Dokl. Akad. Nauk SSSR
Tom. 171 (1966), No. 6

Soviet Math Dokl.
Vol. 7 (1966), No. 6

APPROXIMATION OF ERGODIC DYNAMIC SYSTEMS BY PERIODIC TRANSFORMATIONS

A. B. KATOK AND A. M. STEPIN

1. Let (M, μ) be a Lebesgue space (see [1]), let ξ_n be a partition of (M, μ) into sets C_n^i of measure $1/n$ $(i = 1, \cdots, n)$, and let $\mathfrak{A}_n$ be an algebra of sets measurable with respect to the partition ξ_n [1]. An automorphism S_n of (M, μ) will be called *cyclic with the respect to the partition* ξ_n *if*

P.1. $S_n \xi_n = \xi_n$.

P.2. $S_n^n = E$, $S_n^k \neq E$ for $k < n$.

We shall say that the automorphism T of the space (M, μ) *admits approximation by cyclic transformations at the rate* $o[f(q_n)]$ if for an increasing sequence of natural numbers q_n there exists a sequence of partitions $\xi_{q_n} \to \epsilon$* and a sequence of automorphisms S_{q_n}, cyclic with respect to ξ_{q_n}, such that

$$\sum_{i=1}^{q_n} \mu(TC_{q_n}^i \Delta S_{q_n} C_{q_n}^i) = o[f(q_n)].$$

We indicate the connection of this concept with that of the entropy of an automorphism (for the definition and properties of entropy see, for example, [2]).

Theorem 1. *If the automorphism* T *admits approximation by cyclic transformations at the rate* $o(1/\ln^2 q_n)$, *then* $h(T) = 0$.

Proof. It is well known that for any k_n

$$0 \leqslant H(T, \xi_n) \leqslant H(\xi_n \cdot T\xi_n \cdot \ldots \cdot T^{k_n}\xi_n) \,/\, k_n. \tag{1}$$

Let $\mu_n = \Sigma_{i=1}^{q_n} \mu(TC_{q_n}^i \Delta S_{q_n} C_{q_n}^i)$. From the conditions of the theorem we have:

$$\frac{H(\xi_n \cdot T\xi_n \cdot \ldots \cdot T^{k_n}\xi_n)}{k_n} \leqslant -\frac{\ln(1/q_n) + k_n\mu_n \ln(k_n\mu_n/q_n^{k_n+1})}{k_n}.$$

Assume $k_n = [\ln q_n / \sqrt{\alpha_n}] + 1$. Then

$$H(\xi_n \cdot T\xi_n \cdot \ldots \cdot T^{k_n}\xi_n) \,/\, k_n \to 0. \tag{2}$$

From the results of [5] there follows that if $\xi_n \to \epsilon$ then $H(T, \xi_n) \to h(T)$. Theorem 1 is a result of this observation and formulas (1) and (2).

Remark. Theorem 1 remains valid if we require that the automorphisms S_{q_n} satisfy condition P.1 alone.

We can connect the notion of rate of approximation by periodic transformations with various numerical invariants of dynamic systems, based on the idea of "best" permissible rate of approximations. Here is one of the constructions possible.

Let $X(t)$ be the set of those α for which the automorphism T admits approximation by cyclic

* ϵ is the partition of (M, ξ) into isolated points.

transformations at the rate $o(1/q_n^\alpha)$. Then we assume $d(T) = \sup\{\alpha:\ \alpha \in X(T)\}$.

It follows, for example, from Theorem 1 that if $h(T) > 0$ then $d(T) = 0$. It will be clear from what follows that the invariant $d(T)$ is nontrivial.

2. A higher rate of approximation permits us to draw inferences about the spectral properties of an automorphism T.

Theorem 2. *An automorphism which admits approximation by cyclic transformations at the rate* $o(1/q_n)$ *is ergodic.*

Theorem 3. *If the automorphism* T *admits approximation by cyclic transformations at the rate* $o(1/q_n)$, *then strong convergence* $U_T^{q_n} \Rightarrow E$, *occurs, where* U_T *is the unitary operator in* $L^2(M, \mu)$ *associated with the automorphism* T: $U_T(f(x)) = f(T^{-1}x)$.

Corollary to Theorem 3. *If the automorphism* T *is approximated by cyclic transformations at the rate* $o(1/q_n)$, *then*:

1°. *The maximal spectral type of the operator* U_T *is singular.*

2°. *The automorphism* T *does not have intermixing.*

Now let (M', μ') be the direct product of (M, μ) and the doubleton $Z_2 = (+1, -1)$ with measure $(\frac{1}{2}, \frac{1}{2})$. Let $j \in Z_2$. Let H_1 be the subspace of $L^2(M', \mu')$ consisting of functions $f(x, j)$ for which $f(x, 1) = f(x, -1)$, and let H_{-1} be the subspace of those functions $f(x, j)$ for which $f(x, 1) = -f(x, -1)$. We shall say that the automorphism T' of the space M' is built of the automorphism T of the space M and the function $n(x)$ if $T'(x, j) = (T(x), n(x)j)$, where $n(x)$ is a function on M with values ± 1.

Let the automorphism T satisfy the hypotheses of Theorem 1, and let $N = \{x:\ n(x) = -1\}$ have the following property: there exists a set $N_{q_n} \in \mathfrak{A}_{q_n}$ such that $\mu(N_{q_n} \Delta N) = o(1/q_n)$, where the set N_{q_n} consists of an odd number of elements ξ_{q_n}.

Under these assumptions we have

Theorem 3′. *In the subspace* H_{-1} *the strong convergence* $U_{T'}^{q_n} \Rightarrow E$ *holds.*

To investigate the properties of concrete dynamic systems, we need, in addition to the theorems formulated above, the following lemma.

Lemma 1. *If the normalized measure* σ *on the unit circle is such that* $\int \zeta^{q_n} d\sigma(\xi) \to \zeta_0$; $q_n \to \infty$, $|\zeta_0| = 1$, $\zeta_0 \neq 1$, *then* $\sigma \perp \sigma * \sigma$.

Remark. Lemma 1 remains valid in the case of real time, in the following formulation:

Let $\sigma(\lambda)$ *be a normalized measure on the line such that* $\int e^{it_n\lambda} d\sigma(\lambda) \to \zeta_0$, $t_n \to +\infty$, $|\zeta_0| \to 1$, $\zeta_0 \neq 1$. *Then* $\sigma \perp \sigma * \sigma$.

We shall say that the ordered pair of numbers (A, B) *satisfies condition* C if:

C.1. $0 < A < 1$, $0 < B < 1$, and A is irrational.

C.2. There exists a sequence p_n/q_n of appropriate fractions for A, such that

$$|p_n / q_n - A| = o(1/q_n^2).$$

C.3. There is a $C > 0$ such that $|r/q_n - B| > C/q_n$ for all integers r.

Lemma 2. *If the pair* (A, B) *of numbers satisfies condition* C, *then the function into the unit circle which is* $e^{i\lambda_1}$ *for* $0 \le x < 1 - B$ *and* $e^{i\lambda_2}$ *for* $1 - B \le x \le 1$ $(e^{i\lambda_1} \neq e^{i\lambda_2})$ *cannot be expressed in the form* $h(x + A)/h(x)$, *where* h *is a measurable fraction* (*all equations for functions here and in the sequel are understood to be* mod 0).

3. Let us pass on to examine examples of applications of our general theorems. We begin by constructing examples of an ergodic automorphism and ergodic flow whose maximal spectral types do not subordinate their own convolution.

Let (M, μ) be a circle with Lebesgue measure. $T_\alpha(x) = x + \alpha$, where α is an irrational number. Let us examine the automorphism $T' = T_{\alpha,\gamma}$ of the space $M' = M \times Z_2$, constructed form the automorphism T_α and the function $n(x) = -1$ for $0 \leq x < \gamma$, $n(x) = 1$ for $\gamma \leq x < 1$. When applied to $T_{\alpha,\gamma}$, the theorems and lemmas of part 2 yield:

3.1. *If the pair* (α, γ) *satisfies condition* C, *then the spectrum of the operator* $U_{\alpha,\gamma}$ *in the invariant subspace* H_{-1} *is nonmeasurable* (*Lemma* 2).

3.2. *If there exist sequences,* k_n/m_n *and* l_n/m_n $(m_n \to \infty)$, *of rational numbers such that*

$$|\alpha - k_n / m_n| = o(1/m_n^2); \quad |\gamma - l_n / m_n| = o(1/m_n); \qquad (*)$$

the numbers l_n *are odd*

then the conditions for applying Theorem 3′ *and Lemma* 1 *are fulfilled.*

Thus, if the pair (α, γ) satisfies both condition C and condition (*), then the maximal spectral type σ of the operator $U_{\alpha,\gamma}$, associated with $T_{\alpha,\gamma}$ has the form $\sigma_1 + \sigma_{-1}$, where σ_1 is discrete, concentrated at the points $\{e^{2\pi i n\alpha}\}$; σ_{-1} is continuous; and $\sigma_{-1} * \sigma_{-1} \perp \sigma_{-1}$.

Consequently, σ *does not subordinate* $\sigma * \sigma$.

3.3. If the numbers α and γ are rationally independent, then the multiplicity of the spectrum of $U_{\alpha,\gamma}$ does not exceed 2, since translations of the functions $N(x, j) = jn(x)$ and $M(x, j) = jm(x)$ (where $m(x) = 1$ for $0 \leq x < \alpha$ and $m(x) = 0$ for $\alpha \leq x \leq 1$) generate the entire subspace H_{-1}.

4. Now let (M, μ) be the unit torus with Lebesgue measure; x and y are cyclic coordinates on the torus, $T_t^{(\alpha)}(x, y) = T(x + t, y + \alpha t)$, and α is an irrational number.

For $0 \leq t \leq 1$, we define automorphisms $T_t^{(\alpha,\gamma)}$ of the space $M \times Z_2$ by the formula $T_t^{(\alpha,\gamma)}(x, y, j) = (T_t^{(\alpha)}(x, y), jn_t(x, y))$, where $n_t(x, y) = -1$ for $1 - t \leq x \leq 1$, $1 - \alpha - \gamma \leq y - \alpha x \leq 1 - \alpha$, and $n_t(x, y) = 1$ otherwise. For integers n, we assume $T_n^{(\alpha,\gamma)} = [T_1^{(\alpha,\gamma)}]^n$. Now let $t = n + \tau$ where $0 \leq \tau < 1$. We assume $T_t^{(\alpha,\gamma)} = T_n^{(\alpha,\gamma)} \cdot T_\tau^{(\alpha,\gamma)}$. It is easy to verify that the transformations $T_t^{(\alpha,\gamma)}$ defined in this way comprise a group.

Let $U_t^{(\alpha,\gamma)}$ be the group of unitary operators on $L^2(M \times Z_2)$ associated with the flow $T_t^{(\alpha,\gamma)}$. Applying the results of part 2 to $T_1^{(\alpha,\gamma)}$, and taking into account that the flow $T_t^{(\alpha,\gamma)}$ is ergodic, we obtain:

4.1. *If the pair* (α, γ) *of numbers satisfies condition* C, *then the spectrum of* $U_t^{(\alpha,\gamma)}$ *in the subspace* H_{-1} *is nonmeasurable.*

4.2. *If the pair* (α, γ) *of numbers satisfies condition* (*), *then in the subspace* H_{-1} *strong convergence* $U_{m_n}^{(\alpha,\gamma)} \Rightarrow -E$ *occurs.*

Taking into account the remark following Lemma 1, we conclude that when conditions C and (*) are satisfied simultaneously, *the maximal spectral type of* $U_t^{(\alpha,\gamma)}$ *does not subordinate its convolution.*

4.3. We shall consider the coordinates x, y on the torus to be taken so that the conditions $0 \leq y - \alpha x < 1$, $0 \leq x < 1$ are satisfied. We shall examine the subspace $H^{(n)}$ of functions constant on the intervals $\delta_{k,\beta}$: $\{y - \alpha x = \beta,\ k/2^n \leq x < (k+1)/2^n\}$, $0 \leq \beta < 1$, $k = 0, \cdots, 2^n - 1$. This subspace is invariant relative to the operator $U_2^{(\alpha,\gamma)}$, where $U_{2^{-n}}^{(\alpha,\gamma)}$, restricted to this subspace, is spectrum-isomorphic to the operator $U_{\alpha/2^n,\gamma/2^n}$ of §3. Thus, if α and γ are rationally independent, then the multiplicity of the spectrum of the group $U_t^{(\alpha,\gamma)}$ in invariant subspaces spanned by $H^{(n)}$ does not exceed two. Since

$H^{(n)}/L^2(M \times Z_2)$, it follows that *the maximal multiplicity of the spectral type of the group* $U_t^{(\alpha,\gamma)}$ *does not exceed two.*

5. As another application of our methods, we study the following mapping of the interval $[0, 1]$.

Let $0 < \alpha < \beta < 1$. Suppose $P_{\alpha,\beta}(x) = x + 1 - \alpha$ for $0 \leq x < \alpha$; $P_{\alpha,\beta}(x) = x + 1 - \alpha - \beta$ for $\alpha \leq x < \beta$; and $P_{\alpha,\beta}(x) = x - \beta$ for $\beta \leq x < 1$. This transformation clearly is an interleaving of the segments $[0, \alpha]$, $[\alpha, \beta]$, $[\beta, 1]$ in reverse order. Let $f(x) \in L^2([0, 1])$. As always, we assume $V_{\alpha,\beta}(f(x)) = f(P_{\alpha,\beta}^{-1}(x))$.

Let C be the circle $0 \leq \phi < 1 + \beta - \alpha$ and $T(\phi) = \phi + 1 - \alpha$. The derived transformation [4] induced by the translation T on the arc $[0, 1]$ is isomorphic to the transformation $P_{\alpha,\beta}$. From this it follows that the transformation $P_{\alpha,\beta}$ is or is not ergodic together with T; and consequently a necessary and sufficient condition for $P_{\alpha,\gamma}$ to be ergodic is the irrationality of the number $(1-\alpha)/(1+\beta-\alpha)$ or $\beta/(1-\alpha)$. We write $(1-\alpha)/(1+\beta-\alpha) = A$ and $(\beta-\alpha)/(1+\beta-\alpha) = B$, and apply the results of §2.

5.1. *If the pair* (A, B) *of numbers satisfies condition* C, *then the spectrum of* $V_{\alpha,\beta}$ *is discontinuous.*

For this case, we demonstrate the reduction of such a theorem to Lemma 2.

Let $f(x)$ be an eigenfunction of $V_{\alpha,\beta}^{-1}$; i,e. $V_{\alpha,\beta}^{-1}(f(x)) = e^{i\lambda}f(x)$ ($f(x) \not\equiv$ const, $e^{i\lambda} \neq 1$). We define a function $f^*(\phi)$ on the circle C: $f^*(\phi) = f(\phi)$ for $0 \leq \phi < 1$, $f^*(\phi) = f(\phi + 1 - \alpha)$ for $1 \leq \phi < 1 + \beta - \alpha$. The function $\tilde{f}(z) = f^*(z(1+\beta-\alpha))$ is defined on the circle $0 \leq z < 1$; furthermore, $\tilde{f}(z+A)/\tilde{f}(z) = g(z) = 1$ for $1 - \beta \leq z < 1$ and $g(z) = e^{i\lambda}$ for $0 \leq z < 1 - \beta$. Hence, if $g(z)$ cannot assume the form $h(z+A)/h(z)$, then the operator $V_{\alpha,\beta}$ does not have eigenfunctions.

5.2. *If the pair* (A, B) *of numbers satisfies condition* (*), *the numbers* l_n *being arbitrary, then* $V_{\alpha,\beta}^{m_n - l_n} \Rightarrow E$ *and, consequently, the spectrum of the operator* $V_{\alpha,\beta}$ *is singular, and the transformation* $P_{\alpha,\beta}$ *does not have intermixing.*

5.3. From a theorem proved by V. I. Oseledec [2], it follows that the multiplicity of the spectrum of the automorphism $P_{\alpha,\beta}$ in the ergodic case does not exceed two.

Theorem 4. *There exists a continuum of pairs* (α, β) *of numbers for which the spectrum of the operator* $V_{\alpha,\beta}$ *is single.*

Among this continuum of pairs (α, β), there are those which for pairs $A = (1-\alpha)/(1+\beta-\alpha)$ and $B = (\beta-\alpha)/(1+\beta-\alpha)$, condition C is satisfied. Consequently, among the automorphisms $P_{\alpha,\beta}$, there exist ones with a simple continuous spectrum.

Moscow State University

Received 15/FEB/66

BIBLIOGRAPHY

[1] V. A. Rohlin, Mat. Sb. 25 (67) (1949), 107; English transl., Amer. Math. Soc. Transl. (1) 10 (1962), 1. MR 11, 18.

[2] ———, Uspehi Mat. Nauk 15 (1960), no. 4 (94), 3 = Russian Math. Surveys 15, no. 4, 1. MR 24 #A2002.

[3] V. I. Oseledec, Dokl. Akad. Nauk SSSR 168 (1966), 1009 = Soviet Math. Dokl. 7 (1966), 776.

[4] S. Kakutani, Proc. Imp. Acad. Tokyo 19 (1943), 635. MR 7, 255.

[5] V. A. Rohlin, Dokl. Akad. Nauk SSSR 124 (1959), 980. MR 21 #2037.

Translated by J. E. Israel