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Abstract. Hamiltonian systems arbitrarily close in the C^r topology ($r = 1, 2, \dots$) to a given integrable degenerate Hamiltonian system of class C^∞ which generate an ergodic flow on each manifold of constant energy are constructed. Applications: Small perturbations of a system generated by independent oscillators and Finsler metrics close to standard Riemannian metrics on symmetric spaces of rank 1.

Introduction

1. The concept of an integrable Hamiltonian dynamical system, notwithstanding its respectable age, has no generally accepted formal definition. Exceptionally large variations are possible in the treatment of this concept in the case of not analytical but just differentiable Hamiltonian systems, for example systems of class C^∞ which we shall treat in what follows. Without risking a fall into strong contradictions with traditional presentations we shall call a Hamiltonian system with n degrees of freedom integrable if in the phase space M^{2m} of this system there is an open everywhere dense set M_R which is a locally trivial fibration of class C^∞, with fiber a k-dimensional torus $\mathbf{T}^k$ $(k \leq m)$, and some $(2m-k)$-dimensional manifold N as base, where each fiber of this fibration is invariant with respect to the dynamical system, which induces on it a conditionally periodic motion. If one fixes a basis in the integral homology group $H_1(\mathbf{T}^k_{x_0}, \mathbf{Z})$ of the fiber $\mathbf{T}^k_{x_0}$ over the point $x_0 \in N$ sufficiently close to x_0, then the conditionally periodic motion on $\mathbf{T}^k_x$ will be characterized by the vector of frequencies $\omega(x) = (\omega_1(x), \dots, \omega_k(x))$, which is a smooth function of the point x.

If $k = m$ and x is a regular point of the mapping $\omega: U \to \mathbf{R}^k$, $x \mapsto \omega(x)$ (U is an open subset of N), then we shall say that the considered integrable Hamiltonian dynamical system is nondegenerate on the fiber $\mathbf{T}^m_x$. If, moreover, the vector $\omega(x)$ is not too close to vectors which satisfy integral conditions, then any sufficiently close Hamiltonian system contains an invariant torus, close to the torus $\mathbf{T}^m_x$, on which the

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conditionally periodic motion with the vector of frequencies $\omega(x)$ takes place. Moreover, such tori fill a set of positive measure in M^{2m}, and, for any neighborhood V of the point x in which the mapping ω is regular, as the magnitude of perturbation decreases the measure of the complement of the set $V' = \bigcup_{y \in V} T^m_y$, in the set consisting of the invariant tori of the perturbed system, tends to zero. In the analytic case, i.e. when the original integrable system is indeed an analytic system and the perturbation is sufficiently small in some complex neighborhood of the set V', these statements make up the content of the well-known theorem of A. N. Kolmogorov about the preservation of conditionally periodic motions ([1]; for the proof see [2]), and in the differentiable case when one requires the smallness of the perturbation with some fixed number of derivatives, the basic results are due to Jurgen Moser (cf. [3], [4], [5]).

Degeneracy can appear, roughly speaking, for two reasons: either $k < m$, or the vector of frequencies $\omega(x)$ depends on x in a degenerate way. In these cases, for no perturbation can one guarantee the existence of invariant tori. In the first case, V. I. Arnol'd [6] has shown sufficient conditions which should be satisfied by the perturbation in order that the perturbed system contain invariant tori filled by conditionally periodic motions where these tori turn out to be not k but m-dimensional. These conditions are satisfied in a series of important concrete problems, for example, in the n-body problem.

2. The purpose of this article is to prove that in C^∞ Hamiltonian dynamical systems close in the C^r topology to degenerate integrable systems for an arbitrary a priori given r, there may occur effects for which invariant tori are completely destroyed. We will treat two extreme cases of degeneracy: either $k = 1$, or the vector of frequencies $\omega(x)$ is constant on the manifolds of constant energy. The exact hypotheses about the original degenerate Hamiltonian system are formulated in § 1. Under these assumptions, we shall prove that for any r arbitrarily close, in the C^r topology, to such a degenerate integrable system, there are Hamiltonian systems which on each manifold of constant energy induce an ergodic flow. Results which apply to partial degeneracy will be given in another paper.

If the original Hamiltonian system has some additional structure, in many cases it is interesting to construct perturbations which also have this structure. Thus, for example, giving the manifold M a Riemannian or Finsler metric defines on the cotangent bundle T^*M a Hamiltonian dynamical system for which the Legendre transformation establishes an isomorphism between this dynamical system and the geodesic flow on the tangent bundle TM (cf. § 6). If the original degenerate integrable Hamiltonian system is generated by a Riemannian metric (such a system is generated, for example, by the standard Riemannian metric on the n-dimensional sphere S^n), then it would be interesting to construct a perturbation of the above described kind in the class of Hamiltonian systems generated by Riemannian metrics, since it unknown, for example, whether there exists on S^n $(n \geq 2)$ a Riemannian metric for which the geodesic flow is ergodic on the manifold of unit tangent vectors or whether at least this manifold has an ergodic component of positive measure. The difficulty here consists of the fact that of the known mechanisms which produce ergodicity in geodesic flows, only those which are related to "hyperbolic" behavior of trajectories (we have in mind the condition U of D. V. Anosov [7], or some modification) have been studied, and such phenomena appear on manifolds of negative curvature, or at least in the absence of conjugate points (cf. [7], [8], [9]). In the case of negative curvature one has not only ergodicity but a significantly stronger instability of trajectories, namely, the K-property [10].

Unfortunately the class of Riemannian metrics turns out to be too small to carry out our construction. However, the class of Finsler metrics serves our purpose, and some details of our construction are there precisely in order to adapt it to this case. Our results imply, for example, the existence on S^n ($n \geq 2$) of a Finsler metric of class C^∞ which is close in the natural sense, with an arbitrary given number of derivatives, to the standard Riemannian metric and such that the corresponding geodesic flow has two open ergodic components which fill the manifold of unit tangent vectors except for a set of arbitrarily small measure. Since such Finsler metrics are close to the Riemannian metric of constant positive curvature, the curvature in this metric in the natural sense is positive, even though for Finsler metrics the curvature is not defined by a tensor on the manifold. This is in agreement with the fact that the mechanism of ergodicity, in our case, is quite different from that for metrics of negative curvature. Thus in our examples the geodesic flow in ergodic components has no mixing, even though weak mixing is not excluded. We consider the metric properties of our examples in more detail at the end of § 1.

3. All objects such as manifolds, functions, vector fields, differential forms, etc. are assumed in this article to be smooth of class C^∞, and we shall usually not repeat this. All necessary standard propositions related to the differentiable manifolds, differentiable geometry and symplectic structures can be found in Sternberg's *Lectures on differential geometry* [11]. Our notation, basically, coincides with Sternberg's. The greatest essential difference is that we denote the differential of the map f by the symbol Df and not f_*, saving the symbol f_* for the induced maps of vector fields and measures. All necessary definitions from ergodic theory and explanations can be found in §§ 1–3 of the article of V. A. Rohlin [12].

The author considers it a pleasant duty to express his gratitude to D. V. Anosov, who read the manuscript with exceptional attention. His criticism has allowed us to simplify some proofs, and various notes on the style were very useful.

§ 1. Homogeneous Hamiltonian systems

1. Let M^{2m} be a connected manifold, or a manifold with boundary (not necessarily compact) with a given symplectic structure, i.e. a fixed closed nondegenerate 2-form Ω; let H be a smooth function on M^{2m}. The equality

$$dH = -v_H \lrcorner \Omega$$

defines a vector field v_H which is called the Hamiltonian vector field with the Hamilton function (or Hamiltonian) H. In this paper we shall treat only that case when for each positive number c the set $H^{-1}(c)$ is compact and at all points of the boundary ∂M^{2m} the vector field v_H is tangent to ∂M^{2m}. In that case the vector field v_H is full, i.e. its trajectories can be infinitely continued in both directions with respect to time; the flow generated on M^{2m} will be denoted by $\{S_t^H\}$.

We shall assume that M^{2m} has an additional structure, namely a vector field u which "continuously expands" the symplectic structure. More precisely, this means that

$$\mathcal{L}_u \Omega = \lambda \Omega,$$

where $\mathcal{L}_u$ is the Lie derivative along the vector field u and λ is some positive number. Moreover, we shall make the following assumptions:

(1.3) The vector field u is full and generates a flow $\{\phi_t\}$.

(1.4) The vector field u is nowhere zero.

Remark. If conditions (1.2) and (1.3) are satisfied and (1.4) is not satisfied, then all the constructions can be carried out on the manifold $M^{2m} \backslash \Lambda$ where Λ is the set of zeros of the vector field u on M^{2m}, if, of course, the manifold $M^{2m}\backslash\Lambda$ is connected.

2. The following two examples illustrate the situation and will play an essential role in what follows.

Example 1. Let $p_1, \cdots, p_m, q_1, \cdots, q_m$ be the cartesian coordinates in the Euclidian space $\mathbf{R}^{2m}$, and set

$$\Omega = \sum_{i=1}^{m} dp_i \wedge dq_i,$$

$$u(p, \ldots, p_m, q_1, \ldots, q_m) = \sum_{i=1}^{m} p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i}. \tag{1.5}$$

In accordance with the remark we shall set $M^{2m} = \mathbf{R}^{2m} \backslash \{0\}$. Here $\lambda = 2$.

Example 2. Let M^m be a compact connected m-dimensional manifold, let Ω be the cannonical 2-form on the cotangent bundle T^*M^m (cf. [11], Chapter III, Theorem 7.1), let Γ_0^* be the zero-section of T^*M^m and let u be a vector field on T^*M^m such that for $q \in M^m$ and $p \in T_q^*M^m$ the vector $u(p)$ is tangent to $T_q^*M^m$ and under the natural identification of $T_pT_q^*M^m$ with T_qM^m we have $u(p) = p$. Here $M^{2m} = T^*M^m \backslash \Gamma_0^*$ and $\lambda = 1$.

3. **Definition 1.1.** A smooth function $H: M^{2m} \to \mathbf{R}$ is said to be *homogeneous* if

$$\mathcal{L}_u H = \lambda H. \tag{1.6}$$

In Example 1 the functions homogeneous in the sense of Definition 1.1 are the homogeneous functions of degree 2 in the usual sense. In Example 2 they are the smooth functions on $T^*M^m \setminus \Gamma_0^*$ which on each linear space $T_q^*M^m$ are homogeneous functions of degree 1 in the usual sense.

Lemma 1.1. *If H is a homogeneous function, then the Lie bracket $[u, v_H] = 0$.*

Proof. It suffices to prove that $[u, v_H] \lrcorner \Omega = 0$. We shall use the known formula for the Lie derivative of a differential form along a vector field X (cf. [11], Chapter III, formula (1.9))

$$\mathcal{L}_X \omega = X \lrcorner d\omega + d(X \lrcorner \omega). \tag{1.7}$$

We transform the expression $[u, v_H] \lrcorner \Omega$:

$$\begin{aligned}[u, v_H] \lrcorner \Omega &= -\mathcal{L}_{v_H}(u \lrcorner \Omega) \\ = -v_H \lrcorner d(u \lrcorner \Omega) - d(v_H \lrcorner (u \lrcorner \Omega)) &= -v_H \lrcorner d(u \lrcorner \Omega) + d(u \lrcorner (v_H \lrcorner \Omega)).\end{aligned} \tag{1.8}$$

From (1.2) and (1.7) we obtain

$$\lambda\Omega = \mathcal{L}_u \Omega = u \lrcorner d\Omega + d(u \lrcorner \Omega) = d(u \lrcorner \Omega).$$

The homogeneity of H, (1.7) and (1.1) imply

$$\lambda\, dH = \mathcal{L}_u\, dH = d(u \lrcorner dH) = -d(u \lrcorner (v_H \lrcorner \Omega)).$$

We substitute both expressions into (1.8):

$$[u, v_H] \lrcorner \Omega = -v_H \lrcorner \lambda\Omega - \lambda dH = \lambda dH - \lambda dH = 0.$$

The lemma follows.

Corollary. *Let H be a positive homogeneous function on the manifold M^{2m} without critical points. The restriction of the diffeomorphism ϕ_t to the manifold $H^{-1}(c)$ is a diffeomorphism ϕ_t^c between $H^{-1}(c)$ and $H^{-1}(e^{\lambda t}c)$, and its differential $D\phi_t^c$ takes the vector field v_H into itself, i.e. $D\phi_t^c v_H = v_H \circ \phi_t^c$.*

4. The main proposition which we prove in this paper (Theorem A) applies to the case when the manifold M^{2m} is acted upon by the group $\mathrm{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$ of canonical (i.e. preserving the form Ω) diffeomorphisms and the following conditions (1.9)–(1.11) are satisfied:

(1.9) The action of each one-parameter subgroup of T^2 is generated by a Hamiltonian vector field with a homogeneous Hamilton function.

To the vector $(\alpha, \beta) \in \mathbf{R}^2$ there corresponds a one-parameter subgroup $\{(t\alpha, t\beta) \bmod 1\}$ of the torus T^2. We denote the corresponding homogeneous Hamilton function of this subgroup by $H_{\alpha,\beta}$. Clearly

$$H_{\alpha_1,\beta_1} + H_{\alpha_2,\beta_2} = H_{\alpha_1+\alpha_2,\beta_1+\beta_2}.$$

(1.10) The group $\mathbf{T}^2$ acts effectively, i.e. every element, save for the identity, acts distinctly from the identity transformation.

(1.11) There exists a vector $(\alpha_0, \beta_0) \in \mathbf{R}^2$ such that $H_{\alpha,\beta}$ is a positive function with no critical points with compact level manifolds.

Remark. Any vector (α, β) sufficiently close to (α_0, β_0) also satisfies condition (1.11).

In Example 1 for $m \geq 2$ an action of $\mathbf{T}^2$ satisfying the conditions (1.9)–(1.11) exists; moreover there exists an action of $\mathbf{T}^m$ satisfying analogous conditions. Namely, let $(\alpha_1, \cdots, \alpha_m) \in \mathbf{R}^m$. We set

$$H_{\alpha_1,\ldots,\alpha_m}(p_1, \ldots, p_m, q_1, \ldots, q_m) = 2\pi \sum \alpha_i (p_i^2 + q_i^2).$$

The vector field $vH_{\alpha_1,\cdots,\alpha_m}$ generates an action of the one-parameter subgroup $\{(t\alpha_1, \cdots, t\alpha_m) \bmod 1\}$ of the torus $\mathbf{T}^m$.

In Example 2 this situation occurs, for example, if the manifold M^m has a Riemannian metric ρ with the following properties:

(1.12) All geodesics of this metric are closed, and the lengths of all geodesics are divisors of some positive number.

(1.13) On M^m there is an effective action of the group $\mathbf{S}^1$ which consists of diffeomorphisms $\{\psi_t\}$, $t \in \mathbf{R}$ (ψ_1 is the identity transformation) which preserve the Riemannian metric ρ.

The action of $\mathbf{T}^2$ in this case can be obtained as follows. Let τ_0 be the maximal length of the geodesic line, and set $H_{1,0}(x) = \tau_0 \|x\|_\rho^*$, where $x \in T^*M^m \setminus \Gamma_0^*$ and $\|x\|_\rho^*$ is the norm on T^*M^m dual to the norm on TM^m generated by the Riemannian metric. In § 6 we shall show that the vector field $vH_{1,0}$ generates an action of $\mathbf{S}^1$. Furthermore, let $H_{0,1}$ be the Hamilton function of a vector field on T^*M^m which generates an action of $\mathbf{S}^1$ on T^*M^m by transformations dual to the differentials $D\psi_t$. By (1.13) the vector fields $vH_{1,0}$ and $vH_{0,1}$ commute, and hence they induce an action of $\mathbf{T}^2$ since each of them generates an action of $\mathbf{S}^1$: it is easy to see that conditions (1.10) and (1.11) are satisfied (the last with $(\alpha_0, \beta_0) = (1, 0)$). We shall consider this example in more detail in § 6. Now we shall only remark that compact Riemannian symmetric spaces of rank one, in particular spheres $\mathbf{S}^n$ $(n \geq 2)$ and complex projective spaces $\mathbf{P}^n(\mathbf{C})$, fit into this situation (cf. [15], Chapter IX).

5. We shall show that in the situation just described one can find an arbitrarily small perturbation of the function H_{α_0,β_0} in the class of infinitely differentiable homogeneous functions, where (α_0, β_0) is a vector as in (1.11), and obtain a function $\mathcal{H}$ such that the flow $\{S_t^{\mathcal{H}}\}$ is ergodic on each manifold of constant energy $\mathcal{H}^{-1}(c)$, $c > 0$. We pass to an exact formulation of this statement.

In the space $C^\infty(M^{2m})$ of real C^∞ functions we introduce a system of norms $\| \ \|_r^{\mathcal{K}}$ (r is a nonnegative integer and $\mathcal{K} \subset M^{2m}$ is compact) such that the norm $\| \ \|_r^{\mathcal{K}}$ measures the closeness of r-strings of functions on the compact set $\mathcal{K}$ and all the norms $\| \ \|_r^{\mathcal{K}}$ are coherent in the natural sense.

In view of (1.11) any orbit of the action of $\mathbf{T}^2$ is either a torus or a circle. We denote by M_D the subset of M^{2m} which consists of the points of all periodic orbits of the action of $\mathbf{T}^2$. It is easy to see that

$$M_D = \{x \in M^{2m} : \exists\, (\alpha, \beta) \in \mathbf{R}^2,\ dH_{\alpha,\beta}(x) = 0\}.$$

Theorem A. *Let the symplectic manifold (M^{2m}, Ω) have a vector field u which satisfies the conditions (1.2)–(1.4) and an action of the group $\mathbf{T}^2$ so that the conditions (1.9)–(1.11) are satisfied. Let a positive number δ, a natural number r, a compact set $\mathcal{K} \subset M^{2m}$ and a vector $(\alpha_0, \beta_0) \in \mathbf{R}^2$ satisfying (1.11) be given. There is a vector (α, β) and a positive function $\mathcal{H}$ of class C^∞ on M^{2m} such that the following conditions are satisfied*:

A.1. $$\| \mathcal{H} - H_{\alpha_0,\beta_0} \|_r^{\mathcal{K}} < \delta.$$

A. 2. *At the points of the boundary ∂M^{2m} and the set M_D the function $\mathcal{H}$ coincides with $H_{\alpha,\beta}$ together with differentials of all orders.*

A. 3. *The function $\mathcal{H}$ is homogeneous in the sense of Definition* 1.1.

A. 4. *The flow $\{S_t^{\mathcal{H}}\}$ on each manifold $\mathcal{H}^{-1}(c)$, $c > 0$, is ergodic with respect to the invariant metric μ_c induced by the form Ω.*

A. 5. *For some sequence $t_n \to \infty$ the diffeomorphisms $S_{t_n}^{\mathcal{H}}$ and their differentials of arbitrary order converge to the identity transformation uniformly on arbitrary compact sets.*

A. 6. *The flow $\{S_t^{\mathcal{H}}\}$ has no periodic trajectories outside of the set M_D.*

Remark. The function H_{α_0,β_0} is bounded and nonzero on the compact set $\mathcal{K}$. We chose an N such that $\mathcal{K} \subset \mathcal{K}_N = \mathcal{H}^{-1}_{\alpha_0,\beta_0}([N^{-1}, N])$. If the assertion of Theorem A is satisfied for $\mathcal{K}_N$, then it clearly is satisfied for the original compact set $\mathcal{K}$. Therefore in what follows we shall assume that $\mathcal{K} = \mathcal{K}_N$ for some N.

Theorem A is proved in §§ 2–4. The basis of this proof is the inductive construction described in § 4. In each step of the induction the constructions are made using lemmas which are proved in §§ 2 and 3. Our construction has much in common with the construction which was used in [14] to construct ergodic diffeomorphisms on a manifold with a periodic flow. We remark, however, one essential difference: we do not guarantee that the flow $\{S_t^{\mathcal{H}}\}$ or the separate diffeomorphisms $S_t^{\mathcal{H}}$ admit a sufficiently fast cyclic approximation by periodic transformations (a. p. t.), and therefore we cannot draw conclusions about the metric type of the flow and the diffeomorphisms which make it up. We believe that this distinction is due to the nature of this problem; namely, to the requirement that the function $\mathcal{H}$ be homogeneous, which allows Example 2 to be included in the scheme under consideration when conditions (1.12) and (1.13) are satisfied, where the homogeneity condition is natural (see § 6).

There exists another variant of the construction applicable to some class of Hamiltonian systems. For the situation which arises in Example 1 this variant is described in [13]. The description of this construction in a more general (even though possibly not very natural) situation, together with detailed proofs, will be given in another paper. In this construction we are able to construct an a. p. t. for the flow $\{S_t^{\mathcal{H}}\}$ on each manifold $\mathcal{H}^{-1}(c)$ and even follow the metric type of this flow, but only on almost every manifold $\mathcal{H}^{-1}(c)$, while other difficulties arise because of the necessity of making the constructions of various surfaces of constant energy coherent, which in the homogeneous case is automatic.(1)

In § 5 it is shown how to modify the inductive construction so that in the situation of Example 1 one can get rid of the lack of differentiability at the point zero if one is willing to give up the homogeneity of the function $\mathcal{H}$. The formulation of the corresponding result for more general situations would be too complicated; moreover, I know of no other nontrivial examples for which this result would be of interest. In § 6 Theorem A is applied in the situation of Example 2. A direct application gives examples of nonsymmetric Finsler metrics (i.e. metrics for which the norm $\|x\|$, $x \in TM^m$, is, generally speaking, distinct from $\|-x\|$), close to Riemannian metrics satisfying (1.12) and (1.13) and generating ergodic geodesic flows. For genuine (symmetric) Finsler metrics we are able to obtain not ergodicity but two ergodic components which generate a set whose complement has arbitrarily small measure which is, however, distinct from zero (Theorem C).

§ 2. Lemmas on the canonical action of the group S^1

Let the group S^1 act on a manifold M. A point $x \in M$ will be called regular if its stationary subgroup is trivial. A trajectory of the S^1 action will be called regular if it consists of regular points. If S^1 acts effectively, then the regular points form an open connected set M_R whose complement has measure zero with respect to any measure defined by a smooth density (cf., for example, [14], Proposition 2.1).

Assume now that S^1 acts effectively on a sympletic manifold (M^{2m}, Ω) via canonical diffeomorphisms. Such an action will be said to be canonical. Let a canonical action be generated by a Hamiltonian vector field whose Hamilton function we shall denote by H. We denote by H_R^c and N^c, respectively, the manifolds of regular points and regular trajectories of the action of S^1 on $H^{-1}(c)$; by $\pi\colon H_R^c \to N^c$ we denote the natural projection and by Ω_c the restriction of the 2-form Ω to H_R^c.

(1) In the formulation of the theorem in [13] there is an error. It is claimed there that the Hamilton function H generates a flow which is ergodic and has a discrete spectrum with k equal independent frequencies on each manifold $H^{-1}(c)$. In fact one may only claim that this flow is ergodic on each manifold $H^{-1}(c)$ and for almost all c has a discrete spectrum with k equal independent frequencies.

Lemma 2.1. *On the manifold N^c there is a nondegenerate closed 2-form $\bar{\Omega}_c$ such that*

$$\pi^*\bar{\Omega}_c = \Omega_c. \tag{2.1}$$

Proof. Let $x_1, x_2 \in H^c_R$, $v^1_i, v^2_i \in T_{x_i}$ (T_x denotes the tangent space of H^c_R at the point x) and

$$D\pi v^i_1 = D\pi v^i_2 \quad (i = 1, 2).$$

We shall show that $\Omega_c(v^1_1, v^2_1) = \Omega_c(v^1_2, v^2_2)$. Indeed, $x_2 \in S^H_{t_0} x_1$, $t_0 \in \mathbf{R}$ and

$$(DS^H_{t_0})v^i_1 - v^i_2 = \lambda_i v_H(x_2), \quad i = 1, 2, \quad \lambda_i \in \mathbf{R}.$$

Thus

$$\Omega_c(v^1_1, v^2_1) = (S^H_{t_0})^* \Omega_c(v^1_1, v^2_1) = \Omega_c(DS^H_{t_0}v^1_1, DS^H_{t_0}v^2_1)$$
$$= \Omega_c(v^1_2 + \lambda_1 v_H(x_2), v^2_2 + \lambda_2 v_H(x_2)) = \Omega_c(v^1_2, v^2_2),$$

since $\Omega_c(\xi, v_H(x)) = -dH(\xi) = 0$ for any vector $\xi \in T_x$.

Hence one can define a 2-form $\bar{\Omega}_c$ on N^c, letting for $y \in N^c$, $u_1, u_2 \in T_y N^c$

$$\bar{\Omega}_c(u_1, u_2) = \Omega(v_1, v_2), \tag{2.2}$$

where v_1 and v_2 are two arbitrary vectors with common base for which $D\pi v_i = u_i$, $i = 1, 2$.

The form $\bar{\Omega}_c$ is clearly closed, since $\pi^* d\bar{\Omega}_c = d\Omega_c = 0$ and the operator π^* acting on the differential forms is injective. It remains to show that $\bar{\Omega}_c$ is nondegenerate. To this end consider points $y \in N^c$ and $x \in H^c_R$ such that $\pi x = y$. By the definition of $\bar{\Omega}_c$ (see (2.2)) its values at the point y are completely determined by the values of Ω_c (or Ω) at x. Set $v_H(x) = e_1$ and choose in the tangent space $T_x M^{2m}$ a basis $e_1, \ldots, e_m, f_1, \ldots, f_m$ such that

$$\Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \quad \Omega(e_i, f_j) = \delta_{ij},$$

where

$$i, j = 1, \ldots, m, \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

It is obvious that the subspace $T_x \subset T_x M^{2m}$ is generated by the vectors $e_1, \ldots, e_m, f_2, \ldots, f_m$. The subspace generated by the vectors $e_2, \ldots, e_m, f_2, \ldots, f_m$ on which the $(2m-2)$-form Ω^{m-1} is nonzero is mapped isomorphically by $D\pi$ to the space $T_y N^c$. The lemma follows.

Let $\mathcal{H}$ be the first integral of the flow $\{S^H_t\}$, i.e. the Poisson bracket $[H, \mathcal{H}]$ is equal to 0. Then the restriction of the function $\mathcal{H}$ and the vector field $v_{\mathcal{H}}$ to H^c_R

can be projected to N^c. Indeed, for $y \in N^c$, $x \in H_R^c$ and $\pi x = y$ let

$$\overline{\mathcal{H}}(y) = \mathcal{H}(x), \quad \bar{v}_{\mathcal{H}}(y) = D\pi(v_{\mathcal{H}}(x)).$$

The first definition clearly makes sense, since the function $\mathcal{H}$ is constant on the trajectories of the flow $\{S_t^H\}$. In order to prove that the second definition makes sense, assume that $\pi x_1 = \pi x_2$. Then

$$x_2 = S_{t_0}^H x_1, \quad v_{\mathcal{H}}(x_2) = DS_{t_0}^{\mathcal{H}} v_{\mathcal{H}}(x_1),$$

and therefore

$$D\pi(v_{\mathcal{H}}(x_1)) = D\pi(v_{\mathcal{H}}(x_2)).$$

Lemma 2.2. *The Hamiltonian vector field $v_{\overline{\mathcal{H}}}$ on the symplectic manifold $(N^c, \overline{\Omega}_c)$ coincides with the vector field $\bar{v}_{\mathcal{H}}$.*

Proof. Let $y \in N^c$ and $u \in T_y N^c$. Choose $x \in H_R^c$ and $v \in T_x$ so that $\pi x = y$ and $D\pi v = u$. By the definition of $v_{\mathcal{H}}$ we have

$$\Omega(v, v_{\mathcal{H}}(x)) = d\mathcal{H}(v).$$

By Lemma 2.1 it follows that

$$\Omega(v, v_{\mathcal{H}}(x)) = \overline{\Omega}_c(u, \bar{v}_{\mathcal{H}}(y)).$$

On the other hand, $d\mathcal{H}(v) = d\overline{\mathcal{H}}(u)$. Thus

$$\overline{\Omega}_c(u, \bar{v}_{\mathcal{H}}(y)) = d\overline{\mathcal{H}}(u).$$

But by the definition of $v_{\overline{\mathcal{H}}}$

$$\overline{\Omega}_c(u, v_{\overline{\mathcal{H}}}(y)) = d\overline{\mathcal{H}}(u).$$

Since $\overline{\Omega}_c$ is nondegenerate and u is arbitrary, we have $v_{\overline{\mathcal{H}}} = \bar{v}_{\mathcal{H}}$. The lemma is proved.

We shall denote by μ the measure on M^{2m} (generally σ-finite) induced by the volume element Ω^m. If all manifolds $H^{-1}(c)$ are compact, then μ generates a family of normalized conditional measures μ_c on these manifolds. We denote by $\bar{\mu}_c$ the measure on N^c induced by the volume element $(\overline{\Omega}_c)^{m-1}$.

Lemma 2.3. $\pi_* \mu_c = \lambda_c \bar{\mu}_c$, *where λ_c is a positive constant.*

Proof. If $[H, \mathcal{H}] = 0$, then the vector field $v_{\mathcal{H}}$ preserves the norm μ_c, and hence $\bar{v}_{\mathcal{H}}$ preserves $\pi_* \mu_c$. Since, in view of Lemma 2.2, $v_{\mathcal{H}} = \bar{v}_{\overline{\mathcal{H}}}$, we conclude that $v_{\overline{\mathcal{H}}}$ preserves $\pi_* \mu_c$.

However, the symplectic manifold $(N^c, \overline{\Omega}_c)$ has a unique (up to a positive constant) smooth measure which is invariant under any Hamiltonian vector field, namely the measure $\bar{\mu}_c$ induced by the volume element $(\bar{\Omega}_c)^{m-1}$. This can be easily proven using, for instance, the Darboux theorem ([11], Chapter III, Theorem 6.2), which allows us to introduce in the neighborhood of any point $y \in N^c$ local coordinates $p_1, \ldots, p_{m-1}, q_1, \ldots, q_{m-1}$ such that $\bar{\Omega}_c = \Sigma_1^m dp_i \wedge dq_i$.

§ 3. The basic lemma

1. **Basic Lemma.** *Let A_i and B_i, $i = 1, \ldots, r$, be compact subsets of a symplectic manifold (M^{2m}, Ω) with $\mu(A_i) = \mu(B_i)$ and $A_i \cap A_j = B_i \cap B_j = \varnothing$, $i \neq j$.*

Then for any $\epsilon > 0$ and any open connected set $U \supset \mathfrak{A} = \bigcup_1^r (A_i \cup B_i)$ there exists a canonical diffeomorphism $S\colon M^{2m} \to M^{2m}$ of class C^∞ with the following properties:

O. 1. $\mu(SA_i \triangle B_i) \leqslant \varepsilon, \quad i = 1, \ldots, r.$

O. 2. $S = S_1^{\mathcal{H}^k} \circ \ldots \circ S_1^{\mathcal{H}^1}$, where each Hamilton function $\mathcal{H}^l$, $l = 1, \ldots, k$, is of class C^∞ and zero outside of the set U. From O.2 it obviously follows that S is the identity outside U.

Remark. In order that a diffeomorphism satisfying $O.1$ and $O.2$ with $\epsilon = 0$ exist, first of all some topological conditions on A_i and B_i, which our formulation lacks, are necessary. But even when $r = 1$ and the sets $A_1 = A$ and $B_1 = B$ are diffeomorphic to a closed disc such a diffeomorphism, as a rule, does not exist. The reason is that on the boundary ∂A a smooth one-dimensional distribution (a field of lines) ξ_A is defined which is invariant with respect to canonical transformations (if $T\colon M^{2m} \to M^{2m}$ is a canonical transformation and $TA = B$, then $DT\xi_A = \xi_B$). Namely, the tangent vector $v \in \xi_A$ if $v \lrcorner (\Omega/\partial A) = 0$. For various A the foliations induced by the distributions ξ_A are, as a rule, not even topologically equivalent. It is possible, however, that for two arbitrary subsets $A, B \subset M^{2m}$ diffeomorphic to an open $2m$-disc which have compact closures and the same measure one can construct a canonical diffeomorphism taking A to B or, at least, to B' such that $\mu(B \triangle B') = 0$ and which is, generally, not extendible to the boundary, and a fortiori to a neighborhood of the set A.

2. The proof of the basic lemma consists of two parts. The first and essential part is the proof of a "local" variant of that lemma, i.e. a special case when $M^{2m} = \mathbf{R}^{2m}$, $p_1, \ldots, p_m, q_1, \ldots, q_m$ are the cartesian coordinates in $\mathbf{R}^m$ and $\Omega = \Sigma_1^m dp_i \wedge dq_i$.

We denote by α^s the standard decomposition of $\mathbf{R}^{2m}$ into cubes $\Delta_{l_1, \ldots, l_{2m}}^{k_1, \ldots, k_{2m}}$ with side 2^{-s}. Here $k_i \in \mathbf{Z}$, $l_i = 0, 1, \ldots, 2^s - 1$ and $i = 1, \ldots, 2m$; moreover,

$$\Delta_{l_1, \ldots, l_{2m}}^{k_1, \ldots, k_{2m}} = \Big\{ (p_1, \ldots, p_m, q_1, \ldots, q_m)\colon\ k_i + \frac{l_i}{2^s} \leqslant p_i \leqslant k_i + \frac{l_i + 1}{2^s},$$
$$k_{i+m} + \frac{l_{i+m}}{2^s} \leqslant q_i \leqslant k_{i+m} + \frac{l_{i+m} + 1}{2^s},\ i = 1, \ldots, m \Big\}.$$

In what follows the elements of the decomposition α^s will be called s-cubes.

The θ-kernel of an s-cube Δ is defined to be the closed cube Δ^θ with side $2^{-s}-\theta$, homothetic and concentric with the cube Δ. Let s be so large that, first of all, the $(4m\cdot 2^{-s})$-neighborhood of $\mathfrak{A}$ is contained in U, and, secondly, there exist sets A'_i and B'_i $(i=1,\cdots,r)$ consisting of s-cubes and having the following properties:

1) $A'_i \cap A'_j = B'_i \cap B'_j = \varnothing$, $i \neq j$.

2) For every $i=1,\cdots,r$ the sets A'_i and B'_i consist of the same number of s-cubes.

3) $\mu(A_i \triangle A'_i) < \frac{\varepsilon}{4}$, $\mu(B_i \triangle B'_i) < \frac{\varepsilon}{4}$.

Denote the set of all s-cubes which make up $\bigcup_1^r (A'_i \cup B'_i)$ by J. Let σ be a permutation of the elements of J which takes the s-cubes in A'_i into the s-cubes in B'_i, $i=1,\cdots,r$. We represent σ as a composition of transpositions $\sigma=\sigma_1\cdots\sigma_k$.(2)

The construction of the desired diffeomorphism S rests on the following lemma.

Lemma 3.1. *Let $\Delta, \bar{\Delta} \in J$. For any $\theta > 0$ there exists a C^∞-function $\mathcal{H}_\theta$ on $\mathbf{R}^{2m}$, zero outside of U and on the θ-kernels of all s-cubes of the system J save Δ and $\bar{\Delta}$, such that*

$$S_1^{\mathcal{H}_\theta}\Delta^\theta = \bar{\Delta}^\theta, \quad S_1^{\mathcal{H}_\theta}\bar{\Delta}^\theta = \Delta^\theta.$$

Before we prove this lemma we shall show how to prove the special case of the basic lemma with its aid.

Let the transposition σ_l interchange the cubes $\Delta_{l,1}$ and $\Delta_{l,2}$. Applying Lemma 3.1 to the pair $\Delta_{l,1}$, $\Delta_{l,2}$, we construct a Hamilton function $\mathcal{H}^l_\theta = \mathcal{H}^l$. If the number θ is sufficiently small, then the diffeomorphism

$$S = S_1^{\mathcal{H}^k} \circ \ldots \circ S_1^{\mathcal{H}^1}$$

has the properties O.1 and O.2. Indeed, O.2 is an immediate consequence of Lemma 3.1. In order to check O.1, note that $S\Delta^\theta = (\sigma\Delta)^\theta$ for $\Delta \in J$.

We introduce the notation

$$A_i^\theta = \bigcup_{\Delta \subset A'_i} \Delta^\theta, \quad B_i^\theta = \bigcup_{\Delta \subset B'_i} \Delta^\theta.$$

Then $SA_i^\theta = B_i^\theta \subset B'_i$, and so

(2) We write the composition of transpositions from left to right, as it is usually done, while the composition of transformations we always write from right to left.

$$\mu(SA_i \triangle B_i) \leqslant \mu(SA_i \triangle SA_i^\theta)$$
$$+\mu(B_i \triangle B_i^\theta) = \mu(A_i \triangle A_i^\theta) + \mu(B_i \triangle B_i^\theta)$$
$$\leqslant \mu(A_i \triangle A_i') + \mu(A_i' \setminus A_i^\theta) + \mu(B_i \triangle B_i') + \mu(B_i' \setminus B_i^\theta)$$
$$\leqslant \frac{\varepsilon}{2} + 2\,(2^{-2ms} - (2^{-s} - \theta)^{2m}) \cdot N_i$$

(N_i is the number of s-cubes in A_i').

If θ is sufficiently small, the second summand in the last expression is also smaller than $\epsilon/2$.

3. **Proof of Lemma 3.1.** Let Δ and $\bar{\Delta}$ be two s-cubes with a common $(2m-1)$-face. We shall prove Lemma 3.1 for this case, having changed U to $\mathrm{Int}(\Delta \cup \bar{\Delta})$.

We begin with the following standard diffeomorphism T_ω taking the rectangle

$$\Pi = \left\{(p, q) : |p| \leqslant \frac{\delta}{2}, |q| \leqslant \delta\right\}$$

in $\mathbf{R}^2$ into the semicircle

$$T_\omega(p, q) = ((\rho'(q))^{-1} \cdot p, \rho(q)), \tag{3.1}$$

$$\rho(-q) = -\rho(q), \quad \rho > 0 \text{ for } q > 0, \quad \rho^2 + \frac{\delta^2}{4(\rho')^2} = \left(\sqrt{\frac{2}{\pi}}\delta + \omega\right)^2,$$

where it is assumed that ω is small compared to δ. The relations (3.1) uniquely define a function $\rho(q)$ which for $|q| \leq \delta$ is a monotone increasing real-analytic function. This implies that T_ω is a diffeomorphism. Clearly

$$T_\omega \Pi = \left\{(p, q) : |q| \leqslant \rho(\delta), p^2 + q^2 \leqslant \left(\sqrt{\frac{2}{\pi}}\delta + \omega\right)^2\right\}.$$

From the definition of T_ω it follows that it preserves Lebesgue measure, since the Jacobian $J(T_\omega) \equiv 1$. Moreover, T_ω commutes with the symmetry I with respect to the origin, i.e.

$$T_\omega (I/\Pi)\, T_\omega^{-1} = I/T_\omega \Pi. \tag{3.2}$$

Fix a nonincreasing infinitely differentiable function $\phi(t)$, $t \geq 0$, such that

$$\varphi(t) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant (\rho(\delta) - \omega)^2, \\ 0 & \text{for } t \geqslant \rho(\delta)^2. \end{cases}$$

Set

$$H(p, q) = \frac{\pi}{2}\varphi(p^2 + q^2)(p^2 + q^2)$$

and

$$H'(p,q)=\begin{cases} H(T_\omega(p,q)), & \text{if} \quad (p,q)\in\Pi, \\ 0, & \text{if} \quad (p,q)\notin\Pi. \end{cases}$$

By (3.2) the diffeomorphism $S_1^{H'}$ coincides with the symmetry I on the set $T_\omega^{-1}B_{\rho(\delta)-\omega}$, where B_t is the circle of radius t with center at the origin. It is easy to check that, for any $\theta>0$, ω can be so chosen that $T_\omega^{-1}B_{\rho(\delta)-\omega}$ contains the rectangle

$$\Pi_\theta=\left\{(p,q):\ |p|\leqslant\frac{\delta-\theta}{2},\ |q|\leqslant\delta-\frac{\theta}{2}\right\}.$$

The idea is that $\lim_{\omega\to 0}\rho(\delta)=\sqrt{2/\pi}\,\delta$, the limit of the sets $T_\omega\Pi$ and $B_{\rho(\delta)-\omega}$ is the circle $B_{\sqrt{2/\pi}\,\delta}$ and the limit of $T_\omega\Pi_\theta$ is contained in a circle of smaller radius.

4. Let us return to the cubes Δ and $\bar\Delta$. Making a simple canonical transformation (interchange of coordinates and parallel translation) we take them into cubes whose common $(2m-1)$-face lies in the hyperplane $q_1=0$ and the origin is the center of this face. Set $2^{-s}=\delta$. If the coordinates are changed as was indicated, then

$$\Delta=\Big\{(p_1,\ \dots,p_m,q_1,\ \dots,q_m):|p_i|\leqslant\frac{\delta}{2},\ i=1,\ \dots,m,$$
$$-\delta\leqslant q_1\leqslant 0,\ |q_i|\leqslant\frac{\delta}{2},\ i=2,\ \dots,m\Big\},$$

$$\bar\Delta=\Big\{(p_1,\ \dots,p_m,q_1,\ \dots,q_m):|p_i|\leqslant\frac{\delta}{2},\ i=1,\ \dots,m,$$
$$0\leqslant q_1\leqslant\delta,\ |q_i|\leqslant\frac{\delta}{2};\ i=2,\ \dots,m\Big\}.$$

We introduce another C^∞ function $\psi(t)$ with the property

$$\psi(t)=\begin{cases} 0 & \text{for} \quad |t|>\dfrac{\delta}{2}, \\ 1 & \text{for} \quad |t|\leqslant\dfrac{\delta-\theta}{2}. \end{cases}$$

Set

$$\mathcal{H}(p_1,\ \dots,p_m,q_1,\ \dots,q_m)=H'(p_1,q_1)\psi(p_2)\dots\psi(p_m)\psi(q_2)\dots\psi(q_m).$$

Clearly $\mathcal{H}=0$ outside of $\operatorname{Int}(\Delta\cup\bar\Delta)$. Let

$$\Gamma=\Big\{(p_1,\ \dots,p_m,q_1,\ \dots,q_m):\ |p_i|\leqslant\frac{\delta-\theta}{2},\ i=1,\ \dots,m,$$
$$|q_1|\leqslant\delta-\frac{\theta}{2},\ |q_i|\leqslant\frac{\delta-\theta}{2},\ i=2,\ \dots,m\Big\}.$$

Note that on the set Γ

$$\mathcal{H}(p_1, \ldots, p_m, q_1, \ldots, q_m) = H'(p_1, q_1).$$

Thus on Γ the flow $\{S_t^{\mathcal{H}}\}$ preserves the coordinates $p_2, \ldots, p_m, q_2, \ldots, q_m$, and the coordinates p_1 and q_1 transform as under the flow $\{S_t^{H'}\}$ on $\mathbb{R}^2$. In particular, the diffeomorphism $S_1^{\mathcal{H}}$ coincides on Γ with the reflection in the $(2m-2)$-dimensional hyperplane $p_1 = q_1 = 0$. Since $\Delta^\theta \cup \overline{\Delta}^\theta \subset \Gamma$, it follows that

$$S_1^{\mathcal{H}} \Delta^\theta = \overline{\Delta}^\theta, \quad S_1^{\mathcal{H}} \overline{\Delta}^\theta = \Delta^\theta.$$

5. We now pass to the general situation described by Lemma 3.1. We construct a sequence $\Delta_0 = \Delta, \Delta_1, \ldots, \Delta_k = \overline{\Delta}$ of mutually distinct s-cubes contained in the set U so that any two consecutive cubes have a common $(2m-2)$-face. This can be done because of the choice of δ (generally speaking, not all of these cubes belong to the system J). Applying the construction of subsections 3 and 4 to each pair of consecutive cubes, we obtain functions $\mathcal{H}_0, \ldots, \mathcal{H}_{k-1}$. The function $\mathcal{H}_l$ is constant outside $\operatorname{Int}(\Delta_l \cup \Delta_{l+1})$, and

$$S_1^{\mathcal{H}_l} \Delta_l^\theta = \Delta_{l+1}^\theta, \quad S_1^{\mathcal{H}_l} \Delta_{l+1}^\theta = \Delta_l^\theta. \tag{3.3}$$

Set

$$\hat{S} = S_1^{\mathcal{H}_{k-2}} \circ \ldots \circ S_1^{\mathcal{H}_0}, \quad \mathcal{H}_\theta = \mathcal{H}_{k-1} \circ \hat{S}. \tag{3.4}$$

We shall prove that $\mathcal{H}_\theta$ satisfies the conditions of Lemma 3.1. Indeed, since the diffeomorphism $\hat{S}$ is the identity and the function $\mathcal{H}_{k-1}$ is zero outside of U, we have $\mathcal{H}_\theta = 0$ outside of U. Furthermore

$$\hat{S}\Delta_k^\theta = \Delta_k^\theta, \ \hat{S}\Delta_{k-1}^\theta = \Delta_0^\theta, \ \hat{S}\Delta_l^\theta = \Delta_{l+1}^\theta, \ l = 0, \ldots, k-2;$$

and on the θ-kernels of the remaining s-cubes of the system J the diffeomorphism $\hat{S}$ is the identity. Since $\mathcal{H}_{k-1} = 0$ outside $\Delta_{k-1} \cup \Delta_k$, we conclude that $\mathcal{H}_\theta = 0$ on the θ-kernels of all cubes in J save Δ and $\overline{\Delta}$.

Finally, since

$$S_1^{\mathcal{H}_\theta} = \hat{S}^{-1} S_1^{\mathcal{H}_{k-1}} \hat{S}, \tag{3.5}$$

substituting into (3.5) the formula (3.4) for $\hat{S}$ and using (3.3) we find that $S_1^{\mathcal{H}_\theta} \Delta^\theta = \overline{\Delta}^\theta$ and $S_1^{\mathcal{H}_\theta} \overline{\Delta}^\theta = \Delta^\theta$. Lemma 3.1 follows.

6. **Proof of the basic lemma in the general case.** Let us fix some Riemannian metric on the manifold M^{2m}. Let K be a connected compact set $U \supset K \supset \mathfrak{A}$. In view of the theorem of Darboux, in the neighborhood of any point $x \in M^{2m}$ one can introduce coordinates $p_1, \ldots, p_m, q_1, \ldots, q_m$ so that the form Ω becomes $\Omega = \Sigma_1^m dp_i \wedge dq_i$. We fix a finite system $\mathfrak{A}$ of connected neighborhoods which have the described prop-

erty and which cover K. Let the number $d > 0$ be sufficiently small so that a $3d$-neighborhood (with respect to a given Riemannian metric) of any point in K is contained in U and, moreover, is contained in some neighborhood of the system $\mathfrak{A}$.

Let $\mathfrak{C} = \{C_1, \cdots, C_N\}$ be a system of mutually disjoint closed subsets of the compact set K. We shall call any set which is a union of elements of $\mathfrak{C}$ a *$\mathfrak{C}$-set*. We construct a system $\mathfrak{C}$ with the following properties:

1. $\mu(C_1) = \cdots = \mu(C_N)$ (recall that the measure μ is induced by the symplectic structure and not by the Riemannian metric).

2. $\operatorname{diam} C_i < d$, $i = 1, \ldots, N$.

3. There are mutually disjoint $\mathfrak{C}$-sets $A'_1, \cdots, A'_r$ and mutually disjoint $\mathfrak{C}$-sets $B'_1, \cdots, B'_r$ such that $\mu(A'_i) = \mu(B'_i)$ and

$$\mu(A_i \triangle A'_i) < \frac{\varepsilon}{4}, \quad \mu(B_i \triangle B'_i) < \frac{\varepsilon}{4}$$
$$(i = 1, \ldots, r).$$

4. $d/2$-neighborhoods of the sets C_i, $i = 1, \cdots, N$, cover K.

We omit the simple, though laborious, proof of the existence of such a system $\mathfrak{C}$.

To complete the proof of the basic lemma it suffices to construct for an arbitrary permutation σ of the set $\{1, \cdots, N\}$ and any $\theta > 0$ a diffeomorphism

$$S = S_1^{\mathcal{H}^k} \circ \ldots \circ S_1^{\mathcal{H}^1}$$

such that $\mu(SC_i \Delta C_{\sigma(i)}) < \theta$, $i = 1, \cdots, N$, and the functions $\mathcal{H}^1, \cdots, \mathcal{H}^k$ are zero outside of U. Indeed, we get the statement of the basic lemma by taking for σ any permutation for which

$$A'_i = \bigcup_{j \in N_i} C_j, \quad B'_i = \bigcup_{j \in N_i} C_{\sigma(j)}, \quad i = 1, \ldots, r,$$

and for θ the number $\epsilon/4$.

We shall call a permutation admissible if such a diffeomorphism can be constructed for any $\theta > 0$. The composition of two admissible permutations is clearly an admissible permutation. It is therefore sufficient to prove that all transpositions are admissible.

If the sets C_i and C_j are contained in some neighborhood a of the system $\mathfrak{A}$, then the transposition (i, j) is admissible. Indeed, for any $\theta > 0$ a connected open neighborhood $U_{ij} \supset (C_i \cup C_j)$ such that the closure of U_{ij} is contained in a and $\mu(U_{ij} \cap C_k) < \theta$ for $k \neq i, j$ can be constructed. Applying to the sets C_i, C_j and the neighborhood U_{ij} the local version of the basic lemma proved in subsections 2–5, we find that (i, j) is an admissible transposition.

Now let C_i and C_j be arbitrary sets of the system $\mathfrak{C}$. By condition 4 the $d/2$-

neighborhoods of the sets C_i, $i = 1, \ldots, N$, form a covering of $\mathcal{K}$. Since $\mathcal{K}$ is connected, one can find pairwise distinct numbers $i_0 = i, i_1, \ldots, i_t = j$ such that for $l = 0, \ldots, t-1$ the $d/2$-neighborhoods of the sets C_{i_l} and $C_{i_{l+1}}$ intersect. But then the sets C_{i_l} and $C_{i_{l+1}}$ are contained in a ball of radius $3d$, and so in some neighborhood $a \in \mathfrak{A}$, i.e. the transposition (i_l, i_{l+1}) is admissible.

Since $(i, j) = (i, i_1)(i_1, i_2) \cdots (i_{t-1}, j)(i_{t-1}, i_{t-2}) \cdots (i_1, i)$, the transposition (i, j) is also admissible. This completes the proof of the basic lemma.

§ 4. Proof of Theorem A

1. **The construction.** The function $\mathcal{H}$ satisfying conditions A.1–A.6 will be constructed as the limit of a sequence of homogeneous C^∞ functions $\mathcal{H}^{(n)}$ convergent in the C^∞-topology.

Without loss of generality one can assume that $\alpha_0 \cdot \beta_0^{-1} = r_0$ is a rational number, since otherwise one can replace the vector (α_0, β_0) in the hypothesis of Theorem A by a sufficiently close vector (α_0', β_0') for which $\alpha_0'(\beta_0')^{-1}$ is a rational number.

We set

$$\mathcal{H}^{(0)} = H_{\alpha_0, \beta_0} \tag{4.1}$$

and define the function $\mathcal{H}^{(n)}$ by the recurrence relation

$$\mathcal{H}^{(n)} = \mathcal{H}^{(n-1)} + H_{\delta_n \beta_0, 0} \circ K_n, \tag{4.2}$$

where δ_n is a rational number and $K_n \colon M^{2m} \to M^{2m}$ is a canonical diffeomorphism with

$$K_n = L_n \circ K_{n-1}. \tag{4.3}$$

Thus if the function $\mathcal{H}^{(n-1)}$ is already given it suffices to construct L_n and indicate δ_n to obtain $\mathcal{H}^{(n)}$. The choice of δ_n is made after L_n is constructed. Moreover, the number δ_n will satisfy some arithmetic condition (cf. (4.11) below) which can be satisfied by arbitrarily small numbers. Choosing these numbers δ_n sufficiently small, one can achieve the convergence of $\mathcal{H}^{(n)}$ in the C^∞ topology and the validity of the condition A.1 for each function $\mathcal{H}^{(n)}$, and hence for the limit function $\mathcal{H}$.

The following property is essential for our constructions.

K. *The diffeomorphism L_m commutes with the vector fields $v_{\mathcal{H}^{(n-1)}}$ and u.*

From the fact that L_n and $v_{\mathcal{H}^{(n-1)}}$ commute it follows that

$$\mathcal{H}^{(n)} = H_{\alpha_n, \beta_0} \circ K_n, \tag{4.4}$$

where

$$\alpha_n = \beta_0 \left(r_0 + \sum_{i=1}^{n} \delta_i \right) = \beta_0 r_n. \tag{4.5}$$

Since it is assumed that $\mathcal{H}^{(n-1)}$ is homogeneous and L_n commutes with u, it is clear that $\mathcal{H}^{(n)}$ is homogeneous, and so condition A.3 is also satisfied.

The next condition B together with what was already said assures us that condition A.2 will be satisfied with $\alpha = \lim_{n\to\infty} \alpha_n$ and $\beta = \beta_0$.

B. *The diffeomorphism L_n is the identity in some neighborhood of the set $\partial M^{2m} \cup M_D$.*

Indeed, at the points of $\partial M^{2m} \cup M_D$ the function $\mathcal{H}^{(n)}$ together with its differentials of all orders coincides with $H_{\alpha_n,\beta}$. Since the sequence is convergent in the C^∞ topology, the function $\mathcal{H}$ with its differentials of all orders coincides with $H_{\alpha,\beta}$ at the points of $\partial M^{2m} \cup M_D$.

We shall write for short $\{S_t^{H_{\alpha,\beta}}\} = \{S_t^{\alpha,\beta}\}$. Since $\alpha_n^{-1}\beta$ is rational, $\{S_t^{\alpha_n,\beta}\}$ is a periodic flow. Denote by t_n the largest period of the trajectories of this flow.(3) By (4.4), $S_{t_n}^{\mathcal{H}^{(n)}}$ is the identity map. If the convergence of $\mathcal{H}^{(n)}$ to $\mathcal{H}$ is sufficiently fast (which can be achieved by choosing each time the δ_n to be sufficiently small), then condition A.5 is satisfied.

Condition A.6 can also be satisfied by smallness of δ_n. This is related to the fact that the minimal period of the trajectories of the flow $\{S_t^{\alpha_n,\beta}\}$, and so of $\{S_t^{\mathcal{H}^{(n)}}\}$, which lie outside of the set M_D tends to infinity as $n \to \infty$. We shall see the details in subsection 2.

Thus the formulated conditions, namely (4.1)–(4.3), K, B and smallness of δ_n, assure the convergence of the sequence $\mathcal{H}^{(n)}$ so that the limit function $\mathcal{H}$ satisfies the conditions A.1–A.3, A.5 and A.6. After some preliminary observations we shall go on to construct the diffeomorphism L_n assuming that the function $\mathcal{H}^{(n-1)}$ is already constructed. We shall use only those properties of $\mathcal{H}^{(n-1)}$ which follow from the inductive hypotheses. The proof will be completed by checking the condition A.4, for which, in view of homogeneity of the function $\mathcal{H}$, it will suffice to prove that the flow $\{S_t^{\mathcal{H}}\}$ is ergodic on the manifold $\mathcal{H}^{-1}(1)$ (see the corollary to Lemma 1.1).

2. **Preliminary remarks and notation.** If the homogeneous functions R_1 and R_2 are positive and have no critical points, then there is a standard diffeomorphism between $R_1^{-1}(1)$ and $R_2^{-1}(1)$ which takes the point $x \in R_1^{-1}(1)$ to the unique point in $R_2^{-1}(1)$ which lies on the trajectory of the vector field u passing through the point x. In what follows we shall, without special comment, identify manifolds of the form $R^{-1}(1)$ using this diffeomorphism, and we shall consider any function, measure, vector field, diffeomorphism, flow, differential form, etc., given on one such manifold to be automatically given on all others. Where no confusion can result we shall use the same symbols to denote objects related by the standard diffeomorphism given on various manifolds $R^{-1}(1)$.

We introduce some notation. Let $H_{\alpha,\beta}^{-1}(1) = H^{\alpha,\beta}$. If the number $\alpha\beta^{-1}$ is rational

(3) Let $r_n = p_n/q_n$, where p_n and q_n are relatively prime natural numbers. Then $t_n = q_n \cdot \beta^{-1}$.

we denote by $H_R^{\alpha,\beta}$ the set of regular points of the periodic flow $\{S_t^{\alpha,\beta}\}$, by $N^{\alpha,\beta}$ the manifold of regular trajectories of this flow and by $\pi_{\alpha,\beta}$ the natural projection $\pi_{\alpha,\beta}$: $H_R^{\alpha,\beta} \to N^{\alpha,\beta}$. Furthermore, let $\mathcal{H}_R^{(n-1)}$ be the set of regular points of the flow $\{S_t^{\mathcal{H}^{(n-1)}}\}$ on the manifold $(\mathcal{H}^{(n-1)})^{-1}(1)$, let $\mathfrak{N}^{(n-1)}$ be the corresponding manifold of regular trajectories and let $\pi_{(n-1)}$: $\mathcal{H}_R^{(n-1)} \to \mathfrak{N}^{(n-1)}$ be the projection. We shall deal with manifolds and transformations the relations between which can be conveniently given by the following commutative diagram:

$$\begin{array}{ccccccc} M^{2m} \supset & H^{\alpha_{n-1},\beta} & \supset H_R^{\alpha_{n-1},\beta} & \xrightarrow{\pi_{\alpha_{n-1},\beta}} & N^{\alpha_{n-1},\beta} \\ \uparrow K_{n-1} & \uparrow K_{n-1} & \uparrow K_{n-1} & & \uparrow \bar{K}_{n-1} \\ M^{2m} \supset & (\mathcal{H}^{(n-1)-1})(1) & \supset \mathcal{H}_R^{(n-1)} & \xrightarrow{\pi_{(n-1)}} & \mathfrak{N}^{(n-1)} \end{array}$$

where the symbol $\supset$ denotes natural inclusions and the diffeomorphism $\bar{K}_{n-1}$ is defined so that the diagram will commute. This is well defined in view of (4.4).

Lemma 4.1. *For the action of the torus* T^2 *on a compact manifold* M *there are only a finite number of distinct stationary subgroups* G_x *for the distinct points* $x \in M$.

Proof. Stationary subgroups of points near a given point x must be contained in a small neighborhood of the subgroup G_x. However, for any subgroup $G \subset \mathrm{T}^2$ there is a neighborhood $U_G \supset G$ such that any subgroup $G' \subset U_G$ is contained in G. Thus for any point $x \in M$ there is a neighborhood U_x such that $x' \in U_x$ implies $G_{x'} \subset G_x$. Consider now the action of G_x in the neighborhood of a stationary point x. By a theorem of Bochner ([16], § 5.2, Theorem 1), in some neighborhood U'_x of x there are coordinates in which the group G_x acts via linear transformations. However, for linear actions the stationary subgroup can take only a finite number of values. Choosing from the covering of M by the sets $U_x \cap U'_x$ a finite one, the proposition follows.

Since the manifold $H^{-1}_{\alpha_0,\beta_0}(1)$ is compact, Lemma 4.1 can be applied to actions of T^2 on it; and since the functions $H_{\alpha,\beta}$ are homogeneous, it can also be applied to actions of T^2 on M^{2m}. By this lemma the number of elements of the stationary subgroup G_x for any $x \in M^{2m} \setminus M_D$ is bounded by a single constant which we denote by P. Thus the minimal period τ_{n-1} of the trajectories of the flow $\{S_t^{\alpha_{n-1},\beta}\}$ outside the set M_D is not smaller than t_{n-1}/P.

Another consequence of Lemma 4.1 is that the periods of all periodic orbits of the action of T^2 are bounded. Thus, choosing the number δ_1 sufficiently small, one can achieve that the set $H_R^{\alpha_1,\beta}$, and hence also the sets $H_R^{\alpha_n,\beta}$ for $n = 2, 3, \cdots$, will be disjoint from M_D. We shall assume that δ_1 is chosen so that this condition is satisfied.

Denote by V^{n-1} the operator of averaging functions along the trajectories of the periodic flow $\{S_t^{\mathcal{H}^{(n-1)}}\}$:

$$(V^{n-1}f)(x) = \frac{1}{t_{n-1}} \int_0^{t_{n-1}} f(S_\tau^{\mathcal{H}^{(n-1)}}(x))\, d\tau.$$

The operator V^{n-1} can be thought of as an operator from the space of functions on $\mathfrak{H}_R^{(n-1)}$ to the space of functions on $\mathfrak{N}^{(n-1)}$.

Fix some Riemannian metric invariant under the diffeomorphisms $\{\phi_t\}$ on M^{2m}. This metric induces a Riemannian metric on any submanifold of M^{2m}, in particular on $\mathfrak{H}_R^{(n-1)}$. Averaging the Riemannian metric on $\mathfrak{H}_R^{(n-1)}$ along the trajectories of the flow $\{S_t^{\mathfrak{H}(n-1)}\}$, we obtain a Riemannian metric on $\mathfrak{N}^{(n-1)}$. The distance on $\mathfrak{N}^{(n-1)}$ defined by this metric will be denoted by ρ_{n-1}.

We fix a countable everywhere dense set $\{f_n\}, n = 1, 2, \cdots,$ in $C^\infty(H^{\alpha_0,\beta_0})$. Recall that this fixes a countable everywhere dense set in every space $C^\infty(R^{-1}(1))$, where R is a positive homogeneous function on M^{2m} with no critical point. Let the constant γ_n be such that $\rho_{n-1}(y_1, y_2) < \gamma_n$ for $y_1, y_2 \in \mathfrak{N}^{(n-1)}$ implies the inequality

$$|(V^{n-1}f_i)(y_1) - (V^{n-1}f_i)(y_2)| < \frac{1}{2^n}$$

for $i = 1, \cdots, n$.

3. **Construction of the diffeomorphism L_n.** When constructing L_n we must take care that conditions K and B are satisfied as well as the condition A.4 which does not follow from the inductive hypotheses formulated in subsection 1. The first half of condition K (commuting of L_n with $v_{\mathfrak{H}(n-1)}$) is equivalent to the requirement that the diffeomorphism L_n take each manifold $(\mathfrak{H}^{(n-1)})^{-1}(c)$ into itself. By Lemma 1.1 the second half of this condition (commuting of L_n with u) will be satisfied if L_n is a composition of a finite number of diffeomorphisms which belong to flows generated by Hamiltonian vector fields with homogeneous Hamilton functions. Thus we will be constructing L_n as a composition of a finite number of diffeomorphisms each of which belongs to a flow generated by a Hamiltonian vector field with a homogeneous Hamilton function and leaves the manifold $(\mathfrak{H}^{(n-1)})^{-1}(1)$ invariant.

Since by (4.5) the number $\alpha_{n-1}\beta^{-1}$ is rational, the function $H_{\alpha_{n-1},\beta}$ generates an effective canonical action of S^1 on M^{2m}, and one can use Lemma 2.1 to construct a closed nondegenerate 2-form of the manifold $N^{\alpha_{n-1},\beta}$ which we denote by $\overline{\Omega}_r$. The normalized measure on $N^{\alpha_{n-1},\beta}$ induced by the volume element $\overline{\Omega}_n^{m-1}$ will be denoted by $\overline{\nu}_{n-1}$.

Next we choose in $N^{\alpha_{n-1},\beta}\setminus\partial N^{\alpha_{n-1},\beta}$ some number (denote this number by k_n) of compact mutually disjoint sets $F_1, \cdots, F_{k_n}$ having the following properties:

$$\bar{\nu}_{n-1}(F_1) = \ \dots\ = \bar{\nu}_{n-1}(F_{k_n}), \tag{4.6}$$

$$\bar{\nu}_{n-1}(F_i) > \frac{2^{n+1}-1}{k_n 2^{n+1}}, \tag{4.7}$$

$$\operatorname{diam} F_i < \frac{\gamma_n}{4}, \quad i = 1, \ \dots, \ k_n. \tag{4.8}$$

(The number γ_n is defined at the end of subsection 2.) Here and in what follows we leave out the index n in the symbols of objects which depend on n (in this case the sets F_i) but occur only in one induction step, so that objects introduced for distinct n will not be used simultaneously.

Properties (4.6)–(4.8) can be satisfied only for sufficiently large k_n; moreover, one can find a number κ_n so that any number greater than κ_n will serve as k_n.

Consider the Hamiltonian vector field $v_{\overline{H}_{1,0}}$ on the symplectic manifold $(N^{\alpha_n-1,\beta}, \overline{\Omega}_n)$. Since by Lemma 2.2 we have $v_{\overline{H}_{1,0}} = \overline{v}_{H_{1,0}}$, this vector field induces on $N^{\alpha_n-1,\beta}$ a periodic flow which for brevity will be denoted by $\{\overline{S}_t^{n-1}\}$. The largest period s_{n-1} of the trajectories of this flow is clearly equal to $(\beta t_{n-1})^{-1}$. We construct a system of fundamental domains Δ_q for the transformations

$$\overline{S}^{n-1}_{\frac{s_{n-1}}{q}}, \quad q = 1, 2, \ldots$$

(cf. Proposition 2.3 of [14]); we set

$$\Delta_{k,q} = \overline{S}_{\frac{ks_{n-1}}{q}} \Delta_q$$

and we choose compact sets $G_i \subset \operatorname{Int} \Delta_{i-1,k_n}$, $i = 1, \ldots, k_n$, such that $\overline{\nu}_{n-1}(G_i) = \overline{\nu}_{n-1}(F_i)$.

Let U_n be an open connected subset of $N^{\alpha_n-1,\beta}$ with

(4.9) $U_n \supset \bigcup_{i=0}^{k_n-1} (F_i \cup G_i)$.

(4.10) The closure of U_n is compact and disjoint from the boundary $\partial N^{\alpha_n-1,\beta}$.

We apply the basic lemma to the systems of sets F_i and G_i, $i = 1, \ldots, k_n$, in the symplectic manifold $(N^{\alpha_n-1,\beta}, \overline{\Omega}_n)$, taking $U = U_n$ and $\epsilon = 1/2^{n+1}k_n$. Denote the constructed Hamilton functions by $\overline{h}^l$, $l = 1, \ldots, k$. We lift each function $\overline{h}^l$ to the manifold $H_R^{\alpha_n-1,\beta}$ by letting $h^l(x) = \overline{h}^l(\pi_{\alpha_n-1,\beta}x)$, and extend the function h^l to $H^{\alpha_n-1,\beta}$ by letting $h^l = 0$ on $H^{\alpha_n-1,\beta} \setminus H_R^{\alpha_n-1,\beta}$. Finally we extend h^l to the whole manifold M^{2m} so as to obtain a homogeneous function. Such an extension is unique and is easily obtained by solving the equation $\mathfrak{L}_u h^l = \lambda h^l$ along the trajectories of the vector field u.

Set

$$L_n = S_1^{h^k} \circ \ldots \circ S_1^{h^1}.$$

It is clear that the diffeomorphism L_n commutes with the vector field u, since the h^l are homogeneous functions. From the construction of h^l it is also clear that $\mathcal{H}^{(n-1)}$ is the first integral of the vector fields v_{h^l}; hence each diffeomorphism $S_1^{h^l}$ leaves the manifold $(\mathcal{H}^{(n-1)})^{-1}(1)$ invariant. Thus condition K is satisfied. Condition B is also satisfied, since by (4.10) and the remark on the choice of δ_1 in subsection 2 each function h^l vanishes on some neighborhood of $M_D \cup \partial M^{2m}$, and hence each diffeomorphism $S_1^{h^l}$ is the identity there.

4. **The choice of δ_n.** Let ρ be the distance on M^{2m} given by the fixed Riemannian metric, and let W_n be a $1/2^n$-neighborhood of M_D in this metric. Let

$$\varepsilon_{n-1} = \min_{x \in M^{2m} \setminus W_n} \min_{1 \leqslant \tau \leqslant \frac{\tau_{n-1}}{2}} \min [1, \rho(x, S_\tau^{\mathcal{H}^{(n-1)}} x)].$$

The number δ_n is chosen so that the following hold:

$$\delta_n = \frac{1}{\beta^2 t_{n-1}^2 k_n l_n} = \frac{s_{n-1}^2}{k_n l_n}, \quad l_n \text{ is a natural number,} \tag{4.11}$$

$$\| \mathcal{H}^{(n)} - \mathcal{H}^{(n-1)} \|_{r+n}^{\mathcal{K}_N} < \frac{\delta \cdot \varepsilon_{n-1}}{2^{n+1} (2^{t_{n-1}^2} + 1)}, \tag{4.12}$$

where $\mathcal{K}_N = H_{\alpha_0, \beta}([N^{-1}, N])$. The condition (4.12) does not contain δ_n explicitly, but it is satisfied when δ_n is sufficiently small. Indeed, consider the expression $\mathcal{H}^{(n)}$ in (4.2), where $\mathcal{H}^{(n-1)}$ and K_n are fixed and δ_n varies. Since the functions $H_{\alpha,\beta}$ are C^∞ in α and β, and K_n is a C^∞ diffeomorphism, it follows that $\mathcal{H}^{(n)}$ tends to $\mathcal{H}^{(n-1)}$ together with the differentials of arbitrary high orders on any compact set as $\delta_n \to 0$. Thus to satisfy condition (4.12) the number l_n in (4.11) must be chosen sufficiently large.

The condition (4.12) for $n = 1, 2, \cdots$ assures uniform convergence of the sequence $\mathcal{H}^{(n)}$ together with its differentials of all orders to a C^∞ function $\mathcal{H}$ on the compact set $\mathcal{K}_N$ as $n \to \infty$. Since the functions $\mathcal{H}^{(n)}$ are homogeneous, this convergence takes place at all points of M^{2m} and is uniform on any compact set. The condition A.1 follows immediately from (4.12). Since for a fixed r the distance between the restrictions of $S_t^{\mathcal{H}}$ and $S_t^{\mathcal{H}^{(n)}}$ to a given compact set $\mathcal{K}$ in the metric of C^r-convergence is estimated by CQ^t, where C and Q depend on $\mathcal{H}$ but not on n, (4.12) implies A.5.

It also follows from (4.12) that the distance between $S_t^{\mathcal{H}}x$ and $S_t^{\mathcal{H}^{(n)}}x$, for $x \in \mathcal{H}^{-1}(1) \setminus W_{n-1}$, $1 \leq t \leq \tau_n/2$, for sufficiently large n, is smaller than $\epsilon_n/4$, and hence the flow $\{S_t^{\mathcal{H}}\}$ can have no periodic trajectories outside of W_n whose period does not exceed $\tau_n/2$. Since $\tau_n \to \infty$ as $n \to \infty$ (cf. subsection 2), condition A.6 is also satisfied.

5. **Proof of ergodicity of the flow $\{S_t^{\mathcal{H}}\}$ on $\mathcal{H}^{-1}(1)$.** First recall that we do not distinguish, and denote by the same symbols, functions, measures, flows, etc. on distinct manifolds of the form $R^{-1}(1)$, where R is a positive homogeneous function without critical points, as long as these objects correspond under the standard diffeomorphisms described in subsection 2. Thus, for instance, we denote by $\{S_t^{\mathcal{H}^{(n)}}\}$ not only the flow on the manifold $(\mathcal{H}^{(n)})^{-1}(1)$ but also the corresponding flow on the manifold

$\mathcal{H}^{-1}(1)$. Let ν be a measure on some manifold $R^{-1}(1)$. For brevity the space $L_2(R^{-1}(1), \nu)$ will be denoted by $L_2(\nu)$. We denote by μ and μ_n the corresponding normalized measures induced by the 2-form Ω on the manifolds $\mathcal{H}^{-1}(1)$ and $(\mathcal{H}^{(n)})^{-1}(1)$ and invariant with respect to the flows $\{S_t^{\mathcal{H}}\}$ and $\{S_t^{\mathcal{H}^{(n)}}\}$. Let g be a function on $R^{-1}(1)$. Set

$$(V_t^n g)(x) = \frac{1}{t}\int_0^t g(S_\tau^{\mathcal{H}^{(n)}} x)\, d\tau,$$

$$(V_t g)(x) = \frac{1}{t}\int_0^t g(S_\tau^{\mathcal{H}} x)\, d\tau.$$

In particular, the operator $V_{t_{n-1}}^{n-1}$ coincides with operator V^{n-1} defined in subsection 2. We shall consider the operators V_t^n and V_t in various function spaces.

Let F be a bounded function on $\mathcal{H}^{-1}(1)$ which is measurable with respect to μ and invariant with respect to the flow $\{S_t^{\mathcal{H}}\}$. Fix $\theta > 0$ and choose a number k so that

$$\| F - f_k \|_{L_2(\mu)} < 0. \tag{4.13}$$

Recall that the functions f_k were fixed at the end of subsection 2.

Since $V_t F = F$ and the norm of V_t in $L_2(\mu)$ is 1, for any t we have

$$\| f_k - V_t f_k \|_{L_2(\mu)} \leqslant \| F - f_k \|_{L_2(\mu)} + \| V_t (F - f_k) \|_{L_2(\mu)} < 20. \tag{4.14}$$

Let g be a differentiable function on $\mathcal{H}^{-1}(1)$. Then

$$\begin{aligned} \| V_{t_{n-1}} g - V^{n-1} g \|_{L_2(\mu)} &\leqslant \max_{x \in \mathcal{H}^{-1}(1)} | V_{t_{n-1}} g(x) - V^{n-1} g(x) | \\ &\leqslant \max_x \| (Dg)_x \| \cdot \max_{0 \leqslant t \leqslant t_{n-1}} [\max_x \rho (S_t^{\mathcal{H}} x, S_t^{\mathcal{H}^{(n-1)}} x)], \end{aligned} \tag{4.15}$$

where the norm of the differential Dg is taken in the same Riemannian metric which induces the metric ρ on $\mathcal{H}^{-1}(1)$. It is understood that, in the spirit of the remark at the beginning of subsection 2, the flow $\{S_t^{\mathcal{H}^{(n-1)}}\}$ and the operator V^{n-1} are translated to $\mathcal{H}^{-1}(1)$.

Since the sequence t_n of periods of the flows $\{S_t^{\mathcal{H}^{(n)}}\}$ is increasing and

$$\rho (S_t^{\mathcal{H}} x, S_t^{\mathcal{H}^{(n-1)}} x) \leqslant \sum_{j=n-1}^{\infty} \rho (S_t^{\mathcal{H}^{(j)}}, S_t^{\mathcal{H}^{(j+1)}}),$$

it follows from (4.12) that the second factor in the right-hand side of (4.15) tends to zero as $n \to \infty$. Therefore one can choose N_1 so that for $n > N_1$

$$\| V_{t_{n-1}} f_k - V^{n-1} f_k \|_{L_2(\mu)} < 0,$$

$$\| V^n_{t_{n-1}} f_k - V^{n-1} f_k \|_{L_2(\mu)} < 0. \tag{4.16}$$

Since the measures μ and μ_{n-1} are equivalent, the spaces $L_2(\mu)$ and $L_2(\mu_{n-1})$ consist of the same functions and differ only in their norms. Set

$$\bar{\mu}_{n-1} = (\pi_{(n-1)})_* \mu_{n-1} = (\overline{K}^{-1}_{n-1})_* \bar{\nu}_{n-1}.$$

The projection $\pi_{(n-1)}\colon \mathcal{H}^{(n-1)}_R \to \mathfrak{N}^{(n-1)}$ induces an isometric embedding

$$\pi^*_{(n-1)}\colon L_2(\mathfrak{N}^{(n-1)}, \bar{\mu}_{n-1}) \subset L_2(\mathcal{H}^{(n-1)}_R, \mu_{n-1}) = L_2(\mu_{n-1}).$$

The last equality follows since

$$\mu_{n-1}((\mathcal{H}^{(n-1)})^{-1}(1) \setminus \mathcal{H}^{(n-1)}_R) = 0.$$

Since the flow $\{S^{\mathcal{H}(n-1)}_t\}$ preserves the measure μ_{n-1}, the operator V^{n-1} on $L_2(\mu_{n-1})$ is an orthogonal projection into the space $\pi^*_{(n-1)} L_2(\mathfrak{N}^{(n-1)}, \bar{\mu}_{n-1})$, which we shall identify with the space $L_2(\mathfrak{N}^{(n-1)}, \bar{\mu}_{n-1})$.

Consider the partition η_n of the manifold $\mathfrak{N}^{(n-1)}$ into sets $\overline{K}^{-1}_n \Delta_{i, s_n^{-1} k_n l_n}$, $i = 0, \ldots, s_n^{-1} k_n l_n - 1$, and denote by P_n the orthogonal projection in $L_2(\mathfrak{N}^{(n-1)}, \bar{\mu}_{n-1})$ onto the subspace E_n of functions which are constant on the elements of this partition. We also denote by P_n the orthogonal projection in $L_2(\mu_{n-1})$ onto the subspace $\pi^*_{(n-1)} E_n$. Let us estimate the norm of the function $((\mathrm{Id} - P_k) V^{n-1}) f_k$ in $L_2(\mathfrak{N}^{(n-1)}, \bar{\mu}_{n-1})$ (Id is the identity operator). For brevity we shall denote the norm in this space by $\| \cdot \|$ without any subscripts. Let

$$\Pi_i = \pi^{-1}_{\alpha_{n-1}, \beta} \Delta_{i, k_n}, \quad i = 0, \ldots, k_n - 1,$$

and let $\hat{P}_n$ be the orthogonal projection in $L_2(\mu_{n-1})$ onto the subspace of functions constant on the sets $K^{-1}_n \Pi_i$, $i = 0, \ldots, k_n - 1$. It is clear that

$$\| (\mathrm{Id} - P_n) f \| \leqslant \| (\mathrm{Id} - \hat{P}_n) f \|.$$

Each set $K^{-1}_n \Pi_i$ contains the subset

$$F^n_{i+1} = K^{-1}_{n-1} (\pi^{-1}_{\alpha_{n-1}, \beta} F_{i+1} \cap L^{-1}_n \Pi_i),$$

and by choice of γ_n (see the end of subsection 2) and (4.8) for $n \geq k$ the variation of the function $V^{n-1} f_k$ on each set F^n_{i+1} is smaller than $1/2^n$. On the other hand, by (4.7) and the choice of ϵ in the application of the basic lemma in the construction of the diffeomorphism L_n we have

$$\mu_{n-1}(K_n^{-1}\Pi_i \setminus F_{i+1}^n) < \frac{\tilde{\mu}_{n-1}(\Delta_{i,k_n})}{2^n} = \frac{1}{k_n 2^n}.$$

Thus

$$\|(\mathrm{Id} - \hat{P}_n) V^{n-1} f_k\| \leqslant \max|f_k| \cdot \Big(\mu_{n-1}\big(\mathcal{H}_R^{(n-1)} \setminus \bigcup_{i=1}^{k_n} F_i^n\big)\Big)^{1/2} + \frac{1}{2^n} \leqslant \max|f_k| \cdot 2^{-n/2} + 2^{-n}.$$

We choose N_2 so that for $n > N_2$ the right-hand side of this inequality will be smaller than θ. Then for $n > N_2$

$$\|(\mathrm{Id} - P_n) V^{n-1} f_k\| < \theta.$$

Set $\lambda_{n-1} = d\mu/d\mu_{n-1}$. Then

$$(\min \lambda_{n-1}) \cdot \|f\|_{L_2(\mu_{n-1})} \leqslant \|f\|_{L_2(\mu)} \leqslant (\max \lambda_{n-1}) \|f\|_{L_2(\mu_{n-1})}.$$

From (4.12) it follows that as $n \to \infty$ the distributions λ_{n-1} tend uniformly to 1. (Recall that the manifold $\mathcal{H}^{-1}(1)$ is compact.) Choose N_3 so that for $n > N_3$ we have $\sqrt{2}/2 < \lambda_{n-1} < \sqrt{2}$, and so

$$\frac{\sqrt{2}}{2}\|f\|_{L_2(\mu_{n-1})} \leqslant \|f\|_{L_2(\mu)} \leqslant \sqrt{2}\,\|f\|_{L_2(\mu_{n-1})}.$$

Set $(U_t^{\mathcal{H}^{(n)}} f)(x) = f(S_t^{\mathcal{H}^{(n)}} x)$. The operator $U_t^{\mathcal{H}^{(n)}}$ is a unitary operator in $L_2(\mu_n)$, since the flow $\{S_t^{\mathcal{H}^{(n)}}\}$ preserves the measure μ_n. Thus for $n \geq N_3$ the norm of $U_t^{\mathcal{H}^{(n)}}$ in $L_2(\mu)$ is smaller than 2.

Note that $t_n = s_{n-1}^{-1} k_n l_n t_{n-1}$. Furthermore,

$$V^n = \frac{s_{n-1}}{k_n l_n} \sum_{l=0}^{s_{n-1}^{-1} k_n l_n - 1} U_{lt_{n-1}}^{\mathcal{H}^{(n)}} \circ V_{t_{n-1}}^n.$$

From (4.16) and (4.19) it follows that for $n > \max(N_1, N_3)$

$$\left\| V^n f_k - \frac{s_{n-1}}{k_n l_n} \sum_{l=0}^{s_n^{-1} k_n l_{n-1}} U^{\mathcal{H}^{(n)}}_{lt_{n-1}} V^{n-1} f_k \right\|_{L_2(\mu)}$$
$$< 2 \left\| \frac{s_{n-1}}{k_n l_n} \sum_{l=0}^{s_{n-1}^{-1} k_n l_{n-1}} U^{\mathcal{H}^{(n)}}_{lt_{n-1}} (V^n_{t_{n-1}} - V^{n-1}) f_k \right\|_{L_2(\mu)} < 2\theta. \tag{4.20}$$

Consider now the function $P_n V^{n-1} f_k \in L_2(\mathfrak{N}^{(n-1)}, \bar{\mu}_{n-1})$ constant on the elements of the partition η_n. This partition is invariant under the diffeomorphism $S^{\bar{\mathcal{H}}^{(n)}}_{t_{n-1}}$, and

$$S^{\overline{\mathcal{H}}^{(n)}}_{t_{n-1}} \overline{K}_n^{-1} \Delta_{i, s_{n-1}^{-1} k_n l_n} = \overline{K}_n^{-1} \Delta_{i+1, s_{n-1}^{-1} k_n l_n},$$

where

$$i = 0, 1, \ldots, s_{n-1}^{-1} k_n l_n - 1 \quad \text{and} \quad \Delta_{s_{n-1}^{-1} k_n l_n, s_{n-1}^{-1} k_n l_n} = \Delta_{0, s_{n-1}^{-1} k_n l_n}.$$

Therefore the function

$$\frac{s_{n-1}}{k_n l_n} \sum_{l=0}^{s_{n-1}^{-1} k_n l_{n-1}} U^{\overline{\mathcal{H}}^{(n)}}_{lt_{n-1}} P_n V^{n-1} f_k$$

is a constant which we denote by c_n.

Now let $n > \max(N_1, N_2, N_3)$. From (4.16), (4.17), (4.18) and (4.20) we obtain

$$\| V^n f_k - c_n \|_{L_2(\mu)} \leqslant \left\| V^n f_k - \frac{s_{n-1}}{k_n l_n} \sum_{l=0}^{s_{n-1}^{-1} k_n l_{n-1}} U^{\mathcal{H}^{(n)}}_{lt_n} V^{n-1} f_k \right\|_{L_2(\mu)}$$
$$+ \left\| \frac{s_{n-1}}{k_n l_n} \sum_{l=0}^{s_{n-1}^{-1} k_n l_{n-1}} U^{\overline{\mathcal{H}}^{(n)}}_{lt_n} (\mathrm{Id} - P_n) V^{n-1} f_k \right\|_{L_2(\mu)}$$
$$< 2\theta + 4 \| (\mathrm{Id} - P_n) V^{n-1} f_k \|_{L_2(\mu)} < 8\theta. \tag{4.21}$$

Finally, using (4.13), (4.14), (4.16) and (4.21), we get

$$\| F - c_n \|_{L_2(\mu)} \leqslant \| F - f_k \|_{L_2(\mu)} + \| f_k - V_{t_n} f_k \|_{L_2(\mu)}$$
$$+ \| V_{t_n} f_k - V^n f_k \|_{L_2(\mu)} + \| V^n f_k - c_n \|_{L_2(\mu)} < 12\theta.$$

Since the number θ can be chosen arbitrarily small, the function F is a constant. Thus the ergodicity of $\{S_t^{\mathcal{H}}\}$ on $\mathcal{H}^{-1}(1)$ is proved, and with it Theorem A.

§ 5. Weakly coupled oscillators

1. We turn our attention to Example 1 described in § 1. If all the numbers $\alpha_1, \ldots, \alpha_m$ are positive, then $\pi \Sigma_1^m \alpha_i (p_i^2 + q_i^2) = H_{\alpha_1, \ldots, \alpha_m}(p, q)$ is a positive function

with no critical points on $\mathbf{R}^{2m}\setminus\{0\}$ and the manifolds $H^{-1}_{\alpha_1,\dots,\alpha_m}(1)$ are compact. Now let $\alpha_i > 0$, $i = 1, \dots, m$, and $\alpha_i = \beta r_i$, where the r_i are integers. In the torus $\mathbf{T}^n$ consider the 2-dimensional torus $\{t\alpha_1 + s, t\alpha_2, \dots, t\alpha_n \bmod 1\}$, $t, s \in \mathbf{R}$. The action of this torus on $\mathbf{R}^{2m}\setminus\{0\}$ satisfies conditions (1.9)–(1.11). Theorem A in this situation assures the existence of a homogeneous function of second degree $\mathcal{H}$, arbitrarily close, together with its partial derivatives of degrees up to r, on the sphere $H^{-1}_{1,\dots,1}(1)$ to $H_{\alpha_1,\dots,\alpha_m}$ and generating a flow $\{S_t^{\mathcal{H}}\}$ which is ergodic on each manifold $\mathcal{H}^{-1}(c)$. The function $\mathcal{H}$ may be extended to be continuous at zero by $\mathcal{H}(0) = 0$. This extended function $\mathcal{H}$ is only once differentiable at zero, and so the vector field $v_{\mathcal{H}}$ is only continuous at zero. Replacing the homogeneity of $\mathcal{H}$ by a weaker condition (cf. condition B.3 below), we can remove this defect, and, moreover, we achieve that $\mathcal{H}$ is uniformly close on the whole space $\mathbf{R}^{2m}$ to some function $H_{\alpha_1,\dots,\alpha_m}$. We give a complete formulation of this proposition and indicate the changes necessary in the proof of Theorem A to prove it.

Theorem B. *Let $\alpha_1^0, \dots, \alpha_m^0$ and δ be positive numbers, and let r be a natural number. There exists a vector $(\alpha_1, \dots, \alpha_m)$ and a C^∞ function $\mathcal{H}$ in $\mathbf{R}^{2m}$ such that conditions* A.4–A.6 *and also the following conditions are satisfied*:

B.1. *The function $\mathcal{H} - H_{\alpha_1,\dots,\alpha_m}$ and its partial derivatives of order not exceeding r are bounded by δ on the whole space $\mathbf{R}^{2m}$.*

B.2. $$|\alpha_i - \alpha_i^0| < \delta,\ i = 1, \dots, m.$$

B.3. *For any $c_1, c_2 > 0$ there is a diffeomorphism $\phi_{c_1,c_2}\colon \mathcal{H}^{-1}(c_1) \to \mathcal{H}^{-1}(c_2)$ such that*

$$D\varphi_{c_1,c_2} v_{\mathcal{H}}|_{\mathcal{H}^{-1}(c_1)} = v_{\mathcal{H}}|_{\mathcal{H}^{-1}(c_2)}.$$

2. Let us assume that $\alpha_i^0 = r_i\beta$, $i = 1, \dots, m$, where the r_i are integers and the greatest common divisor of $r_1, \dots, r_m$ is 1. (Otherwise we replace $(\alpha_1^0, \dots, \alpha_m^0)$ by a close vector.) Set

$$\overline{H}_{\alpha,\beta} = H_{\alpha r_1+\beta, \alpha r_2, \dots, \alpha r_m}.$$

As in the proof of Theorem A, we shall inductively construct a sequence $\mathcal{H}^{(n)}$ of functions defined by the following relations:

$$\mathcal{H}^{(0)} = \overline{H}_{0,\beta} = H_{\alpha_1^0,\dots,\alpha_m^0}, \tag{5.1}$$

$$\mathcal{H}^{(n)} = \mathcal{H}^{(n-1)} + \overline{H}_{\delta_n\beta,0} \circ K_n, \tag{5.2}$$

$$K_n = L_n \circ K_{n-1}, \tag{5.3}$$

so that to construct $\mathcal{H}^{(n)}$ one needs to give, besides the function $\mathcal{H}^{(n-1)}$, a canonical diffeomorphism $L_n\colon \mathbf{R}^{2m} \to \mathbf{R}^{2m}$ and a positive number δ_n. The condition K is replaced by the following two conditions:

K.1. *The diffeomorphism* L_n *commutes with the vector field* $v_{\mathcal{H}^{(n-1)}}$.

K.2. *On the set*

$$(\mathcal{H}^{(n-1)})^{-1}([2^{-n}, 2^n])$$

which is invariant with respect to L_n, *the diffeomorphism* L_n *commutes with the vector field u. On the sets*

$$(\mathcal{H}^{(n-1)})^{-1}((0,2^{-n-1}]) \text{ and } (\mathcal{H}^{(n-1)})^{-1}([2^{n+1}, \infty))$$

the diffeomorphism L_n *coincides with the identity.*

Condition K.1 implies (4.4) and (4.5) with $r_0 = 0$.

We wish to construct homogeneous functions $h^l(p, q)$ as described in § 4.3. We set

$$L_n = S_1^{\hat{h}^k} \circ \dots \circ S_1^{\hat{h}^1},$$

where

$$\hat{h}^l = \begin{cases} 0 & \text{for } (p, q) = 0, \\ h^l(p, q) \cdot \rho(\mathcal{H}^{(n-1)}(p, q)), \end{cases}$$

$l = 1, \dots, k$ and $\rho(t)$ is an infinitely differentiable function on the reals such that $\rho(t) = 0$ for $t \in (-\infty, 2^{-n-1}] \cup [2^{n+1}, \infty)$ and $\rho(t) = 1$ for $t \in [2^{-n}, 2^n]$.

The number δ_n is chosen so that (4.11) and the following conditions will be satisfied:

$$\delta_n < \frac{1}{\beta 2^{n+1}}. \tag{5.4}$$

(5.5) The function $\mathcal{H}^{(n)} - \mathcal{H}^{(n-1)}$ and its partial derivatives of all orders not exceeding $r + n$ are bounded on the set $(\mathcal{H}^{(n-1)})^{-1}([2^{-n-1}, 2^{n+1}])$ by the number

$$\frac{\delta\varepsilon_{n-1}}{2^{n+2}(2^{t^2_{n-1}}+1)}.$$

Moreover, the remark about the choice of δ_1 made in § 4.2 still holds. From the conditions (4.11), (5.4), (5.5) for $n = 1, 2, \cdots$ and this remark it follows that the sequence $\mathcal{H}^{(n)}$ converges to a C^∞ function $\mathcal{H}$ and the conditions A.5, A.6, B.1 and B.2 with

$$\alpha_i = \alpha_i^0, \quad i = 2, \dots, m, \quad \alpha_1 = \alpha_1^0 + \beta \sum_{n=1}^{\infty} \delta_n \tag{5.6}$$

are satisfied.

We check condition B.3. Let, for instance, $c_1 < c_2$, and let N be a natural number such that $2^{-N+1} < c_1 < c_2 < 2^{N-1}$. From (5.4) and (5.5) it follows that for $n \geq N$

$$\mathcal{H}^{-1}([c_1, c_2]) \subset (\mathcal{H}^{(n)})^{-1}([2^{-N}, 2^N]).$$

Therefore all diffeomorphisms L_n for $n > N$ are homogeneous in the region $\mathcal{H}^{-1}([c_1, c_2])$, and so the function $\mathcal{H} \circ K_N^{-1}$ is also homogeneous in this region. We set $\tau = \ln(c_2/c_1)$ and $\phi_{c_1,c_2} = K_N^{-1} \circ \phi_\tau \circ K_N$.

Note that the diffeomorphisms ϕ_{c_1,c_2} depend smoothly on c_1 and c_2 and, just as in the homogeneous case, they satisfy the "group" law

$$\varphi_{c_2,c_3} \circ \varphi_{c_1,c_2} = \varphi_{c_1,c_3}.$$

The proof of ergodicity of the flow $\{S_t^{\mathcal{H}}\}$ on the manifold $\mathcal{H}^{-1}(1)$ as given in § 4.5 need not be changed. The ergodicity of $\{S_t^{\mathcal{H}}\}$ on each manifold $\mathcal{H}^{-1}(c)$ follows from B.3. This completes the proof of Theorem B.

3. The function $\mathcal{H}$ constructed in the proof of Theorem B is of the form

$$\mathcal{H} = H_{\alpha_1,\dots,\alpha_m} + \hat{\mathcal{H}}, \tag{5.7}$$

and at the point 0 all the differentials $d\hat{\mathcal{H}}^k(0)$, $k = 0, 1, 2, \dots$, are equal to zero. Thus 0 is a purely elliptical fixed point of the Hamiltonian vector field $v_{\mathcal{H}}$, and the linear part $v_{H_{\alpha_1,\dots,\alpha_m}}$ of this vector field at 0 has the numbers $\{2\pi i\alpha_j\}$, $j = 1, \dots, m$, as spectrum. In view of (5.6) this spectrum is very special; namely, the linear space generated by the numbers $\alpha_1, \dots, \alpha_m$ over the rational numbers has dimension 2.

Our construction can be generalized so that the function $\mathcal{H}$, as before, will satisfy all the assertions of Theorem B and be of the form (5.7) but so that the dimension of the linear space over the rationals generated by $\alpha_1, \dots, \alpha_m$ will be equal to any a priori given number s, $2 \leq s \leq m$. The basic idea is to begin with an action of the s-dimensional torus $\mathbf{T}^s$ and not with $\mathbf{T}^2$. In particular, setting $s = m$, one can achieve that the numbers $\alpha_1, \dots, \alpha_m$ will be rationally independent, as in our construction for $m = 2$. However, even in this case the vector $(\alpha_1, \dots, \alpha_m)$ will not be exceptionally well approximated by vectors $(\alpha_1^n, \dots, \alpha_m^n)$ such that $\alpha_1^n, \dots, \alpha_m^n$ are dependent over the rationals. Thus the following question remains unanswered.

Given a vector $(\alpha_1, \dots, \alpha_m)$ with $\alpha_j > 0$, $j = 1, \dots, m$, does there exist a C^∞ function $\mathcal{H} = H_{\alpha_1,\dots,\alpha_m} + \hat{\mathcal{H}}$ with $d^k\hat{\mathcal{H}}(0) = 0$ for $k = 0, 1, 2$ such that the flow $\{S_t^{\mathcal{H}}\}$ is ergodic on the manifolds $\mathcal{H}^{-1}(c)$ at least for sufficiently small positive c?

4. Another possible modification of our construction is for the case when not all numbers $\alpha_1, \dots, \alpha_m$ in (5.7) are positive, i.e. the manifolds $\mathcal{H}^{-1}(c)$ are not compact. In this case the stability of the fixed point 0 is of interest. It turns out that one can construct a function $\mathcal{H}$ in this case so that 0 is an unstable point and the flow $\{S_t^{\mathcal{H}}\}$ on each submanifold $\mathcal{H}^{-1}(c)$, $c \neq 0$, is ergodic with respect to an infinite invariant measure induced by the Lebesgue measure on $\mathbf{R}^{2m}$. The detailed formula-

tions and proofs in the noncompact case will be given in another paper.

§ 6. Geodesic flows on Finsler manifolds

1. Before applying our results to the situation described in Example 2 we shall recall some known facts about Finsler metrics, geodesic flows and the Legendre transformation. In this connection it is sometimes convenient to use coordinate notation rather than the intrinsic one. Let $\phi: U \to \mathbf{B}^m$ be a coordinate chart, where U is an open subset of the manifold M^m and $\mathbf{B}^m$ is an open m-ball; $(q_1, \cdots, q_m)$ will be the coordinates of a point $x \in M^m$. The differential $D\phi$ is a diffeomorphism between the tangent bundle TU and the product $\mathbf{B}^m \times \mathbf{R}^m$, and allows us to introduce in TU a system of coordinates $(q_1, \cdots, q_m, v_1, \cdots, v_m)$. In other words, $v_1, \cdots, v_m$ are the coefficients of a tangent vector $v \in T_qU$ with respect to the basis $\partial/\partial q_i$ in the tangent space T_qU. Dual local coordinates $(q_1, \cdots, q_m, p_1, \cdots, p_m)$ are determined in the cotangent bundle T^*U. There $p_1, \cdots, p_m$ are the coefficients of a linear form $p \in T^*U$ with respect to the basis dq_i, $i = 1, \cdots, m$. The canonical 2-form Ω on the cotangent bundle T^*M^m can be written as

$$\Omega = \sum_{i=1}^{m} dp_i \wedge dq_i$$

in the coordinates $q_1, \cdots, q_m, p_1, \cdots, p_m$, and is independent of the chart map ϕ (cf. [11], Chapter III, formula (7.7)).

2. Let the manifold M^m be given a Finsler metric σ of class C^∞, i.e. a norm for the tangent vectors $v \in TM^m$, denoted by $\|v\|_\sigma$, is given. This norm is infinitely differentiable on $TM^m \backslash \Gamma_0$, and on each space T_qM^m, $q \in M^m$, it is homogeneous of degree one, convex and symmetric.

Set $L_\sigma(c) = \frac{1}{2}\|v\|_\sigma^2$. The system of Lagrange equations with the Lagrangian (kinetic energy) L_σ in the coordinates $q_1, \cdots, q_m, v_1, \cdots, v_m$ has the form

$$\frac{dq_i}{dt} = v_i, \quad \frac{d}{dt}\left(\frac{\partial L_\sigma}{\partial v_i}\right) = -\frac{\partial L_\sigma}{\partial q_i}. \tag{6.1}$$

The Finsler metric σ is said to be *nondegenerate* if

$$\det\left\| \frac{\partial^2 L_\sigma}{\partial v_i \partial v_j} \right\| \neq 0 \quad \text{for} \quad v \neq 0. \tag{6.2}$$

This condition, which in the calculus of variations is called the strengthened Legendre condition, does not depend on the choice of local coordinates, since the coordinates $v_1, \cdots, v_m$ in each space T_qM^m transform linearly as one goes from one local system to another.

When condition (6.2) is satisfied the Lagrange equations (6.1) can be solved for dv_i/dt, i.e. we can represent them in the form

$$\frac{dq_i}{dt} = v_i, \quad \frac{dv_i}{dt} = F(q, v). \tag{6.3}$$

Indeed, in this case for a fixed $q \in M^m$ the vector $(\partial L_\sigma/\partial v_1, \dots, \partial L_\sigma/\partial v_m)$ depends on $v \in T_q M^m \setminus \{0\}$ in a nondegenerate way, and hence by the implicit function theorem the $v_1, \dots, v_m$ can be locally expressed in terms of $\partial L_\sigma/\partial v_1, \dots, \partial L_\sigma/\partial v_m$. The system (6.3) determines a vector field on $TM^m \setminus \Gamma_0$. The flow induced by this vector field is the geodesic flow of the Finsler metric σ.

3. We define the Legendre transformation $\mathfrak{L}_\sigma \colon TM \to T^*M$ by

$$\mathfrak{L}_\sigma(q, v) = (q, d_v L_\sigma(q, v))$$

or

$$\mathfrak{L}_\sigma(q_1, \dots, q_m, v_1, \dots, v_m) = \left(q_1, \dots, q_m, \frac{\partial L_\sigma}{\partial v_1}, \dots, \frac{\partial L_\sigma}{\partial v_m}\right).$$

We omit the proof of the following, almost obvious proposition.

Proposition 6.1. *If the Finsler metric σ is nondegenerate, then $\mathfrak{L}_\sigma$ is a C^∞ diffeomorphism between $TM^m \setminus \Gamma_0$ and $T^*M^m \setminus \Gamma_0^*$.*

The inverse Legendre transformation is of the form

$$\mathfrak{L}_\sigma^{-1}(q, p) = (q, d_p H_\sigma(q, p))$$

or

$$\mathfrak{L}_\sigma^{-1}(q_1, \dots, q_m, p_1, \dots, p_m) = \left(q_1, \dots, q_m, \frac{\partial H_\sigma}{\partial p_1}, \dots, \frac{\partial H_\sigma}{\partial p_m}\right),$$

where

$$H_\sigma(q, p) = \sum_{i=1}^{m} p_i v_i(q, p) - L_\sigma \circ \mathfrak{L}_\sigma^{-1}(q, p).$$

Since on each space $T_q M^m$ the Lagrangian L_σ is a second degree homogeneous function, we have

$$H_\sigma = L_\sigma \circ \mathfrak{L}_\sigma^{-1}. \tag{6.4}$$

It is easy to show that the function H_σ is half of the square of the norm $\| \ \|_\sigma^*$ on T^*M^m dual to the norm $\| \ \|_\sigma$ on TM^m. By regularity of the inverse Legendre transformation it follows that for $p \neq 0$ the function H_σ also satisfies the strengthened Legendre condition, i.e.

$$\det\left\|\frac{\partial^2 H_\sigma}{\partial p_i\,\partial p_j}\right\| \neq 0. \tag{6.5}$$

Now let H be an arbitrary function on T^*M^m satisfying the following condition.

F. *H is infinitely differentiable on $T^*M^m\setminus\Gamma_0^*$, and on each cotangent space $T_q^*M^m$, $q\in M^m$, the function H is homogeneous of second degree, symmetric and satisfies* (6.5).

Then the transformation

$$\mathfrak{L}_H(q, p) = (q, dH_p(q, p))$$

is a diffeomorphism by Proposition 6.1, and the function $2H\circ\mathfrak{L}_H^{-1}$ determines the square of the norm of some nondegenerate C^∞ Finsler metric σ_H on M^m, and $\mathfrak{L}_H^{-1} = \mathfrak{L}_{\sigma_H}$.

The differential $D\mathfrak{L}_H$ takes the Hamiltonian vector field v_H on T^*M^m into the vector field of the geodesic flow of the metric σ_H. Keeping this in mind we shall directly construct a Hamiltonian system whose Hamiltonian satisfies condition F.

The function L_σ is, by (6.4), the first integral of the geodesic flow; the trajectories of this flow project into geodesic lines of the metric σ by the natural projection TM^m to M^m, and the flow on the hypersurface $L_\sigma^{-1}(c/2)$ can be described as motion with velocity $c^{1/2}$ of the tangent vector along a geodesic line parametrized by the Finsler length.

4. A well-known special case of the described situation occurs when σ is a Riemannian metric on M^m (cf., for example, [11], Chapter IV, § 1). In this case L_σ is a positive definite quadratic form on each space T_qM^m; and condition (6.2), of course, holds. The Legendre transformation $\mathfrak{L}_\sigma$ is linear on each T_qM^m, and the Hamiltonian H_σ as well as L_σ are positive definite quadratic forms. Condition F is satisfied for any infinitely differentiable on $T^*M^m\setminus\Gamma_0^*$ second degree homogeneous function H symmetric on each space $T_q^*M^m$, as long as it is sufficiently close in the C^2 topology to the quadratic form H_σ. Thus the flow $\{S_t^H\}$ on T^*M^m is isomorphic to a geodesic flow of some nondegenerate Finsler metric on M^m.

5. We now turn to Example 2 described in § 1. Recall that we assume that the manifold M^m is given a Riemannian metric ρ which satisfies the following two conditions:

(1.12) All geodesics of this metric are closed and the lengths of all geodesics are divisors of some positive number.

(1.13) On M^m there is an effective action of the group S^1 which consists of diffeomorphisms $\{\psi_t\}$, $t\in\mathbf{R}$, $\psi_1 = \mathrm{Id}$, which preserve the Riemannian metric ρ.

We denote by τ_0 the maximal length of a (closed) geodesic in the metric ρ. It is easy to see that the lengths of all geodesics are divisors of τ_0, i.e. the geodesic flow on the manifold $L_\rho^{-1}(c/2)$ has period $\tau_0 c^{-1/2}$. In view of what was said in subsection 3 the flow $\{S_t^{H_\rho}\}$ on T^*M^m, where $H_\rho(x) = \frac{1}{2}(\|x\|_\rho^*)^2$, $x \in T^*M$, has period $\tau_0 c^{-1/2}$ on the manifold $H_\rho^{-1}(c/2)$, and the flow $\{S_t^{H_{1,0}}\}$, where $H_{1,0}(x) = \tau_0\|x\|_\rho^*$, has period 1 on $T^*M^m \backslash \Gamma_0^*$; and this last flow can be thought of as a canonical action of S^1 on $T^*M^m\backslash\Gamma_0^*$.

We denote by v_ψ the vector field on M^m which induces the flow $\{\psi_t\}$. It is known that the flow $\{D'\psi_t\}$ on T^*M^m, where $D'\psi_t$ is the transformation dual to the differential $D\psi_t$, is induced by a Hamiltonian vector field whose Hamilton function is

$$H_{0,1}(x) = x\,(v_\psi(q)),$$

where $q \in M^m$ and $x \in T_q^*M^m$. The function $H_{\alpha,\beta} = \alpha H_{1,0} + \beta H_{0,1}$ induces an action of the torus $\mathbf{T}^2$ on $T^*M^m\backslash\Gamma_0^*$ satisfying the conditions (1.9)–(1.11) with $(\alpha_0, \beta_0) = (1, 0)$.

Applying Theorem A, we obtain a function $\mathcal{H}$ on $T^*M^m\backslash\Gamma_0^*$ half of whose square $\frac{1}{2}\mathcal{H}^2$ satisfies all hypotheses of condition F, save symmetry, if the number δ in the hypothesis of Theorem A is chosen sufficiently small and $r \geq 2$. In this case the function $\frac{1}{2}\mathcal{H}^2 \circ \mathfrak{L}_{\frac{1}{2}\mathcal{H}^2}^{-1}$ determines the square of the norm for some nonsymmetric Finsler metric with an ergodic geodesic flow, and the closed geodesics in this metric coincide with those closed geodesics of the metric ρ which are invariant under all isometries $\{\psi_t\}$.[4]

In the space of real C^∞ functions on M^m which are homogeneous of first degree for every $q \in M^m$ we introduce a sequence of norms $\|\ \|_r$, $r = 0, 1, \cdots$, such that the norm $\|\ \|_r$ takes into account the closeness of the r-jets of the functions on the manifold $T_\rho = L_\rho^{-1}(\frac{1}{2})$ of unit tangent vectors in the Riemannian metric ρ. Let $A \subset TM^*$. Set $-A = \{x: -x \in A\}$.

Theorem C. *Let ρ be a Riemannian metric on the manifold M^m for which conditions* (1.12) *and* (1.13) *hold. Then for any positive numbers c and δ and any natural number r there exists a Finsler metric σ on M^m and a set $F \subset L_\sigma^{-1}(\frac{1}{2})$ such that*

C.1 $\big\|\|x\|_\rho - \|x\|_\sigma\big\|_r < \delta.$

C.2 $\left(K_{c+\delta} \cap L_\sigma^{-1}\left(\frac{1}{2}\right)\right) \supset F \supset \left(K_{c-\delta} \cap L_\sigma^{-1}\left(\frac{1}{2}\right)\right),$

where

$$K_c = \{x \in TM^m : \tau_0 c\|x\|_\rho \leqslant H_{0,1} \circ \mathfrak{L}_\rho^{-1}(x)\}.$$

C.3. *The set F is invariant with respect to the geodesic flow of the metric σ,*

---

(4) If ρ is the standard metric on S^n, then the minimal number of geometrically distinct closed geodesics in the resulting nonsymmetric Finsler metric is $[(n-1)/2]$. Recall that in the nonsymmetric case it is natural to count each geodesic twice if it is a closed trajectory of the geodesic flow for the motion in both directions.

and this flow is ergodic on both F and $-F$.

C.4. *The geodesic flows of the metrics ρ and σ have the same closed trajectories in F.*

Remark. Choosing the number c sufficiently small, one can make the measure of the set $L_\sigma^{-1}(1/2)\setminus(F\cup -F)$ arbitrarily small.

Proof. Consider the manifold with boundary

$$M_c=\{x\in T^*M^m\setminus\Gamma_0^*: H_{-c,1}(x)\geqslant 0\}.$$

This manifold is invariant under the action of T^2 on $T^*M^m\setminus\Gamma_0^*$, and is homogeneous. Thus conditions (1.2)–(1.4) and (1.9)–(1.11) hold. Applying Theorem A to this situation, we construct a function $\mathcal{H}$ on M_c and a number ϵ so that at the points of the boundary ∂M_c the function $\mathcal{H}$ together with its differentials of all orders coincides with $H_{1,\epsilon}$.

Let $f(t)$ be an infinitely differentiable odd function equal to 1 for $t\geq c$. We extend $\mathcal{H}$ to a function $\overline{\mathcal{H}}$ defined on the whole cotangent bundle T^*M^m as follows:

$$\overline{\mathcal{H}}(x)=\begin{cases}\mathcal{H}(x), & \text{if } x\in M_c,\\ \mathcal{H}(-x), & \text{if } x\in -M_c,\\ H_{1,0}+\varepsilon f\left(\dfrac{H_{0,1}(x)}{H_{1,0}(x)}\right)H_{0,1}(x), & \text{if}\\ \qquad |H_{0,1}(x)|\leqslant cH_{1,0}(x),\ x\notin\Gamma_0^*, & \\ 0, & \text{if } x\in\Gamma_0^*.\end{cases}$$

If the number ϵ is sufficiently small, then the function $\frac{1}{2}\overline{\mathcal{H}}^2$ satisfies condition F.

As was shown in subsection 3, the function $\frac{1}{2}\overline{\mathcal{H}}^2\circ\mathfrak{L}^{-1}_{\frac{1}{2}\overline{\mathcal{H}}^2}$ determines the square of the norm for some nondegenerate Finsler metric σ on M^m. Set

$$F=\mathfrak{L}_\sigma^{-1}\left(M_c\cap L_\sigma^{-1}\left(\frac{1}{2}\right)\right).$$

If in the construction of the function $\mathcal{H}$ the numbers δ and r in the conditions of Theorem A are chosen sufficiently small and sufficiently large, respectively, then statements C.1 and C.2 will hold, since $\mathfrak{L}_\rho^{-1}M_c=K_c$.

Statements C.3 and C.4 hold, as the Legendre transformation $\mathfrak{L}_\sigma$ establishes an isomorphism between the geodesic flow of the metric σ and the flow $\{S_t^{\frac{1}{2}\overline{\mathcal{H}}^2}\}$.

6. In conclusion we consider the conditions (1.12) and (1.13). Not much can be said about manifolds on which such metrics may exist without further restrictions. Thus it is easy to show that condition (1.12) implies that the number of conjugacy classes in the group $\pi_1(M^m)$ must be finite, or, equivalently, the number of free homotopy classes of closed paths in M^m is finite. It is hard to obtain stronger conditions, since

the geodesic lines on M^m may have self-intersections.

The known examples of simply connected Riemannian manifolds on which all geodesics are closed are, up to diffeomorphisms, the compact Riemannian symmetric spaces of rank 1. These are spheres S^n ($n \geq 2$), complex projective spaces $P^n(C)$, $n \geq 1$, quaternionic projective spaces $P^n(Q)$, $n \geq 1$, and the Cayley projective plane K_2. In each of these examples all geodesics are of the same length and have no self-intersections. If one assumes that this property holds for all geodesics passing through some point $x \in M^m$, then, as was shown by Bott [17], either the cohomology ring of M^m is isomorphic to the cohomology ring of one of the listed symmetric spaces and M^m is simply connected, or $\pi_1(M^m) = Z_2$ and M^m has a homology sphere for its universal covering. Since among all symmetric spaces of rank 1 only spheres can be odd-dimensional, this result of Bott and the theorem of Smale [18] (the generalized Poincaré conjecture) imply that a manifold of odd dimension $m \geq 5$ which admits a metric in which all geodesics through some point are closed non-self-intersecting curves of the same length is homeomorphic either to S^m or to the quotient of S^m under the action of some fixed-point-free involution.

All known non-simply-connected Riemannian manifolds with closed geodesics are quotients of symmetric spaces of rank 1 under a free action of a finite group of isometries. These actions are classified up to conjugacy in the group of isometries of the simply connected manifold. Note that all symmetric spaces of rank 1, save for spheres of odd dimension, have positive Euler characteristic, and therefore a fixed-point-free isometry on such a space cannot be homotopic to the identity. From here it is easy to see that the only finite group of isometries that can act without fixed points on S^{2n}, $P^n(C)$, $P^n(Q)$ and K_2 is Z_2, and to describe the isometric actions of Z_2 on these manifolds (see [22]). The case of odd-dimensional spheres is nontrivial. The problem in this case is equivalent to the classification, up to isometries, of odd-dimensional full Riemannian manifolds of constant positive curvature; this is sometimes called "the Klein-Clifford problem of spherical space forms". For dimension 3 the classification was obtained by Seifert and Threlfall in 1930 [19]: in the general case the basic results were obtained by Zassenhaus [20] and Vincent [21], and the full classification was completed by Wolf [22].

Note that all known Riemannian manifolds with closed geodesics have a one-parameter group of isometries, i.e. the condition (1.13) is satisfied. This fact is nontrivial only in the case of quotients of odd-dimensional spheres, and then it follows from the fact that for all cases in the Zassenhaus-Vincent-Wolf classification all transformations of the finite group of isometries G acting on $S^{2m-1} \subset C^m$ are unitary complex transformations, and so commute with multiplication by unimodular complex numbers. However, there is no proof of this fact independent of the classification. Whether or not condition (1.13) is always a consequence of (1.12) is not clear.

Conditions (1.12) and (1.13) can be generalized. Namely, one can assume that

ρ is a nondegenerate Finsler metric and not necessarily a Riemannian metric on M^m. However, I have no examples of Finsler metrics satisfying these conditions save for those obtained from Riemannian metrics by a homogeneous diffeomorphism $f: TM^m \to TM^m$. Note that the results of Bott [17] hold for the case of nondegenerate Finsler metrics, so it is unlikely that this generalization will yield something new, at least from the topological viewpoint.

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