| < \frac{1}{2^n q_n^2}$$

for $k \geq k_{i,j,n}$. Let $k_n = \max_{i,j} k_{i,j,n}$. Then for $k \geq k_n$ one has the inequality

$$\sum_{i,j=1}^{q_n} |\mu(T^{n_k}c_i \cap c_j) - \mu(c_i)\mu(c_j)| < \frac{1}{2^n}.$$

Conversely, suppose 5.1.4 is satisfied. We show that for any two measurable sets $F, G \subset M$ the sequence $\mu(T^{n_k}F \cap G)$ converges to $\mu(F)\mu(G)$. Let $\epsilon > 0$ be fixed. We choose a number K such that for every $k > K$ there exist sets F_k and G_k which are measurable relative to the decomposition ξ_k and for which $\mu(F \triangle F_k) < \epsilon$ and $\mu(G \triangle G_k) < \epsilon$. Then we have

$$\begin{aligned} &|\mu(T^{n_k}F \cap G) - \mu(F)\mu(G)| \leqslant |\mu(T^{n_k}F_k \cap G_k) - \mu(F_k)\mu(G_k)| \\ &\quad + \mu(F \triangle F_k) + \mu(G \triangle G_k) + |\mu(F) - \mu(F_k)| \\ &\quad + |\mu(G) - \mu(G_k)| \leqslant |(T^{n_k}F_k \cap G_k) - \mu(F_k)\mu(G_k)| + 4\varepsilon. \end{aligned} \tag{5.1}$$

Suppose that sets F_k and G_k consist, respectively, of the elements $c_{i_1}, \cdots, c_{i_r}$ and $c_{j_1}, \cdots, c_{j_s}$ of the decomposition ξ_k. We use the abbreviation

$$|\mu(T^{n_k}c_i \cap c_j) - \mu(c_i)\mu(c_j)| = \omega_{ij}.$$

Adding the inequalities

$$\omega_{i_a j_b} \geqslant \mu(T^{n_k}c_{i_a} \cap c_{j_b}) - \mu(c_{i_a})\mu(c_{j_b}) \geqslant -\omega_{i_a j_b}$$

for $a = 1, \cdots, r$ and $b = 1, \cdots, s$ we obtain

$$\sum_{a=1}^{r}\sum_{b=1}^{s} \omega_{i_a j_b} \geqslant \mu(T^{n_k}F_k \cap G_k) - \mu(F_k)\mu(G_k) \geqslant -\sum_{a=1}^{r}\sum_{b=1}^{s} \omega_{i_a j_b}$$

and hence

$$|\mu(T^{n_k}F_k \cap G_k) - \mu(F_k)\mu(G_k)| \leqslant \sum_{i,j=1}^{q_n} \omega_{ij}.$$

It follows from 5.1.4 that here the right-hand side converges to zero as k tends to infinity. Thus we obtain 5.1.3 from (5.1).

2. We show how the parameters in the construction of §3 have to be chosen in order for the limiting automorphism to have property 5.1.4. We shall apply the notation from §3. Inasmuch as in the First Step of the induction the numbers p_{n+1} and q_{n+1} are relatively prime, there exists for any r with $0 \leqq r < q_{n+1}$ a number k^r, $0 \leqq k^r < q_{n+1}$, such that $k^r p_{n+1}/q_{n+1} \equiv r/q_{n+1} \pmod 1$. This means that

$$T_{n+1}^{k^r} = B_{n+1}^{-1} S_{k^r \alpha_{n+1}} B_{n+1} = B_{n+1}^{-1} S_{\frac{r}{q_{n+1}}} B_{n+1}.$$

Now we let $r_n = l_n q_n$. Then we have

$$T_{n+1}^{k^{r_n}} = B_{n+1}^{-1} S_{\frac{1}{k_n q_n}} B_{n+1}.$$

It follows from (3.3) that

$$S_{\frac{1}{k_n q_n}} R^{(n)} = \Big(\bigcup_{i=0}^{k_n-2} \Delta_{a_n(i)k_n+i+1,\, k_n q_n}\Big) \cup \Delta_{(a_n(k_n-1))+1)k_n,\, k_n q_n}. \tag{5.2}$$

We compute the measure of the intersection

$$S_{\frac{1}{k_n q_n}} R^{(n)} \cap S_{\frac{k}{q_n}} R^{(n)}. \tag{5.3}$$

For an arbitrary integer a we denote by $\bar{a}$ the smallest nonnegative number which is congruent to a modulo q_n. Clearly

(5.4) $$S_{\frac{k}{q_n}} R^{(n)} = \bigcup_{i=0}^{k_n-1} \Delta_{\overline{(a_n(i)+k)k_n+i},\, k_n q_n}.$$

We denote by $Q_{n,k}$ the number of elements of the decomposition $\eta_{k_n q_n}$ in the intersection under consideration. Thus

$$\mu\left(S_{\frac{1}{k_n q_n}} R^{(n)} \cap S_{\frac{k}{q_n}} R^{(n)}\right) = \frac{Q_{n,k}}{k_n q_n}$$

and also

$$\mu\left(S_{\frac{1}{k_n q_n}} S_{\frac{l}{q_n}} R^{(n)} \cap S_{\frac{k}{q_n}} R^{(n)}\right) = \frac{Q_{n,\overline{k-l}}}{k_n q_n}.$$

3. We show that for sufficiently large k_n there exists an $a_n(i)$ such that

(5.5) $$\sum_{k=0}^{q_n-1} \left| \frac{Q_{n,k}}{k_n q_n} - \frac{1}{q_n^2} \right| < \frac{3}{q_n^2}.$$

Let $k_n = \lambda_n q_n + \mu_n$, where $0 \leqq \mu_n < q_n$. We define the number $a_n(i)$ for $i = 0, \cdots, k_n - 1$ as follows:

(5.6) $$a_n(i) = \begin{cases} 0, & \text{if } i \text{ is even} \\ & \text{or if } \lambda_n q_n \leqq i \leqq k_n - 1 \text{ and } q_n \text{ is odd;} \\ r, & \text{where } r \text{ is defined by the condition} \\ & (2r-1)\lambda_n \leqslant i < (2r+1)\lambda_n,\ r = 0, \ldots, \left[\frac{q_n}{2}\right], \\ & \text{if } i \text{ is odd.} \end{cases}$$

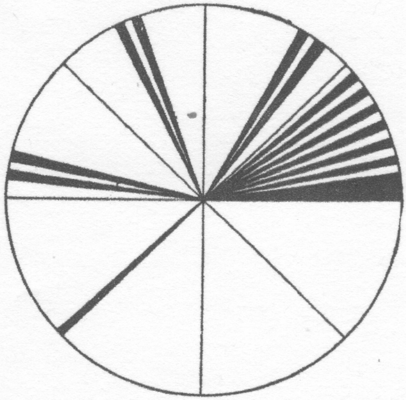

FIGURE 4

In this drawing we have taken $q_n = 8$, $k_n = 16$. The dark sectors belong to the set $R^{(n)}$.

For this system $a_n(i)$ the number $Q_{n,k}$ is λ_n if k is different from 0 and $q_n/2$. For example, for $0<k<q_n/2$ the intersection (5.3) is obtained by only taking the sets $\Delta_{(a_n(i)+k)k_n+i,q_nk_n}$ with $a_n(i)=0$ in (5.4) and $\Delta_{a_n(i)+i+1,k_nq_n}$ with $a_n(i)=k$ in (5.2), where this intersection contains exactly all λ_n sets of the latter type. Conversely, for $q_n/2<k\leqq q_n-1$ the intersection (5.3) consists of all sets of the form $\Delta_{a_n(i)k_n+i+1,k_nq_n}$ with $a_n(i)=0$ in (5.2) which overlap with sets $\Delta_{(a_n(i)+k)k_n+i,k_nq_n}$ with $a_n(i)=q_n-k$ in (5.4). This case also involves all λ_n sets of the latter type. The reader may verify without difficulty that for the excluded values of k the number $Q_{n,k}$ lies in the range between 0 and $2\lambda_n$.

Therefore we have

$$\sum_{k=0}^{q_n-1}\left|\frac{Q_{n,k}}{k_nq_n}-\frac{1}{q_n^2}\right|\leqslant q_n\left|\frac{\lambda_n}{q_n(\lambda_nq_n+\mu_n)}-\frac{1}{q_n^2}\right|+\frac{2}{q_n^2}\leqslant\frac{1}{q_n(\lambda_n+1)}+\frac{2}{q_n^2}.$$

Choosing $\lambda_n>q_n$, we obtain the inequality (5.5).

4. **Lemma** 5.1. *If l_n increases sufficiently rapidly and if ϵ_n decreases sufficiently rapidly, then*

$$\sum_{c_1,c_2\in\xi_{n,n+1}}|\mu(T^{k'r_n}c_1\cap c_2)-\mu(c_1)\mu(c_2)|\to 0.$$

Proof. The left side does not exceed

$$\begin{aligned}(5.7)\qquad &\sum_{c_1,c_2\in\xi_{n,n+1}}|\mu(T_{n+1}^{k'r_n}c_1\cap c_2)-\mu(c_1)\mu(c_2)|\\ &+\sum_{c_1,c_2\in\xi_{n,n+1}}|\mu(T^{k'r_n}c_1\cap c_2)-\mu(T_{n+1}^{k'r_n}c_1\cap c_2)|.\end{aligned}$$

The relations (5.5), (3.1) and the fact B_{n+1} is measure preserving imply that for $\lambda_n>q_n$ the first summation in (5.7) converges to zero as n tends to infinity. On the other hand,

$$|\mu(T^{k'r_n}c_1\cap c_2)-\mu(T_{n+1}^{k'r_n}c_1\cap c_2)|\leqslant\mu((T^{k'r_n}c_1\triangle T_{n+1}^{k'r_n}c_1)\cap c_2).$$

Therefore the second summation in (5.7) does not exceed

$$\sum_{c\in\xi_{n,n+1}}\mu(T^{k'r_n}c\triangle T^{k'r_n}c)\leqslant\sum_{c\in\xi_{n+1}}\mu(T^{k'r_n}c\triangle T_{n+1}^{k'r_n}c).$$

It remains to estimate the summation on the right-hand side of this last inequality. Since $T_{n+1}\xi_{n+1}=\xi_{n+1}$, it is not difficult to see that, for any integer k,

$$(5.8)\qquad \sum_{c\in\xi_{n+1}}\mu(T^kc\triangle T_{n+1}^kc)\leqslant|k|\sum_{c\in\xi_{n+1}}\mu(Tc\triangle T_{n+1}c).$$

In order to estimate the last summation we use the mapping P_∞^{n+1} between the elements of the decompositions ξ_{n+1} and $\xi_{n+1,\infty}$ as well as the inequalities

(3.13) and (3.14):

$$\sum_{c\in\xi_{n+1}} \mu(Tc \triangle T_{n+1}c) \leqslant \sum_{c\in\xi_{n+1}} \mu(Tc \triangle TP_\infty^{n+1}c)$$

$$+ \sum_{c\in\xi_{n+1}} \mu(TP_\infty^{n+1}c \triangle T_{n+1}P_\infty^{n+1}c) + \sum_{c\in\xi_{n+1}} \mu(T_{n+1}c \triangle T_{n+1}P_\infty^{n+1}c)$$

$$\leqslant 2\sum_{c\in\xi_{n+1}} \mu(c \triangle P_\infty^{n+1}c) + \sum_{c\in\xi_{n+1,\infty}} \mu(Tc \triangle T_{n+1}c)$$

$$\leqslant 2\varepsilon_{n+1}' + 4\sum_{k=n+1}^{\infty} \varepsilon_k' + 2\sum_{k=n+1}^{\infty} \frac{1}{q_k l_k}.$$

Thus, the second summation in (5.7) is not larger than

$$k^{r_n}\left(6\sum_{k=n+1}^{\infty} \varepsilon_k' + 2\sum_{k=n+1}^{\infty} \frac{1}{q_k l_k}\right).$$

Since $k^{r_n} < q_{n+1}$, this expression may be made arbitrarily small by a suitable choice of $\epsilon_{n+1}, \epsilon_{n+2}, \cdots$ and l_{n+1} (which we select after having determined q_{n+1}). This completes the proof of Lemma 5.1.

5. It follows easily from (3.11) that $\xi_{n,n+1}$ converges to ϵ (using the fact that ξ_n converges to ϵ and an argument analogous to that given in §3.6 for the convergence of $\xi_{n,\infty}$ to ϵ). By Theorem 5.1, Lemma 5.1 and §3.7 we obtain the following theorem:

THEOREM 5.2. *If $a_n(i)$ is chosen according to (5.6), and if l_n increases sufficiently rapidly and ϵ_n decreases sufficiently rapidly, then the shift operator U_T on the space $L_2(M^m, \mu)$ has a continuous simple singular spectrum but it is not mixing.*

§6. A diffeomorphism which is metrically isomorphic to a shift on the torus

1. In §4 we proved that on any manifold with a periodic flow there exist ergodic diffeomorphisms with discrete spectrum generated by a single eigenvalue. In the present section we generalize this result to the case of a discrete spectrum generated (over **Z**) by an arbitrary number h of linearly independent eigenvalues.

As before, our procedure is based on a special choice of the sets $R^{(n)}$. However, this problem presents some additional difficulties in comparison with the problems solved in §§4 and 5. In §4 the isomorphism which existed between T and a circular rotation was in a certain sense natural. The essence of the version of the limiting process used there lies in the fact that a sequence of decompositions η_n, converging to some decomposition η whose elements are "transversal" to the orbits of the flow $\{S_t\}$ (in the case of $\mathbf{D}^2$ and the flow of rotations this is simply the decomposition into radii), is transformed into a sequence of decompositions ξ_n which converges to ϵ such that the shifts S_{α_n} are transformed into automorphisms T_n (precisely, $A_n\eta_k = \xi_k$ for $k \leq n$). Somehow we have joined together all ergodic components on the flow $\{S_t\}$ into one unit.

It is evident that this approach does not lead to any other automorphisms except those which are isomorphic to circular rotations. (In any case, for that purpose it would be necessary to start from flows with properties other than periodic ones.)

In §5 we obtained an automorphism which is not isomorphic to a circular rotation, but in that case we were concerned not with showing it to be metrically isomorphic to some automorphism of specified form, but only with a specific property of it. Now we shall obtain at each step of the construction an isomorphism between T_n and some periodic shift on the torus such that the sequence of these shifts converges to an ergodic one. Here the isomorphisms at each step will necessarily be of a more "artificial" character than those in §4.

2. By $\mathbf{T}^h$ we denote the h-dimensional torus:

$$\mathbf{T}^h = \{\varphi = (\varphi_1, \ldots, \varphi_h) : \varphi_i \in \mathbf{R}/\mathbf{Z}\}.$$

Furthermore, we denote the group shift on $\mathbf{T}^h$ by T^ψ: $\phi \to \phi + \psi$. We shall be concerned with a periodic flow $\{T^{\gamma t}\}$, where $\gamma = (\gamma_1, \cdots, \gamma_h)$ with relatively prime integers γ_i.

In $\mathbf{T}^h$ there is contained the $(h-1)$-dimensional torus

$$\mathbf{T}^{h-1} = \{\varphi = (\varphi_1, \ldots, \varphi_{h-1}, 0)\}.$$

We consider the restriction of the shift T^{γ/γ_h} to $\mathbf{T}^{h-1}$; this restriction turns out to be a shift on $\mathbf{T}^{h-1}$. Let Γ_0 be an open Dirichlet domain of the point $\bar{0} = (0, \cdots, 0)$. We enlarge Γ_0 by a part of the boundary in such a way as to obtain a fundamental domain, to be denoted by Γ. Obviously Γ is also a fundamental domain for the flow $\{T^{\gamma t}\}$. For each natural number k the set

$$\Gamma_k = \bigcup_{0 \leq t < \frac{1}{k}} T^{\gamma t}\,\Gamma$$

is a fundamental domain of the shift $T^{\gamma/k\gamma_h}$.

We show that under certain conditions a sequence $T^{\alpha^{(n)}}$ on the torus $\mathbf{T}^h$ converging to an ergodic shift T^α has a monotonic sequence of fundamental domains. The conditions which we shall now formulate are of course not necessary, but they are adapted to the inductive process by means of which we will construct on a manifold with a periodic flow a diffeomorphism which is metrically isomorphic to $T^\alpha = \lim T^{\alpha^{(n)}}$. The shift $T^{\alpha^{(n)}}$ will be included in the flow $\{T^{t\gamma^{(n)}}\}$.

LEMMA 6.1. *There exist sequences*

$$\alpha^{(n)} = (\alpha_1^{(n)}, \ldots, \alpha_h^{(n)}) \in \mathbf{T}^h,\ \gamma^{(n)} = (\gamma_1^{(n)}, \ldots, \gamma_h^{(n)}) \in \mathbf{Z}^h,$$

with the following properties:

6.1_n. *The greatest divisor of the* $\gamma_i^{(n)}$ *is* $(\gamma_1^{(n)}, \cdots, \gamma_h^{(n)}) = 1$.

6.2_n. *There exist relatively prime integers* p_n, q_n *such that* $\alpha_i^{(n)} = p_n\gamma_i^{(n)}/q_n \pmod 1$.

6.3_n. *There exists an integer* r_n *such that* $q_n = r_n\gamma_h^{(n)}$.

6.4_n. *There exists an integer* s_{n-1} *such that* $\gamma_h^{(n)} = s_{n-1}\gamma_h^{(n-1)}$.

6.5_n. $\gamma_i^{(n)} \equiv \gamma_i^{(n-1)} \pmod{q_{n-1}}$, $i = 1, \ldots, h$.

6.6_n. *There exists an integer* m_{n-1} *such that*

$$\frac{p_n}{q_n} = \frac{p_{n-1}}{q_{n-1}} + \frac{1}{m_{n-1}\, s_{n-1} q_{n-1}^2}$$

6.7_n. *Let* $\Gamma^{(n)} \subset \mathbf{T}^{h-1}$ *be a fundamental domain of the flow* $\{T^{t\gamma^{(n)}}\}$ *described above. Denote the diameter of* $\Gamma^{(n)}$ *by* d_n, *and by* σ_n *the* $(h-1)$*-dimensional volume of the boundary of* $\Gamma^{(n)}$. *Then*

$$d_n < \frac{1}{2^{n-1}\gamma_h^{(n-1)}\sigma_{n-1}}.$$

6.8_n.

$$\left|\frac{1}{\gamma_h^{(n)}}\gamma^{(n)} - \frac{1}{\gamma_h^{(n-1)}}\gamma^{(n-1)}\right| < \frac{1}{2^{n-1}\sigma_{n-1}q_{n-1}}.$$

(*Distances, diameters and* $(h-1)$*-dimensional volumes are always taken relative to the Euclidean metric.*)

PROOF. Suppose numbers $\alpha^{(j)}$ and $\gamma^{(j)}$ satisfying 6.1_j–6.8_j have already been defined for $j \leq n$. First we construct a number $\gamma^{(n+1)}$ satisfying the conditions 6.1_{n+1}, 6.4_{n+1}, 6.5_{n+1}, 6.7_{n+1} and 6.8_{n+1}. Here we shall use only the conditions 6.1_n, 6.2_n and 6.3_n.

We construct a matrix $A \in \mathbf{SL}(h, \mathbf{Z})$ whose last column coincides with the vector $\gamma^{(n)}$. This is possible in view of 6.1_n (see for example [4], Chapter 1, §2, Corollary 4). Here we can ascertain that the $(h-1)$-dimensional matrix B obtained from the matrix A by canceling the last column and the last row is nonsingular. Indeed, it is easy to see that in the case where $\det B = 0$, the vector $e_h = (0, \cdots, 0, 1)$ belongs to the lattice spanned by the first $h-1$ columns of the matrix A, and in this lattice we may choose a basis beginning with e_h. Let $e_h, a_1, \cdots, a_{k-1}$ be the vectors of this basis. Then we may replace A by the matrix whose columns are

$$e_h + \gamma^{(n)},\ a_1,\ \ldots,\ a_{h-2},\ \gamma^{(n)}.$$

By applying the matrix A, the subspace $\mathbf{R}^{h-1} = \{(x_1, \cdots, x_{h-1}, 0)\}$ is mapped onto a subspace $A\mathbf{R}^{h-1}$ with equations of the form $x_h = \sum_{i=1}^{h-1} b_i x_i$. We observe that, denoting by U_δ a sphere of radius δ in the space $\mathbf{R}^{h-1}$, the cylinder $U_\delta \times \mathbf{R}$ intersects the subspace $A\mathbf{R}^{h-1}$ in an ellipsoid contained in the sphere of radius $b\delta$, where $b^2 = 1 + b_1^2 + \cdots + b_{h-1}^2$. The matrix A transforms the unit cube $0 \leqq x_i \leqq 1$ of the space $\mathbf{R}^h$ into some parallelepiped in such a way that the edge e_h of this cube is mapped onto the vector $\gamma^{(n)}$.

Let

$$\delta = \frac{1}{2^{n+1}\gamma_h^{(n)}\sigma_n b \, \| A^{-1} \|}.$$

We choose any vector $\vartheta = (\vartheta_1, \cdots, \vartheta_{h-1}, 0) \in \mathbf{Z}^h$ with $(\vartheta_1, \cdots, \vartheta_{h-1}) = 1$, using the property that in $\mathbf{T}^{h-1}$ every orbit of a periodic flow $\{T^{t\vartheta}\}$ intersects an arbitrary sphere of radius δ. We let

$$\gamma^{(n+1)} = q_n A^{-1}\vartheta + (cq_n + 1)\gamma^{(n)} = A^{-1}(q_n\vartheta + (cq_n + 1)e_h) \tag{6.1}$$

and we show that the corresponding conditions are satisfied under a suitable choice of the integer c. Condition 6.1_{n+1} follows from (6.1) and the fact that the numbers ϑ_i are relatively prime; condition 6.4_{n+1} follows from (6.1) and 6.3_n, and condition 6.5_{n+1} is an immediate consequence of (6.1). Let c be so large that at the same time when the point $tq_n\vartheta$ is shifted by δ, the point $(cq_n + 1)t$ is moved by not less than one. Then in $\mathbf{T}^h$ every orbit of the flow $\{T^{t(q_n\vartheta + (cq_n+1)e_h)}\}$ intersects with the torus $A\mathbf{T}^{h-1}$ in a $2b\delta$-net. It is easy to see that this ensures the validity of condition 6.7_{n+1}. Finally, it is perfectly clear that in 6.8_{n+1} one has

$$\lim_{c\to\infty} \left| \frac{1}{\gamma_h^{(n+1)}}\gamma^{(n+1)} - \frac{1}{\gamma_h^{(n)}}\gamma^{(n)} \right| = 0.$$

The number m_n which occurs in condition 6.6_{n+1} may be chosen arbitrarily. From 6.6_{n+1} we automatically obtain relatively prime numbers p_{n+1}, q_{n+1}, and then 6.2_{n+1} defines $\alpha^{(n+1)}$. Condition 6.3_{n+1} follows from 6.6_{n+1} and 6.4_{n+1}. This completes the proof of the lemma.

3. The set $\Gamma_{q_n}^{(n)} = \bigcup_{0 \leq t < 1/q_n} T^{t\gamma^{(n)}}\Gamma^{(n)}$ is a fundamental domain for the shift $T^{\gamma^{(n)}/q_n}$, and thus also for $T^{\alpha^{(n)}}$, since the numbers p_n and q_n are relatively prime.

We denote by ζ_n the decomposition of the torus $\mathbf{T}^h$ into the sets

$$\Gamma_{k,n} = T^{\frac{k}{q_n}\gamma^{(n)}}\Gamma_{q_n}^{(n)}, \quad k = 0, \ldots, q_n - 1,$$

and by ζ_n' the decomposition of $\mathbf{T}^{h-1}$ into the sets

$$\Gamma_{k,n}' = T^{\frac{k}{\gamma_h^{(n)}}\gamma^{(n)}}\Gamma^{(n)}, \qquad k = 0, \ldots, \gamma_h^{(n)} - 1.$$

It follows from condition 6.3 that

$$\Gamma_{lr_n, n} = \bigcup_{0 \leqq t < \frac{1}{q_n}} T^{t\gamma^{(n)}}\Gamma_{l,n}', \quad l = 0, \ldots, \gamma_h^{(n)} - 1.$$

Now we consider how the decompositions ζ_n and ζ_n' are related to the de- decompositions ζ_{n+1} and ζ_{n+1}'. We begin with the decompositions ζ_n' and ζ_{n+1}'.

It follows from conditions 6.3 and 6.5 that

$$T^{\frac{1}{\gamma_h^{(n)}}\gamma^{(n+1)}} = T^{\frac{1}{\gamma_h^{(n)}}\gamma^{(n)}}.$$

We shall consider this shift only on the torus $\mathbf{T}^{h-1}$. Denoting it by $V^{(n)}$, we obtain from conditions 6.4 and 6.5 the equation $(V^{(n+1)})^{s_n} = V^{(n)}$.

Let

$$K = \{k : 0 \leqslant k < \gamma_h^{(n+1)},\ \ (V^{(n+1)})^k \bar{0} \in \Gamma^{(n)}\}.$$

We define $\Gamma^{(n,n+1)} = \bigcup_{k\in K} \Gamma'_{k,n+1}$. Then $\Gamma^{(n,n+1)}$ is a fundamental domain for $V^{(n)}$. The decomposition into the sets $V^{(n)l}\Gamma^{(n,n+1)}$, $l = 0, \cdots, s_n - 1$, will be denoted by $\zeta'_{n,n+1}$. Clearly $\zeta'_{n,n+1} < \zeta'_{n+1}$. We estimate the $(h-1)$-dimensional Lebesgue measure $\mu_{h-1}(\Gamma^{(n)} \triangle \Gamma^{(n,n+1)})$. The set $\Gamma^{(n)} \setminus \Gamma^{(n,n+1)}$ is contained in a neighborhood of width d_{n+1} of the boundary of $\Gamma^{(n)}$. Since $\Gamma^{(n)}$ is a convex set, one has

$$\mu_{h-1}(\Gamma^{(n)} \setminus \Gamma^{(n,n+1)}) \leqslant \sigma_n d_{n+1} < \frac{1}{2^n \gamma_h^{(n)}}$$

(in view of condition 6.7). But since $\mu_{h-1}(\Gamma^{(n)}) = \mu_{h-1}(\Gamma^{(n,n+1)})$, this implies the inequality

$$\mu_{h-1}(\Gamma^{(n)} \triangle \Gamma^{(n,n+1)}) < \frac{1}{2^{n-1}\gamma_h^{(n)}}. \tag{6.2}$$

Now we turn to the decompositions ζ_n and ζ_{n+1}. We define

$$\Gamma_{q_n}^{(n,n+1)} = \bigcup_{0 \leqslant t < \frac{1}{r_n \gamma_h^{(n+1)}}} T^{t\gamma^{(n+1)}}\, \Gamma^{(n,n+1)}.$$

The set $\Gamma_{q_n}^{(n,n+1)}$ consists of elements of the decomposition ζ_{n+1} (since q_{n+1} is divisible by $r_n \gamma_h^{(n+1)}$) and is a fundamental domain of the shift $T^{\alpha(n)}$. We note that the set $\Gamma_{q_n}^{(n,n+1)}$ depends only on the choice of the flow $T^{t\gamma^{(n+1)}}$ but not on the choice of m_n. The decomposition into the sets

$$T^{\frac{k}{q_n}\gamma^{(n)}}\, \Gamma_{q_n}^{(n,n+1)},\quad k = 0, \ldots, q_n - 1$$

will be denoted by $\zeta_{n,n+1}$. Evidently $\zeta_{n,n+1} < \zeta_{n+1}$. Now we estimate the h-dimensional Lebesgue measure $\mu_h(\Gamma_{q_n}^{(n,n+1)} \triangle \Gamma_{q_n}^{(n)})$.

We define

$$\hat{\Gamma}_{q_n}^{(n,n+1)} = \bigcup_{0 \leqslant t \leqslant \frac{1}{q_n}} T^{t\gamma^{(n)}}\, \Gamma^{(n,n+1)}.$$

Since

$$\mu_h(\Gamma_{q_n}^{(n)} \triangle \hat{\Gamma}_{q_n}^{(n,n+1)}) = \frac{1}{r_n}\mu_{h-1}(\Gamma^{(n)} \triangle \Gamma^{(n,n+1)}),$$

it follows from (6.2) that

$$\mu_h(\Gamma_{q_n}^{(n)} \triangle \hat{\Gamma}_{q_n}^{(n,n+1)}) < \frac{1}{2^{n-1} q_n}. \tag{6.3}$$

The sets $T^{ts_n\gamma^{(n)}}\Gamma^{(n,n+1)}$ and $T^{t\gamma^{(n+1)}}\Gamma^{(n,n+1)}$ belong to the same torus $\phi_h = \text{const}$ by 6.4. Here the latter one of these sets is obtained from the former through a shift by the vector

$$t\,(\gamma_1^{(n+1)} - s_n\gamma_1^{(n)},\ \ldots,\ \gamma_{h-1}^{(n+1)} - s_n\gamma_{h-1}^{(n)}).$$

In view of 6.8 the length of this vector does not exceed $t\gamma_h^{(n)}(2^n\sigma_n q_n)^{-1}$. For $t \leq 1/q_n$ this length is not greater than $(2^n\sigma_n q_n r_n)^{-1}$. Thus for these t the set $T^{ts_n\gamma^{(n)}}\Gamma^{(n,n+1)} \setminus T^{t\gamma^{(n+1)}}\Gamma^{(n,n+1)}$ is contained in the $(2^n\sigma_n q_n r_n)^{-1}$-neighborhood of the first of these sets, and the last one, as is evident from what was said above, lies the $(2^n\gamma_h^{(n)}\sigma_n)^{-1}$-neighborhood. Therefore we have

$$\mu_{h-1}\,(T^{ts_n\gamma^{(n)}}\,\Gamma^{(n,n+1)} \triangle T^{t\gamma^{(n+1)}}\,\Gamma^{(n,n+1)}) \leqslant \frac{4r_n}{2^n q_n}$$

and

$$\mu_h\,(\Gamma_{q_n}^{(n,n+1)} \triangle \widehat{\Gamma}_{q_n}^{(n,n+1)}) \leqslant \frac{1}{2^{n-2} q_n}$$

(remembering that the hth coordinate is shifted by r_n). Finally, comparing this with (6.3), we find that

$$\mu_h\,(\Gamma_{q_n}^{(n,n+1)} \triangle \Gamma_{q_n}^{(n)}) < \frac{1}{2^{n-3} q_n}. \tag{6.4}$$

4. Now it is already easy to obtain from the sequence of decompositions ζ_n a monotonic sequence of decompositions $\zeta_{n,\infty}$ by a procedure which is analogous to that used in the proof of Lemma 3.1. For this purpose we establish a correspondence Q_{n+1}^n between the elements of ζ_n and those of $\zeta_{n,n+1}$ by putting

$$Q_{n+1}^n\,\Gamma_{k,n} = T^{\frac{k}{q_n}\gamma^{(n)}}\,\Gamma_{q_n}^{(n,n+1)}. \tag{6.5}$$

In analogy to §3.6 we define for any set E an image $Q_{n+1}^n E$ which consists of complete elements from the decomposition ζ_n; furthermore, we introduce

$$Q_n^m = Q_n^{n-1}\ \ldots\ Q_{m+2}^{m+1}Q_{m+1}^m \text{ and } \zeta_{m,n} = Q_n^m\,\zeta_m \text{ for } n > m.$$

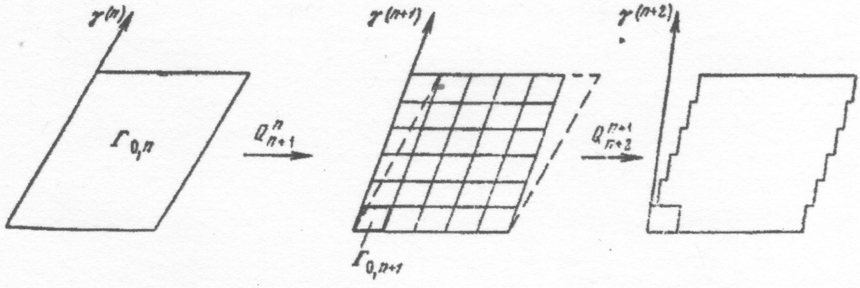

FIGURE 5

LEMMA 6.2. *For given m, the sequence of decompositions $\zeta_{m,n}$ converges to a decomposition $\zeta_{m,\infty}$. The sequence $\zeta_{m,\infty}$ has the following properties:*

6.2.1. $\zeta_{m+1,\infty} > \zeta_{m,\infty}$, $m = 1, 2, \cdots$.

6.2.2. $T^{\alpha^{(m)}}\zeta_{m,\infty} = \zeta_{m,\infty}$, *where the factor-automorphism* $T^{\alpha^{(m)}}/\zeta_{m,\infty}$ *is a cyclic permutation of the elements of the decomposition* $\zeta_{m,\infty}$.

6.2.3. $\zeta_{m,\infty} \to \epsilon$ *as* $m \to \infty$.

The proof of this lemma is completely analogous to the proof of Lemma 3.1, with the inequality (6.4) playing the role of (3.4).

We make some remarks which will be important in the sequel. The element $\lim_{n\to\infty} Q_n^m \Gamma_{k,m}$ of the decomposition $\zeta_{m,\infty}$ will be denoted by $Q_\infty^m \Gamma_{k,m}$. In the same manner we define the mapping

$$Q_\infty^m : \mathbf{T}^h/\zeta_m \to \mathbf{T}^h/\zeta_{m,\infty},$$

which is measure preserving and commutes with $T^{\alpha^{(m)}}$.

5. We shall construct simultaneously a sequence of shifts $T^{\alpha^{(n)}}$ of the torus $\mathbf{T}^h$ which satisfies Conditions 6.1–6.8 and a sequence of diffeomorphisms T_n on the manifold M^m with a periodic flow which satisfy the conditions of the First through the Fourth Step of §3. These sequences will be related to each other as follows: We introduce a mapping

$$K_n : \mathbf{T}^h/\zeta_n \to M^m/\xi_n$$

by the equations

$$K_n \Gamma_{k,n} = B_n^{-1} \Delta_{k,q_n}, \quad k = 0, 1, \dots, q_n - 1. \tag{6.6}$$

Then the factor-automorphism satisfies

$$K_{n+1}/\zeta_{n,n+1} = P_{n+1}^n K_n (Q_{n+1}^n)^{-1}. \tag{6.7}$$

We show that in this case the limiting diffeomorphism T on M^m is metrically isomorphic to the limiting shift T^α on the torus $\mathbf{T}^h$. For this purpose we consider the mapping

$$\widetilde{K}_n : \mathbf{T}^h/\zeta_{n,\infty} \to M^m/\xi_{n,\infty},$$

where

$$\widetilde{K}_n = P_\infty^n K_n (Q_\infty^n)^{-1}.$$

It is easy to see that

1. $\widetilde{K}_n T^{\alpha^{(n)}} = T_n \widetilde{K}_n$.

2. The restriction of $\widetilde{K}_{n+1}$ to $\mathbf{T}^h/\zeta_{n,\infty}$ coincides with $\widetilde{K}_n$.

It follows from Lemma 4.1 that the automorphism T^α is metrically isomorphic to T.

It remains to construct the sequences T_n and $T^{\alpha^{(n)}}$. We proceed by induction. Assuming the sequences to be defined for some n, we show how to define T_{n+1} and $T^{\alpha^{(n+1)}}$ such that condition (6.7) is satisfied.

In order to define the shift $T^{\alpha^{(n+1)}}$ it suffices to construct a vector $\gamma^{(n+1)}$ and to exhibit a number m_n. Let the vector $\gamma^{(n+1)}$ be constructed by the relevant part of Lemma 6.1. Then, in particular, a number s_n is defined by condition 6.4. In the First Step in the induction of §3 we let $k_n = s_n \gamma_h^{(n+1)}$.

We define the fundamental domain

$$\hat{\Gamma}^{(n+1)} = \bigcup_{0 \leq i < \frac{1}{k_n q_n}} T^{i\gamma^{(n+1)}} \Gamma^{(n+1)} \quad \text{of the shift } T^{\frac{1}{k_n q_n}\gamma^{(n+1)}}$$

and the decomposition $\hat{\zeta}_n$ into the sets

$$\hat{\Gamma}_{k,n} = T^{\frac{k}{k_n q_n}\gamma^{(n+1)}} \hat{\Gamma}^{(n+1)}, \quad k = 0, \ldots, k_n q_n - 1.$$

The set $\Gamma_{q_n}^{(n,n+1)}$, as we mentioned, does not depend on the choice of m_n; furthermore, it is easy to verify that this set consists of elements of the decomposition $\hat{\zeta}_n$, and thus $\hat{\zeta}_n > \hat{\zeta}_{n,n+1}$. We define a mapping $\hat{K}_n \colon \mathbf{T}^h/\hat{\zeta}_n \to M^m/\eta_{k_n q_n}$ by letting

$$\hat{K}_n \hat{\Gamma}_{k,n} = \Delta_{k,k_n q_n}, \quad k = 0, \ldots, k_n q_n - 1. \tag{6.8}$$

Now we introduce

$$R^{(n)} = \hat{K}_n \Gamma_{q_n}^{(n,n+1)}. \tag{6.9}$$

The set $R^{(n)}$ is a fundamental domain for S_{1/q_n}, such that we have

$$\hat{K}_n T^{\frac{1}{q_n}\gamma^{(n)}} / \hat{\zeta}_n = S_{\frac{1}{q_n}} \hat{K}_n \tag{6.10}$$

by (6.8). The quantities k_n and $R^{(n)}$ thus defined determine the numbers $a_n(i)$ for $i = 0, \cdots, k_{n-1}$. Now we define a mapping A_{n+1} and we choose a number l_n such that the conditions of the Second, Third and Fourth Steps [9] are satisfied.

We introduce $m_n = \gamma_h^{(n+1)} l_n$ and we construct a decomposition ζ_{n+1}. We have $\zeta_{n+1} > \hat{\zeta}_n$, where the fact that every element of the decomposition $\hat{\zeta}_n$ contains elements of the decomposition ζ_{n+1} may be accomplished in such a way that each following element is obtained from the preceding element by applying the mapping $T^{1/q_{n+1}\gamma^{(n+1)}}$. Similarly the fact that every element of the decomposition $\eta_{k_n q_n}$ contains elements of the decomposition $\eta_{q_{n+1}}$ may be accomplished in such a way that each following element is obtained from the preceding element by applying the mapping $S_{1/q_{n+1}}$. Hence it follows from (6.8) that the diagram

$$\begin{array}{ccc} \mathbf{T}^h/\zeta_{n+1} & \xrightarrow{\overline{K}_n} & M^m/\eta_{q_{n+1}} \\ \downarrow & & \downarrow \\ \mathbf{T}^h/\hat{\zeta}_n & \xrightarrow{\hat{K}_n} & M^m/\eta_{k_n q_n} \end{array} \tag{6.11}$$

is commutative; here $\overline{K}_n \Gamma_{k,n+1} = \Delta_{k,q_{n+1}}$, and the vertical arrows stand for natural embeddings of the decompositions.

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[9] Even though l_n does not occur explicitly in these conditions, the possibility of satisfying them is related to the choice of l_n in §3.5.

Now let K_{n+1} be a mapping defined according to (6.6) with $n+1$ instead of n. We have to verify that K_{n+1} satisfies (6.7), i.e.

$$K_{n+1}Q_{n+1}^{n}\Gamma_{k,n} = P_{n+1}^{n}K_{n}\Gamma_{k,n}. \tag{6.12}$$

Here we rewrite the left-hand side, using the relations $K_{n+1} = B_{n+1}^{-1}\overline{K}_n$, (6.5), $\hat{\zeta}_n > \zeta_{n,n+1}$ and (6.11), in the form

$$B_{n+1}^{-1}\overline{K}_n T^{\frac{k}{q_n}\gamma^{(n)}}\Gamma_{q_n}^{(n,n+1)} = B_{n+1}^{-1}\widehat{K}_n T^{\frac{k}{q_n}\gamma^{(n)}}\Gamma_{q_n}^{(n,n+1)}.$$

To the right-hand side of (6.12) we apply (6.6), the definitions of P_{n+1}^{n} and C_n from §3.6, and (6.9), thus finding it equal to

$$P_{n+1}^{n}B_n^{-1}\Delta_{k,q_n} = B_{n+1}^{-1}C_n\Delta_{k,q_n} = B_{n+1}^{-1}S_{\frac{k}{q_n}}R^{(n)} = B_{n+1}^{-1}S_{\frac{k}{q_n}}\widehat{K}_n\Gamma_{q_n}^{(n,n+1)}.$$

Now the equation (6.12) follows from (6.10).

Thus the following theorem has been proved.

THEOREM 6.1. *Let M^m be a manifold on which there exists a periodic flow of class C^∞, let h be an arbitrary positive integer and let μ be a positive measure of class C^∞ on M^m. Then there exists a diffeomorphism $T\in \mathrm{Diff}^\infty(M^m,\mu)$ which is metrically isomorphic to some ergodic shift of the h-dimensional torus $\mathbf{T}^h$.*

REMARK 6.1. By means of minor modifications in our construction we may also obtain a diffeomorphism $T\in \mathrm{Diff}^\infty(M^m,\mu)$ which is metrically isomorphic to some ergodic shift of the infinite-dimensional torus $\mathbf{T}^\infty$ (i.e. a diffeomorphism with discrete spectrum generated over $\mathbf{Z}$ by a countable set of independent eigenvalues).

6. Finally, it is possible to construct also a diffeomorphism $T\in \mathrm{Diff}^\infty(M^m,\mu)$ which is metrically isomorphic to the direct product of some ergodic shift on $\mathbf{T}^h$ or $\mathbf{T}^\infty$ and some automorphism with continuous simple singular spectrum which is a realization of a diffeomorphism $\hat{T}\in \mathrm{Diff}^\infty(N^k,\nu)$ of an arbitrary manifold N^k with a periodic flow by means of the construction from §5. In order to accomplish this one has to construct simultaneously a sequence $T^{a(n)}$ of shifts of the torus as described in Lemma 6.1, a sequence T of periodic diffeomorphisms of manifold of N^k as described in §5, and a sequence $\hat{T}_n$ of periodic diffeomorphisms of a manifold M^m as described in §3. For this purpose one has to apply a certain lemma on the existence of a sequence $\epsilon_n \nearrow \epsilon$ of finite measurable decompositions which are invariant with respect to the automorphisms $T_n = T_n^1\times T_n^2$ of a Lebesgue space and which furthermore have the property that T_n/ξ_n is cyclic if the sequences T_n^i have invariant finite decompositions $\xi_n^i \nearrow \epsilon$, if these sequences themselves converge sufficiently rapidly under the weak topology and if the mappings T_n^i/ξ_n^i are cyclic. Such a decomposition enables us to choose $a_n(i)$ for T_{n+1} if $\gamma^{(n+1)}$ and q_{n+1} have already been determined for $T^{a(n+1)}$ and if the corresponding quantities $\hat{a}_n(i)$ have been chosen for $\hat{T}_{n+1}$. Thereafter the number l_n (for T_{n+1}) must be chosen sufficiently large in order to guarantee that the condition of the Third Step is satisfied simultaneously for T_{n+1} and $\hat{T}_{n+1}$.

§7. Ergodic diffeomorphisms contained in the closure of periodic diffeomorphisms

We consider all possible periodic flows of class C^∞ on M^m which preserve a given positive measure μ of class C^∞, and we denote by $\mathfrak{P}(M^m, \mu)$ the set of all diffeomorphisms contained in such flows. The closure $\overline{\mathfrak{P}(M^m, \mu)}$ of this set in the space $\mathrm{Diff}^\infty(M^m, \mu)$ is a nowhere dense, perfect subset of this space. Any diffeomorphism T obtained by the construction of §3 obviously belongs to the set $\overline{\mathfrak{P}(M^m, \mu)}$.

Lemma 7.1. *The set of diffeomorphisms which as automorphisms of the Lebesgue space (M^m, μ) possess a cyclic APM with a given speed of approximation $f(n)$ is everywhere dense in $\mathfrak{P}(M^m, \mu)$.*

Proof. It suffices to show that in any neighborhood of a diffeomorphism $S \in \mathfrak{P}(M^m, \mu)$ there exist diffeomorphisms which have a cyclic APM with a speed of approximation $f(n)$. Let $S = S_\alpha$, where $\{S_t\}$ is a periodic flow. We choose a fixed open neighborhood $\mathfrak{U}$ of the diffeomorphism S in the space $\mathrm{Diff}^\infty(M^m, \mu)$ and we determine a rational number α_0 such that $S_{\alpha_0} \in \mathfrak{U}$. Then we let $T_0 = S_{\alpha_0}$ in the construction of §3. The diffeomorphism T may be constructed in such a manner that it is contained in any neighborhood of the automorphism T_0 given in advance (in order to accomplish this is only necessary to choose ϵ_n in the conditions of the Third Step sufficiently small). If, furthermore, ϵ_n and l_n are suitably chosen such that the expression on the right-hand side of (3.14) becomes less than $f(q_n)$, then the diffeomorphism $T = \lim T_n$ possesses a cyclic AMP with the speed of approximation $f(n)$.

Theorem 7.1. *The set of diffeomorphisms from $\overline{\mathfrak{P}(M^m, \mu)}$ which as automorphisms of the Lebesgue space (M^m, μ) possess a cyclic AMP with a given speed of approximation $f(n)$ is a set of the second category within $\overline{\mathfrak{P}(M^m, \mu)}$ (it contains an everywhere dense G_δ-set).*

(We note that $\mathrm{Diff}^\infty(M^m, \mu)$ may be considered as a complete metric space with the metric

$$\rho(S, T) = \sum_{n=1}^{\infty} \frac{\rho_n(S, T)}{2^n(1 + \rho_n(S, T))};$$

hence Baire's Theorem holds for any closed subset in the space.)

Proof. We consider all possible sequences of diffeomorphisms $\{T_n\}$ which satisfy the inductive hypothesis of §3 with values of ϵ_n and l_n for which the expression on the right-hand side of (3.14) does not exceed $f(q_n)$. Let $\mathfrak{U}(T_n)$ be a neighborhood of the diffeomorphism T_n defined as follows:

$$\mathfrak{U}(T_n) = \{S : S \in \mathrm{Diff}^\infty(M^m, \mu),$$

$$\rho_{\left[\frac{1}{\varepsilon_n}\right]}(T_n, S) < 2\varepsilon_n, \quad \sum_{c \in \xi_{n,\infty}} \mu(Sc \,\triangle\, T_n c) < f(q_n)\}.$$

We denote by $\mathfrak{G}_n$ in the union of the neighborhoods $\mathfrak{U}(T_n)$ for all T_n which occur in the sequence described above, and we denote by $\mathfrak{G}$ the set

$$\mathfrak{G} = \bigcap_n \bigcup_{s \geqslant n} \mathfrak{G}_s.$$

The set $\mathfrak{G}$ is a G_δ set since the set $\mathfrak{G}_n$ is open for every n. For any of the sequences $\{T_n\}$ under consideration, the limit T belongs to $\mathfrak{G}$ (since the Third Step and (3.14) imply that $T \in \mathfrak{U}(T_n)$ for all n). Hence by Lemma 7.1 the set $\mathfrak{G}$ is everywhere dense in $\mathfrak{P}(M^m, \mu)$. We show that any diffeomorphism $S \in \mathfrak{G}$ possesses a cyclic APM with speed of approximation $f(n)$. Indeed, there exists a sequence $n_k \to \infty$ for which $T \in \mathfrak{G}_{n_k}$, i.e. $T \in \mathfrak{U}(T^{(k)}_{n_k})$, $k = 1, 2, \cdots$, where $T^{(k)}_{n_k}$ is the n_kth term of the sequence $\{T^{(k)}_n\}$.

All decompositions which correspond to the sequence $\{T^{(k)}_n\}$ will be labeled by the upper index (k). By the conditions of the Second Step we have $\xi^{(k)}_{n_k} \to \epsilon$, and (3.13) implies that also $\xi^{(k)}_{n_k, \infty} \to \epsilon$. Hence it follows from the definition of $\mathfrak{U}(T^{(k)}_{n_k})$ that S possesses a cyclic APM with speed of approximation $f(n)$.

By combining the theorem just proved with the results of [8] we obtain the following corollary:

Corollary. *The ergodic but nonmixing diffeomorphisms for which the shift operator U_T in the space $L^2(M^m, \mu)$ has a simple singular spectrum lie in the space $\mathfrak{P}(M^m, \mu)$ and form a subset of the second category there.*

Remark. The results of the present section carry over to the case of diffeomorphisms with finite smoothness.

The present article is not a joint paper in the usual sense; it was written by A. B. Katok. However, the construction described in §3 was preceded by a construction of a topologically transitive, (Lebesgue-) measure-preserving C^∞-diffeomorphism of the two-dimensional circle due to D. V. Anosov. That construction involved the inductive conditions of the First and Third Step from §3, but instead of the Second and Fourth Step another argument was used. Mindful of this fact and also of the help rendered by D. V. Anosov for the writing of this paper through invaluable advise and comments, we thought it most fitting to publish the article under joint authorship. The ordering of the authors' names is alphabetical and does not have any additional significance.

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Bibliography

[1] D. V. Anosov and A. B. Katok, *New examples of ergodic diffeomorphisms of smooth manifolds*, Uspehi Mat. Nauk 25 (1970), no. 4 (154), 173–174.

[2] A. S. Besicovitch, *A problem on topological transformations of the plane*, Fund. Math. 28 (1937), 61–65.

[3] ———, *A problem on topological transformations of the plane*. II, Proc. Cambridge Philos. Soc. 47 (1951), 38–45. MR 12, 519.

[4] J. W. S. Cassels, *An introduction to the geometry of numbers*, Springer-Verlag, Berlin, 1959; Russian transl., "Mir", Moscow, 1965. MR 28 # 1175; 31 # 5841.

[5] W. H. Gottschalk, *Orbit-closure decompositions and almost periodic properties,* Bull. Amer. Math. Soc. **50** (1944), 915–919. MR **6**, 165.

[6] P. R. Halmos, *Lectures on ergodic theory,* Publ. Math. Soc. Japan, no. 3, Math. Soc. Japan, Tokyo, 1956; reprint, Chelsea, New York, 1960; Russian transl., IL, Moscow, 1959. MR **20** # 3958; **22** # 2677.

[7] ———, *Approximation theories for measure preserving transformations,* Trans. Amer. Math. Soc. **55** (1944), 1–18. MR **5**, 189.

[8] A. B. Katok and A. M. Stepin, *Approximations in ergodic theory,* Uspehi Mat. Nauk **22** (1967), no. 5 (137), 81–106 = Russian Math. Surveys **22** (1967), no. 5, 77–102. MR **36** # 2776.

[9] ———, *Metric properties of homeomorphisms that preserve measure,* Uspehi Mat. Nauk **25** (1970), no. 2 (152), 193–220 = Russian Math. Surveys **25** (1970), no. 2, 193–220. MR **41** # 5594.

[10] A. N. Kolmogorov, *Théorie générale des systèmes dynamiques et mécanique classique,* Proc. Internat. Congress Math. (Amsterdam, 1956), vol. 1, North-Holland, Amsterdam, 1957, pp. 315-333; Russian transl., Fizmatgiz, Moscow, 1961, pp. 187-208. MR **20** # 4066.

[11] ———, *On dynamical systems with an integral on the torus,* Dokl. Akad. Nauk SSSR **93** (1953), 763–766. (Russian) MR **16**, 36.

[12] D. Montgomery and L. Zippin, *Topological transformation groups,* Interscience Tracts in Pure and Appl. Math., no. 1, Interscience, New York, 1955. MR **17**, 383.

[13] J. Moser, *On invariant curves of area-preserving mappings of an annulus,* Sympos. (Colorado Springs, 1961), Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **1962**, 1-20; Russian transl., Matematika **6** (1962), no. 5, 51–67. MR **26** # 5255.

[14] ———, *A rapidly convergent iteration method and nonlinear differential equations.* I, II, Ann. Scuola Norm. Sup. Pisa (3) **20** (1966), 265–315, 499–535; Russian transl., Uspehi Mat. Nauk **23** (1968), no. 4 (142), 179–238. (Russian) MR **33** # 7667; **34** # 6280; **37** # 4382.

[15] ———, *On the volume elements on a manifold,* Trans. Amer. Math. Soc. **120** (1965), 286-294. MR **32** # 409.

[16] V. V. Nemyckiĭ and V. V. Stepanov, *Qualitative theory of differential equations,* 2nd ed., GITTL, Moscow, 1949; English transl., Princeton Math. Series, no. 22, Princeton Univ. Press, Princeton, N. J., 1960. MR **10**, 612; **22** # 12258.

[17] J. C. Oxtoby and S. Ulam, *Measure-preserving homeomorphisms and metrical transitivity,* Ann. of Math. (2) **42** (1941), 874-920. MR **3**, 211.

[18] R. S. Palais, *Local triviality of the restriction map for embeddings,* Comment. Math. Helv. **34** (1960), 305–312. MR **23** # A666.

[19] H. Poincaré, *Sur les courbes définies par les équations différentiels* III, IV, J. Math. Pures Appl. (4) **1** (1885), 167–244; ibid. **2** (1886), 151–217; Russian transl., GITTL, Moscow, 1947.

[20] L. G. Šnirel'man, *Example of a transformation of the plane,* Izv. Donsk. Politehn. Inst. Novočerkasske **14**, Naučn. Otdel. Fiz.-Mat. Čast' (1930), 64–74. (Russian)

[21] E. A. Sidorov, *Smooth topologically transitive dynamical systems,* Mat. Zametki **4** (1968), 751–759. (Russian) MR **39** # 939.

[22] M. D. Sklover, *Classical dynamical systems on the torus with continuous spectrum,* Izv. Vysš. Učebn. Zaved. Matematika **1967**, no. 10 (65), 113–124. (Russian) MR **37** # 1737.

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