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PERIODIC CYCLE FUNCTIONALS AND COCYCLE RIGIDITY FOR CERTAIN PARTIALLY HYPERBOLIC $\mathbb{R}^k$ ACTIONS

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Abstract. We give a proof of cocycle rigidity in Hölder and smooth categories for Cartan actions on $SL(n, \mathbb{R})/\Gamma$ and $SL(n, \mathbb{C})/\Gamma$ for $n \geq 3$ and Γ cocompact lattice, and for restrictions of those actions to subspaces which contain a two-dimensional plane in general position. This proof does not use harmonic analysis, it relies completely on the structure of stable and unstable foliations of the action. The key new ingredient is the use of the description of generating relations in the group SL_n.

1. Cocycles and cocycle rigidity.

1.1. **Definitions.** Let $\alpha : \mathbb{R}^k \times M \to M$ be an action of $\mathbb{R}^k$ on a compact Riemannian manifold M. If H is a topological group then a *cocycle* (or an one-cocycle) over the action α with values in H is a continuous function $\beta : \mathbb{R}^k \times M \to H$ satisfying:

$$\beta(a + b, x) = \beta(a, \alpha(b, x))\beta(b, x) \tag{1.1}$$

for any $a, b \in \mathbb{R}^k$. A cocycle is *cohomologous to a constant cocycle* (cocycle not depending on x) if there exists a homomorphism $\pi : \mathbb{R}^k \to H$ and a continuous *transfer map* $h : M \to H$ such that for all $a \in \mathbb{R}^k$

$$\beta(a, x) = h(\alpha(a, x))\pi(a)h(x)^{-1}. \tag{1.2}$$

In particular, a cocycle is a *coboundary* if it is cohomologous to a trivial cocycle $\pi(a) = id_H$ i.e. if for all $a \in \mathbb{R}^k$ the following equation holds:

$$\beta(a, x) = h(\alpha(a, x))h(x)^{-1} \tag{1.3}$$

Finally, in the last section we will use a somewhat weaker property than that given by (1.2) whose implications are not completely clear. Let $\tilde{M}$ be the universal cover of M. We will say that a cocycle β is *virtually cohomologous to a constant cocycle* if there exists a homomorphism $\pi : \mathbb{R}^k \to H$ and a continuous map from the universal cover $h : \tilde{M} \to H$ such that for all $a \in \mathbb{R}^k$

$$\beta(a, x) = h(\alpha(a, x))\pi(a)h(x)^{-1}. \tag{1.4}$$

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Definition 1. An action α is $C^{p,q}$ *H-cocycle rigid* if *any* C^p cocycle over α with values in H is cohomologous to a constant cocycle via a C^q transfer map. We will write p=Lip when we wish to refer to Lipschitz cocycles or functions.

In case $p = q = \infty$ we will say that the action α is C^∞ *H-cocycle rigid*.

We will say that α is *Hölder H-cocycle rigid* if any Hölder cocycle over α with values in H is cohomologous to a constant cocycle via a Hölder transfer map.

In this paper, with the exception of the last section, we will assume that the cocycles are real-valued and hence use additive notation.

Let $G = SL(n, \mathbb{K})$ where $\mathbb{K}$ equals either $\mathbb{R}$ or $\mathbb{C}$. Let $D_n^+ \subset G$ be the subgroup of diagonal matrices with positive elements. It is convenient to parametrize D_n^+ as

$$D_n^+ = \{diag(e^{t_1}, \dots, e^{t_n}) : \quad \mathbf{t} = (t_1, .., t_n) \in \mathbb{R}^n, \quad \sum_{i=1}^n t_i = 0\}$$

The $n-1$-dimensional subspace of $\mathbb{R}^n$ of all $\mathbf{t} = (t_1, .., t_n)$ with $\sum_{i=1}^n t_i = 0$ will be denoted by $\mathbb{D}_n$.

We will say that a two-dimensional subspace $P \subset \mathbb{D}_n$ is *in general position* if P intersects each hyperplane given by the equation $t_i = t_j,\ i \neq j$ by a different line. There is a natural correspondence between subspaces of $\mathbb{D}_n$ and closed connected subgroups of D_n^+ via exponentiation.

Let $\Gamma \subset G$ be a cocompact lattice (a discrete subgroup of finite co-volume so that the quotient space is a compact manifold). We will consider the action of D_n^+ on the space G/Γ by left translations (those are sometimes called *Cartan actions*) and restrictions of that action to various subgroups of D_n^+. Cartan actions on $SL(n, \mathbb{R})/\Gamma$ are Anosov actions of $\mathbb{R}^{n-1}$ and for $n \geq 3$ they are particular cases of Weyl chamber flows [11]. Cartan actions on $SL(n, \mathbb{C})/\Gamma$, as well as restrictions of Cartan actions on both $SL(n, \mathbb{R})/\Gamma$ and $SL(n, \mathbb{C})/\Gamma$ to proper closed subgroups of D_n^+, are partially hyperbolic actions.

1.2. **The main result.** The purpose of this paper is to develop a new method for proving cocycle rigidity for various classes of cocycles. Usefulness of this method is illustrated by the following result.

Theorem 1. *Let $S \subset \mathbb{D}_n$ be any subspace which contains a two-dimensional subspace in general position and let $\exp S \subset D_n^+$ be the corresponding subgroup of D_n^+. Then the action of $\exp S$ on G/Γ by left translations is both Hölder $\mathbb{R}$-cocycle rigid and C^∞ $\mathbb{R}$-cocycle rigid.*

In the last section we partially extend this result to cocycles with values in other groups.

The C^∞ part of Theorem 1 is not new although the method of proof is. In fact, we deduce the C^∞ result from the Hölder result. The latter is new in all partially hyperbolic cases, i.e. in all situations which we consider here with the exception of the Weyl chamber flows on $SL(n, \mathbb{R})/\Gamma$.

1.3. **History, method and motivation.** In their first paper on rigidity of Abelian actions [11], Katok and Spatzier proved C^∞ $\mathbb{R}$-cocycle rigidity for the class of standard Anosov actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$, $k \geq 2$ which includes all Weyl chamber flows. Their proof is based on harmonic analysis of semisimple Lie groups. Specifically, exponential decay of matrix coefficients is used to show that every cocycle over the action is cohomologous to a constant cocycle. Smoothness of the transfer function follows then from the fact that it is smooth along stable, unstable and orbit directions and that those generate the tangent space at every point. In their second paper [12] the same harmonic analysis argument is used to show cocycle rigidity for a large class of partially hyperbolic homogeneous actions. All actions considered in that paper satisfy the following property which is also essential for our proof of Theorem 1:

($\mathfrak{N}$) *Stable and unstable directions of regular elements and the orbit direction of the action are totally non-integrable together, i.e. their brackets of all orders generate the tangent space.*

In this case a general, albeit more sophisticated than in the hyperbolic case, elliptic operator theory argument is used to obtain smoothness of the transfer function.

The harmonic analysis approach is restricted to algebraic actions, therefore in order to study cohomological equations over general actions different techniques are needed. For hyperbolic maps and flows some geometric techniques have been available for a long time beginning from the classical work of A. Livshitz (also spelled Livšic) from the early seventies.

In [9] Katok and Kononenko introduced a geometric approach for studying cocycles over partially hyperbolic systems satisfying a certain version of accessibility property. That property was introduced by Brin and Pesin [1] and has proved very fruitful in the study of dynamics and ergodic theory of partially hyperbolic systems. For classical systems (actions of $\mathbb{Z}$ and $\mathbb{R}$) cocycle rigidity is very rare and is known to appear only for Diophantine translations and linear flows on the torus [8, Section 11]. The purpose of [9] was to show a weaker property of cocycle stability for such systems. The cohomology invariants described in [9], the *periodic cycle functionals,* provide an infinite (and complete) set of obstructions for solving the cohomology equation in these cases.

In this paper we extend the geometric approach of [9] to a class of partially hyperbolic actions of higher rank abelian groups. Although at present we apply this approach to prove cocycle rigidity for some *algebraic* models as described in Theorem 1, the general criteria we present such as Proposition 6, unlike the harmonic analysis methods, may be applied with proper modifications to more general situations including perturbations of algebraic actions. This opens a prospect of proving a version of *local differentiable rigidity* for new classes of partially hyperbolic actions, e.g. restrictions of the Weyl chamber flow on $SL(n,\mathbb{R})/\Gamma$ described in Theorem 1.

This should work as follows: by the foliation rigidity, which can be proven by the non-stationary normal forms method [5, 13], every perturbation is differentiably conjugate to a perturbation which preserves the orbit foliation of the Weyl chamber flow, i.e the neutral foliation of the unperturbed action. This foliation has linear structure and the conjugacy equation reduces to a vector-valued cocycle equation over the *perturbed* action. Such an equation reduces to a system of $\mathbb{R}$ valued equations, one for each coordinate. Thus, Hölder cocycle rigidity for the perturbed action would provide a Hölder conjugacy with the restriction of the Weyl chamber flow to a nearby subspace. Then again elliptic operator theory would guarantee smoothness of the conjugacy.

Notice that so far a version of local differentiable rigidity have been proven only for one class of partially hyperbolic actions which are not Anosov, namely for actions by automorphisms of a torus and for suspensions of such actions [3, 2]. In those cases geometric methods are not applicable because condition ($\mathfrak{N}$), the prerequisite for accessibility which is the starting point for the geometric arguments, does not hold. The method used in [3, 2] combines KAM-type iteration process with tame estimates for solutions of cohomological equations obtained by Fourier analysis methods. In order to pursue a similar approach in the semisimple cases (like those considered in the present paper) one would need detailed information about unitary representations of simple Lie groups involved such as $SL(n,\mathbb{R})$ for $n \geq 4$ which does not seem to be readily available.

A method somewhat similar to an infinitesimal version of the method presented here appeared earlier in the unpublished and unfinished paper [4]. In that paper smooth cocycle rigidity was proved for Weyl chamber flows for simple Lie groups without any use of harmonic analysis. There the derivative of the future solution of the cohomological equation is constructed as a differential form which is then proved to be first closed and finally exact.

The algebraic structure of a simple Lie group is essential for the arguments in [4] so those arguments do not look suitable for an extension to perturbations. It is also not clear whether this method can be extended to restrictions of Weyl chamber flows to subgroups. Another difference is that due to its infinitesimal nature of the method it is restricted to smooth cocycles.

On the other hand, the Lie group information used in [4] is less specific and the argument works for the Weyl chamber flow on any simple Lie group while our method requires specific information for each case and groups other than $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$ have not been treated yet.

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2. **Coarse Lyapunov foliations.** For an action α of $\mathbb{R}^k$ on a compact manifold M by diffeomorphisms preserving an ergodic probability measure μ there are finitely many linear functionals λ on $\mathbb{R}^k$, a set of full measure Λ and a measurable splitting of the tangent bundle $T_\Lambda M = \bigoplus_\lambda E^\lambda$, such that for $v \in E^\lambda$ and $a \in \mathbb{R}^k$ the Lyapunov exponent of v with respect to a is $\lambda(a)$.

If χ is a non-zero Lyapunov functional then we define its *coarse Lyapunov subspace* by $E_\chi := \bigoplus_{\{\lambda = c\chi : c>0\}} E^\lambda$.

For every $a \in \mathbb{R}^k$ we define stable, unstable and neutral subspaces for a by: $E_a^s = \bigoplus_{\lambda(a)<0} E^\lambda$, $E_a^u = \bigoplus_{\lambda(a)>0} E^\lambda$ and $E_a^0 = \bigoplus_{\lambda(a)=0} E^\lambda$. In particular, for any $a \in A := \bigcap_{\chi \neq 0} (Ker\chi)^c$ the subspace E_a^0 is the same and thus can be denoted simply by E^0. Hence we have for any such a:

$$TM = E_a^s \oplus E^0 \oplus E_a^u.$$

Now assume that E^0 is a continuous distribution uniquely integrable to a foliation $\mathcal{N}$ with smooth leaves, and that for some $a \in A$ the map $\alpha(a)$ is normally hyperbolic with respect to $\mathcal{N}$ in the sense of the Hirsch-Pugh-Shub theory [6]. We will call such an element a an*$\mathcal{N}$-normally hyperbolic element* of the action. Then by stability theorems of [6] the set of $\mathcal{N}$-normally hyperbolic elements is open in $\mathbb{R}^k$. We call such actions *$\mathcal{N}$-normally hyperbolic*. In particular, if E^0 is the tangent distribution to the orbit foliation $\mathcal{O}$ of the action and some element of the action is $\mathcal{O}$-normally hyperbolic then the element is called *Anosov* or *regular* and the action is Anosov. Otherwise the action is only partially hyperbolic.

It is not known whether the set of $\mathcal{O}$-normally hyperbolic elements is necessarily dense in $\mathbb{R}^k$. All known rigidity results for $\mathbb{R}^k$ ($k \geq 2$) actions are obtained for actions for which the set of normally hyperbolic elements is dense.

Therefore we assume that the set $\tilde{A}$ of $\mathcal{N}$-normally hyperbolic elements for the action α is dense in $\mathbb{R}^k$. Stable and unstable distributions E^s_a and E^u_a for any element $a \in \tilde{A}$ are Hölder and they integrate to Hölder foliations $\mathcal{W}^s_a$ and $\mathcal{W}^u_a$ with smooth leaves. We will need however certain finer foliations preserved by all elements of the action. For $a \in \tilde{A}$ any other $b \in \tilde{A}$ preserves $\mathcal{W}^s_a$ and $\mathcal{W}^u_a$ as well as $\mathcal{W}^s_b$ and $\mathcal{W}^u_b$ because of commutativity and a dynamical characterization of these foliations. Then both a and b preserve the intersections. These intersections of stable leaves for all normally hyperbolic elements of the action which lie in the same Lyapunov half space for some non-zero Lyapunov functional, are actually leaves of a foliation whose tangent distribution is the coarse Lyapunov distribution corresponding to that Lyapunov functional. This is proved in [7] for Anosov actions whose set of Anosov elements is dense, and the same proof can be carried out *verbatim* for the $\mathcal{N}$-normally hyperbolic case.

Proposition 1. *Let α be an $\mathbb{R}^k$ action preserving an ergodic probability measure μ with full support and such that the set $\tilde{A}$ of $\mathcal{N}$-normally hyperbolic elements is dense. Then:*

1. *For each non-zero Lyapunov exponent χ and every $p \in \Lambda$ the coarse Lyapunov distribution is:*
$$E_\chi(p) = \bigcap_{\{a \in \tilde{A} : \chi(a) < 0\}} E^s_a(p).$$
The right-hand side is Hölder continuous and thus E^χ can be extended to a Hölder distribution tangent to the foliation $\mathcal{F}_\chi := \bigcap_{\{a \in \tilde{A} : \chi(a) < 0\}} \mathcal{W}^s_a$ with C^∞ leaves.
2. *The hyperplanes $Ker\chi$ for non-zero χ and the connected components of the set A (Weyl chambers) agree for all invariant measures.*
3. $A = \tilde{A}$

We denote by $\chi_1, .., \chi_r$ a maximal collection of non-zero Lyapunov exponents that are not positive multiples of one another and by $\mathcal{F}_1, ..., \mathcal{F}_r$ the corresponding coarse Lyapunov foliations.

3. Periodic cycle functionals.

3.1. **Definition and properties.** For a Riemannian manifold M and $x, y \in M$ denote by $d_M(x, y)$ the infimum of lengths of smooth curves in M connecting x and y. Similarly, for a smooth submanifold M' of M and $x, y \in M'$ denote by $d_{M'}(x, y)$ infimum of lengths of smooth curves in M' connecting x and y.

Denote the δ-ball in M centered at $x \in M$ by

$$B_M(x,\delta) := \{y \in M : d_M(x,y) < \delta\}$$

and call an $\mathcal{F}_{1,..,r}$-*path* any ordered set of points in M such that each two consecutive points lie in a single leaf of one of the foliations: $\mathcal{F}_1, .., \mathcal{F}_r$.

Definition 2. Let $\mathcal{F}_1, .., \mathcal{F}_r$ be continuous foliations of a compact manifold M with smooth leaves. The collection $\mathcal{F}_1, .., \mathcal{F}_r$ is called *locally transitive* if there exists $N \in \mathbb{N}$ such that for any $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in M$ and for every $y \in B_M(x,\delta)$ there are points $x = x_1, x_2, .., x_{N-1}, x_N = y$ in the ball $B_M(x,\epsilon)$ such that $x_{i+1} \in \mathcal{F}_{j(i)}(x_i)$ and $d_{\mathcal{F}_{j(x_i)}}(x_{i+1}, x_i) < 2\epsilon$ for $i = 1, .., N$, and $j(i) \in \{1, .., r\}$.

In other words, any two sufficiently close points can be connected by a $\mathcal{F}_{1,..,r}$-path of not more than N pieces of a given bounded length.

Definition 3. An ordered set of points $x_1, .., x_N, x_{N+1} = x_1 \in M$ is called an $\mathcal{F}_{1,..,r}$-*cycle* of length N if for every $i = 1, .., N$, there exists $j(i) \in \{1, .., r\}$ such that $x_{i+1} \in \mathcal{F}_{j(i)}(x_i)$.

Let α be an $\mathbb{R}^k$ action satisfying conditions of Proposition 1 with the coarse Lyapunov foliations $\mathcal{F}_1, .., \mathcal{F}_r$. Let $\beta : \mathbb{R}^k \times M \to \mathbb{R}$ be a Hölder cocycle over α. To simplify notations we will often denote the action $\alpha(a,x)$ by ax , $\alpha(ka,x)$ by kax for $k \in \mathbb{N}$, and for $a \in \mathbb{R}^k$ the function $\beta(a,\cdot) : M \to \mathbb{R}$ will be denoted by f_a.

Definition 4. Let $j \in \{1, .., r\}$, $a \notin Ker\chi_j$, $y \in \mathcal{F}_j(x)$ and β be a Hölder cocycle cocycle over α. The β-*potential of* y *with respect to* x is defined by

$$P_a^j(x,y)(\beta) = \begin{cases} \displaystyle\sum_{k=0}^{\infty}(f_a(kax) - f_a(kay)), & \chi_j(a) < 0, \\ \displaystyle -\sum_{-\infty}^{k=-1}(f_a(kax) - f_a(kay)), & \chi_j(a) > 0. \end{cases}$$

This can be written in the more compact form as follows:

$$P_a^j(x,y)(\beta) = * \sum^{*}(f_a(kax) - f_a(kay)), \tag{3.1}$$

where $ax := \alpha(a,x)$, $* := *(j,a) := -sgn(\chi_j(a)) \in \{+,-\}$,

$$\sum^{+} := \sum_{k=0}^{\infty}, \text{ and } \sum^{-} := \sum_{k=-1}^{-\infty}.$$

We will use this notation in certain calculations below.

Each sum above converges absolutely for any Hölder cocycle β and for any $a \notin Ker\chi_j$ because the terms decay exponentially. Moreover, if β is C^∞ then the series can be differentiated implying that the potential $P_a^j(x,y)(\beta)$ is C^∞ with respect to y, along the leaves of the foliation $\mathcal{F}_j$. These and other important properties of the potential function $P_a^j(x,y)(\beta)$ are summarized as follows.

Proposition 2.
1. *If β is p-Hölder continuous (smooth) then $P_a^j(x,y)(\beta)$ is continuous in both variables and p-Hölder continuous (smooth) in $y \in \mathcal{F}_j(x)$ i.e. along the leaves of $\mathcal{F}_j$.*
2. $P_a^j(x,y)(\beta) = P_b^j(x,y)(\beta)$ *for any* $a, b \notin Ker\chi_j$.

Proof. For the statement (1) continuity in x and y follows from the uniform contraction of stable foliations. The Hölder part follows from the inequalities

$$|P_a^j(x,y)(\beta)| \leq \sum^* |f_a(kax) - f_a(kay)| \leq \sum^* C(\beta) d_M(kax, kay)^p$$
$$\leq \sum^* C(\beta)\rho^{*kp} d_M(x,y)^p \leq C(\beta, a) d_M(x,y)^p,$$

where $\rho < 1$ is a contraction coefficient for the contracting directions for a and $-a$. Smoothness follows from the fact that each term in the series in (3.1) can be differentiated, and, since the foliation $\mathcal{F}_j$ is contracted under the action a, the series for the derivatives converges exponentially and can be differentiated again.

To show (2) we first notice that if β is a cocycle over α then for any $a, b \in \mathbb{R}^k$ we have:

$$f_b \circ a - f_b = f_a \circ b - f_a. \tag{3.2}$$

Substituting this into the definition of $P_b^j(ax, ay)$ it is easy to compute that for a fixed χ_j and $a, b \notin Ker\chi_j$ we have

$$P_b^j(ax, ay)(\beta) = P_b^j(x,y)(\beta) - (f_a(x) - f_a(y))$$

for any $y \in \mathcal{F}_j(x)$ (since foliations $\mathcal{F}_j$ are invariant under the action this implies $ay \in \mathcal{F}_j(ax)$). It follows form the definition of P_a^j, that:

$$P_a^j(ax, ay)(\beta) = P_a^j(x,y)(\beta) - (f_a(x) - f_a(y))$$

Thus the difference function $R_{a,b}^j(x,y) := P_b^j(x,y) - P_a^j(x,y)$ satisfies:

$$R_{a,b}^j(ax, ay)(\beta) = R_{a,b}^j(x,y)(\beta).$$

Therefore:

$$R_{a,b}^j(nax, nay)(\beta) = R_{a,b}^j(x,y)(\beta)$$

for every $n \in \mathbb{Z}$ and since by (1) $R_{a,b}^j(x,y)(\beta)$ is Hölder, by letting $n \to \infty$ $(n \to -\infty)$ if $\chi_j(a) < 0$ $(\chi_j(a) > 0)$, the left hand side of the expression above tends to zero. Thus $R_{a,b}^j(x,y)(\beta) = 0$ and this implies (2). $\square$

Definition 5. Given $a \in \tilde{A} = A := \bigcap_{j=1}^{r} (Ker\chi_j)^c$ and an $\mathcal{F}_{1,..,r}$-cycle $\mathcal{C}$ of length N, following [9], define the *periodic cycle functional* on the space of Hölder cocycles over α by

$$F_a(\mathcal{C})(\beta) = \sum_{i=1}^{N} P_a^{j(i)}(x_i, x_{i+1})(\beta)$$

where as in Definition 2 for each $i = 1,..,N$, $j(i) \in \{1,..,r\}$ is such that $x_{i+1} \in \mathcal{F}_{j(i)}(x_i)$.

Remark 1. Due to (2) of Proposition 2 for any $a, b \in A$ we have $F_a(\mathcal{C})(\beta) = F_b(\mathcal{C})(\beta)$ so we may write $F(\mathcal{C})(\beta)$ instead of $F_a(\mathcal{C})(\beta)$. This also implies invariance of periodic cycle functionals, namely:

$$F_a(\mathcal{C})(\beta) = F_a(b\mathcal{C})(\beta) \tag{3.3}$$

for any $a, b \in A$ since by definition $F_b(\mathcal{C})(\beta) = F_b(b\mathcal{C})(\beta)$. In fact, it is easy to see by direct computation that in (3.3) it is enough to assume that a is regular i.e. that $a \in A$.

3.2. **Vanishing of periodic cycle functionals and trivialization of cocycles.**

Proposition 3. *If β is C^0 cohomologous to a constant cocycle then all periodic cycle functionals vanish i.e. for every $\mathcal{F}_{1,..,r}$-cycle $\mathcal{C}$ we have $F(\mathcal{C})(\beta) = 0$.*

Proof. Suppose β is C^0 cohomologous to a constant cocycle i.e. there is a continuous transfer function h on M and a map $\pi : \mathbb{R}^k \to \mathbb{R}$ such that for all $a \in \mathbb{R}^k$:

$$\beta(a, x) = h(ax) - h(x) + c(a) \tag{3.4}$$

Let $\mathcal{C} = (x_1, ..., x_{N+1} = x_1)$ be a $\mathcal{F}_{1,..,r}$-cycle $\mathcal{C}$ in M. Then for $a \in A$:

$$\begin{aligned} P_b^{j(i)}(x_i, x_{i+1})(\beta) &= * \sum_k^* f_a(kax_i) - f_a(kax_{i+1}) \\ &= * \sum^* [h((k+1)ax_i) - h(kax_i) - h((k+1)ax_{i+1}) + h(kax_{i+1})] \\ &= h(x_i) - h(x_{i+1}) \end{aligned}$$

Therefore $F(\mathcal{C})(\beta) = h(x_1) - h(x_2) + h(x_2) - h(x_3) + .. + h(x_N) - h(x_1) = 0$. $\square$

As we shall see, local transitivity of the foliations $\mathcal{F}_1, .., \mathcal{F}_r$ along with vanishing of all periodic cycle functionals will be sufficient for C^0-cocycle rigidity of the action. However, to obtain better regularity results we will need a stronger transitivity assumption on the foliations $\mathcal{F}_1, .., \mathcal{F}_r$.

Definition 6. Let $0 < p < 1$.

The foliations $\mathcal{F}_1, .., \mathcal{F}_r$ are *locally p-Hölder transitive* if there exists $\delta > 0$ and a constant $L > 0$ such that for every $x \in M$ and every $y \in B_M(x, \delta)$ there is an $\mathcal{F}_{1,..,r}$-path $x = x_1, x_2, .., x_N = y$ such that:

$$\sum_{i=1}^{i=N-1} d_{\mathcal{F}_{j(i)}}(x_i, x_{i+1}) < L d_M(x, y)^p.$$

The foliations $\mathcal{F}_1, .., \mathcal{F}_r$ are *weakly locally p-Hölder transitive* if there exists $\delta > 0$ and a constant $L > 0$ such that for every $x \in M$ and every $y \in B_M(x, \delta)$ there is an $\mathcal{F}_{1,..,r}$-path $x = x_1, x_2, .., x_N = y$ such that:

$$d_{\mathcal{F}_{j(i)}}(x_i, x_{i+1}) < L d_M(x, y)^p \quad \text{for all} \quad i \in \{1, .., N-1\}.$$

The following statement is the counterpart in our setting of Theorems 1 and 2 of [9]. The proof follows the same line.

Proposition 4. *Let α be an $\mathbb{R}^k$ action by diffeomorphisms on a compact Riemannian manifold M such that a dense set of elements of $\mathbb{R}^k$ acts normally hyperbolically with respect to an invariant foliation. Let β be a p-Hölder cocycle over the action α such that $F(\mathcal{C})(\beta) = 0$ for all $\mathcal{F}_{1,..,r}$-cycles $\mathcal{C}$. Then the following hold:*

1. *If the foliations $\mathcal{F}_1, .., \mathcal{F}_r$ are locally transitive then β is cohomologous to a constant cocycle via a continuous transfer function.*
2. *If the foliations $\mathcal{F}_1, .., \mathcal{F}_r$ are locally q-Hölder transitive then the transfer function is pq-Hölder.*
3. *If the foliations $\mathcal{F}_1, .., \mathcal{F}_r$ are weakly locally q-Hölder transitive and β is Lipschitz then the transfer function is q-Hölder.*

Proof. After fixing a point $x \in M$ and a normally hyperbolic element $a \in A$, for arbitrary $y \in M$ we define

$$F_a(S(x,y))(\beta) := \sum_{i=1}^{m} P_a^{j(i)}(x_i, x_{i+1})(\beta)$$

where $S(x,y)$ is some $\mathcal{F}_{1,..,r}$-path connecting x and y. For any two points x and y in M such a path exists due to the (local) transitivity of foliations $\mathcal{F}_1, .., \mathcal{F}_r$. Then from the assumption that $F_a(\mathcal{C})(\beta) = 0$ for any such closed path i.e. for any $\mathcal{F}_{1,..,r}$-cycle $\mathcal{C}$, we deduce that $F_a(S(x,y))(\beta)$ depends only on the points x and y and not on the choice of the path $S(x,y)$ and so we may write $F_a(x,y)$ instead. From now on we will drop dependence on β in out notations. Thus we can uniquely define a function $h : M \to \mathbb{R}$ by:

$$h(y) = F_a(x,y) \tag{3.5}$$

Because of the property (2) of Proposition 2 and *local* transitivity of the foliations the map h is continuous (*only* continuous, since length of a path connecting x and y may be large). To obtain Hölder regularity, Hölder local transitivity is needed and statements (2) and (3) are obtained exactly as in [9]).

To check that h satisfies the coboundary equation (3.4) we fix a $\mathcal{F}_{1,..,r}$-path $S(x,y) = (x = x_1, .., y = x_m)$ from x to y and note that $aS(x,y) = (ax = ax_1, .., ay = ax_m)$ is a $\mathcal{F}_{1,..,r}$-path connecting ax and ay due to the invariance of the foliations. Then

$$\begin{aligned} h(ay) &= h(ax) + F_a(ax, ay) \\ &= h(ax) + \sum_{i=1}^{m-1} P_a^{j(i)}(ax_i, ax_{i+1}) \\ &= h(ax) + \sum_{i=1}^{m-1} [* \sum_{k}^{*} f_a((k+1)ax_i) - f_a((k+1)ax_{i+1})] \\ &= h(ax) + F_a(x,y) - \sum_{i=1}^{m-1} (f_a(x_i) - f_a(x_{i+1})) \\ &= h(y) + f_a(y) + c(a) \end{aligned}$$

where $c(a) := h(ax) - f_a(x)$. This implies that h is a (continuous, Hölder) transfer function for $f_a = \beta(a, \cdot)$.

Now we show that for any $b \in \mathbb{R}^k$ the same transfer function makes f_b cohomologous to a constant and thus makes β cohomologous to a constant map.

First, assume that b is also a regular element. Since potentials for a and b coincide by Proposition 2(2), the function F_b defined as in (3.5) with b instead of a is the same. By the above argument the cohomology equation (3.4) is satisfied for any regular b. But since regular elements are dense by Proposition 1(3) and both sides of (3.4) are continuous in both variables, this equation is satisfied identically. □

3.3. **Vanishing of the functionals and allowed substitutions.**

Proposition 5. *If a $\mathcal{F}_{1,..,r}$-cycle $\mathcal{C}$ is completely contained in a stable leaf for some element of the action then for any Hölder continuous cocycle β, $F(\mathcal{C})(\beta) = 0$.*

Proof. This is an immediate consequence of the definition of $F(\mathcal{C})(\beta)$ and the invariance property (3.3). □

This motivates the following construction. Consider a $\mathcal{F}_{1,..,r}$-path $\mathcal{P}$: $x = x_1, x_2, \ldots, x_{N-1}, x_N = y$. Suppose for some i, j $1 \leq i < j \leq N$ all points x_k, $k = i, i+1, \ldots, j$ lie in the same stable leaf for some element of the action. Let $x_i = x'_1, x'_2, \ldots, x'_s = x_j$ be a $\mathcal{F}_{1,..,r}$-path which lies in the same stable manifold as $x_i, \ldots, x_j$. Define the path $\mathcal{P}'$ as $x, \ldots, x_i, x'_2, \ldots, x'_{s-1}, x_j, \ldots, y$.

Definition 7. A substitution of a $\mathcal{F}_{1,..,r}$-path $\mathcal{P}$ from x to y by any $\mathcal{F}_{1,..,r}$-path $\mathcal{P}'$ connecting x and y obtained as above is called an *allowed substitution*.

Now we introduce the key ingredient which distinguishes the higher rank situation from the rank one case considered in [9].

Definition 8. A sequence of $\mathcal{F}_{1,..,r}$-cycles $\mathcal{C} = \mathcal{C}_1, \ldots \mathcal{C}_m$ constitutes a *reduction via allowed substitutions* of $\mathcal{C}$ if the substitution of $\mathcal{C}_i$ by $\mathcal{C}_{i+1}$ is an allowed substitution for $i = 1, \ldots, m-1$.

In particular, if $\mathcal{C}_m$ is a trivial one-point cycle the reduction is called a *trivialization via allowed substitutions.*

Now based on Proposition 5 above we conclude that trivializing a cycle via allowed substitutions implies vanishing of the corresponding periodic cycle functional, thus by Proposition 4 we have:

Proposition 6. *Let α be an action as described in Proposition 4 with coarse Lyapunov foliations $\mathcal{F}_1, .., \mathcal{F}_r$. Assume that every $\mathcal{F}_{1,..,r}$-cycle can be trivialized via allowed substitutions. Then the action α is:*

1. $C^{p,0}$ $\mathbb{R}$*-cocycle rigid if the foliations $\mathcal{F}_1, .., \mathcal{F}_r$ are locally transitive,*
2. $C^{p,pq}$ $\mathbb{R}$*-cocycle rigid if the foliations $\mathcal{F}_1, .., \mathcal{F}_r$ are locally q-Hölder transitive,*
3. $C^{Lip,q}$ $\mathbb{R}$*-cocycle rigid if the foliations $\mathcal{F}_1, .., \mathcal{F}_r$ are weakly locally q-Hölder transitive.*

This proposition indicates a method of establishing cocycle rigidity by showing that any cocycle trivializes via vanishing of periodic cycle functionals. There are two principal classes of actions to which this method can be successfully applied. We briefly describe the first class now, and give a detailed description and a proof of cocycle rigidity for the second class of examples in subsequent sections.

Let $a \in A$ and let $\mathcal{C} = (x_1, ..x_N, x_{N+1} = x_1)$ be a $\mathcal{F}_{1,..,r}$-cycle such that $x_{i+1} \in \mathcal{F}_{j(i)}(x_i)$. For any two distinct $j_1, j_2 \in \{1, .., r\}$ the hyperplanes $\chi_{j_1} = 0$ and $\chi_{j_2} = 0$ divide $\mathbb{R}^k$ either into four regions (if χ_{j_1} and χ_{j_2} are not proportional) or into two regions (if χ_{j_1} and χ_{j_2} are negatively proportional). In the first case for $\chi_{j(i)}$ and $\chi_{j(i+1)}$ it is clearly not possible to have $\chi_{j(i)}(a)\chi_{j(i+1)}(a) < 0$ for all $a \in A$. Thus for some $a \in A$ both $F_{j(i)}$ and $F_{j(i+1)}$ are stable. Therefore if within this stable leaf one can substitute an $\mathcal{F}_{j(i),j(i+1)}$-path by a $\mathcal{F}_{j(i+1),j(i)}$-path, this is an allowed substitution.

This approach works for TNS (totally non-symplectic) actions on tori and nilmanifolds. Namely, the TNS condition asserts that no two Lyapunov exponents are negatively proportional. It follows from this condition that the local structure of stable foliations for different elements of the action implies that allowed substitutions described above trivialize every small contractible cycle. For large cycles one passes to the universal cover and uses the fact that Anosov maps on tori and nilmanifolds have no invariant elements in the first homology.

See [10] for precise definitions and a discussion of properties of TNS actions, as well as for an infinitesimal version of the proof of cocycle rigidity outlined above.

For actions which do have negatively proportional Lyapunov functionals the linear algebra of Lyapunov exponents is not sufficient and one has to use non-commutativity of Lyapunov foliations. This is possible for Weyl chamber flows and some partially hyperbolic actions derived from those. However, we will need to use not just allowable substitutions but their limits also. So in fact we will not be referring to Proposition 6 but directly to Proposition 4.

4. **Generic $\mathbb{R}^2$ subactions.** We return to the setting of Section 2. Let α be an action of $\mathbb{R}^k$ by diffeomorphisms on a compact Riemannian manifold M such that the set A consists of $\mathcal{N}$-normally hyperbolic elements, where $\mathcal{N}$ is an invariant foliation. Consider a two-dimensional subspace $P \subset \mathbb{R}^k$. If P intersects each two distinct Lyapunov hyperplanes along distinct lines (as in Section 1) we will say that this subspace is *in general position*. Denote the action restricted to P by γ . Then γ is also a $\mathcal{N}$-normally hyperbolic action. Moreover, all elements outside of the intersection lines are $\mathcal{N}$-normally hyperbolic. We will show that elements in such a plane P are sufficient to determine all the coarse Lyapunov distributions for α and, moreover, that for any cycle along the corresponding Lyapunov foliations certain allowed substitutions for the action α are also allowed substitutions for the subaction γ.

Proposition 7. *Given an $\mathbb{R}^k$ $\mathcal{N}$-normally hyperbolic action α, for a two-plane $P \subset \mathbb{R}^k$ in general position the following holds:*

1. *For any non-zero Lyapunov exponent χ of the action α the corresponding coarse Lyapunov subspace E_χ coincides with the intersection of the stable subspaces for all $a \in P \cap A$ with $\chi(a) < 0$.*
2. *If an $\mathcal{F}_{i_1,i_2}$-path $(i_1, i_2 \in \{1,..,r\})$ lies in the stable leaf for some element $a \in A$ then it also lies in the stable leaf for some element $b \in P \cap A$.*

Proof. (1) Let χ be a non-zero Lyapunov exponent for α. Then by definition of the coarse Lyapunov subspace corresponding to χ we have

$$E_\chi(p) \subset \bigcap_{a \in P \cap A, \chi(a)<0} E^s_a(p)$$

for any $p \in M$. Now assume that there is a vector $v \in \bigcap_{a \in P \cap A, \chi(a)<0} E^s_a(p)$ so that v is not in $E_\chi(p)$. Then $v = \sum_\lambda v_\lambda$ where $v_\lambda \in E^\lambda$ (See section 2) and since it is not in $E_\chi(p)$ there is some (non-zero) Lyapunov exponent λ not positively proportional to χ, such that $v_\lambda \neq 0$. Since our generic plane P intersects any two distinct Lyapunov hyperplanes along distinct lines, if λ and χ are not positively proportional there is a point $a \in P \cap A$ such that $\lambda(a) > 0$ and $\chi(a) < 0$. Therefore v_λ is not in the stable subspace for a while by assumption it is contained in $\bigcap_{a \in P \cap A, \chi(a)<0} E^s_a(p)$ which gives us a contradiction. If λ and χ are negatively proportional the same conclusion follows.

(2) Suppose a path $\mathcal{C} = (x_1, ..., x_{k+1})$ consists of pieces which lie in leafs of $\mathcal{F}_{i_1}, \mathcal{F}_{i_2}$ and all the points $x_1, .., x_{k+1}$ are in the same stable leaf for some element $a \in A$. The generic plane P intersects the hyperplanes $Ker(\chi_{i_1})$ and $Ker(\chi_{i_2})$ along two distinct lines (since two exponents which are negatively proportional cannot have the corresponding Lyapunov directions stable for any element of the action we get

exactly two lines), therefore in one of the four quadrants in P bounded by these two lines, both exponents χ_{i_1} and χ_{i_2} take negative values. Therefore for any element b in this quadrant in P we have that the initial path $\mathcal{C}$ is contracted by b. $\square$

5. Cocycle rigidity for Cartan actions on $SL(n,\mathbb{R})/\Gamma$ and $SL(n,\mathbb{C})/\Gamma$.

5.1. **Scheme of the proof.** We will prove the statement of Theorem 1 for Hölder cocycles by checking conditions of Proposition 4(2). We will first do that for the full Cartan actions on $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$ for $n \geq 3$ and use Proposition 7 to extend the result for subactions.

The key idea is that for the Cartan actions on $SL(n,\mathbb{R})/\Gamma$ and $SL(n,\mathbb{C})/\Gamma$, many cycles along Lyapunov foliations are generated by a finite set of elementary cycles which are given by commutator relations between elementary unipotent matrices and their conjugates. Each elementary cycle lies in a stable manifold for some element of the action. Thus canceling such a cycle is an allowed substitution, and consequently periodic cycle functionals for all such cycles vanish. Canceling additional relations in the group is possible due to the fact that the periodic cycle functional depend continuously on the cycle. Thus, functionals corresponding to cycles which lift to the universal cover as closed cycles, all vanish. The remaining argument is provided by a reference to the Margulis Normal Subgroup Theorem which guarantees vanishing of all periodic cycle functionals and hence allows us to use Proposition 4.

5.2. **Cartan actions on $SL(n,\mathbb{R})/\Gamma$ and $SL(n,\mathbb{C})/\Gamma$.** Let $d(\cdot,\cdot)$ denote a right invariant metric on $SL(n,\mathbb{R})$ and the induced metric on $SL(n,\mathbb{R})/\Gamma$. A foliation F is isometric under $\alpha^{\mathbf{t}}$ if $d(\alpha^{\mathbf{t}}x,\alpha^{\mathbf{t}}y) = d(x,y)$ whenever $y \in F(x)$. Let $1 \leq i,j \leq n$, $i \neq j$ be two fixed different indices, and let exp be the exponentiation map for matrices. Let $v_{i,j}$ be the elementary $n \times n$ matrix with only one nonzero entry equal to one, namely, that in the row i and the column j. With this we define the foliation $F_{i,j}$, for which the leaf through x

$$F_{i,j}(x) = \big\{\exp(sv_{i,j})x : s \in \mathbb{R}\big\} \tag{5.1}$$

consists of all left multiples of x by matrices of the form $\exp(sv_{i,j}) = \mathrm{Id} + sv_{i,j}$.

The foliation $F_{i,j}$ is invariant under α, in fact a direct calculation shows that

$$\alpha^{\mathbf{t}}\big(\mathrm{Id} + sv_{i,j}\big)x = \big(\mathrm{Id} + se^{t_i - t_j}v_{i,j}\big)\alpha^{\mathbf{t}}x. \tag{5.2}$$

Hence the leaf $F_{i,j}(x)$ is mapped onto $F_{i,j}(\alpha^{\mathbf{t}}x)$ for any $\mathbf{t} \in \mathbb{D}_n$. Consequently the foliation $F_{i,j}$ is contracted (corr. expanded or neutral) under $\alpha^{\mathbf{t}}$ if $t_i < t_j$ (corr. $t_i > t_j$ or $t_i = t_j$). If the foliation $F_{i,j}$ is neutral under $\alpha^{\mathbf{t}}$, it is in fact isometric under $\alpha^{\mathbf{t}}$.

The leaves of the orbit foliation $\mathcal{O}(x) = \{\alpha^{\mathbf{t}}x : \mathbf{t} \in \mathbb{D}_n\}$ can be described similarly using the matrices

$$u_{i,j} = diag(0,\dots,1,\dots,-1,\dots,0)$$

having non-zero entries only at diagonal positions (i,i) and (j,j), $i \neq j$. In fact $\exp(u_{i,j}) = \alpha^{\mathbf{t}}$ for some $\mathbf{t} \in \mathbb{D}_n$.

The tangent vectors to the leaves in (5.1) for various pairs (i,j) together with the orbit directions form a basis of the tangent space at every $x \in X$.

For every $i \neq j$ the equation $t_i = t_j$ defines a Lyapunov hyperplane in $\mathbb{D}_n$ which will be denoted by $H_{i,j}$. Any element of this hyperplane acts on the foliation $F_{i,j}$

by isometries. Notice that $H_{i,j} = H_{j,i}$ and hence each of these subgroups acts by isometries on two foliations: $F_{i,j}$ and $F_{j,i}$. The connected components of

$$A = \mathbb{D}_n \setminus \bigcup_{i \neq j} H_{i,j}$$

are the Weyl chambers of α. For every $\mathbf{t} \in A$ only the orbit directions are neutral; hence $\mathbf{t}$ is a regular element.

Let $I = \{(i,j) : i < j\}$, and let M_I be the span of $v_{i,j}$ for $(i,j) \in I$ (in the Lie algebra $\mathfrak{sl}_n$ of $SL(n,\mathbb{R})$). For the invariant foliation F_I the leaf through x is defined by

$$F_I(x) = \{\exp(w)x : w \in M_I\}. \tag{5.3}$$

Furthermore, there exists a Weyl chamber C such that for every $\mathbf{t} \in C$, the leaf $F_I(x)$ is the unstable manifold for $\alpha^{\mathbf{t}}$. In fact $C = \{\mathbf{t} \in \mathbb{D}_n : t_i > t_j \text{ for all } i < j\}$. This Weyl chamber is called the *positive Weyl chamber*.

For the action of D_n^+ on $SL(n,\mathbb{C})/\Gamma$ the individual expanding and contracting foliations are similarly given by:

$$F_{i,j}^{\mathbb{C}}(x) = \{\exp(sv_{i,j})x : s \in \mathbb{C}\} \tag{5.4}$$

while the neutral direction comes from the complex diagonal and thus contains the orbit direction as described above and the corresponding compact part coming from diagonal matrices with imaginary entries.

Thus we can give the following description of the Lyapunov exponents.

Proposition 8.
1. *Non-zero Lyapunov exponents for the Weyl chamber flow on $SL(n,\mathbb{R})/\Gamma$ are $t_i - t_j$ where $i \neq j$ and $1 \leq i,j \leq n$. Zero Lyapunov exponent comes only from the orbit foliation and hence has multiplicity $n-1$. Consequently any matrix $d \in D_n^+$ whose diagonal entries are pairwise different acts normally hyperbolically on $SL(n,\mathbb{R})/\Gamma$ with respect to the orbit foliation and hence is an Anosov element of the action.*
2. *Non-zero Lyapunov exponents for the Cartan action on $SL(n,\mathbb{C})/\Gamma$ are $t_i - t_j$ where $i \neq j$ and $1 \leq i,j \leq n$ and each has multiplicity two. Zero Lyapunov exponent comes from the neutral foliation and has multiplicity $2(n-1)$. Any matrix $d \in D_n^+$ whose diagonal entries are pairwise different acts normally hyperbolically on $SL(n,\mathbb{C})/\Gamma$ with respect to the neutral foliation.*

Except for the multiplicity, the picture of Weyl chambers in $\mathbb{D}_n$ is the same for the action on $SL(n,\mathbb{C})/\Gamma$ as for the action on $SL(n,\mathbb{R})/\Gamma$.

5.3. **Generating relations and Steinberg symbols.** For the contents of this section we refer to [17] and [19].

The group $SL(n,\mathbb{K})$, where $\mathbb{K}$ is a field, is generated by elements $e_{ij}(t) := \exp tv_{i,j}$ with generators satisfying the commutator relations:

$$[e_{ij}(t), e_{kl}(s)] = \begin{cases} 1, & j \neq k, i \neq l \\ e_{il}(st), & j = k, i \neq l \\ e_{kj}(-st), & j \neq k, i = l \end{cases}$$

where $[\cdot,\cdot]$ denotes the commutator. Following [17], [19] for $n \geq 3$ the abstract Steinberg group $St_n(\mathbb{K})$ over the field $\mathbb{K}$ is defined by generators $x_{ij}(t), t \in \mathbb{K}$, $i,j \in \{1,..,n\}, i \neq j$ subject to relations

$$x_{ij}(t)x_{ij}(s) = x_{ij}(t+s)$$

and

$$[x_{ij}(t), x_{kl}(s)] = \begin{cases} 1, & j \neq k, i \neq l \\ x_{il}(st), & j = k, i \neq l \\ x_{kj}(-st), & j \neq k, i = l \end{cases}$$

The natural map $\phi : St_n(\mathbb{K}) \to GL(n, \mathbb{K})$ defined by $\phi(x_{ij}(t)) = e_{ij}(t)$ is a homomorphism whose kernel is denoted by $K_2(\mathbb{K})$. The group $K_2(\mathbb{K})$ is precisely the center of the Steinberg group, and the group $St_n(\mathbb{K})$ is actually the universal central extension of the group $SL(n, \mathbb{K})$. In other words, the kernel $K_2(\mathbb{K})$ can be identified with the Schur multiplier H_2G where $G = SL(n, \mathbb{K})$. Aside for proving triviality of the Schur multiplier in the case of finite fields, in [19] Steinberg also obtained the following presentation of $SL(n, \mathbb{K})$ for an arbitrary field $\mathbb{K}$ in terms of generators and relations :

Theorem 2 (Steinberg)**.** *For $n \geq 3$ and any field $\mathbb{K}$ the group $SL(n, \mathbb{K})$ is generated by the $e_{ij}(t)$'s subject to the relations:*

$$e_{ij}(t)e_{ij}(s) = e_{ij}(t + s) \tag{5.5}$$

$$[e_{ij}(t), e_{kl}(s)] = \begin{cases} 1, & j \neq k, i \neq l \\ e_{il}(st), & j = k, i \neq l \\ e_{kj}(-st), & j \neq k, i = l \end{cases} \tag{5.6}$$

and the following extra relations

$$h_{12}(t)h_{12}(s) = h_{12}(ts) \tag{5.7}$$

where $h_{12}(t) := e_{12}(t)e_{21}(-t^{-1})e_{12}(t)e_{12}(-1)e_{21}(1)e_{12}(-1)$ *for each* $t \in \mathbb{K}^*$.

The kernel $K_2(\mathbb{K})$ in this case comes from the relation (5.7) and for an arbitrary field $\mathbb{K}$ its structure has been established by Matsumoto [16]:

Theorem 3 (Matsumoto)**.** *For $SL(n, \mathbb{K})$ where $n \geq 3$ and $\mathbb{K}$ is a field, the kernel $K_2(\mathbb{K})$ is generated by symbols $(t, s) \in (\mathbb{K}^*)^2$, corresponding to the relations in* (5.7), *subject to:*

1. (t, s) *is bi-multiplicative*
2. $(t, 1 - t) = 1$ *if* $t \neq 0, 1$

Any such bi-multiplicative map $c : \mathbb{K}^* \times \mathbb{K}^* \to A$ into an abelian group A satisfying $c(t, 1 - t) = 1_A$ is called a *Steinberg symbol* on the field $\mathbb{K}$. Steinberg symbols on the field $\mathbb{K}$ with values in A are in one-to-one correspondence with central extensions of $SL(n, \mathbb{K})$ with kernel A. When the field $\mathbb{K}$ has a topology which makes $SL(n, \mathbb{K})$ a topological group, if A is a Hausdorff space and if c is a continuous map then it is possible to say more about the Steinberg symbols. We will use the following result about continuous Steinberg symbols for the fields $\mathbb{R}$ and $\mathbb{C}$ [17]:

Theorem 4 (Milnor)**.** *a) Every continuous Steinberg symbol on the field $\mathbb{C}$ of complex numbers is trivial.*
b) If $c(t, s)$ is a continuous Steinberg symbol on the field $\mathbb{R}$, then $c(t, s) = 1$ if s or t are positive, and $c(t, s) = c(-1, -1)$ has order at most 2 if s and t are both negative.

5.4. **Hölder cocycle rigidity for the Cartan actions.**

Theorem 5. *Any Hölder cocycle over the Weyl chamber flow on $SL(n,\mathbb{R})/\Gamma$ or over the Cartan (real diagonal) action on $SL(n,\mathbb{C})/\Gamma$, $n \geq 3$, is cohomologous to a constant cocycle via a Hölder transfer function.*

Proof. The invariant foliations that we consider are $F_{i,j}$ $(i \neq j)$ (corr. $F^{\mathbb{C}}_{i,j}$, for the second action) as defined in (5.1) (corr.(5.4)). Notice that those foliations are smooth and their brackets generate the whole tangent space. This implies that this system of foliations is locally 1/2-Hölder transitive ([9, Section 4, Proposition 1]). Every path represents a product of elements of the form $e_{ij}(t)$. In particular, if a path is a cycle, this product is an element of Γ.

Let us consider first contractible cycles, i.e. those for which this element is the identity. Every such cycle represents a relation in the group. Hence by Theorem 2 the word represented by this cycle can be written as a product of conjugates of basic relations (5.5), (5.6) and (5.7). Now we will show that we can cancel any basic relation without changing the value of the periodic cycle functional thus implying that the value was zero to begin with. For the relations of the type (5.5) and (5.6) we use the fact that cycle defined by such a relation is contained in a leaf of the stable manifold for some element of the action and thus provides an allowable substitution.

For relations (5.5) this fact is obvious.

To see this for cycles given by relations (5.6) notice that any two elements $e_{ij}(t)$ and $e_{kl}(t)$ with $i \neq j$ and $k \neq l$ are in the stable leaf for any element of the action with $t_i < t_j$ and $t_k < t_l$. If $j = k, i \neq l$ one takes $t_i < t_j < t_l$ and similarly for the remaining case.

Now consider relations (5.7). For any pair $(t,s) \in (\mathbb{K}^*)^2$ where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ at any point $x \in X$ we may consider the cycle $\mathcal{C}_x(t,s)$ at x given by the relation $h_{12}(t)h_{12}(s)h_{12}^{-1}(ts) = 1$. Such path goes along unipotent foliations $F_{1,2}$ and $F_{2,1}$. On any such path we can compute the periodic cycle functional along this path. Thus we can define a map $c : \mathbb{K}^* \times \mathbb{K}^* \to \mathbb{R}$ by assigning to any (t,s) the value of the periodic cycle functional on the cycle $\mathcal{C}_x(t,s)$. As this map is continuous and is a homomorphism (thus preserves the structure of $K_2(\mathbb{K})$) in the context of Section 5.3, it defines a continuous Steinberg symbol over the field $\mathbb{K}$. If $\mathbb{K}$ is equal to $\mathbb{C}$ or $\mathbb{R}$ by Theorem 4 such a map must be trivial. Therefore the periodic cycle functional vanishes on paths given by the relations (5.7).

Finally, to cancel conjugations one notices that cancelling $e_{ij}(t)e_{ij}(t^{-1}) = \mathrm{id}$ is also an allowed substitution and each conjugation can be cancelled inductively using that.

Thus, the value of the periodic cycle functional for any Hölder cocycle β depends only on the element of Γ this cycle represents. Furthermore, these values provide a homomorphism of Γ to $\mathbb{R}$. By the Margulis Normal Subgroup Theorem [14], [15, Theorem 4', Introduction] any such homomorphism is trivial; hence all periodic cycle functionals vanish on β. Now Proposition 4(2) implies that β is cohomologous to a constant cocycle via a Hölder transfer function. □

5.5. **Smooth cocycle rigidity.**

Theorem 6. *The Weyl chamber flow on $SL(n,\mathbb{R})/\Gamma$ is C^∞ $\mathbb{R}$- cocycle rigid, i.e. any C^∞ cocycle over this action is cohomologous to a constant cocycle via a C^∞*

transfer function. Similarly, the real diagonal action on $SL(n,\mathbb{C})/\Gamma$ is C^∞ $\mathbb{R}$-cocycle rigid.

Proof. For a C^∞ cocycle, the transfer function h constructed using periodic cycle functionals, as in the proof of Proposition 4, is C^∞ along the foliations $F_{i,j}$ (corr $F^{\mathbb{C}}_{i,j}$ in the complex case), $i \neq j$ tangent to the generating *smooth* distributions. Now a general result stating that in case the smooth distributions along with their Lie brackets generate the tangent space at any point of a manifold a function smooth along corresponding foliations is necessarily smooth (see [12] for a detailed discussion and references to proofs), implies that the transfer map h is C^∞. □

6. **Proof of Theorem 1.** First let us notice that it is sufficient to assume that $S = P$, a two-dimensional subspace in general position since the restriction of any cocycle over the action of $\exp S$ to $\exp P$ is a cocycle over the action of $\exp P$. Thus, consider the restriction γ of the Weyl chamber flow α on $SL(n,\mathbb{R})/\Gamma$ or of the D_n^+ action α' on $SL(n,\mathbb{C})/\Gamma$ to $\exp P$ where $P \subset \mathbb{D}_n$ is a two-dimensional subspace in general position, i.e. a plane which intersects all the hyperplanes $H_{i,j}$ along distinct lines. The number of connected components of $P \setminus \bigcup_{i\neq j} H_{i,j}$ is twice the number of the Lyapunov hyperplanes $H_{i,j}$ i.e. $n(n-1)$. Then every element in a Weyl chamber for γ on P acts normally hyperbolically with respect to the orbit (corr. neutral) foliation for α (corr. α') which is the neutral (isometric) foliation for the restricted action γ. Thus γ is a partially hyperbolic action.

We use Proposition 7 to show that the picture of coarse Lyapunov foliations is exactly the same for the action of $\exp P$ as for the whole Cartan action. The first part of Proposition 7 assures that all the unipotents are obtained as intersections of stable directions for various elements in P while the second part of Proposition 7 assures that all the commutator relations are obtained as well. The relations (5.7) depend only on the structure of the unipotent subgroups which remains the same. Thus all the arguments from the proofs of Theorems 5 and 6 apply. This finishes the proof of Theorem 1. □

To gain a better insight into the combinatorics involved we also provide an explicit calculation of coarse Lyapunov foliations for the action of $\exp P$ in the case of the algebraic actions that are under considerations here.

Arrange intersections of Weyl chambers with P a in cyclic order. For each pair of indexes i, j take the four Weyl chambers bordering the intersection of P with the hyperplane $t_i = t_j$. They form two pairs of adjacent ones and their opposites. Let W be one of those Weyl chambers for which $t_i > t_j$ and let W' be its neighbor. Now take the intersection of the stable manifolds for W and $-W'$. Since $-W'$ borders $-W$ along the line $t_i = t_j$ all exponents other than $t_i - t_j$ and $t_j - t_i$ have opposite signs in W and $-W'$. But these two exponents have the same signs in the two chambers. In particular, the only exponent negative in these two Weyl chambers is $t_j - t_i$. Since i and j are arbitrary, the direction of $v_{j,i}$ is the only one being contracted by all the regular elements in P having $t_j < t_i$.

However, it is not sufficient to show that every one-dimensional unipotent appears as an intersection of stable manifolds for different Weyl chambers. In order to apply Steinberg's result we need to know that for every triple i, j, k there is a Weyl chamber in P for which the order is $t_i > t_j > t_k$ i.e. both exponents are positive.

Take four Weyl chambers bordering the line $t_i = t_j$. On that line t_k is not equal to either t_i or t_j (because of the general position of P). Then on one half-line $t_k > t_i = t_j$ and on the other $t_i = t_j > t_k$. Let W and W' be two Weyl chambers bordering the second half-line. Since $t_i - t_j$ changes sign on the line in one of these Weyl chambers we have $t_i > t_j > t_k$.

Hence all three unipotents $v_{i,j}$, $v_{j,k}$ and $v_{i,k}$ lie in the stable manifold for that Weyl chamber and the periodic cycle functional corresponding to this relation vanishes for the two-dimensional action.

Here is a specific example of the situation described above.

Example 1. Consider the Weyl chamber flow on $SL(4,\mathbb{R})/\Gamma$ and take the plane

$$P : t_2 + 3t_3 + 7t_4 = 0$$

(as usual: $t_1 + t_2 + t_3 + t_4 = 0$). Now consider intersections of this plane with $t_i = t_j, i, j = 1, 2, 3, 4$. The intersections are 6 distinct lines therefore the plane P intersects 12 Weyl chambers. The 12 Weyl chambers which appear in the intersection with this plane (4321, 4312, 4132, 4123,1423, 1243 and their opposites, where $ijkl$ stands for the ordering $t_i > t_j > t_k > t_l$) have stable manifolds whose intersections give all 12 one-dimensional unipotents.

Remark 2. If a two-dimensional subspace $P \subset \mathbb{D}_n$ is not in general position the above arguments do not apply and hence our proof of cocycle rigidity does not work. Still for most such subspaces, one would expect that cocycle rigidity holds. A good concrete example for the $SL(4,\mathbb{R})/\Gamma$ case is the plane given by the equation $t_2 + 2t_3 + 3t_4 = 0$. In this case there are two unipotents v_{14} and v_{23} which are not both stable for any element of the action so there are more periodic cycles than those coming from the groups relations. Still the picture looks closer to the general position picture than to rank one or product pictures which produce non-rigid situations.

7. Cocycles with values in Lie groups.

7.1. Non-abelian potential and periodic cycle functionals.

For an action α (as described in Section 2) of $\mathbb{R}^k$ on a compact manifold M with coarse Lyapunov foliations $\mathcal{F}_1, .., \mathcal{F}_r$ and for a cocycle $\beta : \mathbb{R}^k \times M \to H$ where H is a Lie group we define the *H-valued potential* of β as

$$\begin{cases} P_a^j(y,x) = \lim\limits_{n\to+\infty} \beta(na,y)^{-1}\beta(na,x), & \chi_j(a) < 0 \\ P_a^j(y,x) = \lim\limits_{n\to-\infty} \beta(na,y)^{-1}\beta(na,x),, & \chi_j(a) > 0 \end{cases}$$

As before, we introduce the following compact expression for $P_a^j(y,x)$:

$$P_a^j(y,x) = \lim_{n\to*\infty} \beta(na,y)^{-1}\beta(na,x) \tag{7.1}$$

where $j \in \{1, \cdots, r\}$, $a \notin Ker\chi_j$, $x \in M$, $y \in \mathcal{F}_j(x)$ and $* := *(j,a) := -sgn(\chi_j(a)) \in \{+,-\}$.

If H is an abelian group or, more generally, if it possesses a bi-invariant metric (e.g. if it is compact) the limits in the right hand part of (7.1) always exist.

For a general Lie group H there may be a "competition" between the exponential speed of decay for the distance between nax and nay on the one hand, and the exponential growth of the cocycle norm on the other. To guarantee convergence one needs to introduce certain extra conditions on the cocycle. One possible version is the condition of "smallness" defined below.

We assume here that H is a closed subgroup of $GL(d,\mathbb{R})$ with the matrix norm $\|\cdot\|$ and we denote the induced metric by d_H.

Assuming convergence and using these potentials, as in Section 3, for any $\mathcal{F}_{1,..,r}$-cycle $\mathcal{C}:(x_1,..,x_{N+1}=x_1)$ we can define the corresponding periodic cycle functional:

$$F_a(\mathcal{C})(\beta)=\prod_{i=1}^{N}P_a^{j(i)}(x_i,x_{i+1})(\beta). \tag{7.2}$$

(Here the order of multiplication is of course essential).

For $a\in A$ we define λ_a as the exponential of the largest negative Lyapunov exponent of a, i.e.

$$\lambda_a=\lim_{n\to\infty}\|(na)_*|_{E_a^s}\|^{\frac{1}{n}}$$

where na denotes the map $\alpha(na,\cdot)$ on M. For a cocycle β its *asymptotic growth* (as in [10, Section 2]) is defined by

$$\mu_-(a,\beta):=\lim_{n\to\infty}\inf_{x\in M}\|\beta(na,x)^{-1}\|^{-\frac{1}{n}}$$

and

$$\mu_+(a,\beta):=\lim_{n\to\infty}\inf_{x\in M}\|\beta(na,x)\|^{\frac{1}{n}}$$

For a p-Hölder cocycle β over an $\mathbb{R}^k$ partially hyperbolic action, define a subset $\mathcal{R}_\beta$ of the acting group as

$$\mathcal{R}_\beta:=\{a\in A:\lambda_a^p<\mu_+(a,\beta)^{-1}\cdot\mu_-(a,\beta)\}$$

If β is C^∞ then the set $\mathcal{R}_\beta$ is defined as above with $p=1$. We note that if a belongs to $\mathcal{R}_\beta$ then so does any ta for $t>0$. Now we state the condition for the smallness of the cocycle.

Definition 9. A p-Hölder (smooth) cocycle β taking values in H is *small* if the set $\mathcal{R}_\beta$ intersects every Weyl chamber.

As before, we are assuming that α is an $\mathbb{R}^k$ action satisfying the assumptions of Proposition 1. We formulate now a counterpart of Proposition 2 for cocycles with values in Lie groups.

Proposition 9. *Let β be a small cocycle. Then for $a\in\mathcal{R}_\beta$ the limit in the right hand part of* (7.1) *exists and furthermore*

1. *If β is p-Hölder continuous and $a\in\mathcal{R}_\beta$ then $P_a^j(y,x)(\beta)$ is continuous in both variables and p-Hölder continuous in $y\in\mathcal{F}_j(x)$ i.e. along the leaves of the foliation $\mathcal{F}_j$. If β is smooth then $P_a^j(y,x)(\beta)$ is smooth along $\mathcal{F}_j(x)$.*
2. *$P_a^j(y,x)(\beta)=P_b^j(y,x)(\beta)$ for any $a,b\in\mathcal{R}$.*

Remark 3. If the group H has a metric (and hence a Riemannian metric) which is both left-invariant and right-invariant then the smallness condition is superfluous and the argument works for any cocycle and any element of the action as in the real-valued case simply by interchanging factors under the sign of the norm.

For the proof of this proposition and further references we refer the reader to [10, Lemma 4.1] and [18, Theorem 6.1].

7.2. **Cocycle rigidity.** With the properties of the potential as in Proposition 9 the rest of the geometric argument for the cocycle rigidity of Cartan actions and their generic subactions on $SL(n,\mathbb{R})/\Gamma$ and $SL(n,\mathbb{C})/\Gamma$ follows exactly as in the case of real valued cocycles until the last step which involved the Margulis Normal Subgroup Theorem or could have been deduced from his superrigidity theorem. First we have the following counterpart of Proposition 4.

Proposition 10. *Let α be an $\mathbb{R}^k$ action by diffeomorphisms on a compact Riemannian manifold M such that a dense set of elements of $\mathbb{R}^k$ acts normally hyperbolically with respect to an invariant foliation, $\tilde{M}$ be the universal cover of M and β be a small p-Hölder cocycle over the action α such that $F(\mathcal{C})(\beta) = 0$ for any $\mathcal{F}_{1,..,r}$-cycle $\mathcal{C}$ which lifts to to a cycle on $\tilde{M}$ Then the following hold:*

1. *If the foliations $\mathcal{F}_1,..,\mathcal{F}_r$ are locally transitive then β is* virtually *cohomologous to a constant cocycle via a continuous map $h : \tilde{M} \to H$.*
2. *If the foliations $\mathcal{F}_1,..,\mathcal{F}_r$ are locally q-Hölder transitive then the map h pq-Hölder.*
3. *If the foliations $\mathcal{F}_1,..,\mathcal{F}_r$ are weakly locally q-Hölder transitive and β is Lipschitz then the map h is q-Hölder.*

Proof. We will repeat the arguments for the proof of Proposition 4 in the non-abelian setting and on the universal cover $\tilde{M}$. Fix a point $x \in \tilde{M}$ and for an arbitrary y consider a $\mathcal{F}_{1,..r}$ path $\mathcal{C} : x = x_1,..,x_N = y$ connecting x and y. Existence of such a path follows from transitivity of the system of foliations on M and from compactness of M. Define

$$h(y) = F_a(x,y) = P_a^{j(N-1)}a(x_N,x_{N-1}) \cdot ... \cdot P_a^{j(1)}(x_2,x_1)$$

for some $a \in \mathcal{R}_\beta$. Here $x_i \in \mathcal{F}_{j(i)},\ j = 1,\dots,N-1$. Due to the Proposition 9, this definition does not depend on $a \in \mathcal{R}_\beta$ and defines a continuous map $\tilde{M} \to H$. Moreover

$$\begin{aligned} h(ay) &= F_a(ay,ax)h(ax) \\ &= \Big(\prod_{i=N-1}^{1} \lim_{n\to *\infty} \beta(na,ax_{i+1})^{-1}\beta(na,ax_i)\Big)h(ax) \\ &= \Big(\prod_{i=N-1}^{1} \lim_{n\to *\infty} \beta(a,x_{i+1})\beta((n+1)a,x_{i+1})^{-1}\beta((n+1)a,x_i)\beta(a,x_i)^{-1}\Big)h(ax) \\ &= \beta(a,y)\Big(\prod_{i=N-1}^{1} \lim_{n\to *\infty} \beta(na,x_{i+1})^{-1}\beta(na,x_i)\Big)\beta(a,x)^{-1}h(ax) \\ &= \beta(a,y)h(y)\beta(a,x)^{-1}h(ax) \end{aligned}$$

Since x is fixed, denoting $c(a) = h(ax)^{-1}\beta(a,x)$ this implies:

$$h(ay)c(a)h(y)^{-1} = \beta(a,y)$$

for any $a \in \mathcal{R}_\beta$ and $y \in \tilde{M}$. For all other elements of the action the same transfer function h applies since the set $\mathcal{R}_\beta$ contains lines in all the Weyl chambers and thus generates $\mathbb{R}^k$. Namely, if $a,b \in \mathcal{R}_\beta$ then $h(ay)c(a)h(y)^{-1} = \beta(a,y)$, $h(by)c(b)h(y)^{-1} = \beta(b,y)$ and thus by using the cocycle property $\beta(a+b,y) = \beta(a,by)\beta(b,y) = h((a+b)y)c(a)h(by)^{-1}h(by)c(b)h(y)^{-1}$, therefore $\beta(a+b,y) =$

$h((a+b)y)c(a+b)h(y)^{-1}$ where the map c is just extended to a homomorphism on $\mathbb{R}^k$.

By the first part of Proposition 9 h is Hölder for a Hölder cocycle and smooth along generating foliations if the cocycle is smooth, and this implies that the transfer map is Hölder (corr. smooth) as in the proof of Theorem 5 (corr. Theorem 6). □

Theorem 7. 1. *Every small Hölder (corr. C^∞) cocycle with values a connected matrix Lie group H over the Weyl chamber flow on $SL(n,\mathbb{R})/\Gamma$ is virtually cohomologous to a constant cocycle via a Hölder (corr. C^∞) transfer map. Similarly, every small Hölder (C^∞) H- valued cocycle over the real diagonal action on $SL(n,\mathbb{C})/\Gamma$ is virtually cohomologous to a constant cocycle via a Hölder (corr. C^∞) transfer map.*

2. *Every small Hölder (C^∞) H- valued cocycle over the action of* $\exp S \subset D_n^+$, *on $SL(n,\mathbb{R})/\Gamma$ and on $SL(n,\mathbb{C})/\Gamma$, where the subspace $S \subset \mathbb{D}_n$ contains a two-dimensional subspace in general position, is virtually cohomologous to a constant cocycle via a Hölder (corr. C^∞) transfer map.*

Proof. Let β be a small Hölder (corr. C^∞) cocycle. The second part of Proposition 9 implies the invariance of periodic cycle functionals for an H valued cocycle as in the real case: $F_a(b\mathcal{C})(\beta) = F_a(\mathcal{C})(\beta)$ for $a, b \in \mathcal{R}_\beta$ which implies the vanishing of the periodic cycle functionals for cycles contained in the stable leaf for an element in $\mathcal{R}_\beta$ (thus for some element in each Weyl chamber, since $\mathcal{R}_\beta$ intersects each Weyl chamber). Therefore the same argument as in the proof of Theorem 5 based on the sufficiency of generating relations in $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$, implies the vanishing of the periodic cycle functionals for all cycles which lift to closed cycles on the universal cover which we will denote by G and which in the cases under consideration is equal to either $SL(n,\mathbb{R})$ or $SL(n,\mathbb{C})$ for some $n \geq 3$. Now the statement of the theorem in the Hölder case follows from Proposition 10. In the smooth case the smoothness of h along the foliations follows from Proposition 9(1) and global smoothness from the elliptic theory exactly as in the case or real-valued cocycles. □

If for an H-valued cocycle β the periodic cycle functionals vanish for any cycle $\mathcal{C}$ which lifts to a cycle on the universal cover, the values of the functionals for all cycles are determined by a homomorphism of the fundamental group of the manifold to H, i.e in our case by a homomorphism $\Gamma \to H$. If any such homomorphism is trivial, as in the case $H = \mathbb{R}$, then virtual cohomology to a constant cocycle implies cohomology to a constant cocycle. Furthermore, not every homomorphism may serve to define values of the periodic cycle functional. Margulis Superrigidity Theorem [14, 15] gives detailed information about such homomorphisms for different target groups H. In certain cases one can show that no such cocycles exist or that the cocycles obtained are still cohomologous to constant one. Implication to the cohomology over the Cartan actions and their restrictions will be considered in a subsequent paper.

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