

A Conjecture about Entropy

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1. In the Introduction to [1], M. Shub cites a definition of the topological entropy of a continuous map T of a compact metric space in terms of the asymptotics of the number of elements in (n, ε) -separated sets. We supplement this with some remarks which will be useful in what follows.

We set

$$d_n(x, y) = \max_{0 \leq i \leq n} d(T^i x, T^i y)$$

and let $r_n(T, \varepsilon)$ be the minimum number of elements in an ε -net in the space X with metric d_n . It is clear that

$$r_n(T, \varepsilon/2) \geq S_n(T, \varepsilon) \geq r_n(T, \varepsilon), \quad (1)$$

where $S_n(T, \varepsilon)$ is the maximal number of elements in an (n, ε) -separated subset of X .

In fact, the right-hand inequality follows from the fact that a maximal (n, ε) -separated set in X is an ε -net with respect to the metric d_n , and the left holds because any (n, ε) -separated set of balls with respect to the metric d_n centered at the points of this set are pairwise disjoint and any $\varepsilon/2$ -net must meet such a ball. Inequality (1) implies the following equivalent definition of the topological entropy:

$$h(T) = \lim_{\varepsilon \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log r_n(T, \varepsilon). \quad (2)$$

We let $B_n(x, \varepsilon)$ denote the ball in X centered at x and having radius ε with respect to the metric d_n .

In [1], Shub stated the following conjecture.

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THE ENTROPY CONJECTURE (see [1], §V). For any C^1 map f of a compact manifold M to itself,

$$h(f) \geq \log s(f_*). \quad (3)$$

Here $f_*: H_*(M, \mathbf{R}) \rightarrow H_*(M, \mathbf{R})$ denotes the linear map induced by f on the total homology of M ,

$$H_*(M, \mathbf{R}) = \bigoplus_{i=0}^{\dim M} H_i(M, \mathbf{R}), \quad (4)$$

and $s(f_*)$ is the spectral radius of f_* . That is, $s(f_*) = \lim_{n \rightarrow \infty} (\|f_*^n\|)^{1/n}$, which coincides with the maximum of the moduli of the eigenvalues of f_* .

We allow ourselves a little abuse of language and say that a map or class of maps *satisfies the entropy conjecture* if we can establish that (3) holds for such maps.

We will investigate partial results, counterexamples, and some suggestive considerations related to the entropy conjecture and its possible generalizations. We feel that the conjecture has been a very fruitful problem and that attempts to prove it have been very helpful for the development of that direction in the theory of dynamical systems, connected in the first place with the work of Smale, Shub, and Sullivan, which tends to unite the methods of investigation of smooth maps used in differential topology and the theory of smooth dynamical systems (differentiable dynamics).

The available partial results can be divided into three groups: assertions weaker than the entropy conjecture which have been proved for arbitrary smooth or even continuous maps, a proof of the entropy conjecture for special classes of manifolds, and a proof of it for special classes of maps. We begin with some general remarks, then examine the available results in the above order, and conclude with a discussion of when the equality $h(f) = \log s(f_*)$ is attained.

2. Since the expansion (4) is clearly invariant under f_* , we have

$$s(f_*) = \max_{1 \leq i \leq \dim M} s(f_{*i}),$$

where f_{*i} denotes the restriction of f_* to the space $H_i(M, \mathbf{R})$.

Thus, (3) is equivalent to the system of inequalities

$$h(f) \geq \log s(f_{*i}), \quad i = 1, \dots, \dim M. \quad (5)$$

Throughout what follows, we will suppose that the manifold M has a fixed Riemannian metric, and we let $d(x, y)$, $x, y \in M$, denote the distance function on M induced by this metric. Bounding from above the action of f on homology (and this is necessary for the entropy conjecture) can be carried out by the following considerations.

We shall consider k -dimensional chains generated by smooth singular simplexes. If σ^k is such a chain, let $\lambda_k(\sigma^k)$ denote its k -dimensional Riemannian volume. Define the volume of the k -dimensional chain $c^k = \sum a_i \sigma_i^k$, $a_i \in \mathbf{R}$, to be

$\lambda_k(c^k) = \sum_i |a_i| \lambda_k(\sigma_i^k)$. Define the norm $\|\alpha\|$ of a homology class $\alpha \in H_k(M, \mathbf{R})$ to be the infimum of the volumes of all cycles representing the class α . It is clear that $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$ and that $\|a\alpha\| = |a| \cdot \|\alpha\|$ for $a \in \mathbf{R}$. In order to verify that $\|\cdot\|$ actually defines a norm on $H_k(M, \mathbf{R})$, we need only check that $\|\alpha\| = 0$ implies $\alpha = 0$.

To do this, observe that for any k -dimensional differential form γ , the inequality $\int_{\sigma^k} \gamma \leq c(\gamma) \lambda_k(\sigma^k)$ holds for some constant $c(\gamma)$. Choose a basis of $H^k(M, \mathbf{R})$ and represent the elements of the basis by differential forms $\gamma_1, \dots, \gamma_s$. If $\|\alpha\| = 0$, then for any $\varepsilon > 0$ there exist cycles $c_\varepsilon \in \alpha$ such that $|\int_{c_\varepsilon} \gamma_i| < \varepsilon$ for $i = 1, \dots, s$. But, since these integrals do not depend on the choice of cycles representing α , the de Rham theorem implies that $\alpha = 0$.

Thus, to establish the inequality $\log s(f_{*k}) \leq h(f)$ it suffices to show that for any sufficiently small, smooth, singular k -dimensional simplex σ^k there is a chain c_n homologous to $f^n \sigma^k$ such that

$$\lim_{n \rightarrow \infty} \frac{\log \lambda_k(c_n)}{n} \leq h(f).$$

In particular, this inequality holds if we have

$$\overline{\lim}_{n \rightarrow \infty} \frac{\log \lambda_k(f^n \sigma^k)}{n} \leq h(f) \quad (6)$$

for a generic simplex σ^k .

Of course, when f is not one-to-one the quantity $\lambda_k(f^n \sigma^k)$ must be computed counting multiplicity. The stipulation following (6) that the simplex be generic is crucial. As G. A. Margulis has remarked, without this stipulation, inequality (6) can fail even for Morse-Smale diffeomorphisms. It is true that in his examples either the diffeomorphisms or the simplexes are only finitely differentiable. We note that inequality (6) pertains exclusively to differentiable dynamics, and were one to successfully prove it, then the entropy conjecture would be established without recourse to the methods of differential topology.

3. THEOREM 1 (A. MANNING [2]). *If f is any continuous map of a smooth compact manifold M , then*

$$h(f) \geq \log s(f_{*1}).$$

PROOF. The smooth one-dimensional simplexes are just the paths on M . We let $*$ denote the usual composition of paths. A homotopy of a path which is not closed shall be understood to be a homotopy which fixes the endpoints. By the discussion above, it suffices to show that for any sufficiently small path σ and for some $\varepsilon > 0$, the image $f^n \sigma$ is homotopic to a path whose length is bounded by a constant multiple of $r_n(f, \varepsilon)$.

Choose $\delta > 0$ such that any ball of radius δ on M is contractible. Choose $\varepsilon > 0$ so that any two points x and y whose distance from one another does not exceed 4ε can be joined by a path of length less than δ/K lying in a ball of radius δ/K , where $K = \max_{x \in M} \|Df_x\|$.

Now suppose that x and y are the endpoints of a path σ which lies entirely in a ball of radius ε . Let Q_n be an ε -net with respect to the metric d_n on M which consists of $r_n(f, \varepsilon)$ elements. Choose points $x_0, x_1, \dots, x_s \in Q_n$ and points $z_0 = x, z_1, \dots, z_s = y$ on σ such that $z_i \in B_n(x_i, \varepsilon)$ and the segment $\{z_i, z_{i+1}\}$ of the path σ between the points z_i and z_{i+1} lies in the union of the balls $B_n(x_i, \varepsilon)$ and $B_n(x_{i-1}, \varepsilon)$, $i = 0, 1, \dots, s$. Connect the points z_i to x_i by means of a path $\{z_i, x_i\}$ lying entirely in a ball of radius δ/K . Consider the path

$$\sigma' = \{z_0, x_0\} * \{x_0, z_0\} * \{z_0, z_1\} * \dots * \{z_{s-1}, z_s\} * \{z_s, x_s\} * \{x_s, z_s\},$$

where $\{x_i, z_i\}$ denotes the path running in the opposite direction to $\{z_i, x_i\}$. It is clear that σ' is homotopic to σ . If two of the points x_0, \dots, x_s coincide then σ' contains a loop beginning and ending at these points; this loop is clearly null-homotopic. Eliminating such loops from σ' gives a new path σ'' which is homotopic to σ and σ' and consists of elements of the form $\{z_i, z_{i-1}\}$, $\{x_i, z_i\}$, and $\{z_i, x_i\}$. Moreover, the total number of such elements does not exceed $3r_n(f, \varepsilon)$. We show by induction that, for each member $\kappa = \{y, y'\}$ of the path σ'' and for each $i = 0, 1, \dots, n$, the image $f^i \kappa$ is homotopic to a path κ_i of length less than δ/K which lies in a ball of radius δ/K . In fact, if $f^i \kappa$ is homotopic to κ_i , then $f^{i+1} \kappa$ is homotopic to $f \kappa_i$; this path lies in a contractible ball of radius δ and, consequently, is homotopic to a path κ_{i+1} of length less than δ/K which lies in a ball of radius δ/K . The latter exists because the distance between the points $f^{i+1} y$ and $f^{i+1} y'$ is less than 4ε . Thus, the path $f^n \sigma''$, and consequently the path $f^n \sigma$, is homotopic to a path of length no greater than $3\delta r_n(f, \varepsilon)$. This proves the theorem.

We remark that we have, in fact, proved a stronger assertion than Theorem 1.

With the same hypotheses as Theorem 1, suppose that $\gamma_1, \dots, \gamma_m$ is a system of generators of the group $\pi_1(M)$, and let $\gamma \in \pi_1(M)$. Consider all possible representations of $f_* \gamma$ in the form

$$\gamma_1^{i_1} \gamma_2^{i_2} \dots \gamma_m^{i_m} \gamma_1^{i_{m+1}} \dots \gamma_m^{i_{km}}, \quad i_j \in \mathbf{Z}$$

(where f_* is the endomorphism of $\pi_1(M)$ induced by f) and let $\varphi_n(\gamma)$ be the minimum of $\sum_1^{km} |i_j|$ over all such representations. Then

$$\overline{\lim}_{n \rightarrow \infty} \frac{\log \varphi_n(\gamma)}{n} \leq h(f).$$

Another way to generalize Theorem 1 is to relax the restrictions on the space on which f acts.

It is shown in [2] that it is not necessary to suppose that M is a manifold. It suffices that the following conditions be met:

1. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that any two points x and y for which $d(x, y) < \delta$ can be joined by a path of diameter less than ε .

2. There exists an $\varepsilon_0 > 0$ such that any loop of diameter less than ε_0 is contractible in M .

4. The following useful remark pertains to the fact that we may, without loss of generality, assume that the manifold M is orientable when dealing with considerations relating to the entropy conjecture.

PROPOSITION 1. *Let M be a nonorientable manifold and $f: M \rightarrow M$ a continuous map. Let \tilde{M} be the two-sheeted orientable cover of M and $\tilde{f}: \tilde{M} \rightarrow \tilde{M}$ the map covering f . Then $h(f) = h(\tilde{f})$ and $s(f_{*i}) \leq s(\tilde{f}_{*i})$ for $i = 1, \dots, \dim M$.*

PROOF. We let $\pi: \tilde{M} \rightarrow M$ denote the projection and assume that the Riemannian metric on \tilde{M} is induced from that on M . Clearly, $h(f) \leq h(\tilde{f})$. On the other hand, if ε is sufficiently small and if Q_n is an ε -net for the map f with respect to the metric d_n , then $\pi^{-1}(Q_n)$ is obviously an ε -net for the map \tilde{f} with respect to the metric d_n . Thus $r_n(\tilde{f}, \varepsilon) \leq 2r_n(f, \varepsilon)$, and hence $h(\tilde{f}) = h(f)$.

The map $\pi_k^*: H^k(M, \mathbf{R}) \rightarrow H^k(\tilde{M}, \mathbf{R})$ on cohomology induced by f is injective, and $\pi_k^* f_k^* = \tilde{f}_k^*$, where $f_k^*: H^k(M, \mathbf{R}) \rightarrow H^k(M, \mathbf{R})$ is the map induced by f . Since $s(f_{*k}) = s(f_k^*)$, the proposition follows.

When M is a compact orientable m -dimensional manifold, Poincaré duality gives maps $D_i: H_i(M, \mathbf{R}) \rightarrow H^{m-i}(M, \mathbf{R})$ which are defined as follows. If $\alpha \in H_i(M, \mathbf{R})$ and $\beta \in H_{m-i}(M, \mathbf{R})$, then

$$(D_i \alpha)(\beta) = \langle \alpha, \beta \rangle,$$

where $\langle \alpha, \beta \rangle$ denotes the intersection index of the cycles. Since

$$\langle f_{*i} \alpha, f_{*m-i} \beta \rangle = \deg f \langle \alpha, \beta \rangle,$$

we obtain the relation

$$f_{m-i}^* = \deg f D_i f_{*i}^{-1} D_i^{-1},$$

when $f: M \rightarrow M$ is a homeomorphism. Consequently,

$$s(f_{*m-i}) = s(f_{m-i}^{-1*}) = s(f_{i*}^{-1}). \quad (7)$$

Theorem 1, Proposition 1, formula (7), and the equality $h(f) = h(f^{-1})$ now imply the following assertion (see [2]).

COROLLARY 1. 1. *If $f: M \rightarrow M$ is a homeomorphism and $\dim M = m$, then $\log s(f_{*m-1}) \leq h(f)$.*

2. *The entropy conjecture holds for any homeomorphism of manifolds with dimension less than or equal to three.*

We mention the following assertion related to homeomorphisms which was formulated in [3] as "Theorem 2" (the quotation marks are those of the authors) and for which a heuristic proof was sketched.

If $\dim M \geq 5$, the entropy conjecture holds for all homeomorphisms belonging to an open dense set of the space of all homeomorphisms of M with the C^0 -topology.

In [3], the authors propose to imitate for homeomorphisms the "Markov approximation" procedure for diffeomorphisms described by Shub and Sullivan in [4], §1, and thereby construct a dense set of homeomorphisms for which the entropy conjecture holds. They then assume that one can establish that such

homeomorphisms are semistable.⁽¹⁾ Nitecki in [5] established this for diffeomorphisms with a hyperbolic structure. This would imply that any sufficiently small perturbation in the C^0 -topology of such a homeomorphism could not decrease the topological entropy.

For arbitrary homeomorphisms, the entropy conjecture does not hold (see Theorem 3, below).

5. The next case after the one-dimensional case in which inequality (5) has been successfully proved is for the top-dimensional homology groups; that is, the case when the dimension is equal to that of the manifold.

If $\dim M = m$, then $s(f_{*m}) = |\deg f|$, and thus the inequality $\log s(f_{*m}) \leq h(f)$ is trivially satisfied for homeomorphisms. Almost as trivial is the case when $f: M \rightarrow M$ is a continuous covering map. For, in this case, $|\deg f|$ is equal to the number of preimages of an arbitrary point $x \in M$ and the total preimage $\{f^{-n}x\}$ of x is an (n, ε) -separated set for any sufficiently small ε .

However, the inequality $h(f) \geq \log |\deg f|$ may fail to hold for an arbitrary continuous map. The simplest example of this sort was constructed by Shub in [6] and is based on the same idea as the suspension of the homeomorphism of cell complexes (not manifolds) described in §V of [1]. We consider the sphere S^n ($n \geq 2$) as the suspension of S^{n-1} and we choose a map $g: S^{n-1} \rightarrow S^{n-1}$ such that $|\deg g| > 1$. Compose the suspension of g with a map which moves each point along a "meridian" from the "north pole" N of the sphere S^n towards the "south pole" S. As a result we obtain a map $f: S^n \rightarrow S^n$ satisfying $\deg f = \deg g$, whose nonwandering set $\Omega(f)$ consists of the poles N and S. Thus,

$$h(f) = h(f | \Omega) = 0 < \log |\deg f|.$$

We shall consider this example in a little more detail. If the map g is chosen to be smooth and if the map which pushes down along the meridians is smooth along the meridians, then f is trivially smooth everywhere except at the unstable pole N and the stable pole S. Moreover, in a neighborhood of the stable pole, we can make the map f a strict contraction. By the same token, it is possible to make f infinitely differentiable (and, in some cases, even analytic) at S. The index of f and all its iterates at this point is equal to 1.

It turns out that the map f fails to be smooth in an essential way at N. In fact, f is not even a local homeomorphism at this point. Thus, if we were able to "smooth" f at N, the Jacobian of f would have to be equal to zero in spite of the fact that N is a repelling point.

Another interpretation, due to Shub, of the essential nonsmoothness of f at N is (somewhat freely translated) as follows. A direct computation shows that the index of f^n at N is equal to $(\deg g)^n$. On the other hand, Shub and Sullivan proved in [7] that the index at a fixed point of any iterate of a smooth map is

⁽¹⁾A map f is called *semistable* if, for any map g sufficiently close to f in the C^0 -topology, there exists a continuous map h for which $f \circ h = h \circ g$.

bounded. Thus, were f to be a limit in the C^0 -topology of smooth maps, the point N would have to “absorb” an infinite set of periodic points in the limit. This would guarantee smooth maps with sufficiently large topological entropy.

It turns out that two of the properties of smooth maps, boundedness of the Jacobian (the expansion coefficient of the Riemannian volume) and being local homeomorphisms at points where the Jacobian is different from zero, are already sufficient to prove the inequality $h(f) \geq \log |\deg f|$ by imitating the elementary arguments for covers cited in the beginning of this section. This result is due to Misiurewicz and Przytycki, who proved it first for two-dimensional manifolds in [8] and later for the general case, in a considerably simpler fashion, in [9]. We reproduce the latter proof here.

THEOREM 2 (MISIUREWICZ AND PRZYTICKI). *If $f: M \rightarrow M$ is any C^1 -map of a smooth compact manifold M , then $h(f) \geq \log |\deg f|$.*

PROOF. Suppose that $0 < \alpha < 1$ and let L be the lowest upper bound of the absolute value of the Jacobian of f . Set $\varepsilon = L^{-1/(1-\alpha)}$ and let B be the compact set for which the absolute value of the Jacobian is no less than ε . Cover B by open sets on which f is a local diffeomorphism. Let δ be the Lebesgue number of this cover. Thus, if $x, y \in B$ and $d(x, y) \leq \delta$, then $f(x) \neq f(y)$.

Fix a natural number n and consider the set $A = A_n$ consisting of those points x, y for which no more than αn of the images $x, f(x), \dots, f^{n-1}(x)$ belong to B . If $x \in A_n$, then the absolute value $|Jf^n(x)|$ satisfies the estimate

$$|Jf^n(x)| = \prod_{j=0}^{n-1} |Jf(f^j(x))| < \varepsilon^{(1-\alpha)n} L^{\alpha n} \leq (\varepsilon^{1-\alpha} L)^n = 1$$

and, consequently, the Riemannian volume of the set $f^n A$ is less than the volume of the whole manifold M .

Using this fact and Sard’s theorem, we choose a regular value $x \in M \setminus f^n(A)$ of f^n . Now choose a sufficiently large (n, δ) -separated set from among the n preimages of x . We argue as follows. Each regular value y of the map f has no more than $N = |\deg f|$ preimages. If among these preimages there are N preimages from the set B , we choose them. Otherwise, choose a preimage which does not belong to B . Beginning at the point x and inductively applying this procedure, we obtain in turn some subset of the set $f^{-1}(\{x\})$, then some subset of $f^{-2}(\{x\})$, and so forth, until we obtain a subset Q_n of $f^{-n}(\{x\})$. By the choice of δ and the description of the procedure, it is clear that Q_n is an (n, δ) -separated set. We bound from below the number of elements of this set. Suppose $y \in Q_n$. Since $x \notin f^n(A)$ and $y \in f^{-n}(\{x\})$, we have $y \notin A$. In the construction of Q_n , we considered two sorts of transitions to preimages: taking N “good” preimages or a single “bad” preimage. Since $y \notin A$, there are no more than αn numbers k between 0 and $n - 1$ for which $f^k(y) \in B$. This means that in passing from x to y there are at least $m = [\alpha n] + 1$ “good” transitions. Since this

holds for any $y \in Q_n$, there are at least $N^m \geq N^{\alpha n}$ elements, and consequently $S_n(f, \delta) \geq N^{\alpha n}$, from which it follows that $h(f) \geq \alpha \log N$. Since α can be chosen arbitrarily close to 1, we have $h(f) \geq \log N$.

COROLLARY 2. *The entropy conjecture holds for any smooth map of*

- 1) *spheres S^n , or*
- 2) *any manifold of dimension no more than two.*⁽²⁾

6. We now turn to the case of invertible maps. Recall that here the case of m -dimensional homology is trivial ($m = \dim M$) and inequality (5) has already been proved for arbitrary homeomorphisms in the 1- and $(m - 1)$ -dimensional cases (Theorem 1 and Corollary 1). The situation turns out to be much more complicated in the intermediate dimensions. We cite a negative result obtained as a result of discussions at the Warwick symposium on dynamical systems in 1974. An account is given in a report by Pugh [10] (the title of which lists the participants in the discussions).

THEOREM 3 [10]. *There exists a homeomorphism f of a compact, smooth, 8-dimensional manifold M for which $\Omega(f)$ is a finite set and $s(f_{*2}) > 1$.*

Thus, $0 = h(f) < \log s(f_{*2})$ and the entropy conjecture does not hold for the homeomorphism f . The construction of f is based on a further development of the idea of suspension using some results and methods of piecewise linear topology.

We first construct a suspension K of the two-dimensional torus. By composing the suspension of an Anosov diffeomorphism of the torus with a translation map on K which moves points from the north pole to the south, we obtain a homeomorphism $B: K \rightarrow K$ for which the set $\Omega(B)$ consists of two points and for which $s(f_{*2}) > 1$.

We then construct a piecewise linear inclusion $i: K \rightarrow \mathbf{R}^8$. It turns out that the homeomorphism induced by B on the image iK can be extended to a homeomorphism \bar{B} of the Euclidean space \mathbf{R}^8 . Let N be the star neighborhood of iK in the second barycentric subdivision of a triangulation of \mathbf{R}^8 which contains the polyhedron iK . By composing the homeomorphism $\bar{B} | N$ with a homeomorphism $h: \bar{B}N \rightarrow N$ which is the identity on iK (and which exists by a result of Hirsch [11]), we obtain an extension C of the homeomorphism B to the manifold with boundary N . Now consider the double M of the manifold N with the map C . This map has two invariant sets K_+ and K_- . The manifold M possesses a smooth structure. Furthermore, it is possible to perturb the map C , without changing it on K_+ or K_- , so that the points are moved from one pole of M to

⁽²⁾Assertion 2) actually only uses Theorem 2 in the case of spheres. In fact, by Proposition 1, it suffices to restrict attention to orientable surfaces. Theorem 1 then gives the result for the one-dimensional homology. For manifolds of genus greater than 1, the modulus of an iterate of any map does not exceed 1. Finally, for the torus, other considerations yield a more general result pertaining to arbitrary continuous maps (see Theorem 4).

the other, repelled from the set K_- , and attracted to K_+ . Thus, the homeomorphism f of M constructed in this way has a finite number of nonwandering points. Consequently its topological entropy is zero.

It remains only to verify that, under the inclusion of the polyhedron K_+ into M , the two-dimensional cycles on K_+ do not become trivial. But this follows because $\dim M > 2 \dim K_+ + 1$ and K_+ is a retract of the set $M \setminus K_-$.

Shub's explanation in terms of indexes (see §5) is also applicable to this example.

7. On some manifolds, the entropy conjecture turns out to hold for arbitrary continuous maps.

THEOREM 4 (MISIUREWICZ AND PRZYTYCKI [12]). *The entropy conjecture holds for any continuous map f of the m -dimensional torus \mathbf{T}^m .*

PROOF. For definiteness, we will assume that the torus $\mathbf{T}^m = \mathbf{R}^m / \mathbf{Z}^m$ is endowed with the fixed standard Euclidean metric.

We first establish the inequality,

$$h(f) \geq \log |\deg f|, \quad (8)$$

for any map of the torus. For the torus, we have

$$\deg f = \det f_{1*}. \quad (9)$$

We let $\pi_m: \mathbf{R}^m \rightarrow \mathbf{T}^m$ denote the standard projection and consider the map $\tilde{f}: \mathbf{R}^m \rightarrow \mathbf{R}^m$ covering f . Suppose that $\Delta_n \subset \mathbf{R}^m$ is the standard fundamental domain in \mathbf{R}^m (the Euclidean cube). From (9) and the fact that f is homotopic to an algebraic endomorphism of the torus induced by the linear operator f_{1*} on \mathbf{R}^m , it follows that the volume of the set $\tilde{f}^n \Delta_n$ is no smaller than $|\deg f|^n$. Thus, by a theorem of Minkowski, there exists a point $x \in \mathbf{T}^m$ such that the set $f^n \Delta_n \cap \pi_m^{-1}(x) = Q_n(x)$ contains no less than $|\deg f|^n$ points. For each point $y \in Q_n(x)$, choose a point $z(y) \in \tilde{f}^{-n}(y)$ and set $K_n = \bigcup_{y \in Q_n(x)} \{\pi_m z(y)\}$. We shall prove that, for sufficiently small $\varepsilon > 0$, K_n is (n, ε) -separated. Suppose that $x_1, x_2 \in K_n$ and the distance between x_1 and x_2 is sufficiently small. Consider points $z_1 \in \pi_m^{-1}(x_1)$ and $z_2 \in \pi_m^{-1}(x_2)$ such that $d(z_1, z_2) = d(x_1, x_2)$. Since $\tilde{f}^n z_1, \tilde{f}^n z_2 \in \pi_m^{-1}(x)$ are distinct, we have $d(\tilde{f}^n z_1, \tilde{f}^n z_2) \geq 1$. Thus, there exists a constant δ_0 which depends only on f , and not on n , for which the inequality $\delta_0 < d(\tilde{f}^k z_1, \tilde{f}^k z_2) < 1/2$ is satisfied for some k , $1 \leq k \leq n - 1$. But, then

$$d(f^k x_1, f^k x_2) = d(\pi_m \tilde{f}^k z_1, \pi_m \tilde{f}^k z_2) = d(\tilde{f}^k z_1, \tilde{f}^k z_2) > \delta_0$$

and, consequently, K_n is an (n_1, δ_0) -separated set.

In order to establish the inequality $h(f) \geq \log s(f_{*k})$ for any k , we proceed in a similar fashion. We fix a standard basis γ for the group $H_k(\mathbf{T}^m, \mathbf{R})$ and realize each element α of this basis by the standard inclusion $i_\alpha: \mathbf{T}^k \rightarrow \mathbf{T}^m$ of the k -dimensional "coordinate" torus into \mathbf{T}^m . Suppose that $h_\alpha: \mathbf{T}^m \rightarrow \mathbf{T}^k$ is the standard projection, so that $h_\alpha \circ i_\alpha = \text{id}_{\mathbf{T}^k}$. The matrix of f_{*k}^n with respect to the basis γ has elements of the form $c_{\alpha\beta}^n = \deg(h_\beta f^n i_\alpha)$ where $\alpha, \beta \in \gamma$.

Using Minkowski's theorem again, we find a point $x \in \mathbf{T}^k$ for which the set $Q_n(x) = \tilde{h}_\beta \tilde{f}^n \tilde{i}_\alpha \Delta_k \cap \pi_k^{-1}(x)$ contains no fewer than $c_{\alpha\beta}$ elements. Then, for each $y \in Q_n(x)$, we choose a point $z(y) \in (h_\beta \tilde{f}^n \tilde{i}_\alpha)^{-1}y$. An argument similar to the one above shows that the set $K_n = \bigcup_{y \in Q_n(x)} i_\alpha \pi_k(z(y))$ is (n, ε) -separated for sufficiently small ε .

Since

$$\log s(f_*) \leq \lim \frac{\log \sum_{\alpha, \beta \in \gamma} |c_{\alpha, \beta}^n|}{n},$$

the inequality $h(f) \geq \log(f_{*k})$ follows from (8).

We remark that for any algebraic endomorphism f_A of the torus generated by the matrix $A \in \text{GL}(n, \mathbf{Z})$, we have

$$h(f_A) = \log s(f_{A*}) = \sum_{\substack{\lambda \in \text{sp} A \\ |\lambda| > 1}} \log |\lambda|.$$

Since any Anosov automorphism of the torus is topologically conjugate to an algebraic automorphism (see [13] and [14]), we obtain the following assertion.

COROLLARY 3. *If f is an Anosov diffeomorphism of the torus \mathbf{T}^m , then $h(f) = \log s(f_*)$.*

We also mention the following property. If the map f_{*s} is invertible and has no eigenvalues on the unit circle, then, by Proposition 2.1 of Franks [13], there exists a continuous map $h: \mathbf{T}^m \rightarrow \mathbf{T}^m$ such that $hf = gh$, where g is the algebraic automorphism of the torus generated by f_{*1} . Thus, $h(f) \geq h(g) = \log s(f_{*1})$.

We remark that the entropy conjecture is established in [12] for homeomorphisms of orientable manifolds of the form $\mathbf{T}^m \times X$, where $\dim X = n$ and $H_i(X, \mathbf{R}) = 0$ for $0 < i \leq (m+n)/2$.

The arguments used in constructing (n, ε) -separated sets in the preimages of points which were successfully applied in the proofs of Theorems 2 and 4, or modifications of these arguments, might also turn out to be useful in some other cases. For example, they might be useful in proving the following.

CONJECTURE. *If M is a manifold whose universal covering space is homeomorphic to Euclidean space, then any continuous map $f: M \rightarrow M$ satisfies the entropy conjecture.*

8. Shub formulated the entropy conjecture in connection with the problem of defining the simplest diffeomorphisms in each isotopy class of diffeomorphisms (see [1] and [4]). From this standpoint, it is important to prove the entropy conjecture for "good" (for example, structurally stable) diffeomorphisms. In [4], Shub and Sullivan described an open and dense (in the C^0 -topology) subset of the set of structurally stable diffeomorphisms for which the entropy conjecture holds. This subset consisted of the diffeomorphisms which were "Markov-fitted with respect to some handle decomposition of the manifold M ." The structure of such a diffeomorphism accords well with the structure of the cell complex

generated by the handle decomposition. As a result, $s(f_*)$ can be computed in terms of the algebraic intersection matrices and $h(f)$ can be bounded from below in terms of matrices consisting of the moduli of the elements of the intersection matrices. It does not follow from this situation that the entropy conjecture holds for the diffeomorphisms constructed by Shub and Sullivan. However, these diffeomorphisms satisfy the conditions of a theorem of Bowen [15] establishing the entropy conjecture for Axiom A, no-cycles diffeomorphisms which have nonwandering sets of dimension zero.

Later, Shub and Williams [16] obtained a more general result by eliminating the restriction $\dim \Omega(f) = 0$ and thus proved the entropy conjecture for all well-known (and possibly all) Ω -stable diffeomorphisms.

THEOREM 5 (SHUB AND WILLIAMS [16], announced in [6]). *The entropy conjecture holds for any Axiom A, no-cycles diffeomorphism.*

PROOF. We follow the outline of the proof in [16] with one essential difference. Let k be the dimension of the unstable subfoliation on a basic set Ω_i . Shub and Williams used Markov decompositions (to compute the topological entropy) and an extension of the system of stable manifolds to a neighborhood of the basic set to compute the volume of the part of the image of a k -dimensional simplex lying in Ω_i . Instead of computing the topological entropy from the intersection matrix associated to the Markov decomposition, we compute it directly from the asymptotics of the number of elements in (n, ε) -separating sets; and instead of extending the stable manifolds (which is always a delicate matter), we extend semi-invariant systems of cones around the stable and unstable subspaces (a procedure that presents no difficulties).

Recall that if $f: M \rightarrow M$ is an axiom A, no-cycles diffeomorphism, there exists a filtration $M_0 \subset M_1 \subset \dots \subset M_m = M$ such that $f(M_i) \subset \text{Int}(M_i)$ and $\bigcap_{n \in \mathbf{Z}} f^n(M_i \setminus M_{i-1}) = \Omega_i$ is a basic set of f for $i = 0, \dots, m$. Let

$$\begin{aligned} f_*^{(i)} &: H_*(M_i, M_{i-1}, \mathbf{R}) \rightarrow H_*(M_i, M_{i-1}, \mathbf{R}), \\ f_{*j}^{(i)} &: H_j(M_i, M_{i-1}, \mathbf{R}) \rightarrow H_j(M_i, M_{i-1}, \mathbf{R}) \end{aligned} \tag{10}$$

denote the maps induced by f . It is known that

$$h(f) = \sup_i h(f | \Omega_i).$$

On the other hand, from the exact homology sequences of the pairs (M_i, M_{i-1}) , $i = 0, \dots, m$, it is not difficult to show that

$$s(f_*) \leq \sup_i s(f_*^{(i)}).$$

Thus, to prove Theorem 5, it suffices to establish that

$$h(f | \Omega_i) \geq \log s(f_*^{(i)})$$

for each i .

Furthermore, instead of using the relative homology of the pair (M_i, M_{i-1}) to compute $s(f_*^{(i)})$, we can use the homology of the pair (X, A) where $X = f^N M_i$,

$A = X \cap f^{-N}M_{i-1}$, and N is an arbitrary positive integer. We will choose N such that the set $X \setminus A$ is contained in a sufficiently small neighborhood U of Ω_i which will be specified below. Observe that the set $X \setminus A$ is convex with respect to trajectories. This means that if $x \in X \setminus A$ and $f^n x \in X \setminus A$ for some $n > 0$, then $f^k x \in X \setminus A$ for $k = 0, 1, \dots, n-1, n$. This property follows at once from the properties of the filtration.

Suppose that we are given a cone K_x in the tangent space $T_x M$ for each x in a neighborhood U of Ω_i . We say that a submanifold $N \subset U$ is *compatible* with the system of cones K_x if $T_x N \subset K_x$ for every $x \in N$.

At each point $x \in \Omega_i$ we construct cones $K_x^s \supset E_x^s$ and $K_x^u \supset E_x^u$ which are "narrow" enough so that

$$Df^{-1}K_x^s \subset \text{Int } K_{f^{-1}x}^s \quad \text{and} \quad DfK_x^u \subset \text{Int } K_{fx}^u. \quad (11)$$

We do this in such a way that K_x^s and K_x^u depend continuously on x .

For example, we can choose a sufficiently small number $\gamma > 0$ and set

$$K_x^s = \{v \in T_x M : v = v_1 + v_2, v_1 \in E_x^s, v_2 \in E_x^u, \|v_2\| \leq \gamma \|v_1\|\},$$

$$K_x^u = \{v \in T_x M : v = v_1 + v_2, v_1 \in E_x^s, v_2 \in E_x^u, \|v_1\| \leq \gamma \|v_2\|\}.$$

If U is chosen sufficiently small, then we can extend the system of cones to a neighborhood U so that the formulas (11) are satisfied for $x \in U$ (provided $f^{-1}x$ or fx , as the case may be, lies in U) and so that there exist constants c_1 and $\varepsilon > \delta > 0$ such that any k -dimensional submanifold N consistent with the system of cones K_x^u possesses the following properties.

1. The volume of the sphere of radius ε in N does not exceed 1.
2. If $x, y \in N$ and the distance $d_N(x, y)$ between x and y in the intrinsic metric on N does not exceed δ , then $d_N(x, y) \leq c_1 d(x, y)$.

We now prove that $h(f|\Omega_i) \geq \log s(f_{*k}^{(i)})$. Using a relative variant of the arguments in §2, we easily see that it suffices to show that

$$\overline{\lim}_{n \rightarrow \infty} \frac{\log \lambda_k(f^n \sigma^k \cap (X \setminus A))}{n} \leq h(f|\Omega_i), \quad (12)$$

for any sufficiently small k -dimensional simplex $\sigma^k \subset X \setminus A$.

In addition, we can restrict ourselves to those simplexes for which the tangent space at each point x intersects the cone K_x^s only at the origin. Because $X \setminus A$ is convex with respect to trajectories in this case, there exists an $s > 0$ such that the manifold $f^s \sigma^k \cap (X \setminus A)$ is compatible with the cones K_x^u . Set $c_2 = \max_{x \in M} \|Df_x\|$ and cover $f^s \sigma^k \cap (X \setminus A)$ by a finite number of balls of radius $\delta/2c_2$ in the intrinsic metric. Let N be any such ball. We estimate the volume of the manifold $f^n N \cap (X \setminus A)$. Choose a system of points on $f^n N \cap (X \setminus A)$ with the property that the distance between any two such points in the metric on $f^n N$ is larger than ε . Let S denote the n th preimage of this set.

We show that S is an $(n, \delta/c_1 c_2)$ -separating set. Suppose that $x, y \in S$ and set $a_l = d_{f^l N}(f^l x, f^l y)$. Since $x, y \in N$, we have $a_0 \leq \delta/c_2$. On the other hand, $a_n \geq \varepsilon > \delta$ and, clearly, $a_{l+1} < c_2 a_l$. Therefore, there exists an l , $0 \geq l \geq n$,

such that $\delta/c_2 \leq a_i \leq \delta$. We may assume that δ and the neighborhood of $X \setminus A$ have been chosen so small that the δ -neighborhood of $X \setminus A$ is contained in a neighborhood U for which properties 1 and 2 hold for the system of cones K_x^u . In particular, we can use the compatibility of $f^l N \cap U$ with the system K_x^u and property 2 to conclude that

$$d(f^l x, f^l y) \geq \delta/c_1 c_2.$$

Now suppose that the set S contains a maximal number of points. Then $f^n S$ is an ε -net on the set $f^n N \cap (X \setminus A)$ in the metric on $f^n N$. Since the system of cones K_x^u possesses property 1, the number of elements in such an ε -net is no less than the k -dimensional volume of the set $f^n N \cap (X \setminus A)$. Thus,

$$\lambda_k(f^n N \cap (X \setminus A)) \leq r_n(f, \delta/c_1 c_2).$$

Inequality (12) follows.

We will not deal in as much detail with the proof of the inequality $h(f | \Omega_i) \geq \log s(f_{*j}^{(i)})$ for $j \neq k$.

In this case, it is shown in [16] that the inequality is always strict. When $j < k$, it is possible to show that the volume of a j -dimensional simplex cannot grow faster than the volume of a k -dimensional simplex since there are more “free” directions in which to expand. For $j > k$, it is also necessary to use the result just established when $j \leq n - k$ for the dual filtration

$$\overline{M \setminus M_{m-1}} \subset \overline{M \setminus M_{m-2}} \subset \cdots \subset \overline{M \setminus M_0} \subset M$$

of the diffeomorphism f^{-1} .

The required result follows from duality between the $(n - j)$ -dimensional cohomology of the pair $(\overline{M \setminus M_{i-1}}, \overline{M \setminus M_i})$ and the j -dimensional homology of the pair (M_i, M_{i-1}) (see [15]).

9. In this section we consider the question of which diffeomorphisms satisfy the equation $h(f) = \log s(f_*)$ or its local variants. In the case of diffeomorphisms with a hyperbolic structure, it is possible to find natural sufficient conditions. Although these conditions are not necessary, they cannot “essentially” be dispensed with.

We begin with Anosov diffeomorphisms.

PROPOSITION 2. *If $f: M \rightarrow M$ is an Anosov diffeomorphism and if the unstable subfoliation E^u of the tangent bundle TM is orientable, then $h(f) = \log s(f_*)$.*

PROOF. By Theorem 5, it suffices to prove that $h(f) \leq \log s(f_*)$. For hyperbolic sets (and, in particular, for Anosov diffeomorphisms), one can compute the topological entropy from the asymptotics of the number $N_n(f)$ of periodic points of f of period n . More explicitly,

$$h(f) = \overline{\lim} \frac{\log N_n(f)}{n}.$$

Let $P_n = \{x \in M, f^n x = x\}$ and let $i_{f^n}(x)$ be the index of x as a fixed point of f^n .

By the Lefschetz formula, we have

$$L(f^n) = \sum_{x \in P_n} i_{f^n}(x) = \sum_{i=0}^{\dim M} (-1)^i \operatorname{tr}(f_{*i}^n).$$

Since all periodic points of f^n are hyperbolic, the index equals $+1$ or -1 and depends only on the dimension. Thus, the map Df_x^n either preserves or reverses the orientation on the invariant expanding subspace E_x^u . Since M is connected and the subfoliation E^u is orientable, the spaces E_x^u can be oriented in a consistent manner such that the differential $Df_x^n|_{E_x^u}: E_x^u \rightarrow E_{f^n x}^u$ is either orientation-preserving at all points or orientation-reversing at all points. In particular, this applies to the fixed points of f^n and, thus, the indexes $i_{f^n}(x)$ are equal for every $x \in P_n$. That is, $|L(f^n)| = N_n(f)$. Furthermore,

$$|L(f^n)| = \left| \sum_{i=0}^{\dim M} (-1)^i \operatorname{tr}(f_{*i}^n) \right| \leq \sum_{i=0}^{\dim M} |\operatorname{tr}(f_{*i}^n)| \leq \dim H_*(M, \mathbf{R})(s(f_*))^n.$$

Therefore,

$$\overline{\lim} \frac{\log N_n(f)}{n} \leq s(f_*).$$

This proves the proposition.

The question of whether the unstable foliation E^u of an Anosov diffeomorphism is always orientable has been studied for more than ten years and has not yet been solved. In [19], Smale referred to this question in connection with the problem of the rationality of the zeta function of an Anosov diffeomorphism. This latter problem was subsequently solved using Markov decompositions [22] in the more general setting of hyperbolic sets where orientability may fail to hold (see below).

CONJECTURE. *If $f: M \rightarrow M$ is an Anosov diffeomorphism, then $h(f) = \log s(f_*)$.*

We now suppose that both the invariant subfoliations E^u and E^s of a given Anosov diffeomorphism are orientable (this is equivalent to the orientability of E^u and the manifold M itself). In this case, it is possible to amplify Proposition 2. In fact, let k be the dimension of E^u . Then, we can find a functional $\bar{\alpha}$ on the k -dimensional differential forms on M which geometrically realizes a nonzero element $\alpha \in H_k(M, \mathbf{R})$ for which $f_* \alpha = \lambda \alpha$ and $\log |\lambda| = h(f)$. The construction is a particular case of a construction due to Ruelle and Sullivan [17].

On the global stable submanifolds W^s of M , there exists a family of σ -finite Borel measures μ_{W^s} (in general, singular) possessing the following properties (see [23] and [18]).

1. The measure of any compact subset of W^s is finite.
2. The measures pass into one another under translation along the local unstable submanifolds.

3. $f\mu_{W^s} = \lambda^{-1}\mu_{fW^s}$, where $\log|\lambda| = h(f)$.

Now let ω^k be a k -dimensional differential form on M . We let $\bar{\alpha}$ denote the functional whose value on ω^k is calculated as follows. Cover M by small open sets which have a local product structure (see [13]) and use a partition of unity to represent ω^k as a sum of forms with support in these sets. In each such open set, integrate the form along each local unstable manifold (taking into account the orientation, which is assumed to be chosen consistently), and then integrate the resulting integrals with respect to the measure μ_{W^s} along any transverse local stable manifold. Add the resulting contributions from each local form to obtain the value $\bar{\alpha}(\omega^k)$. We shall show that $\bar{\alpha}$ is a cycle; that is, $\bar{\alpha}(\partial\omega^{k-1}) = 0$ for any $(k-1)$ -form ω^{k-1} . Let $\omega^{k-1} = \sum_i \omega_i^{k-1}$, where the support of each ω_i^{k-1} lies in the interior of some open set U_i with a local product structure. It is clear that

$$\bar{\alpha}(\partial\omega^{k-1}) = \sum_i \bar{\alpha}(\partial\omega_i^{k-1}).$$

But $\bar{\alpha}(\partial\omega_i^{k-1})$ is equal to the integral of the values of the form ω_i^{k-1} on the boundaries of the local unstable manifolds in U_i . Since these boundaries lie outside the support of ω_i^{k-1} , we have $\bar{\alpha}(\partial\omega_i^{k-1}) = 0$.

Let α denote the homology class of $\bar{\alpha}$. We need to show that $f_{*k}\alpha = \lambda\alpha$, $\log|\lambda| = h(f)$, and $\alpha \neq 0$.

Suppose, for definiteness, that the diffeomorphism f preserves the orientations of E^u and E^s (the other case can be handled by making the obvious modifications to the argument following).

We compute the intersection index of $\bar{\alpha}$ with $(n-k)$ -dimensional chains. To do this, it suffices to find the intersection index of $\bar{\alpha}$ with any sufficiently small singular simplex σ^{n-k} transverse to the subfoliation E^u . Suppose U is an open set which has a local product structure and which contains σ^{n-k} . The simplex σ^{n-k} intersects each local unstable manifold at no more than one point. Thus, it follows from the definition of $\bar{\alpha}$ that the absolute value of the intersection index $\langle \bar{\alpha}, \sigma^{n-k} \rangle$ is equal to the measure μ_{W^s} of the projection (along the local unstable manifolds) of σ^{n-k} onto any local stable manifold in U . The intersection index takes a plus or minus sign according to whether the orientation of the projection does or does not coincide with the orientation of the local stable manifolds.

From property 3 of the measures μ_{W^s} it follows that

$$\langle \bar{\alpha}, f\sigma^{n-k} \rangle = \lambda^{-1}\langle \bar{\alpha}, \sigma^{n-k} \rangle.$$

Linearity of the intersection index implies that a similar equation holds for any smooth singular $(n-k)$ -dimensional chain. Thus, for any $\gamma \in H_{n-k}(M, \mathbf{R})$, we have

$$\langle \alpha, f_{*n-k}\gamma \rangle = \lambda^{-1}\langle \alpha, \gamma \rangle.$$

Since $\langle \alpha, \gamma \rangle$ is equal to the value on γ of $D_k\alpha \in H^{n-k}(M, \mathbf{R})$, we have $f_{*n-k}^*D_k\alpha = \lambda D_k\alpha$. Since $f_{*k} = D_k^{-1}f_{*n-k}^*D_k$, it follows that $f_{*k}\alpha = \lambda\alpha$, where $\log|\lambda| = h(f)$. Interchanging the roles of the stable and unstable manifolds, we can construct an $(n-k)$ -dimensional cycle $\bar{\beta}$. From the definitions it is easy to

see that $\langle \alpha, \beta \rangle \neq 0$, where β denotes the cohomology class of $\bar{\beta}$. Thus, $\alpha \neq 0$ and $\beta \neq 0$.

We now pass to the more general class of Axiom A, no-cycles diffeomorphisms. In this case, there are local variants of the propositions cited above. The orientation condition for the foliation E^u on a basic set Ω_i is as follows: There exists an orientation of the subfoliation E^u on Ω_i such that for every $x \in \Omega_i$ the differential

$$Df_x | E_x^u: E_x^u \rightarrow E_{f_x}^u$$

is either simultaneously orientation-preserving or simultaneously orientation-reversing. In this case,

$$h(f | \Omega_i) = \log s(f_*^{(i)}).$$

Shub and Williams proved this in [16] by an index argument which carries over word for word to this case by using a relative variant of the Lefschetz formula. In the case when the stable subfoliation E^s on Ω_i also satisfies the analogue of the orientability condition, Ruelle and Sullivan [17] constructed an eigenvector in $H_k(X, A, \mathbf{R})$ with eigenvalue λ , where $\log |\lambda| = h(f | \Omega_i)$. Their construction is similar to the one above for Anosov diffeomorphisms. The measures μ_{W^s} on the stable manifolds are constructed using conditional measures induced by the invariant measure with maximal entropy on Ω_i .

We give an example to show that the orientability conditions are essential. Smale, in the now classical horseshoe example (see [19], §I.5, especially Figures 7 and 13), constructed a diffeomorphism of the two-dimensional sphere S^2 with completely disconnected basic set. In this case, the unstable foliation on the basic set naturally possesses infinitely many orientations, but no orientation compatible with the action of the diffeomorphism exists. The topological entropy in Smale's example is equal to $\log 2$, while the spectral radius of the induced operator on the homology (in both the absolute and relative cases) is equal to 1.

Plikin [20] subsequently constructed an Axiom A diffeomorphism of S^2 with a one-dimensional attracting basic set on which the unstable subfoliation is not orientable. In this example the topological entropy is also positive, while a neighborhood of the basic set is contractible and, therefore, the spectral radius of the operator on homology is also equal to 1.

Finally, we remark that even if the orientability condition is satisfied on all the basic sets, it can happen that $h(f) > \log s(f_*)$, because

$$s(f_*) < \max_i s(f_*^{(i)}).$$

Gibbons [24] gives an example of this type on the three-dimensional sphere in which there are two basic sets, an attracting and a repelling solenoid, and the topological entropy is positive. The nontrivial relative one-dimensional cycles vanish under passage to absolute homology.

10. We conclude by mentioning some other unsolved problems connected with the topological entropy. In [4] it was shown that some isotopy classes of

diffeomorphisms may fail to contain an Axiom A diffeomorphism f satisfying the strong transversality condition and the equality $h(f) = \log s(f_*)$.

PROBLEM. *Does every isotopy class of diffeomorphisms contain a diffeomorphism f for which $h(f) = \log s(f_*)$? In particular, does each isotopy class of diffeomorphisms for which $s(f_*) = 1$ contain a diffeomorphism with topological entropy equal to zero?*

It is known [25] that the topological entropy is neither continuous nor even upper or lower semicontinuous on $\text{Diff}^r(M)$.

PROBLEM (see [21], Problem 41). *Is the topological entropy continuous on a second category set in $\text{Diff}^r(M)$?*

We have already mentioned that the topological entropy of "good" diffeomorphisms can be computed in terms of the asymptotics of the number of periodic points. This is not true for arbitrary diffeomorphisms (see [1]).

PROBLEM. *Does the inequality*

$$h(f) \leq \overline{\lim}_{n \rightarrow \infty} \frac{\log N_n(f)}{n}$$

hold for diffeomorphisms in a second category set in $\text{Diff}^r(M)$?

Many interesting unsolved problems pertaining to smooth dynamical systems are contained in the list of 50 problems compiled by Palis and Pugh [21]. These problems reflect the main directions in the theory of dynamical systems and the various questions discussed at the symposium at the University of Warwick in 1974.

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