Invariant Measures on G/Γ for Split Simple Lie Groups G

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In memory of Jürgen Moser

Abstract

We study the left action α of a Cartan subgroup on the space $X = G/\Gamma$, where Γ is a lattice in a simple split connected Lie group G of rank $n > 1$. Let μ be an α-invariant measure on X. We give several conditions using entropy and conditional measures, each of which characterizes the Haar measure on X. Furthermore, we show that the conditional measure on the foliation of unstable manifolds has the structure of a product measure. The main new element compared to the previous work on this subject is the use of noncommutativity of root foliations to establish rigidity of invariant measures.

1 Introduction

1.1 Overview

This paper is a part of an ongoing effort to understand the structure of invariant measures for algebraic (homogeneous and affine) actions of higher-rank abelian groups ($\mathbb{Z}^n$ and $\mathbb{R}^n$ for $n \geq 2$) with hyperbolic behavior. It started from attempts to answer the question raised in Furstenberg's paper [4] concerning common invariant measures for multiplication by p and q on the circle, where $p^n \neq q^m$ unless $n = m = 0$. (See [2, 3, 5, 6, 7, 8, 12, 14, 15, 18] and the references thereof; in particular, for an account of results on Furstenberg's question, see the introduction to [12].) This study of invariant measures can in turn be viewed as a part of a broader program of understanding rigidity properties of such actions, including local and global differentiable rigidity, cocycle rigidity, and such [9, 10, 11, 13].

The principal conjecture concerning invariant measures (see "main conjecture" in the introduction of [12]) asserts that unlike the "rank-1 case" (actions of $\mathbb{Z}$ and $\mathbb{R}$), the collection of invariant measures is rather restricted; in a somewhat imprecise way, one might say that those measures are of algebraic nature unless a certain degeneracy appears that essentially reduces the picture to a rank-1 situation. Notice, however, that the picture is intrinsically considerably more complicated than

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in the case of unipotent actions, which has been fully understood in the landmark work by Ratner [22, 23]. In the latter case the leading paradigm is unique ergodicity, although other measures of algebraic nature may be present. Such measures appear only in finitely many continuous parametrized families such as closed horocycles on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$. In the case of higher-rank hyperbolic actions, there are always infinitely many isolated compact orbits of increasing complexity that are dense in the phase space so that in the ideal situation (which corresponds to unique ergodicity in the unipotent case) one expects the δ-measures on those orbits to be the only ergodic measures other than Haar.

All the progress made, both for Furstenberg's question and for the general case, concerns invariant measures that have positive entropy with respect to at least some elements of the action. The reason for this is that the available techniques introduced explicitly in [12] (and implicitly present already in earlier work on the Furstenberg question) are based on the consideration of families of conditional measures for various invariant foliations, which are contracted by some elements of the action. For a zero-entropy measure every such conditional measure is atomic, so no further information can be obtained from these methods. In the positive entropy case the algebraic nature of conditional measures is established first, and this serves as a basis of showing rigidity of the global measure.

There are several methods for showing that conditional measures are algebraic, which all involve various additional assumptions. The starting point of all considerations is showing that conditional measures are invariant under certain isometries that appear as restrictions of certain singular elements of the action (see Proposition 5.1 below). In order to produce a sufficiently rich collection of such isometries, typical leaves of the foliation in question should be contained in ergodic components of the corresponding singular elements. This, of course, follows from ergodicity of the element. However, ergodicity of the whole action with respect to an invariant measure in general does not imply ergodicity of individual elements, and singular elements are particularly prone to be exceptional in this case. Many of the basic results in the works mentioned above include explicit ergodicity assumptions for singular elements.

There are several ways to verify these conditions. For special measures in some applications such as isomorphism rigidity, these assumptions are satisfied automatically [7, 8] due to the properties of Haar measure. For actions by automorphisms of the torus, which have been studied most extensively so far, there are specific tools from linear algebra and algebraic number theory. One possible assumption, which is used in [12, 14] (see also [5]) to prove rigidity of ergodic measures with nonvanishing entropy for some actions by toral automorphisms, is that no two Lyapunov exponents are negatively proportional. Such actions are called "totally nonsymplectic" (TNS).

First results for other cases including the Weyl chamber flows were obtained in [12, sect. 7] and [14], where certain errors in the original presentation were corrected. For Weyl chamber flows and similar examples, Lyapunov exponents always appear in pairs of opposite sign (this is a corollary of the structure of root systems for semisimple Lie groups), so the TNS condition never holds. Accordingly, in the work of Katok and Spatzier quoted above, as well as in the recent, more advanced work in that direction [7], some form of ergodicity for certain singular directions is assumed.

In this paper we take a different course. We study a more restricted class of the Weyl chamber flows than in those papers, but we manage to find sufficient conditions for algebraicity that avoid any ergodicity-type assumptions for individual elements.

Our main technical innovation is the observation that noncommutativity of the foliations corresponding to roots allows us to produce the desired translations in some directions; assuming the conditional measures for two directions are non-atomic, translation invariance is obtained for the commutator direction. This key observation is carried out in Lemma 6.1 for the $\mathrm{SL}(n,\mathbb{R})$ case and in Proposition 7.1 for the general split case.

1.2 The Setting

We consider a simple, ($\mathbb{R}$-)split connected Lie group G of rank $n > 1$, a lattice Γ in G, and the action α of a maximal Cartan subgroup of G on the quotient G/Γ.

1.2.1 The $\mathrm{SL}(n,\mathbb{R})$ Case

To illustrate, let us first consider the special case given by the subgroup α of $\mathrm{SL}(n+1,\mathbb{R})$ consisting of all diagonal matrices with positive entries. The left action of α on $X = \mathrm{SL}(n+1,\mathbb{R})/\Gamma$ is often called the "Weyl chamber flow" and is defined by

$$\alpha^{\mathbf{t}} x = \begin{pmatrix} e^{t_1} & & \\ & \ddots & \\ & & e^{t_{n+1}} \end{pmatrix} x$$

where $x \in X$ and

$$\mathbf{t} \in R = \left\{ \mathbf{t} \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} t_i = 0 \right\} \cong \mathbb{R}^n .$$

Let m be the Haar measure on X, which is invariant under the left action of $\mathrm{SL}(n+1,\mathbb{R})$ and, in particular, α-invariant. We study ergodic α-invariant measures μ on X. Aside from Haar measure, there are also δ-measures on compact orbits of α; such orbits are dense (see [20, 21] or [1] for a quantitative statement). We say the measure is standard if it is the Haar measure or a measure supported on a compact orbit. Among the standard measures, only Haar has positive entropy with respect to individual elements.

We will give several conditions, each of which characterizes the Haar measure on X.

In the unpublished manuscript by M. Rees [24] (see Section 9) a lattice Γ in $\mathrm{SL}(3,\mathbb{R})$ is constructed for which there exist nonstandard invariant measures on X. With respect to some of these measures, the entropies $\mathrm{h}_\mu(\alpha^{\mathbf{t}})$ of some individual elements of the Weyl chamber flow are positive. Furthermore, the conditional measure for the foliation into unstable manifolds for $\alpha^{\mathbf{t}}$ is supported on a single line. Thus unlike the TNS case for toral automorphisms, positive entropy for some elements of the Weyl chamber flow alone is in general not sufficient to deduce that $\mu = m$.

We show that for $\mathrm{SL}(3,\mathbb{R})$, in terms of conditional measure structure, the picture that appears in Rees's example is the only one possible for measures other than Haar. A part of this result has been outlined without proof at the end of [12].

There are other assumptions (see Theorem 4.1) about entropy or the conditional measures that are sufficient in order to show that $\mu = m$. For instance, if all elements of the flow have positive entropy

$$\mathrm{h}_\mu(\alpha^{\mathbf{t}}) > 0 \quad \text{for all } \mathbf{t} \in R \setminus \{0\}$$

for the action on $\mathrm{SL}(n+1,\mathbb{R})/\Gamma$ or some elements have sufficiently large entropy, then $\mu = m$.

1.2.2 The General Split Case

Let now G be a simple, split connected Lie group of rank greater than 1, let Γ be a lattice in G, and let α be a maximal Cartan subgroup that acts from the left on $X = G/\Gamma$. We write again $\alpha^{\mathbf{t}}$ for the action of an individual element of the Cartan subgroup, where $\mathbf{t} \in \mathbb{R}^n$ and $n > 1$ is the (real) rank of the Lie group. We denote the (nonzero) roots of G with respect to α by the letter λ and the set of all roots by Φ. Furthermore, we let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{g}_\lambda$ be the root space corresponding to the root λ. Then $\mathfrak{g}_\lambda$ is one-dimensional since G is simple and split. For any $\mathbf{t}$ the root space $\mathfrak{g}_\lambda$ is an eigenspace for the adjoint action of $\alpha^{\mathbf{t}}$ on the Lie algebra $\mathfrak{g}$; the corresponding eigenvalue is $e^{\lambda(\mathbf{t})}$.

In this case the Haar measure is characterized as the only Borel probability measure invariant and ergodic with respect to the left action of α for which the conditional measures on the one-dimensional foliations corresponding to all root spaces are nonatomic, or as the measure with a sufficiently large entropy (see Theorem 4.1).

Before presenting the result and its proof in Sections 4 through 8, we will give in the next two sections a short description of the foliations of X and the conditional measures for μ.

1.3 Extensions

1.3.1 Implications for Diophantine Approximation

There are interesting connections between number theory and dynamics of higher-rank actions. For example, the famous Littlewood conjecture on Diophantine approximation would follow if one could show that any bounded orbit of the

Weyl chamber flow on $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$ is closed. See [16, sect. 5.3]. This statement is a topological version of the rigidity of invariant measures. Our results may be applied to obtain certain partial results along the line of showing that possible counterexamples to the Littlewood conjecture form a "thin" set.

1.3.2 Infinite Volume Factors

While we present our results in the setting of a finite volume factor G/Γ, finiteness of the volume (i.e., the fact that Γ is a lattice) does not play any role in our considerations. One can consider left actions of a maximal Cartan subgroup on factor spaces of more general kind G/Λ where Λ is a discrete subgroup of G. It is the finiteness of the measure μ that is important. Thus if the factor space has infinite volume, our conditions properly modified will imply that an invariant measure with one of the additional properties (see Theorem 4.1) simply does not exist.

1.3.3 Nonsplit Groups

The assumption that G is split is technically important for our argument, since it guarantees that the root spaces are one-dimensional and hence the only non-atomic algebraic measures are Lebesgue. In the case of more general (nonsplit) semisimple groups of higher rank, considerations of conditionals within the root spaces are more involved. Still, one can find conditions guaranteeing algebraicity without invoking ergodicity directly. These results will appear in a separate paper subsequently.

2 Foliations of X

2.1 $\mathrm{SL}(n, \mathbb{R})$ case

We begin by briefly describing some foliations for the special case $X = \mathrm{SL}(n+1, \mathbb{R})/\Gamma$, their expanding and contracting behavior, and the Weyl chamber picture in R. Let $d(\cdot, \cdot)$ denote a right-invariant metric on $\mathrm{SL}(n+1, \mathbb{R})$ and the induced metric on X.

Recall that a foliation F is contracted under $\alpha^{\mathbf{t}}$ for a fixed $\mathbf{t} \in R$ if for any $x \in X$ and any $y \in F(x)$, $d(\alpha^{n\mathbf{t}}x, \alpha^{n\mathbf{t}}y) \to 0$ for $n \to \infty$. In other words, the leaf through x is a part of the stable manifold at x. A foliation is expanded under $\alpha^{\mathbf{t}}$ if it is contracted under $\alpha^{-\mathbf{t}}$, or each leaf is a part of an unstable manifold. A foliation F is isometric under $\alpha^{\mathbf{t}}$ if $d(\alpha^{\mathbf{t}}x, \alpha^{\mathbf{t}}y) = d(x, y)$ when $y \in F(x)$ is close to x (in the metric of the submanifold $F(x)$).

Let $1 \leq a, b \leq n+1$ always denote two different fixed indices, and let exp be the exponentiation map for matrices. Define the matrix

$$v_{a,b} = (\delta_{(a,b)(i,j)})_{(i,j)} ,$$

where $\delta_{(a,b)(i,j)}$ is 1 if $(a, b) = (i, j)$ and 0 otherwise. So $v_{a,b}$ has only one nonzero entry, namely, that in row a and column b. With this we define the foliation $F_{a,b}$, for which the leaf

$$F_{a,b}(x) = \{\exp(sv_{a,b})x : s \in \mathbb{R}\} \tag{2.1}$$

through x consists of all left multiples of x by matrices of the form $\exp(sv_{a,b}) = \mathrm{Id} + sv_{a,b}$.

The foliation $F_{a,b}$ is invariant under α; in fact, a direct calculation shows

$$\alpha^{\mathbf{t}}(\mathrm{Id} + sv_{a,b})x = \big(\mathrm{Id} + se^{t_a - t_b}v_{a,b}\big)\alpha^{\mathbf{t}}x\,, \tag{2.2}$$

the leaf $F_{a,b}(x)$ is mapped onto $F_{a,b}(\alpha^{\mathbf{t}}x)$ for any $\mathbf{t} \in R$. Consequently, the foliation $F_{a,b}$ is contracted (respectively, expanded or neutral) under $\alpha^{\mathbf{t}}$ if $t_a < t_b$ (respectively, $t_a > t_b$ or $t_a = t_b$). If the foliation $F_{a,b}$ is neutral under $\alpha^{\mathbf{t}}$, it is in fact isometric under $\alpha^{\mathbf{t}}$.

The leaves of the orbit foliation $O(x) = \{\alpha^{\mathbf{t}}x : \mathbf{t} \in R\}$ can be described similarly using the matrices

$$u_{a,b} = (\delta_{(a,a)(k,l)} - \delta_{(b,b)(k,l)})_{k,l}\,.$$

In fact, $\exp(u_{a,b}) = \alpha^{\mathbf{t}}$ for some $\mathbf{t} \in R$.

Clearly the tangent vectors to the leaves in (2.1) for various pairs (a, b) together with the orbit directions form a basis of the tangent space at every $x \in X$.

For every $a \neq b$ the equation $t_a = t_b$ defines a hyperplane $H_{a,b} \subset R$. The connected components of

$$A = R \setminus \bigcup_{a \neq b} H_{a,b}$$

are the Weyl chambers C of the flow α. For every $\mathbf{t} \in A$ only the orbit directions are neutral; such a $\mathbf{t}$ is called a regular element.

Let $I = \{(a, b) : a < b\}$, and let M_I be the span of $v_{a,b}$ for $(a, b) \in I$ (in the Lie algebra of $\mathrm{SL}(n+1, \mathbb{R})$). For the invariant foliation F_I the leaf through x is defined by

$$F_I(x) = \{\exp(w)x : w \in M_I\}\,. \tag{2.3}$$

Furthermore, there exists a Weyl chamber C such that for every $\mathbf{t} \in C$, the leaf $F_I(x)$ is the unstable manifold for $\alpha^{\mathbf{t}}$. In fact, $C = \{\mathbf{t} \in R : t_a > t_b \text{ for all } a < b\}$; this Weyl chamber is called the "positive Weyl chamber."

2.2 General Split Case

Let $\exp : \mathfrak{g} \to G$ be the exponential map from the Lie algebra to the Lie group. For any root λ and $w \in \mathfrak{g}_\lambda$, it follows that $\alpha^{\mathbf{t}}\exp(w) = \exp(e^{\lambda(\mathbf{t})}w)\alpha^{\mathbf{t}}$. We define the foliation F_λ on G/Γ by its leaf through x

$$F_\lambda(x) = \{\exp(w)x : w \in \mathfrak{g}_\lambda\}\,.$$

By construction F_λ is an invariant foliation that is contracted by $\alpha^{\mathbf{t}}$ if $\lambda(\mathbf{t}) < 0$, expanded if $\lambda(\mathbf{t}) > 0$, and isometric if $\lambda(\mathbf{t}) = 0$. For every root λ, fix a nontrivial element $v_\lambda \in \mathfrak{g}_\lambda$. The tangent space of G/Γ at any fixed point is isomorphic to $\mathfrak{g}$ and splits into the sum of $\mathfrak{g}_\alpha$ and the various root spaces $\mathfrak{g}_\lambda$ for $\lambda \in \Phi$. Here $\mathfrak{g}_\alpha$

is the Lie algebra of the Cartan subgroup—and corresponds to the orbit directions in G/Γ. Any Lie subalgebra $\mathfrak{h}$ that is a sum of root spaces defines similarly an invariant foliation $F_{\mathfrak{h}}$. For a fixed $\mathbf{t}$ and the subalgebra generated by the root spaces $\{\mathfrak{g}_\lambda : \lambda(\mathbf{t}) > 0\}$, this foliation is the foliation into unstable manifolds for $\alpha^{\mathbf{t}}$. The notions of Weyl chambers and regular elements apply similarly. In the context of $\mathrm{SL}(n+1,\mathbb{R})$ the set Φ and the root spaces are naturally described by pairs $\lambda = (a, b)$ of different indices and the corresponding matrix spaces as above.

In what follows the foliation F will always be the orbit foliations of a unipotent subgroup normalized by the Cartan subgroup such that the foliation F is expanded by a single element of the action. More precisely, there is a Lie subalgebra $\mathfrak{h}$ that is a sum of root spaces such that $H = \exp \mathfrak{h}$ is a unipotent subgroup, the leaf $F(x) = Hx$ is the orbit of $x \in X$ under H, and there is a $\mathbf{t} \in \mathbb{R}^n$ such that F is expanded under $\alpha^{\mathbf{t}}$. Notice that, under those assumptions, the parametrization map $\varphi_x^{\mathfrak{h}}(w) = \exp(w)x$ for $w \in \mathfrak{h}$ and most fixed $x \in X$ is injective. In fact, if the map is not injective at x, then discreteness of Γ implies that $\alpha^{-n\mathbf{t}}x$ eventually stays outside every compact set of X when $n \to \infty$. This is impossible if X is compact. Furthermore, for a general X we see that the map is injective for a.e. $x \in X$ with respect to any $\alpha^{\mathbf{t}}$-invariant probability measure. We say $A \subset F(x)$ is open (bounded, etc.) if A is open (bounded, etc.) in the topology of the submanifold $F(x)$, i.e., if and only if there is an open (bounded, etc.) subset $B \subset \mathfrak{h}$ with $A = \varphi_x^{\mathfrak{h}}(B)$.

As in the case of $\mathrm{SL}(n+1,\mathbb{R})$, we let $d(\cdot\,,\cdot)$ be a fixed, right-invariant metric on G and use the induced metric on $X = G/\Gamma$.

3 The Conditional Measures

Throughout this paper the standing assumption on μ is that μ is a Borel probability measure, which is ergodic with respect to the action α. We will study μ by means of its conditional measures μ_x^F for various foliations F of the kind described in Section 2. We recall some of the basic facts; see also section 4 of [12]. We write F_λ (respectively, F) for a foliation whose leaves are one-dimensional corresponding to a root space (respectively, to a Lie subalgebra $\mathfrak{h}$ that is a sum of root spaces). As noted before, we assume that there is a map $\alpha^{\mathbf{t}}$ that expands the foliation F.

3.1 Construction of Conditional Measures for Foliations

First we recall the basic notion of conditional measures with respect to a σ-algebra $\mathcal{A}$ or a measurable partition. All statements (or characterizations) below should be understood a.e. with respect to μ. The conditional measures $\mu_x^{\mathcal{A}}$ are a family of probability measures satisfying the following characterizing properties:

(i) The assignment $x \to \mu_x^{\mathcal{A}}$ is $\mathcal{A}$-measurable, where we use the weak* topology in the space of probability measures.

(ii) For any integrable function f and $A \in \mathcal{A}$

$$\int_A f(x)\,\mathrm{d}\mu(x) = \int_A \left[\int_X f(y)\,\mathrm{d}\mu_x^{\mathcal{A}}(y) \right] \mathrm{d}\mu(x)\,.$$

Using the conditional expectation, the above properties together are equivalent to

$$E(f \mid \mathcal{A})(x) = \int_X f(y)\,\mathrm{d}\mu_x^{\mathcal{A}}(y)$$

for all integrable functions f. If $\mathcal{A}$ is countably generated by $A_1, \dots, A_i, \dots$, the atom of x is defined by

$$[x] = [x]_{\mathcal{A}} = \bigcap_{i:x\in A_i} A_i \cap \bigcap_{i:x\notin A_i} X \setminus A_i\,.$$

Then $\mu_x^{\mathcal{A}}$ is a probability measure on the atom $[x]$ for a.e. $x \in X$.

The foliations of the kind we are considering are usually not measurable with respect to a measure μ; hence the above construction cannot be applied directly. Typically those foliations have dense leaves. Conditional measures in such a setting are defined by approximation; there are measurable partitions whose elements are large pieces of the leaves of the foliations. While this construction can be carried out in greater generality, we will restrict our description to the particular case of homogeneous foliations on factor spaces of Lie groups.

Let F be the foliation defined by the Lie algebra $\mathfrak{h}$ (as always a sum of root spaces), and let

$$T = \sum_{\lambda:v_\lambda\notin\mathfrak{h}} \mathfrak{g}_\lambda + \mathfrak{g}_\alpha$$

be the subspace transversal to $\mathfrak{h}$. Let $z \in X$, $O \subset \mathfrak{h} \times T$, be a bounded open set such that

$$\varphi_{O,z}(w, w') = \exp(w)\exp(w')z \text{ for } (w, w') \in O \text{ is injective.}$$

Define the σ-algebra for the "foliated set" $U = \varphi_{O,z}(O)$ to be

$$\mathcal{A}(O, F) = \{X \setminus U, \varphi_{O,z}(O \cap B) : B = \mathfrak{h} \times C \text{ and } C \subset T \text{ is measurable}\}\,.$$

Notice that the atoms $[x]$ of x with respect to the σ-algebra $\mathcal{A}(O, F)$ are $X \setminus U$ for $x \notin U$, and $\varphi_{O,z}(O \cap (\mathfrak{h} \times \{w'\}))$ for $x = \varphi_{O,z}(w, w') \in U$ and $(w, w') \in O$. In the second case, the atom is an open subset of the leaf $F(x)$. The conditional measure μ_x^F for a foliation is a family of σ-finite measures with the following characterizing properties:

(i) For a.e. pair of points $x, y \in X$, the conditional measures agree up to a multiplicative constant $C > 0$

$$\mu_x = C\mu_y \quad \text{whenever } F(x) = F(y) \text{ is their common leaf.} \tag{3.1}$$

(ii) The complement of the leaf $F(x)$ is a null set with respect to μ_x^F for a.e. $x \in X$.

(iii) The conditional measure μ_x^F satisfies for a.e. $x \in U$ that the conditional measures for $\mathcal{A}(O, F)$ and μ_x^F are up to a multiplicative constant equal when restricted to $[x]$. More precisely, the function $g(x) = \mu_x^F([x])$ is finite and measurable, and

$$\mu_x^{\mathcal{A}(O,F)}(C) = \frac{\mu_x^F(C \cap [x])}{\mu_x^F([x])}$$

for any measurable $C \subset X$ and a.e. $x \in X$.

The existence of the conditional measures for a foliation F can be shown by using the conditional measure for various σ-algebras $\mathcal{A}(O, F)$. There is a sequence of open sets as in property (iii) above whose images cover X. Furthermore, by applying the powers of the automorphism $\alpha^{\mathbf{t}}$ that expands F, one can make the atoms $[x]$ for $\mathcal{A}(O, F)$ become larger and larger pieces of the leaf $F(x)$. This produces a (doubly infinite) sequence of σ-algebras $\mathcal{A}(O, F)$ with the property that for a.e. point $x \in X$ and any bounded set $C \subset F(x)$ there is a σ-algebra $\mathcal{A}(O, F)$ of that sequence such that $C \subset [x] \subset F(x)$. The conditional measure μ_x^F is a limit of scalar multiples of the conditional measures for such σ-algebras $\mathcal{A}(O, F)$. It is necessary to use multiples since otherwise the conditional measure on a larger atom might not extend the conditional measure on the smaller one—one fixes for every point x a bounded set $D_x \subset F(x)$ and uses D_x as a normalizing set so that $\mu_x^F(D_x) = 1$. This can be carried out so that the measurability part of property (iii) is satisfied.

We summarize the most important properties. The conditional measure μ_x^F is a σ-finite measure on the leaf $F(x)$, and locally finite when considered as a measure on the manifold $F(x)$. Although this measure is not canonically defined, the ratios

$$\frac{\mu_x^F(A)}{\mu_x^F(O)}$$

for a measurable $A \subset F(x)$ and an open bounded set $O \subset F(x)$ that contains x are canonical. The invariance of the measure under the flow implies that

$$\mu_{\alpha^{\mathbf{t}}x}^F(\alpha^{\mathbf{t}}A) = C\mu_x^F(A) \tag{3.2}$$

for any $\mathbf{t} \in R$, a.e. $x \in X$, some constant $C > 0$ (depending on x and $\mathbf{t}$), and any measurable $A \subset F(x)$.

For a foliation F_λ into one-dimensional leaves defined by a root λ (or a pair of different indices $\lambda = (a, b)$), we write μ_x^λ for the family of conditional measures and impose the following normalization:

$$\mu_x^\lambda\big(\{\exp(sv_\lambda)x : s \in [-1, 1]\}\big) = 1\,, \tag{3.3}$$

where $v_\lambda \in \mathfrak{g}_\lambda$ is a fixed nontrivial vector in the root space (or the matrix $v_\lambda = v_{a,b}$). This and equation (3.2) imply

$$\mu_{\alpha^{\mathbf{t}}x}^\lambda(\alpha^{\mathbf{t}}A) = \mu_x^\lambda(A) \quad \text{if } F_\lambda \text{ is isometric under } \alpha^{\mathbf{t}}. \tag{3.4}$$

A close relation of μ and its conditional measures is how null sets of the first behave under the second. We note the following lemma for later use:

LEMMA 3.1 *Let $F_1, \dots, F_j$ be several foliations as in Section 2. Let N be a null set; then there exists a null set $N' \supset N$ with $\mu_x^{F_i}(N') = 0$ for all $x \notin N'$ and $i = 1, \dots, j$.*

PROOF: For a conditional measure with respect to a σ-algebra $\mathcal{A}$, it follows from property (ii) of the characterizing properties that $\mu_x^{\mathcal{A}}(N) = 0$ for a.e. $x \in X$. Using the characterizing properties for μ_x^F, we get similarly $\mu_x^{F_i}(N) = 0$ for a.e. $x \in X$ and all i, say $C_i(N) \supset N$ is a null set for which $\mu_x(N) = 0$ for $x \notin C_i(N)$.

Let $N_0 = N$ and $N_{k+1} = C_1(C_2(\cdots C_j(N_k)\cdots))$ for $k = 0, \dots$. Clearly $N' = \bigcup N_k$ is a null set. If $x \notin N'$ and $k \geq 0$, we have $\mu_x^{F_i}(N_k) = 0$ because $x \notin C_i(N_k) \subset N_{k+1}$. Since μ_x^F is a measure, we conclude $\mu_x^F(N') = 0$. $\square$

3.2 Some Dynamical Properties

We say μ_x^λ is Lebesgue a.e. if it is invariant under left multiplication with $\exp(sv_\lambda)$ for a.e. $x \in X$ and all $s \in \mathbb{R}$. We say μ_x^F is atomic a.e. (respectively, trivial a.e.) if μ_x^F is an atomic measure (respectively, $\mu_x^F = \delta_x$) for a.e. $x \in X$.

LEMMA 3.2 *Let μ be an α-invariant ergodic measure on X. Either μ_x^λ is Lebesgue a.e. or the conditional measure is almost never Lebesgue. Similarly, either μ_x^F is atomic a.e.* (*respectively, trivial a.e.*) *or μ_x^F has no atoms* (*respectively, is not trivial*) *a.e. Furthermore, if μ_x^F is atomic a.e., it is in fact trivial a.e.*

SKETCH OF PROOF: The first statement follows from equation (3.2), which implies that the set of points $x \in X$ where μ_x^λ is Lebesgue (atomic or trivial) is α-invariant. For the second statement, let $\varepsilon > 0$ and let x be an atom of μ_x^F. Then there is a small neighborhood $U \subset \mathfrak{h}$ of the origin such that $\mu_x^F(\exp(U)x \setminus \{x\}) \leq \varepsilon\mu_x^F(\{x\})$. Let A be the set of points where this inequality for a fixed ε and U holds. We claim that A is measurable. First, we can divide X into countably many pieces X_k such that for $x \in X_k \subset \varphi_{O_k,z_k}(O_k)$ we have $\exp(U)x \subset [x]_{\mathcal{A}(O_k,F)}$. Then $A \cap X_k$ is measurable because of property (iii) of μ_x^F and property (i) of $\mu_x^{\mathcal{A}(O_k,F)}$. Let $\alpha^{\mathbf{t}}$ be such that F is expanded. By Poincaré recurrence a.e. $x \in A$ returns infinitely often to A; i.e., there are infinitely many positive integers n with $\alpha^{n\mathbf{t}}x \in A$. We see from equation (3.2) that

$$\mu_x^F(F(x) \setminus \{x\}) \leq \varepsilon\mu_x^F(\{x\})$$

for a.e. $x \in A$. The set B of points x with the above property is α-invariant, and so $B = X$. However, by varying U and ε we see that μ_x^F is trivial a.e. $\square$

The conditional measure μ_x^F for various foliations F cannot be used to describe μ uniquely (locally one would need a transversal factor measure as well). However, the measure μ and the measures μ_x^F for $x \in X$ are closely related, and some properties of μ can be characterized by the conditionals.

PROPOSITION 3.3 *Let μ be an ergodic α-invariant measure on X. If μ_x^λ is Lebesgue a.e. for all roots $\lambda \in \Phi$ (respectively, pairs of different indices $\lambda = (a, b)$), then $\mu = m$ is the Haar measure. For a fixed map $\alpha^{\mathbf{t}}$ the entropy $\mathrm{h}_\mu(\alpha^{\mathbf{t}})$ is trivial if and only if the conditional measure μ_x^F is trivial for the foliation F into unstable manifolds.*

SKETCH OF PROOF: For the first statement consider an element $g = \exp(sv_\lambda)$ for a small s and some $B \subset X$ with small diameter. There exists an open set O as in property (iii) for conditional measures on leaves that contains B and gB. By property (iii) the conditional measure for the σ-algebra $\mathcal{A}(O, F_\lambda)$ and $x \in \varphi_{O,z}(O)$ is the restriction of μ_x^λ to the atom $[x]$ almost surely. From this we conclude $\mu(B) = \mu(gB)$. Clearly, every set B can be partitioned into at most countably many sets of small diameter, so the same holds for any measurable B. Furthermore, the tangent space of X is spanned by the tangent vectors to the various F_λ and the orbit directions, so that the measure is left-invariant under any small $g \in G$. It follows that $\mu = m$. The second statement is taken from [12, prop. 4.1]. □

Notice that, if the conditional measures with respect to a foliation are δ-measures, then the foliation is in fact measurable, the corresponding σ-algebra is the σ-algebra of all measurable sets, and hence the measurable partition corresponding to the σ-algebra is the partition ε whose elements are single points: Almost every leaf has only one "significant" point, the support of the conditional measure.

It follows from the second statement of Proposition 3.3 that, if F is the foliation into unstable manifolds and F_λ is a one-dimensional subfoliation with nonatomic conditional measure, then μ_x^F is nonatomic a.e. and the entropy of $\alpha^{\mathbf{t}}$ is positive. We will study in Section 8 how the conditional measures μ_x^λ for $\lambda \in \Phi$ determine the conditional measure μ_x^F for the foliation F into unstable manifolds. Corollary 7.2 will give a closer connection between the conditional measures on the one-dimensional foliations and entropy, in particular, a converse to the above.

4 Formulation of Results

As we noted in the introduction (see also Section 9) positive entropy for certain maps of the Weyl chamber flow is not sufficient to deduce that μ is the Haar measure.

THEOREM 4.1 *Let G be a simple, split connected Lie group of rank $n > 1$ and let $\Gamma \subset G$ be a lattice. Let α be the left action of a maximal Cartan subgroup on $X = G/\Gamma$. For a fixed regular element $\alpha^{\mathbf{t}}$ there exists a number $q < 1$ such that for any ergodic α-invariant probability measure μ on X the following conditions are equivalent:*

(i) *$\mu = m$ is the Haar measure on X.*
(ii) *For every root $\lambda \in \mathcal{R}$ the conditional measure μ_x^λ is nonatomic a.e.*
(iii) *The entropies of $\alpha^{\mathbf{t}}$ with respect to μ and m satisfy the inequality $\mathrm{h}_\mu(\alpha^{\mathbf{t}}) > q\mathrm{h}_m(\alpha^{\mathbf{t}})$.*

If $G = \mathrm{SL}(n+1, \mathbb{R})$ *we also have the condition*

(iv) *The entropy* $\mathrm{h}_\mu(\alpha^{\mathbf{s}}) > 0$ *is positive for all nontrivial* $\alpha^{\mathbf{s}}$.

We will see in Example 9.5 that in general condition (iv) does not characterize the Haar measure (or a Haar measure on an invariant homogeneous submanifold).

As a model for the proof, and since stronger statements are available in the case of $\mathrm{SL}(3, \mathbb{R})$, we have the following theorem.

THEOREM 4.2 *Let* $G = \mathrm{SL}(3, \mathbb{R})$ *and let* α *be the* $\mathbb{R}^2$ *Weyl chamber flow on* $X = G/\Gamma$, *where* Γ *is a lattice in* G. *Let* μ *be an ergodic* α*-invariant probability measure on* X. *The following are equivalent:*

(i) $\mu = m$ *is the Haar measure on* X.
(ii) *For every* $a \neq b$ *the conditional measure* $\mu_x^{(a,b)}$ *is nonatomic a.e.*
(iii) *For at least three different pairs of indices* (a, b) *with* $a \neq b$, *the conditional measure* $\mu_x^{(a,b)}$ *is nonatomic a.e.*
(iv) *The entropy* $\mathrm{h}_\mu(\alpha^{\mathbf{t}}) > 0$ *is positive for all* $\mathbf{t} \in R \setminus \{0\}$.
(v) *For some* $\mathbf{t} \in R \setminus \{0\}$ *the entropies with respect to* μ *and* m *satisfy*

$$\mathrm{h}_\mu(\alpha^{\mathbf{t}}) > \frac{1}{2}\mathrm{h}_m(\alpha^{\mathbf{t}}) .$$

(vi) *The entropy function* $\mathbf{t} \mapsto \mathrm{h}_\mu(\alpha^{\mathbf{t}})$ *does not agree with a linear map on a half-space.*

Furthermore, if μ *is not Haar and the entropy with respect to an element* $\alpha^{\mathbf{t}}$ *of the action is positive, there exists a pair of indices* (a, b) *such that the following holds*:

(*) *For any element of the action* $\alpha^{\mathbf{s}}$ *the conditional measure on its stable manifold is supported by a single leaf of* $F_{a,b}$ *or* $F_{b,a}$.

Notice that for every regular element of the action and every pair of indices (a, b) either $F_{a,b}$ or $F_{b,a}$ is expanded and the other one is contracted.

In Rees's example (see [24] and Section 9), a nonstandard ergodic invariant measure is supported on a compact homogeneous subspace M that fibers over a compact manifold, and the action on M splits into an $\mathbb{R}$-action and a rotation on the fibers. In this case the product of any $\mathbb{R}$-invariant measure in the base and Lebesgue measure in the fibers is α-invariant. Here the statements of Theorem 4.2 can be checked easily.

The main conjecture in [12] implies in this case that this is the only possible picture for an invariant measure as in (*) in Theorem 4.2.

We indicate some possible strengthenings of Theorem 4.1: First, as we noticed in the introduction, it is not necessary for our proofs to assume that Γ is a lattice. However, if μ satisfies either condition (ii) or (iii), then Γ is a lattice and $\mu = m$. The particular value of q in condition (iii) in Theorem 4.1 does not depend on the lattice Γ, only on the roots and root spaces of the Cartan action α. An optimal choice of q requires an analogue of condition (iii) in Theorem 4.2 for general groups. One not optimal such analogue is the following: It is enough to

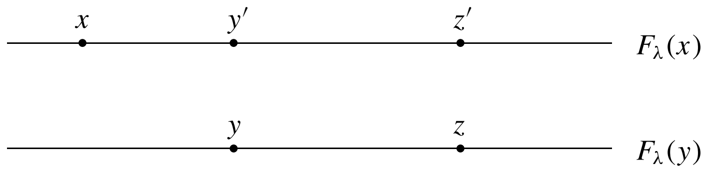

FIGURE 5.1. The two leaves $F_\lambda(x)$ and $F_\lambda(y)$ approach each other when $\alpha^{n\mathbf{t}}$ is applied to them and $n \to \infty$.

assume that μ_x^λ is nonatomic a.e. for a set of roots λ whose root spaces together generate $\mathfrak{g}$.

5 Beginning of the Proof: Translation Invariance of Conditional Measures

For the proofs of the theorems we need more information about the conditional measures. The next proposition will be used, similarly to lemma 5.4 in [12], to show that the conditional measures are invariant under translations. The letter λ will in the following always denote a root of the Lie algebra $\mathfrak{g}$ of G. In the case of $\mathrm{SL}(n+1,\mathbb{R})$ we can identify the roots with the pairs $\lambda = (a, b)$ of different indices. Recall that μ is assumed to be an α-ergodic probability measure on X.

PROPOSITION 5.1 *Let $\alpha^{\mathbf{t}}$ be such that F_λ is an isometric foliation of X. There exists a null set $N \subset X$ such that the following holds: For any two $x, y \notin N$ for which there exists*

$$y' = \exp(s v_\lambda)x \in F_\lambda(x)$$

with

$$d(\alpha^{n\mathbf{t}} y, \alpha^{n\mathbf{t}} y') \to 0 \quad \textit{for } n \to \infty\,, \tag{5.1}$$

we define the map

$$\phi : \begin{array}{c} F_\lambda(y) \to F_\lambda(x) \\ \exp(r v_\lambda)y \mapsto \exp((r+s)v_\lambda)x\,, \end{array}$$

which maps $z \in F_\lambda(y)$ to the unique $z' \in F_\lambda(x)$ satisfying equation (5.1) *with z and z'. The conditional measure μ_x^λ coincides with the image of μ_y^λ under ϕ up to a multiplicative constant.* (*See Figure* 5.1.)

Sometimes it is more convenient to use the following locally finite measure ν_x^λ on $\mathbb{R}$, which is just an isomorphic copy of μ_x^λ. For a measurable set $A \subset \mathbb{R}$ we define

$$\nu_x^\lambda(A) = \mu_x^\lambda(\{\exp(r v_\lambda)x : r \in A\})\,.$$

With this notation the conclusion of Proposition 5.1 can be expressed as

$$\nu_y^\lambda(A) = C\nu_x^\lambda(A+s)$$

for a multiplicative constant $C > 0$ and any measurable $A \subset \mathbb{R}$. Also note that in the case of $y' = x$, the number s vanishes and by the normalization in equation (3.3) the multiplicative constant equals 1. This leads to the following corollary:

COROLLARY 5.2 *Let λ and ξ be two roots (or pairs of indices). Assume $\lambda \neq \pm\xi$ (or that the pair λ is neither the pair ξ nor the reversed pair); then for a.e. $x \in X$ and μ_x^ξ-a.e. $y \in F_\xi(x)$,*

$$\nu_x^\lambda = \nu_y^\lambda .$$

PROOF: From the discussion above, the corollary follows at once if we find a map $\alpha^{\mathbf{t}}$ contracting F_ξ and stabilizing F_λ. In the case of $\mathrm{SL}(n+1, \mathbb{R})$ and the pairs $\lambda = (a, b)$ and $\xi = (c, d) \neq (a, b), (b, a)$, it is easy to find $\mathbf{t}$ satisfying $t_a = t_b$ and $t_c < t_d$.

For the general case, note that λ and ξ cannot be multiples of each other, since this can only be if $\lambda = \pm\xi$. However, this means that λ and ξ are linearly independent linear functionals on the Lie algebra $\mathfrak{g}_\alpha$ corresponding to the Cartan subgroup. There exists an element $\alpha^{\mathbf{t}}$ of the Cartan subgroup that contracts F_ξ and acts isometrically on F_λ. □

PROOF OF PROPOSITION 5.1: The proof is a variation of Hopf's argument. Let $\mathfrak{g}' \subset \mathfrak{g}$ be the Lie subalgebra whose elements are contracted by the adjoint action of $\alpha^{\mathbf{t}}$. Then the points x, y, and y' as in the proposition satisfy $y' = \exp(w)y$ for some $w \in \mathfrak{g}'$ and $y' = \exp(s v_\lambda)x$ for some $s \in \mathbb{R}$. If $z = \exp(r v_\lambda)y \in F_\lambda(y)$, then

$$z' = \exp((r+s)v_\lambda)x = \exp(r v_\lambda)\exp(w)\exp(-r v_\lambda)z \in F_\lambda(x)$$

satisfies equation (5.1) with z and z'. Here we use that the metric d is the induced metric of a right-invariant metric on G. Since F_λ is an isometric foliation, there can only be one such z'.

The map $x \mapsto \nu_x^\lambda$ is measurable, where we use the weak* topology on the set of locally finite measures. More precisely, we claim $x \mapsto \int f \, d\nu_x^\lambda$ is measurable for any continuous function with compact support $f : \mathbb{R} \to \mathbb{R}$. Let $O \subset \mathfrak{h} \times T$ be as in property (iii) of the characterizing properties for μ_x^F (p. 1191), write $\mathcal{A} = \mathcal{A}(O, F_\lambda)$ for the corresponding σ-algebra, and fix the function f with support in $[-N, N]$. Let $O' \subset O$ be an open set such that $[-N, N] \times \{0\} + O' \subset O$. There is a sequence of such open sets $O_i' \subset O_i$ for which $\varphi_{O_i, z_i}(O_i')$ covers X. For this reason it is enough to show measurability for $x \in \varphi_{O,z}(O')$ for a fixed such $O' \subset O$.

Using a local inverse of $\varphi = \varphi_{O,z}$, one can define a uniformly continuous function $k : \varphi(O') \times \varphi(O)$ satisfying $k(\exp(s v_\lambda)x, x) = f(s)$ for all $|s| \leq N$ and $x \in \varphi(O')$. For this function

$$\int f \, d\nu_x^\lambda = \frac{1}{C(x)} \int k(y, x) \, d\mu_x^{\mathcal{A}}(y) \quad \text{for any } x \in \pi(U),$$

where $C(x)$ is the normalizing constant. We consider first the measurability of the integral.

Note that $u \mapsto \mu_u^{\mathcal{A}}$ is measurable by one of the properties of the conditional measures. This shows that

$$K(u,x) = \int k(y,x)\,\mathrm{d}\mu_u^{\mathcal{A}}(y)$$

is measurable in $u \in \pi(O')$ for any fixed $x \in \pi(O)$. If x' is close to x, then $K(u,x')$ is uniformly close to $K(u,x)$. We cover $\pi(O)$ by a sequence of ε-balls $B_\varepsilon(x_i)$ and produce a measurable partition $\{P_1, P_2, \dots\}$ with $P_i \subset B_\varepsilon(x_i)$. Then $K_\varepsilon(u,x) = K(u,x_i)$ for $x \in P_i$ defines a measurable function K_ε on $\varphi(O') \times \varphi(O)$. Letting ε tend to zero, K_ε tends to K. This shows that K is measurable as a function in two variables. Therefore $K(x,x)$ is a measurable function in x, which is exactly the integral term above. The multiplicative constant $C(x)$ is equal to $K(x,x)$ if f is the characteristic function of the interval $[-1,1]$—approximating the characteristic function by continuous functions from above shows in the limit that $C(x)$ is measurable.

Let N_0 be a null set such that for any two points not in N_0 properties (3.1) and (3.4) hold for all powers of $\alpha^{\mathbf{t}}$.

By Luzin's theorem there exists a compact subset $K_j \subset X \setminus N_0$ with measure

$$\mu(K_j) > 1 - \frac{1}{j} \tag{5.2}$$

such that the restriction of ν_x^λ to K_j is continuous. We can assume the sequence K_j is increasing. Let

$$f_j(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{K_j}(\alpha^{k\mathbf{t}}x)$$

be the limit of the Birkhoff averages of the characteristic function 1_{K_j}. Define

$$L_j = \left\{ x \in X : f_j(x) \le \frac{1}{2} \right\};$$

clearly L_j is decreasing. We claim $\lim_{j\to\infty} \mu(L_j) = 0$. From equation (5.2) and the ergodic theorem we get

$$1 = \lim_{j\to\infty} \int_X f_j \,\mathrm{d}\mu \le \lim_{j\to\infty} \left(\mu(X \setminus L_j) + \frac{1}{2}\mu(L_j) \right) \le 1 - \frac{1}{2} \lim_{j\to\infty} \mu(L_j)\,.$$

This proves the claim. We obtain from this the null set $N = N_0 \cup \bigcap_j L_j$.

Now suppose $x, y \notin N$ are such that there exists a y' as in the assumptions of the proposition. By the definition of N we can find j such that $x, y \notin L_j$. Therefore both points return to K_j infinitely often and with a frequency higher than $\frac{1}{2}$. So one can find a single sequence of integers $n_i \to \infty$ such that

$$\alpha^{n_i\mathbf{t}}x, \alpha^{n_i\mathbf{t}}y \in K_j \quad \text{for all } i.$$

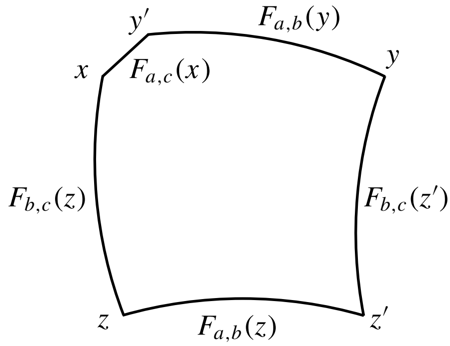

FIGURE 6.1. The rectangle with sides parallel to $F_{a,b}$ and $F_{c,d}$ only closes up with another side parallel to $F_{a,c}$.

Since K_j is compact, we can find a subsequence (again denoted by n_i) and two points $\bar{x}, \bar{y} \in K_j$ such that

$$\alpha^{n_i \mathbf{t}} x \to \bar{x}$$

and the same for y and $\bar{y}$ (along the same sequence).

From equation (3.4) we get

$$\nu_x^\lambda = \nu_{\alpha^{n_i \mathbf{t}} x}^\lambda \quad \text{and by the continuity on } K_j \quad \nu_x^\lambda = \nu_{\bar{x}}^\lambda .$$

The same argument for y shows

$$\nu_y^\lambda = \nu_{\bar{y}}^\lambda .$$

Since y and y' satisfy equation (5.1), the limit points satisfy $\bar{y} \in F_\lambda(\bar{x})$ and $\mu_{\bar{y}}^\lambda = C\mu_{\bar{x}}^\lambda$ by Property (3.1). This shows the required equality of measures. □

6 Proof in the SL(n, $\mathbb{R}$) Case

One main ingredient of the proof of Theorem 4.1 is the non-abelian structure of the foliations. The next lemma makes use of this. We will first turn to the case of $G = \mathrm{SL}(n+1, \mathbb{R})$ and consider the general case later.

LEMMA 6.1 *Let* $G = \mathrm{SL}(n+1, \mathbb{R})$ *and let* α *be the Weyl chamber flow on* $X = G/\Gamma$. *Suppose* μ *is an invariant ergodic measure and* $1 \le a, b, c \le n+1$ *are three different indices. If* $\mu_x^{(a,b)}$ *and* $\mu_x^{(b,c)}$ *are nonatomic a.e., the conditional measure* $\mu_x^{(a,c)}$ *is Lebesgue a.e.*

The idea of the proof is to translate the measure $\nu_x^{(a,c)}$ along a rectangle with sides parallel to the foliations $F_{a,b}$ and $F_{b,c}$, and use the fact that due to the commutation relations such a rectangle does not close up. The two endpoints are in the same $F_{a,c}$-leaf; see Figure 6.1. We will show that the measure $\nu_x^{a,c}$ on $\mathbb{R}$ does not change under such a translation. However, for this it is necessary to avoid non-typical points. Here we will use Proposition 5.1.

PROOF: Find a null set N_0 such that Proposition 5.1, Corollary 5.2, and property (3.1) hold for all pairs of indices and all $x \notin N_0$. Enlarge N_0 to N such that $\mu_x^{(i,j)}(N) = 0$ for all $x \notin N$ and $i \neq j$ as in Lemma 3.1. Let $z \notin N$; we are going to define all points in Figure 6.1. Since $\mu_z^{(a,b)}$ is nonatomic, there exists a point $z' = (\mathrm{Id} + r v_{a,b}) z \in F_{a,b}(z) \setminus N$ with $r \in \mathbb{R} \setminus \{0\}$. Since every neighborhood of z has positive measure with respect to $\mu_z^{(a,b)}$ the number r can be chosen arbitrarily small. Since $z, z' \notin N$, we have

$$\mu_z^{(b,c)}(N) = \mu_{z'}^{(b,c)}(N) = 0.$$

By Corollary 5.2, $\nu_z^{b,c} = \nu_{z'}^{b,c}$. Since this measure is nonatomic but the preimages of N under the parametrization maps at z and z' are null sets, there exists an arbitrarily small $s \in \mathbb{R} \setminus \{0\}$ with $x = (\mathrm{Id} + s v_{b,c}) z \notin N$ and $y = (\mathrm{Id} + s v_{b,c}) z' \notin N$. For x and y we see that

$$y = (\mathrm{Id} + s v_{b,c})(\mathrm{Id} + r v_{a,b}) z = (\mathrm{Id} + s v_{b,c})(\mathrm{Id} + r v_{a,b})(\mathrm{Id} + s v_{b,c})^{-1} x\,.$$

By the commutation relation for $v_{a,b}$ and $v_{b,c}$ we get

$$y = (\mathrm{Id} + r v_{a,b})(\mathrm{Id} - r s v_{a,c}) x = (\mathrm{Id} + r v_{a,b}) y'$$

where $y' = (\mathrm{Id} - r s v_{a,c}) x \in F_{a,c}(x)$. Choosing $\alpha^{\mathbf{t}}$ such that $F_{a,b}$ is contracted and $F_{a,c}$ is isometric we get from Proposition 5.1

$$\nu_z^{a,c} = \nu_{z'}^{a,c} \quad \text{and} \quad \nu_x^{a,c}(A) = D \nu_y^{a,c}(A - rs)$$

for some $D > 0$ and any measurable $A \subset \mathbb{R}$. Furthermore, $\nu_x^{a,c} = \nu_z^{a,c}$ and $\nu_y^{a,c} = \nu_{z'}^{a,c}$ by Corollary 5.2.

We have shown that for $z \in X \setminus N$ there are arbitrarily small non-vanishing $t \in \mathbb{R}$ and a constant $D > 0$ such that

$$\nu_z^{a,c}(A + t) = D \nu_z^{a,c}(A) \quad \text{for any measurable } A \subset \mathbb{R}, \tag{6.1}$$

in other words $\nu_z^{a,c}$ is invariant under t in the affine sense. This situation also appeared in the proof of measure rigidity in case of automorphisms of the torus $\mathbb{T}^n$; the arguments in [12, 14] or [5] could be used to complete the proof. We present here a slightly different proof for completeness.

We claim that $D = 1$ a.e., so that $\nu_z^{a,c}$ is invariant under t in the strict sense. We assume by contradiction that $D \neq 1$ on a set of positive measure. We define the measurable function

$$f(z) = \limsup_{r \to \infty} \frac{\log \nu_x^{a,c}([-r, r])}{2r}$$

which measures the exponential growth of the measure $\nu_x^{a,c}$. (Measurability of the fraction follows easily by using a decreasing sequence of continuous functions approximating the characteristic function of $[-r, r]$ from above.) Replacing t with $-t$ if necessary, we can assume $D > 1$. Iterating equation (6.1) starting with $A = [-1, 1]$ we get

$$\nu_x^{a,c}([-1 + nt, 1 + nt]) = D^n.$$

If $t > 0$, let $r_n = 1 + nt$. Then

$$\log \nu_x^{a,c}([-r_n, r_n]) \geq n \log D \quad \text{shows that } f(x) > 0.$$

The other case and the proof that $f(x) < \infty$ are similar; together we have shown that $f(x) \in (0, \infty)$ whenever $D \neq 1$.

Suppose $\alpha^{\mathbf{t}}$ expands $F_{a,c}$, then $e^{t_a - t_c} > 1$. The function f satisfies

$$\begin{aligned} f(\alpha^{\mathbf{t}} x) = \limsup_{r\to\infty} \frac{\log \nu_{\alpha^{\mathbf{t}} x}^{a,c}([-r, r])}{2r} &= \limsup_{r\to\infty} \frac{\log C \nu_x^{a,c}(e^{t_c - t_a}[-r, r])}{2r} \\ &= e^{t_a - t_c} \limsup_{r\to\infty} \frac{\log \nu_x^{a,c}(e^{t_c - t_a}[-r, r])}{2e^{t_c - t_a} r} \\ &= e^{t_a - t_c} f(x) . \end{aligned}$$

In the second line we used the analog to formula (3.2) for the measures $\nu_x^{a,c}$. However, this contradicts Poincaré recurrence since $f(\alpha^{n\mathbf{t}} x) \nearrow \infty$. This shows that $D = 1$ and $\nu_z^{a,c}$ is invariant under t.

For every z define

$$G_z = \{t \in \mathbb{R} : \nu_z^{a,c} \text{ is invariant under } t\} .$$

Obviously G_z is a subgroup of $\mathbb{R}$. Therefore there are three cases, either $G_z = \mathbb{R}$ and $\nu_z^{a,c}$ is the Lebesgue measure. Or G_z is a discrete subgroup, or G_z is a dense subgroup. We show that the first case happens a.e.

Since t can be made arbitrarily small, G_z cannot be a discrete subgroup. Suppose G_z is dense and $\nu_z^{a,c}$ has atoms, then by invariance under G_z there is a dense set of atoms which all have the same mass. However, this contradicts that $\nu_z^{a,c}$ is locally finite. Suppose now G_z is dense and $\nu_z^{a,c}$ does not have any atoms, let I be any interval in $\mathbb{R}$. The function $g(t) = \nu_z^{a,c}(I + t)$ is continuous in t and constant on G_z. Therefore g is constant. Since this holds for any interval the measure is invariant under any $t \in \mathbb{R}$ and $G_z = \mathbb{R}$.

Since $\nu_z^{a,c}$ is the isomorphic copy of $\mu_z^{(a,c)}$ using the parametrization of $F_{a,c}$, this completes the proof that $\mu_z^{(a,c)}$ is Lebesgue a.e. □

We proceed to the proof of Theorem 4.2 and part of Theorem 4.1 in the case of $G = \mathrm{SL}(n+1, \mathbb{R})$. It is clear that the Haar measure always satisfies the other conditions in the theorem, so we only have to prove one direction.

PROOF THAT (ii) IMPLIES (i) FOR $G = \mathrm{SL}(n+1, \mathbb{R})$: Suppose condition (ii) holds, let a, c be two different indices. There exists b different from a and c, by assumption $\mu_x^{(a,b)}$ and $\mu_x^{(b,c)}$ are nonatomic a.e. By Lemma 6.1 $\mu_x^{(a,c)}$ is Lebesgue a.e. This holds for all $a \neq c$ and Proposition 3.3 concludes the proof. □

We will use the following lemma (a version of the Ledrappier-Young entropy formula, see [17]) to conclude the proof of the case $\mathrm{SL}(n+1, \mathbb{R})$ and postpone the proof of the lemma to Section 8.

LEMMA 6.2 *There are constants* $s_{a,b}$ *with*

$$s_{a,b} = 0 \text{ if } \mu_x^{(a,b)} \text{ is atomic a.e.} \quad \text{and} \quad s_{a,b} \in (0,1] \text{ otherwise}$$

such that for any $\mathbf{t} \in R$

$$\mathrm{h}_\mu(\alpha^{\mathbf{t}}) = \sum_{a,b} s_{a,b}(t_a - t_b)^+. \tag{6.2}$$

Here $(r)^+ = \max(0, r)$ *denotes the positive part of* $r \in \mathbb{R}$. *In particular the entropy* $\mathrm{h}_\mu(\alpha^{\mathbf{t}})$ *is positive if and only if there is a pair* (a, b) *whose foliation is expanded and has nonatomic conditional measure. In the case of the Haar measure* $s_{a,b} = 1$ *for all* (a, b).

Clearly the last statement of the lemma holds for foliations which are contracted as well. Below we say the pair (a, b) of different indices is *nonatomic*, if the conditional measure $\mu_x^{(a,b)}$ is nonatomic a.e.

PROOF THAT (iv) IMPLIES (ii) IN THEOREMS 4.1 AND 4.2: Consider an element $\mathbf{t} \in R$ with $t_1 > t_2 = t_3 = \cdots = t_{n+1}$. Clearly $(t_a - t_b) > 0$ only for the pairs $(1, 2), \ldots, (1, n + 1)$. Since the entropy $\mathrm{h}_\mu(\alpha) > 0$ is positive by assumption, Lemma 6.2 shows that one of the pairs $(1, 2), \ldots, (1, n + 1)$ is nonatomic. By rearranging the indices from 2 to $n + 1$ we can assume $(1, 2)$ is nonatomic; this does not change the set of pairs $\{(1, 2), \ldots, (1, n + 1)\}$.

We proceed by induction and show the pairs $(1, 2), \ldots, (1, n + 1)$ are nonatomic. Assume $(1, 2), \ldots, (1, k)$ are nonatomic. Let $\mathbf{t}$ satisfy $t_1 = \cdots = t_k > t_{k+1} = \cdots = t_{n+1}$. As above we see that there is a nonatomic pair (a, b) whose foliation is expanded by $\alpha^{\mathbf{t}}$ – so $a \le k < b$. If $a > 1$ we know $(1, a)$ and (a, b) are nonatomic, and Lemma 6.1 shows that $(1, b)$ is nonatomic. Without loss of generality $b = k+1$, and we have shown that $(1, 2), \ldots, (1, k+1)$ are nonatomic.

Repeating the argument for contracting foliations we get $(2, 1), \ldots, (n + 1, 1)$ are nonatomic pairs. Let $a \ne b$; we want to show that (a, b) is nonatomic. If $a = 1$ or $b = 1$ we already know that. Otherwise $(a, 1)$ and $(1, b)$ are two nonatomic pairs, and Lemma 6.1 shows that (a, b) is nonatomic. □

PROOF THAT (iii) IMPLIES (ii) IN THEOREM 4.2: Assume that

$$(a_1, b_1), (a_2, b_2), (a_3, b_3)$$

are three different nonatomic pairs. For one of the three — say (a_1, b_1) — the reversed pair (b_1, a_1) is not among the three. Let c be the index different from a_1 and b_1. If (b_1, c) is among the list, Lemma 6.1 shows that (a_1, c) is nonatomic. If (c, a_1) appears in the list, similarly (c, b_1) is nonatomic. If none of the above cases takes place, the other indices must be

$$\{(a_2, b_2), (a_3, b_3)\} = \{(a_1, c), (c, b_1)\}$$

because (b_1, a_1) is not in the list. In all three cases we found indices d, e, f such that

$$(d, e), (d, f), (e, f) \tag{6.3}$$

are nonatomic.

Assume for simplicity $d = 1, e = 2, f = 3$. Then the pairs in list (6.3) are all pairs from the upper triangle. We claim that the remaining pairs

$$(2, 1), (3, 1), (3, 2)$$

are also nonatomic. Let $\mathbf{t} \in R$ be such that

$$t_1 = t_2 > t_3 .$$

Then $\alpha^{\mathbf{t}}$ expands exactly the two one-dimensional foliations for the pairs $(1, 3)$ and $(2, 3)$. Since those pairs are nonatomic, Lemma 6.2 implies that the entropy $\mathrm{h}_\mu(\alpha^{\mathbf{t}}) > 0$ is positive. The inverse map expands only the foliations corresponding to the pairs $(3, 1)$ and $(3, 2)$. By Lemma 6.2 we see that at least one of the two pairs is nonatomic. Using a different map it follows similarly that at least one of the pairs $(3, 1)$ and $(2, 1)$ is nonatomic.

If $(3, 1)$ is nonatomic, Lemma 6.1 implies $(3, 2)$ is nonatomic since $(1, 2)$ is nonatomic. Similarly $(2, 1)$ must be nonatomic as well, showing that all pairs are nonatomic.

The only other case to consider would be that $(3, 2)$ and $(2, 1)$ are nonatomic. However, Lemma 6.1 implies immediately that $(3, 1)$ is nonatomic as well. Therefore we have shown condition (ii). □

PROOF THAT (v) IMPLIES (iii) AND (vi) IMPLIES (iii) IN THEOREM 4.2: We prove that if Condition (iii) fails, Conditions (v) and (vi) fail as well.

If there is no nonatomic pair, entropy vanishes and the other conditions fail trivially. Furthermore, it is not possible to have exactly one nonatomic pair. For if (a, b) is nonatomic and $\alpha^{\mathbf{t}}$ expands the corresponding foliation, the entropy $\mathrm{h}_\mu(\alpha^{\mathbf{t}})$ must be positive by Lemma 6.2, and using the contracting foliations there must be another nonatomic pair.

LEMMA 6.3 *Suppose now there are exactly two different nonatomic pairs and assume they are*

$$(1, 2), (a, b) .$$

Then $(a, b) = (2, 1)$.

PROOF: If (a, b) is equal to $(2, 3)$ or $(3, 1)$ Lemma 6.1 implies immediately that there are three nonatomic pairs. If (a, b) is equal to $(1, 3)$ (resp. $(3, 2)$) the element $\alpha^{\mathbf{t}}$ for $t_1 > t_2 = t_3$ (resp. $t_1 = t_3 > t_2$) expands both nonatomic pairs and therefore the entropy $\mathrm{h}_\mu(\alpha^{\mathbf{t}})$ must be positive by Lemma 6.2. However, this shows that there must be another nonatomic pair for which the foliation is contracted which is again a contradiction. So the only possible case is $(a, b) = (2, 1)$, as claimed. □

Lemma 6.2 shows that for $\mathbf{t}$ satisfying $t_1 > t_2$ the right-hand side of equation (6.2) is linear in $\mathbf{t}$. Therefore the entropy coincides with a linear map on a half-space showing that (vi) cannot hold.

For Condition (v) let $\mathbf{t} \in R$ be arbitrary. Lemma 6.2 shows $\mathrm{h}_\mu(\alpha^{\mathbf{t}}) \le |t_1 - t_2|$. By the triangle inequality

$$2\mathrm{h}_\mu(\alpha^{\mathbf{t}}) \le 2|t_1 - t_2| \le |t_1 - t_2| + |t_3 - t_1| + |t_2 - t_3| = \mathrm{h}_m(\alpha^{\mathbf{t}})\,.$$

Therefore Condition (v) cannot hold. □

PROOF OF (*): From Lemma 6.3 we know that the only nonatomic pairs are (a, b) and (b, a) for some a, b. Note that if $F_{a,b}$ is contracted by some $\alpha^{\mathbf{t}}$, then $F_{b,a}$ is expanded. Let F be the foliation into unstable manifolds for $\alpha^{\mathbf{t}}$. By Proposition 8.3 the conditional measure μ_x^F is the product measure of the conditionals on three pairs, of which only one is not a δ-measure. Hence the measure μ_x^F is supported by a single one-dimensional leaf of either $F_{a,b}$ or $F_{b,a}$. □

We have completed the proof of Theorem 4.2 (up to Lemma 6.2 and Proposition 8.3).

7 Proof in the General Case

We turn our attention to the general case in Theorem 4.1. The main argument above was the repeated use of Lemma 6.1 which we replace by the next proposition.

Recall that the roots $\lambda \in \mathcal{R}$ of the simple split Lie algebra $\mathfrak{g}$ are elements of the dual space of the Lie algebra of the Cartan subgroup. Since G is simple and split, the root spaces $\mathfrak{g}_\lambda \subset \mathfrak{g}$ are one-dimensional eigenspaces for the adjoint action of $\alpha^{\mathbf{t}}$ with eigenvalue $e^{\lambda(\mathbf{t})}$. For a fixed basis of the Cartan subgroup the dual is isomorphic to $\mathbb{R}^n$ where n is the rank of G. For a root λ the only multiple of λ which is also a root is $-\lambda$. If $\lambda_1 \neq -\lambda_2$ are two roots, the sum $\lambda_1 + \lambda_2$ is a root if and only if $v = [v_{\lambda_1}, v_{\lambda_2}]$ is nontrivial, in which case $v = s v_\lambda$ ($s \in \mathbb{R}$) is in the root space of $\lambda = \lambda_1 + \lambda_2$ (see [25, pg. 268 and 282]).

PROPOSITION 7.1 *Let $\lambda_1, \lambda_2 \in \Phi$ be roots such that $\lambda = \lambda_1 + \lambda_2$ is a root and $\mu_x^{\lambda_i}$ is nonatomic a.e. for $i = 1, 2$. Then μ_x^λ is Lebesgue a.e. Furthermore, the same is true for every root ξ different from λ_1 and λ_2 with v_ξ belonging to the Lie algebra generated by v_{λ_1} and v_{λ_2}.*

PROOF: The proof is similar to the proof of Lemma 6.1. The difference in the general situation here is that $[v_\lambda, v_{\lambda_i}]$ might not be zero for some i. Let $\Phi' \supset \{\lambda, \lambda_1, \lambda_2\}$ be the set of roots ξ with v_ξ belonging to the Lie algebra generated by v_{λ_1} and v_{λ_2}. By the discussion above, every root $\xi \in \Phi'$ can be expressed as $\xi = n_1\lambda_1 + n_2\lambda_2$ with integers $n_1, n_2 \ge 0$ (see Figure 7.1). The space $\mathfrak{g}' = \sum_{\lambda\in\Phi'} \mathfrak{g}_\lambda$ is a nilpotent Lie algebra. For $r, s \neq 0$ the commutator

$$\exp(s v_{\lambda_2}) \exp(r v_{\lambda_1}) \exp(-s v_{\lambda_2}) \exp(-r v_{\lambda_1}) = \exp(s_1 v_{\xi_1}) \cdots \exp(s_\ell v_{\xi_\ell}) \tag{7.1}$$

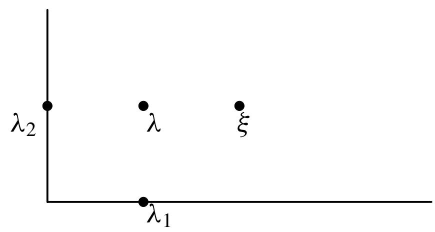

FIGURE 7.1. A possible configuration for Φ'.

can be expressed as a product over various exponentials of elements of root spaces. We use a fixed ordering of the elements in $\Phi' = \{\lambda_1, \lambda_2, \xi_1, \dots, \xi_\ell\}$, which will be further specified later. Note that λ_1 and λ_2 do not appear on the right side of equation (7.1). If r, s are small and nonzero, the elements s_i are small, and s_k for $\xi_k = \lambda$ is small and nonzero. This follows for instance from the Campbell-Baker-Hausdorff formula.

Let N be a null set satisfying Proposition 5.1 for all root spaces (and finitely many elements of the action which are used in the proof). Enlarge N so that $\mu_x^\xi(N) = 0$ for every root ξ and $x \notin N$ by using Lemma 3.1. Let $z \notin N$; as before we will define the points as in Figure 6.1. Since $\mu_z^{\lambda_1}$ is nontrivial and $\mu_z^{\lambda_1}(N) = 0$, there exists a point $z' = \exp(rv_{\lambda_1})z \in F_{\lambda_1}(z)\backslash N$. From Corollary 5.1 we have $\nu_z^\xi = \nu_{z'}^\xi$ for any $\xi \in \Phi' \setminus \{\lambda_1\}$. The assumption that $\mu_x^{\lambda_2}$ is nonatomic a.e. implies now that there exists $s \neq 0$ such that $x = \exp(sv_{\lambda_2})z$ and $y = \exp(sv_{\lambda_2})z'$ do not belong to N. The point $x \notin N$ satisfies that $y = \exp(sv_{\lambda_2})\exp(rv_{\lambda_1})\exp(-sv_{\lambda_2})x \notin N$. This is similar to Figure 6.1, but the fifth line might not be part of a single leaf.

To overcome this problem we proceed by induction, and show for Φ' step by step that μ_x^ξ is Lebesgue for more and more roots in $\Phi' \setminus \{\lambda_1, \lambda_2\}$ until we reach $\xi = \lambda$. This will prove the first statement, and the second follows from the first.

We order the elements $\xi = n_1\lambda_1 + n_2\lambda_2 \in \Phi'$ with $n_1, n_2 > 0$ (or equivalently $\xi \neq \lambda_1, \lambda_2$) by the quotient n_2/n_1 starting with the biggest. The chosen order of the roots implies for every $i \leq \ell$ that

$$v_\lambda, v_{\xi_i} \quad \text{do not appear as commutators in } \sum_{j \geq i} \mathfrak{g}_{\xi_j} + \mathfrak{g}_{\lambda_1}, \tag{7.2}$$

which we will use below in the following form: changing the order of multiplication for terms involving only roots with $j \geq i$ can be compensated for by changing some coefficients without effecting those for λ and ξ_i.

For the induction it is convenient to prove first the following fact. Suppose $x', y' \notin N$, $\nu_{x'}^\xi = \nu_{y'}^\xi$ for all $\xi \in \phi' \setminus \{\lambda_1, \lambda_2\}$, and

$$y' = \exp(s_i' v_{\xi_i}) \cdots \exp(s_\ell' v_{\xi_\ell}) \exp(rv_{\lambda_1})x'.$$

Then $\nu_{x'}^{\xi_i}$ is invariant under translation by s_i' in the affine sense.

Notice first that $\xi_i \neq \lambda_1$. Therefore there exists an element $\alpha^{\mathbf{t}}$ of the action such that F_{ξ_i} is isometric and F_{λ_1} is contracted. By the choice of the order on Φ' the foliations F_{ξ_j} for $j > i$ are contracted as well. Proposition 5.1 implies the claim.

Assume we already showed that μ_x^{ξ} is Lebesgue a.e. for all roots ξ in a subset $\Phi'' \subset \Phi'$. From the commutator relation in (7.1) we see

$$\begin{aligned} y &= \exp(s v_{\lambda_2}) \exp(r v_{\lambda_1}) \exp(-s v_{\lambda_2}) x \\ &= \exp(s_1 v_{\xi_1}) \cdots \exp(s_\ell v_{\xi_\ell}) \exp(r v_{\lambda_1}) x\,. \end{aligned} \tag{7.3}$$

Let

$$\xi_i = n_1\lambda_1 + n_2\lambda_2 \in \Phi' \setminus (\Phi'' \cup \{\lambda_1, \lambda_2\})$$

be such that $q = n_2/n_1$ is biggest.

We claim there are points $x', y' \notin N$ with the same conditionals $\nu_{x'}^{\xi} = \nu_x^{\xi}$ and $\nu_{y'}^{\xi} = \nu_y^{\xi}$ for $\xi \in \Phi'$ such that

$$y' = \exp(s_i' v_{\xi_i}) \cdots \exp(s_\ell' v_{\xi_\ell}) \exp(r v_{\lambda_1}) x', \tag{7.4}$$

where $s_k = s_k'$ for $\xi_k = \lambda$. If the first element $\xi_1 \in \Phi'$ in (7.3) does not belong to Φ'', there is no difference between (7.3) and (7.4), we set $x' = x$ and $y' = y$. So suppose $\xi_1 \in \Phi''$ then $\mu_x^{\xi_1}$ and $\mu_y^{\xi_1}$ are Lebesgue. Furthermore, $\mu_x^{\xi_1}(N) = \mu_y^{\xi_1}(N) = 0$, there exists $t \in \mathbb{R}$ with $x' = \exp(t v_{\xi_1}) x$, $y' = \exp((t - s_\xi) v_{\xi_1}) y \notin N$. Since $x' \in F_{\xi_1}(x)$, Corollary 5.2 implies that $\nu_x^{\xi} = \nu_{x'}^{\xi}$ other than for $\xi = \xi_1$. However, $\mu_x^{\xi_1}, \mu_{x'}^{\xi_1}$ are both Lebesgue, so these conditionals agree as well. The same applies to y and y'. Changing to the new points equation (7.3) becomes

$$\begin{aligned} y' &= \exp(t v_{\xi_1}) \big(\exp(s_2 v_{\xi_2}) \cdots \exp(s_\ell v_{\xi_\ell}) \exp(r v_{\lambda_1})\big) \exp(-t v_{\xi_1}) x' \\ &= \exp(s_2' v_{\xi_2}) \cdots \exp(s_\ell' v_{\xi_\ell}) \exp(r v_{\lambda_1}) x', \end{aligned}$$

where we used the statement after (7.2) to rewrite the product, which also implies that $s_k' = s_k$ does not change. Repeating the argument if necessary we finally find x', y' as claimed.

Clearly s and r can be chosen arbitrarily small; we fix a sequence $s(n), r(n) \to 0$. If the term s_i' in equation (7.4) does not vanish for infinitely many n in the sequence a.e., we conclude that $\nu_x^{\xi_i}$ is invariant in the affine sense under translation by elements of a dense subgroup. As in the proof of Lemma 6.1 this implies that $\nu_x^{\xi_i}$ is Lebesgue. If s_i' vanishes for almost all n, we remove this term in the product equation (7.4) and proceed to $i + 1$. Since $s_k' = s_k$ remains unchanged — and so nonzero — in this procedure, we have shown that μ_x^{ξ} is Lebesgue for some $\xi \in \Phi' \setminus \Phi''$. This concludes the inductive argument; after finitely many steps we see that μ_x^{λ} is Lebesgue a.e. □

For the proof of Theorem 4.1 we will use the following generalization of Lemma 6.2 — a corollary to Proposition 8.3.

COROLLARY 7.2 *Let α be the left action of a maximal Cartan subgroup on G/Γ, where Γ is a lattice in the simple split connected Lie group G. Assume μ is an ergodic α-invariant measure on X. Let Φ be the set of roots and let μ_x^λ be the conditional measure for the one-dimensional invariant foliation F_λ corresponding to $\lambda \in \Phi$. There exists a set of numbers $s_\lambda \in [0, 1]$ for $\lambda \in \Phi$ with the following properties. If μ_x^λ is nonatomic a.e. then s_λ is positive and zero otherwise. For any element $\alpha^{\mathbf{t}}$ of the action the entropy with respect to μ equals*

$$\mathrm{h}_\mu(\alpha^{\mathbf{t}}) = \sum_{\lambda\in\Phi} s_\lambda(\lambda(\mathbf{t}))^+.$$

Here $(a)^+ = \max(a, 0)$ denotes the positive part of the number a and $e^{\lambda(\mathbf{t})}$ is the eigenvalue of the adjoint of $\alpha^{\mathbf{t}}$ on the root space $\mathfrak{g}_\lambda$.

In the case $\mu = m$ we have $s_\lambda = 1$ for every $\lambda \in \Phi$.

PROOF OF THEOREM 4.1: Clearly the Haar measure $\mu = m$ satisfies all the other conditions. For the converse note that we only have to consider the Conditions (ii) and (iii). In the case of $G = \mathrm{SL}(n+1, \mathbb{R})$ Condition (iv) has been shown to be sufficient earlier.

Suppose every conditional measure μ_x^λ is nonatomic a.e. as in Condition (ii). Let $\lambda \in \Phi$ be a root. Since G is simple and the rank is $n > 1$, there exists an element $\xi \in \Phi$ such that $\lambda + \xi \in \Phi$. Otherwise we have $[v_{\pm\lambda}, v_{\pm\xi}] = 0$ for all $\xi \neq \pm\lambda$, v_λ and $v_{-\lambda}$ generate an ideal $\mathfrak{g}_1$, $\Phi \setminus \{\lambda, -\lambda\}$ generate an ideal $\mathfrak{g}_2$, and this contradicts $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ being simple. By assumption the conditional measures $\mu_x^{\lambda+\xi}$ and $\mu_x^{-\xi}$ are nonatomic a.e., so Proposition 7.1 shows that the conditional measure μ_x^λ is Lebesgue a.e. Since this holds for all roots, Proposition 3.3 implies that $\mu = m$.

Fix a regular element $\alpha^{\mathbf{t}}$ of the action as in Condition (iii). Let Φ' be the set of roots with $\lambda(\alpha^{\mathbf{t}}) > 0$. From Corollary 7.2 we know that

$$\mathrm{h}_\mu(\alpha^{\mathbf{t}}) = \sum_{\lambda\in\Phi'} s_\lambda\lambda(\mathbf{t}) \le \sum_{\lambda\in\Phi'} \lambda(\mathbf{t}) = \mathrm{h}_m(\alpha^{\mathbf{t}})\,.$$

From this it is easy to find a number q such that $\mathrm{h}_\mu(\alpha^{\mathbf{t}}) > q\mathrm{h}_m(\alpha^{\mathbf{t}})$ forces all s_λ to be positive for $\lambda \in \Phi'$. Applying the same to the inverse map $\alpha^{-\mathbf{t}}$ shows that $s_\lambda > 0$ for all $\lambda \in \Phi \setminus \Phi'$. However, this shows that every conditional measure is nonatomic a.e. and we already know this implies $\mu = m$. □

This completes the proof of Theorem 4.1 assuming Corollary 7.2. The corollary (which includes Lemma 6.2) will be proved at the end of the next section.

8 Product Structure of the Conditional Measure on the Expanding Manifolds

In this section we show that the conditional measure for foliations into higher-dimensional leaves is a product measure of the measures on the one-dimensional leaves. For this we need some preliminaries.

Let ν be a locally finite measure on $\mathbb{R}^k$. Let $V = \mathbb{R} \times \{0\}^{k-1}$ be the subspace generated by the first basis vector. Define $\mathcal{B}^{(1)}$ to be the σ-algebra whose atoms are the cosets $\mathbf{a} + V$ for $\mathbf{a} \in \mathbb{R}^k$. There exists a collection of locally finite conditional measures $\nu_{\mathbf{a}}^{(1)}$ on the subspaces $\mathbf{a} + V$ that are defined uniquely a.e. up to a multiplicative constant such that the following holds: For every rectangle $Q = [-M, M]^k$ let ν_Q be the probability measure coming from ν by normalizing the restriction of ν to Q. Let $\mathcal{B}_Q^{(1)}$ be the restriction of $\mathcal{B}^{(1)}$ to Q. For a.e. $\mathbf{a} \in Q$ the measures $\nu_{\mathbf{a}}^{(1)}$ restricted to the intersection $(\mathbf{a} + V) \cap Q$ and normalized to be a probability measure is the conditional measure of ν_Q with respect to $\mathcal{B}_Q^{(1)}$. The conditional measures are easily constructed from the latter probability measures for the σ-algebra $\mathcal{B}_Q^{(1)}$ for $M \to \infty$.

The above construction of the conditional measures $\nu_{\mathbf{a}}^{(1)}$ is similar to the construction of μ_x^F for a foliation F of X and in fact is also related to it by the next lemma. If the foliation $F = F_{\mathfrak{h}}$ is defined by the Lie subalgebra $\mathfrak{h}$, we write $\mu_x^{\mathfrak{h}} = \mu_x^F$ for the conditional measure.

Note that, in general, a measure can only be pushed forward under a measurable map. However, the functions we are using are injective for a.e. base point $x \in X$, and so we can use the pullback.

LEMMA 8.1 *Let Φ' be a set of roots such that $\mathfrak{g}' = \sum_{\lambda \in \Phi'} \mathfrak{g}_\lambda$ is a Lie subalgebra that is contracted by the adjoint of some element of the Cartan subgroup. Fix an order on $\Phi' = \{\lambda_1, \dots, \lambda_k\}$. If ν is the pullback of the conditional measure $\mu_x^{\mathfrak{g}'}$ under the map φ_x defined by*

$$\mathbf{s} \in \mathbb{R}^k \mapsto \exp(s_1 v_{\lambda_1}) \cdots \exp(s_k v_{\lambda_k}) x \in F_{\mathfrak{g}'}(x)\,, \tag{8.1}$$

then the conditional measure $\nu_{\mathbf{a}}^{(1)}$ is (up to a multiplicative constant) a.e. the pullback of the conditional measure $\mu_{\varphi_x(\mathbf{a})}^{\lambda_1}$.

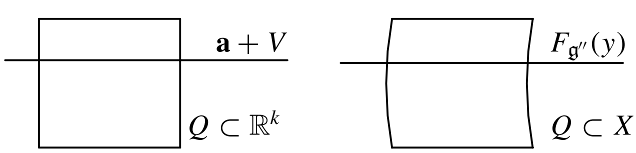

FIGURE 8.1. Both conditional measures $\mu_x^{\lambda_1}$ and $\nu_{\mathbf{a}}^{(1)}$ are characterized by how they restrict to large rectangle-like sets Q.

PROOF: Clearly the parametrization map ψ_y corresponding to F_{λ_1} for $y \in F_{\mathfrak{g}'}(x)$ with $y = \varphi_x(\mathbf{a})$ for $\mathbf{a} \in \mathbb{R}^k$ agrees with the restriction of φ_x to the coset $\mathbf{a} + V$. In other words, the leaves for the foliation F_{λ_1} are the images of the cosets $\mathbf{a} + V$ for $\mathbf{a} \in \mathbb{R}^k$.

We consider the restriction of μ to a set $\varphi_{O,z}(O) \subset X$ as in the characterizing properties of the conditional measures in Section 3. Let $M > 0$ and choose an open subset $O' \subset O$ with the following properties: The preimage Q under $\varphi_{O,z}$ of the atom $[x]$ for the σ-algebra $\mathcal{A}(O, F_{\mathfrak{g}'})$ and $x \in \pi(O')$ contains the rectangle $[-M, M]^k$. Because X can be covered by a sequence of images of such sets O', it is enough to consider the points $x \in \varphi_{O,z}(O')$ of one such set. Using the same open set for the foliation F_{λ_1} gives a σ-algebra $\mathcal{A}(O, F_{\lambda_1}) \supset \mathcal{A}(O, F_{\mathfrak{g}'})$. The conditional expectation with respect to two such σ-algebras satisfies

$$\mathrm{E}(f \mid \mathcal{A}(O, F_{\mathfrak{g}'})) = \mathrm{E}\big(\mathrm{E}(f \mid \mathcal{A}(O, F_{\lambda_1})) \mid \mathcal{A}(O, F_{\mathfrak{g}'})\big)$$

for any integrable function f. For a characteristic function f of a measurable set B and typical points x, this implies

$$\mu_x^{\mathcal{A}(O,F_{\mathfrak{g}'})}(B) = \int \mu_y^{\mathcal{A}(O,F_{\lambda_1})}(B)\, \mathrm{d}\mu_x^{\mathcal{A}(O,F_{\mathfrak{g}'})}(y)\,.$$

Up to a scalar multiple, $\mu_x^{\mathcal{A}(O,F_{\mathfrak{g}'})}$ and $\mu_x^{\mathfrak{g}'}$ agree on $[x]$, and the same holds, respectively, for λ_1 on the atoms for the σ-algebra $\mathcal{A}(O, F_{\lambda_1})$. Taking preimages under φ_x shows that the normalized restriction ν_Q of ν to Q, and the pullbacks $\nu''_{\mathbf{a}}$ of $\mu_{\varphi_x(\mathbf{a})}^{\mathcal{A}(O,F_{\lambda_1})}$ satisfy

$$\nu_Q(B) = \int \nu''_{\mathbf{a}}(B)\, \mathrm{d}\nu_Q(\mathbf{a}) \quad \text{for any measurable } B \subset Q\,.$$

Since the same holds for the properly normalized $\nu_{\mathbf{a}}^{(1)}$ instead of $\nu''_{\mathbf{a}}$, those two measures agree ν_Q a.e. This is the same as saying that the lemma holds for the restrictions of the measures to the set Q. We let M go to infinity, and the lemma follows. □

We can use the conditional measures $\nu_{\mathbf{a}}^{(1)}$ to characterize product measures. Let $W = \{0\} \times \mathbb{R}^{k-1}$. Clearly if $\nu = \nu_V \times \nu_W$ is a product measure on $\mathbb{R} \times \mathbb{R}^{k-1}$, the conditional measure on a.e. fiber $\mathbf{a} + V$ is up to a multiplicative constant a copy of ν_V.

LEMMA 8.2 *Let ν be a locally finite measure on $\mathbb{R}^k$. Assume the conditional measures $\nu_{\mathbf{a}}^{(1)}$ are equal up to a multiplicative constant in the following sense: On almost every coset $\mathbf{a} + V$ we have $\nu_{\mathbf{a}}^{(1)}(\mathbf{a} + A) = C\nu_V(A)$ where $\mathbf{a} \in W = \{0\}^p \times \mathbb{R}^q$, ν_V is a fixed measure on V, and $C = C_{\mathbf{a}}$ may depend on $\mathbf{a}$. Then the same holds for the conditional measures on the cosets $\mathbf{b} + W$ and for a fixed measure ν_W on W, and ν is up to a multiplicative constant the product measure of ν_V and ν_W.*

PROOF: Let $M > 0$ and ν_Q be a probability measure on $Q = [-M, M]^k$. First we show the lemma in this case.

Let $\mathcal{A} = \{[-M, M], \varnothing\} \times \mathcal{B}$ be the product of the trivial σ-algebra and the Borel σ-algebra in $[-M, M]^{k-1}$. The atoms for $\mathcal{A}$ are planes of the form $[-M, M] \times (a_2, \ldots, a_k)$ with $(a_2, \ldots, a_k) \in [-M, M]^{k-1}$. Assume the conditional measure

$\nu_{\mathbf{a}}^{(1)}$ on the atoms $[-M, M] \times (a_2, \ldots, a_k)$ for $\mathcal{A}$ are equal to $\nu_{V,Q}$ in the mentioned sense. We claim ν_Q is the product measure of $\nu_{V,Q}$ and some measure $\nu_{W,Q}$ on $[-M, M]^{k-1}$.

Let B and C be measurable sets in $[-M, M]$ and $[-M, M]^{k-1}$. The properties of conditional measures in Section 3 state that

$$\begin{aligned}\nu_Q(B \times C) &= \int_Q \nu_{V,M}(B) 1_{[-M,M]\times C}(\mathbf{a})\, \mathrm{d}\nu_Q(\mathbf{a}) \\ &= \nu_{V,M}(B)\nu_Q([-M, M] \times C)\,.\end{aligned}$$

Define $\nu_{W,Q}(C) = \nu_Q([-M, M] \times C)$. Varying B and C, it follows that $\nu_{W,Q}$ is a probability measure, and that ν_Q is the product measure of $\nu_{V,Q}$ and $\nu_{W,Q}$. The conditional measures for the atoms $a_1 \times [-M, M]^{k-1}$ agree in the mentioned sense with $\nu_{W,Q}$.

For ν the above shows that for every M the restriction to $[-M, M]^k$ equals a product measure up to a multiplicative constant. However, if $\nu([M, M]^k) > 0$, this constant has to remain the same for every $M' > M$ and the lemma follows. □

PROPOSITION 8.3 *Let $\Phi' \subset \Phi$ be a set of roots such that $\mathfrak{g}' = \sum_{\lambda\in\Phi'} \mathfrak{g}_\lambda$ is a Lie subalgebra satisfying that the foliation $F_{\mathfrak{g}'}$ is expanded by some element of the flow. For any order $\lambda_1, \ldots, \lambda_k$ of the elements of Φ' we let φ_x be the map defined in* (8.1) *parametrizing the leaf $F_{\mathfrak{g}'}(x)$ through x. The pullback $\nu_x^{\mathfrak{g}'}$ of the conditional measure $\mu_x^{\mathfrak{g}'}$ under the map φ_x is up to a multiplicative constant the product measure*

$$\prod_{i=1}^{k} \nu_x^{\lambda_i}\,.$$

PROOF: First we prove by induction that there is an order for the elements of Φ' for the which the statement is true. After the inductive argument we will show, using Proposition 7.1, that changing the order does not affect the statement. The case $|\Phi'| = 1$ holds by definition of ν_x^λ. For the induction we need to find $\lambda_1 \in \Phi'$ so that $\Phi'' = \Phi' \setminus \{\lambda_1\}$ satisfies the assumptions of the proposition, so that additionally F_{λ_1} is isometric and F_{λ_i} for $i > 1$ are contracted for some fixed $\alpha^{\mathbf{s}}$.

Let Φ' be a set of roots satisfying the assumptions of the proposition and $|\Phi'| = k$. There is an element $\alpha^{\mathbf{t}}$ with $\lambda(\alpha^{\mathbf{t}}) > 0$ for all $\lambda \in \Phi'$. We identify the dual of the Lie algebra of the Cartan subgroup α with $\mathbb{R}^n$, so that $\lambda(\alpha^{\mathbf{t}}) = \lambda \cdot \mathbf{t}$ can be written as an inner product. Let C be the set of all vectors in $\mathbb{R}^n$ that can be expressed as a linear combination $\sum_{\lambda\in\Phi'} c_\lambda \lambda$ with nonnegative coefficients $c_\lambda \geq 0$. Then C is a cone and the intersection $C \cap P$ with the hyperplane $P = \{\mathbf{a} : \mathbf{a}\cdot\mathbf{t} = 1\}$ is a convex set in P.

Let $\mathbf{a}$ be an extremal element of $C \cap P$; then $\mathbf{a}$ must be a multiple of some element $\lambda = \lambda_1 \in \Phi'$. By the construction λ cannot be expressed as a sum of elements of $\Phi'' = \Phi' \setminus \{\lambda\}$. Furthermore, since $\mathbf{a}$ is extremal, there exists a linear map on $\mathbb{R}^n$ whose maximal value on $C \cap P$ is 0 and is achieved only at $\mathbf{a}$. In other words, there exists $\alpha^{\mathbf{s}}$ such that F_λ is isometric and F_ξ for $\xi \in \Phi''$ are contracted.

The comment before Proposition 7.1 implies that $\mathfrak{g}'' = \sum_{i=2}^k \mathfrak{g}_{\lambda_i}$ is a Lie ideal in $\mathfrak{g}'$. In particular, Φ'' satisfies the assumptions of the proposition and has fewer elements. By the inductive assumption, there exists an order on Φ'' so that the following holds: If ψ_x is the map from $\mathbb{R}^{k-1}$ to $F_{\mathfrak{g}''}(x)$ defined analogously to (8.1), the preimage ν'' of $\mu_x^{\mathfrak{g}''}$ under ψ_x is the product measure of $\nu_x^{\lambda_i}$ for $i = 2, \ldots, k$ for a.e. $x \in X$. Let N be a null set such that this property, Proposition 5.1 for λ and $\alpha^{\mathbf{s}}$, and property (3.1) hold for all $x, y \notin N$. By Lemma 3.1 we can enlarge N so that $\mu_x^\xi(N) = 0$ for all $x \notin N$ and $\xi \in \Phi$. Notice that the restriction of φ_x to $W = \{0\} \times \mathbb{R}^{k-1}$ agrees with ψ_x. For that reason we will not distinguish between this restriction and ψ_x, and identify ν'' with a measure on W.

Let ν be the pullback of the conditional measure $\mu_x^{\mathfrak{g}'}$ under the map φ_x. The lines $\mathbf{a} + V$ for $V = \mathbb{R} \times \{0\}^{k-1}$ are mapped onto the leaves $F_\lambda(\varphi_x(\mathbf{a}))$. By Lemma 8.1 the conditional measure $\nu_{\mathbf{a}}^{(1)}$ is the pullback of $\mu_{\varphi_x(\mathbf{a})}^\lambda$ under the map φ_x. Since $F_{\mathfrak{g}''}$ is contracted under $\alpha^{\mathbf{s}}$, we can apply Proposition 5.1 for any point $y \in F_{\mathfrak{g}'}(x)$. Let $y = \varphi_x(\mathbf{a}) \notin N$; then μ_x^λ is the image of μ_y^λ under ϕ as in Proposition 5.1. It is easy to check that ϕ corresponds to the translation mapping from $\mathbf{a} + V$ to V along the orthogonal subspace W. However, this shows the assumption of Lemma 8.2 for ν and $\nu_V = \nu_x^\lambda$. We will show that the measure ν_W equals the pullback ν'' of $\mu^{\mathfrak{g}''}$ under φ_x.

In case μ_x^λ is atomic a.e., then $\nu_V = \delta_0$ and for a.e. $x, y \notin N$ with $y \in F_{\mathfrak{g}'}(x)$, we have in fact $y \in F_{\mathfrak{g}''}(x)$. Therefore the two σ-algebras in the characterizing properties of $\mu_x^{\mathfrak{g}'}$ and $\mu_x^{\mathfrak{g}''}$ are on the complement of N equal; the same holds for the conditional measures with respect to these σ-algebras, and $\mu_x^{\mathfrak{g}'} = \mu_x^{\mathfrak{g}''}$ a.e. follows. Therefore $\nu = \nu''$, and the inductive assumption shows that ν is the product measure as stated.

Assume now μ_x^λ is nonatomic a.e. Let $y = \varphi_x(\mathbf{b}) \notin N$. Let $\nu'_{\mathbf{b}}$ be the conditional measure for the coset $\mathbf{b} + W$ (defined analogously to $\nu_x^{(1)}$ as in the beginning of the section). We claim that $\nu'_{\mathbf{b}}$ transported back to W is equal to ν'' for a.e. $\mathbf{b}$. By Lemma 8.2 the same holds for ν_W, so that ν_W agrees with ν''. Therefore the claim and the inductive assumption imply $\nu = \nu_x^\lambda \times \nu_W$ is again the product measure.

In case $\mathbf{b} \in W$ the claim is trivial. In case $\mathbf{b} \in V \setminus \varphi_x^{-1}(N)$ Corollary 5.2 implies that $\nu_x^{\lambda_i} = \nu_y^{\lambda_i}$ for $1 < i \leq k$. By assumption ν'' is the product measure of $\nu_x^{\lambda_i}$, and the same holds for the preimage of $\mu_y^{\mathfrak{g}''}$ under ψ_y. However, the restriction of φ_x to the plane $\mathbf{b} + W$ might not be equal to $\psi_{y'}$. In fact,

$$\varphi_x(\mathbf{b} + \mathbf{w}) = \exp(b_1 v_\lambda) \prod_{i=2}^k \exp(w_i v_{\lambda_i}) x \,,$$

and in $\psi_y(\mathbf{w})$ the first term would be right in front of x since $y = \exp(b_1 v_\lambda) x$. If $\exp(b_1 v_\lambda)$ commutes with the other terms, moving the first term to x does not

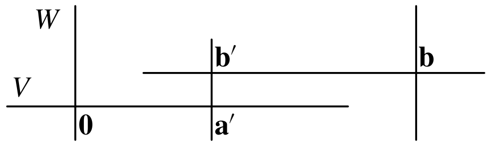

FIGURE 8.2. We can reach $\mathbf{b}$ from $\mathbf{0}$ in two steps.

change the terms in between, and ψ_y is the restriction of φ_x. If $\mu_x^{\lambda_i}$ is trivial a.e., then $w_i = 0$ and the terms commute whenever $\mathbf{w} \in W$ with $\mathbf{b} + \mathbf{w} \notin \varphi_x^{-1} N$. Otherwise, whenever $\exp(w_1\lambda)$ does not commute with $\exp(b_i v_{\lambda_i})$, moving the first behind the latter produces a shearing along some directions $\exp(v_{\lambda_j})$. For j like that, Proposition 7.1 shows that $\nu_x^{\lambda_j}$ is Lebesgue and the mentioned shearing does not change the product measure. We see that the conditional measure on the hyperplane $\mathbf{b} + W$ is a translate of ν''. Similarly, one shows that the conditional measure for the planes $\mathbf{b}+W$ and $\mathbf{b}'+W$ are translates of each other if $\mathbf{b}, \mathbf{b}' \notin \varphi_x^{-1} N$ and $\mathbf{b}' - \mathbf{b} \in V$.

Let now $\mathbf{b} \in \mathbb{R}^k$ be arbitrary. The measures $\nu_{\mathbf{0}}^{(1)}$ and $\nu_{\mathbf{b}}^{(1)}$ are translates of each other along $(0, b_2, \ldots, b_k) \in W$. This point might belong to the null set $\varphi_x^{-1} N$ and we cannot argue using it. By the assumption on N we have $\nu_0^{(1)}(\varphi^{-1} N) = \nu_{\mathbf{b}}^{(1)}(\varphi^{-1} N) = 0$. Therefore there exist two points $\mathbf{b}' \in \mathbf{b} + V$ and $\mathbf{a}' \in V$ that do not belong to $\varphi_x^{-1} N$ such that $\mathbf{b}' \in \mathbf{a} + W$; see Figure 8.2. From the two cases before, we know that the conditional measures for the planes W, $\mathbf{a}' + W = \mathbf{b}' + W$, and $\mathbf{b} + W$ are all translates of ν_x'' along V. This proves the claim and concludes the inductive argument.

To conclude the proof of the proposition, we need to show that the statement holds for any order of the roots. Without loss of generality, we consider a new order where only two neighboring indices are swapped. Let φ_x be the parametrization of $F_{\mathfrak{g}'}(x)$ defined by (8.1) and define $\tilde{\varphi}_x$ by swapping the two terms $\exp(s_i v_{\lambda_i})$ and $\exp(s_{i+1}) v_{\lambda_{i+1}}$ in the product (leaving their order in the parameter space unchanged). In case $\lambda_i + \lambda_{i+1}$ is not a root, those two elements commute and the two maps agree. Otherwise the two maps differ but have the same image $F_{\mathfrak{g}'}(x)$. We show that in all cases the pullbacks of the measure $\mu_x^{\mathfrak{g}'}$ under φ_x and $\tilde{\varphi}_x$ agree.

In case $\mu_x^{\lambda_j}$ is atomic a.e. for some $j \in \{i, i+1\}$, the product measure $\nu_x^{\mathfrak{g}'}$ is supported on a hyperplane $\{\mathbf{m} \in \mathbb{R}_k : s_j = 0\}$, and the maps φ_x and $\tilde{\varphi}_x$ are equal a.e. with respect to $\nu_x^{\mathfrak{g}'}$. Since $\nu_x^{\mathfrak{g}'}$ is the pullback of $\mu_x^{\mathfrak{g}'}$ under φ_x, the same is true for $\tilde{\varphi}_x$. Assume now the two conditional measures are nonatomic a.e. The commutator of $\exp(s_i v_{\lambda_i})$ and $\exp(s_{i+1} v_{\lambda_{i+1}})$ can be expressed as a product of various $\exp(s_j v_{\lambda_j})$. From Proposition 7.1 for every such j the conditional measure $\mu_x^{\lambda_j}$ is

Lebesgue a.e. Therefore the maps φ_x and $\tilde{\varphi}_x$ differ by an application of a measure-preserving action on $\mathbb{R}^k$—a shearing along some directions where $\nu_x^{\lambda_j}$ is Lebesgue. The product measure again equals the pullback of $\mu_x^{\mathfrak{g}'}$. □

Let F be the foliation into unstable manifolds for $\alpha^{\mathbf{t}}$. Since μ_x^F is a product measure, we can proceed to the proof of Corollary 7.2, which gives a uniform description of the entropy $\mathrm{h}_\mu(\alpha^{\mathbf{t}})$.

PROOF OF COROLLARY 7.2: We will use the Ledrappier-Young entropy formula [17]. Note that our space X might not be compact, but due to the algebraic nature of α this is not a necessary assumption (see also [19, sect. 9] for part of the Ledrappier-Young theory in this setting).

For a root $\lambda \in \Phi$ we define for a.e. x

$$\delta_\lambda(x) = \lim_{\varepsilon \to 0} \frac{\log \nu_x^\lambda([-\varepsilon, \varepsilon])}{\log \varepsilon} \quad \text{and} \quad s_\lambda = \int \delta_\lambda(x)\, \mathrm{d}\mu\,.$$

The existence of the limit follows from the arguments in [17, sects. 9–10], where one uses a partition subordinate to the foliation F_λ instead of the foliation mentioned there and an element of the action that expands F_λ. Furthermore, $s_\lambda \in [0, 1]$, and $s_\lambda = 0$ if and only if μ_x^λ is atomic a.e. In case μ_x^λ is Lebesgue a.e., we see immediately $s_\lambda = 1$. The number s_λ can be interpreted as the dimension of μ along the leaves of F_λ.

Let $\alpha^{\mathbf{t}}$ be a fixed element of the action, and let $\Phi' = \{\lambda \in \Phi : \lambda(\alpha^{\mathbf{t}}) > 0\}$ be the set of roots whose foliations are expanded. We order $\Phi' = \{\lambda_1, \ldots, \lambda_k\}$ such that

$$\lambda_1' = \lambda_1(\alpha^{\mathbf{t}}) = \cdots = \lambda_{k_1}(\alpha^{\mathbf{t}}) > \lambda_2' = \lambda_{k_1+1}(\alpha^{\mathbf{t}}) = \cdots = \lambda_{k_2}(\alpha^{\mathbf{t}}) > \\ \cdots > \lambda_r' = \lambda_{k_{r-1}+1}(\alpha^{\mathbf{t}}) = \cdots = \lambda_k(\alpha^{\mathbf{t}}) > 0\,.$$

The set of roots $\Phi_1 = \{\lambda_1, \ldots, \lambda_{k_1}\}$ defines a Lie subalgebra $\mathfrak{h}_1$; in fact, the elements of the root spaces commute with each other (this follows from the statement before Proposition 7.1). The foliation F_1 defined by $\mathfrak{h}_1$ is the foliation for the biggest Lyapunov exponent λ_1' of $\alpha^{\mathbf{t}}$. In the notation of [17], the dimension of μ along F_1 is

$$\delta_1(x) = \lim_{\varepsilon \to 0} \frac{\log \mu_x^{F_1}(B_\varepsilon(x, F_1))}{\log \varepsilon}$$

for a.e. x. Here the set $B_\varepsilon(x, F_1)$ is the ε-ball around x in the leaf $F_1(x)$. It is easy to see that the exact shape of the ε-ball is not important. If we use a product of ε-balls in the one-dimensional leaves instead, Proposition 8.3 implies

$$\delta_1(x) = \delta_{\lambda_1}(x) + \cdots + \delta_{\lambda_{k_1}}(x)\,.$$

Proceeding similarly, it follows for any j that $\Phi_j = \{\lambda_1, \ldots, \lambda_{k_j}\}$ defines a Lie subalgebra, and that the dimension of μ along the corresponding foliation F_j is given by

$$\delta_j(x) = \sum_{i=1}^{k_j} \delta_{\lambda_i}(x) .$$

Now [17, theorem C] reads

$$\mathrm{h}_\mu(\alpha^{\mathbf{t}}) = \int \sum_{\lambda\in\Phi'} \lambda(\alpha^{\mathbf{t}})\delta_\lambda(x)\,\mathrm{d}\mu = \sum_{\lambda\in\Phi'} \lambda(\alpha^{\mathbf{t}})s_\lambda = \sum_{\lambda\in\Phi} (\lambda(\alpha^{\mathbf{t}}))^+ s_\lambda ,$$

which concludes the proof of the corollary and the theorems. □

9 Nonstandard and Nonalgebraic Measures

In the unpublished manuscript [24], M. Rees gives an example of a uniform lattice $\Gamma \subset \mathrm{SL}(3,\mathbb{R})$ that allows compact invariant submanifolds and nonstandard invariant measures. We will reproduce the construction and give further examples along those lines. Some of the measures below are homogenous measures, i.e., Haar measures on homogenous submanifolds. More important for our discussion is the fact that there are huge varieties of nonalgebraic measures that are supported by some of those invariant homogeneous submanifolds.

Let $\mathbb{Z}[\sqrt[4]{2}]$ be the ring generated by $\sqrt[4]{2}$, and write the elements of $\mathbb{Z}[\sqrt[4]{2}]$ as $s = s_1 + s_2\sqrt[4]{2}$ with $s_1, s_2 \in \mathbb{Z}[\sqrt{2}]$; then $\overline{s} = s_1 - s_2\sqrt[4]{2}$ is the Galois conjugate of s. For $w_1, w_2 \in \mathrm{Mat}_n(\mathbb{Z}[\sqrt{2}])$ we define similarly

$$\overline{w_1 + w_2\sqrt[4]{2}} = w_1 - w_2\sqrt[4]{2}$$

and

$$\Gamma = \left\{ w = w_1 + w_2\sqrt[4]{2} \in \mathrm{SL}(n,\mathbb{R}) : w_1, w_2 \in \mathrm{Mat}_n(\mathbb{Z}[\sqrt{2}]) \text{ and } \overline{w^{\mathsf{T}}}w = \mathrm{Id} \right\} .$$

We will show later that Γ is a uniform lattice in $\mathrm{SL}(n,\mathbb{R})$. Another feature of this lattice is that it contains many diagonal matrices. A diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$ belongs to Γ if λ_i are units of $\mathbb{Z}[\sqrt[4]{2}]$ with $\lambda_i\overline{\lambda_i} = 1$ and $\lambda_1 \cdots \lambda_n = 1$. A direct calculation shows that $\tau = 3+2\sqrt[4]{2}+2\sqrt{2}+2\sqrt[4]{8}$ is such a unit. Therefore the diagonal matrix with entries τ^{k_i} belongs to Γ if $\sum_i k_i = 0$.

Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ denote a partition of $\{1, \ldots, n\}$ into sets of consecutive indices; in particular,

$$\mathcal{P}_1 = \{\{1,2\},\{3\}\} \quad \text{and} \quad \mathcal{P}_2 = \{\{1,2,3\},\{4,5\}\} .$$

For a partition element $P_i \in \mathcal{P}$, we define $\mathrm{SL}(P_i,\mathbb{R})$ to be the subgroup of $\mathrm{SL}(n,\mathbb{R})$ whose elements are block matrices

$$\begin{pmatrix} \mathrm{Id} & & \\ & w & \\ & & \mathrm{Id} \end{pmatrix} \quad \text{with } w \in \mathrm{SL}(|P_i|,\mathbb{R}) .$$

Here the matrix w is placed on the diagonal corresponding to the partition element P_i. For every partition $\mathcal{P}$, we define $G_{\mathcal{P}}$ as the subgroup generated by

$\mathrm{SL}(P_i, \mathbb{R})$ for $i = 1, \dots, k$ and the diagonal matrices $\alpha^{\mathbf{t}}$ for $\mathbf{t} \in R$. Since the various $\mathrm{SL}(P_i, \mathbb{R})$ commute with each other, the group $G_{\mathcal{P}}$ is isomorphic to a product of subgroups isomorphic to $\mathrm{SL}(|P_i|, \mathbb{R})$ and a diagonal subgroup D. In particular,

$$G_{\mathcal{P}_1} = \left\{ \begin{pmatrix} w_{1,1} & w_{1,2} & 0 \\ w_{2,1} & w_{2,2} & 0 \\ 0 & 0 & w_{3,3} \end{pmatrix} \in \mathrm{SL}(3, \mathbb{R}) : w_{3,3} > 0 \right\} \simeq \mathrm{SL}(2, \mathbb{R}) \times D_1$$

where

$$D_1 = \left\{ \begin{pmatrix} e^t \, \mathrm{Id}_2 & \\ & e^{-2t} \end{pmatrix} \in \mathrm{SL}(3, \mathbb{R}) : t \in \mathbb{R} \right\}$$

and similarly

$$G_{\mathcal{P}_2} \simeq \mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \times D_2$$

where

$$D_2 = \left\{ \begin{pmatrix} e^{2t} \, \mathrm{Id}_3 & \\ & e^{-3t} \, \mathrm{Id}_2 \end{pmatrix} \in \mathrm{SL}(5, \mathbb{R}) : t \in \mathbb{R} \right\}.$$

Here $\mathrm{Id}_j \in \mathrm{Mat}_j(\mathbb{R})$ denotes the identity matrix. In general, D will be the subgroup of all diagonal matrices $\alpha^{\mathbf{t}}$ with equal entries $t_a = t_b$ for different elements $a, b \in P \in \mathcal{P}$ in the same partition element. Notice that D is a finite index subgroup of the center of $G_{\mathcal{P}}$.

PROPOSITION 9.1 [24, M. Rees] *The subgroup* Γ*, as above, is a uniform lattice in* $\mathrm{SL}(n, \mathbb{R})$*. More generally, for a fixed partition* $\mathcal{P}$ *and any* $P \in \mathcal{P}$*, the subgroups* $\Gamma \cap \mathrm{SL}(P, \mathbb{R})$*,* $\Gamma \cap D$*, and* $\Gamma \cap G_{\mathcal{P}}$ *are uniform lattices in* $\mathrm{SL}(P, \mathbb{R})$*,* D*, and* $G_{\mathcal{P}}$*, respectively.*

The proof will rely on the Borel-Harish-Chandra theorem.

PROOF: For any two $w_1, w_2 \in \mathrm{Mat}_n(\mathbb{C})$ and a fixed number $r \neq 0$, we define the conjugation map

$$\phi(w_1, w_2) = \frac{1}{2} \begin{pmatrix} \mathrm{Id} & r \, \mathrm{Id} \\ \frac{1}{r} \mathrm{Id} & -\mathrm{Id} \end{pmatrix} \begin{pmatrix} w_1 & \\ & w_2 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & r \, \mathrm{Id} \\ \frac{1}{r} \mathrm{Id} & -\mathrm{Id} \end{pmatrix}.$$

A direct calculation shows that

$$\phi(w_1, w_2) = \begin{pmatrix} A & r^2 B \\ B & A \end{pmatrix} \tag{9.1}$$

where

$$A = \frac{w_1 + w_2}{2} \quad \text{and} \quad B = \frac{w_1 - w_2}{2r}$$

are two matrices in $\mathrm{Mat}_n(\mathbb{C})$. Exchanging w_1 and w_2 clearly leaves A unchanged and replaces B by $-B$. If we transpose w_1 and w_2, this transposes A and B.

We set $r = \sqrt[4]{2}$ and define $\varphi(w) = \phi(w, (w^{\mathsf{T}})^{-1})$ for any $w \in \mathrm{SL}(n, \mathbb{R})$. Let $G_{\mathbb{R}}$ be the image of φ in $\mathrm{SL}(2n, \mathbb{R})$. From the properties of ϕ, it follows that a matrix in $G_{\mathbb{R}}$ can be decomposed into matrices A and B as in (9.1), which additionally have the property

$$\begin{pmatrix} A & \sqrt{2}B \\ B & A \end{pmatrix} \begin{pmatrix} A^{\mathsf{T}} & -\sqrt{2}B^{\mathsf{T}} \\ -B^{\mathsf{T}} & A^{\mathsf{T}} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & \\ & \mathrm{Id} \end{pmatrix}. \tag{9.2}$$

The inverse of φ is given by

$$\begin{pmatrix} A & \sqrt{2}B \\ B & A \end{pmatrix} \mapsto A + \sqrt[4]{2}B\,. \tag{9.3}$$

Define the two polynomials p and q whose variables are the matrix coefficients of A and B and whose coefficients are in $\mathbb{Z}[\sqrt{2}]$ by

$$p(A, B) + \sqrt[4]{2}q(A, B) = \det(A + \sqrt[4]{2}B)\,;$$

they can be found by expanding the right-hand side and collecting the terms with (respectively, without) the factor $\sqrt[4]{2}$ to $\sqrt[4]{2}q$ (respectively, p). For any element of $G_{\mathbb{R}}$

$$\begin{aligned} p(A, B) + \sqrt[4]{2}q(A, B) &= \det(w) = 1 \\ p(A, B) - \sqrt[4]{2}q(A, B) &= \det((w^{\mathsf{T}})^{-1}) = 1 \end{aligned} \tag{9.4}$$

and therefore $p(A, B) = 1$ and $q(A, B) = 0$. The second equation in (9.4) follows by taking the conjugate of the first. This shows that $G_{\mathbb{R}}$ is the set of matrices that satisfy a certain set of polynomial equations. The equations ensure that the matrix decomposes into A and B and satisfies equation (9.2) and that the determinant of the preimage is 1; the latter corresponds to $p = 1$ and $q = 0$. All coefficients of the polynomials needed belong to $\mathbb{Q}[\sqrt{2}]$. Let $G_{\mathbb{Z}[\sqrt{2}]} = G_{\mathbb{R}} \cap \mathrm{SL}(2n, \mathbb{Z}[\sqrt{2}])$ and $G_{\mathbb{Q}[\sqrt{2}]} = G_{\mathbb{R}} \cap \mathrm{SL}(2n, \mathbb{Q}[\sqrt{2}])$. The isomorphism φ maps Γ exactly to $G_{\mathbb{Z}[\sqrt{2}]}$.

Similarly to the above, we can define an isomorphism φ' from $\mathrm{SU}(n)$ to the group $G'_{\mathbb{R}}$ using $r = i\sqrt[4]{2}$. The group $G'_{\mathbb{R}}$ consists of all matrices

$$\begin{pmatrix} C & -\sqrt{2}D \\ D & C \end{pmatrix} \tag{9.5}$$

with $C, D \in \mathrm{Mat}_n(\mathbb{R})$ satisfying the equations $\overline{p}(C, D) = 1$, $\overline{q}(C, D) = 0$ and

$$\begin{pmatrix} C & -\sqrt{2}D \\ D & C \end{pmatrix} \begin{pmatrix} C^{\mathsf{T}} & \sqrt{2}D^{\mathsf{T}} \\ -D^{\mathsf{T}} & C^{\mathsf{T}} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & \\ & \mathrm{Id} \end{pmatrix}.$$

Since $\mathrm{SU}(n)$ and $G'_{\mathbb{R}}$ are isomorphic, both are compact. Note that the equations defining $G'_{\mathbb{R}}$ are exactly the images of the equations for $G_{\mathbb{R}}$ under the Galois automorphism of $\mathbb{Q}[\sqrt{2}]$ defined by $\sqrt{2} \mapsto -\sqrt{2}$. The subgroup $G'_{\mathbb{Q}[\sqrt{2}]}$ of matrices in $G'_{\mathbb{R}}$ with entries in $\mathbb{Q}[\sqrt{2}]$ is isomorphic to $G_{\mathbb{Q}[\sqrt{2}]}$ using the same automorphism. And since every unitary matrix is diagonalizable, every matrix in $G_{\mathbb{Q}[\sqrt{2}]}$ is diagonalizable as well.

We claim $G_{\mathbb{R}} \times G'_{\mathbb{R}} \simeq \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SU}(n)$ is isomorphic to a group H that is algebraic over $\mathbb{Q}$; i.e., H is the set of all elements of $\mathrm{Mat}_{4n}(\mathbb{R})$ satisfying a certain set of polynomial equations with rational coefficients. Let $w \in \mathrm{SL}(n,\mathbb{R})$ and $v \in \mathrm{SU}(n)$, and define A, B, C, and D by the maps φ and φ' as above. Using ϕ for $r = \sqrt{2}$, we define

$$\psi(w,v) = \phi\begin{pmatrix}\varphi(w) & \\ & \varphi'(v)\end{pmatrix} = \begin{pmatrix} a & 2b \\ b & a\end{pmatrix} \in \mathrm{Mat}_{4n}(\mathbb{R})$$

where we used equation (9.1). From the shape of the matrices in (9.3) and (9.5), we see furthermore

$$a = \frac{1}{2}\begin{pmatrix} A+C & \sqrt{2}(B-D) \\ B+D & A+C\end{pmatrix} \quad \text{and} \quad b = \frac{1}{2\sqrt{2}}\begin{pmatrix} A-C & \sqrt{2}(B+D) \\ B-D & A-C\end{pmatrix}.$$

With the abbreviations

$$e_1 = \frac{1}{2}(A+C)\,, \quad f_1 = \frac{1}{2\sqrt{2}}(B-D)\,, \quad e_2 = \frac{1}{2\sqrt{2}}(A-C)\,, \quad f_2 = \frac{1}{2}(B+D)\,,$$

the above becomes

$$a = \begin{pmatrix} e_1 & 2f_1 \\ f_2 & e_1\end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} e_2 & f_2 \\ f_1 & e_2\end{pmatrix}.$$

We know in $G_\mathbb{R}$ (and similarly in $G'_\mathbb{R}$) that the inverse is given by the pair $A^\top$ and $-B^\top$. This is the same as saying that for $\psi(w^{-1}, v^{-1}) = \psi(w,v)^{-1}$ the four matrices are $e_1^\top, e_2^\top, -f_1^\top, -f_2^\top$; this statement corresponds to a set of polynomial equations with rational coefficients. The only other equations defining $G_\mathbb{R}$ are $p(A,B) = 1$ and $q(A,B) = 0$, and similarly $\overline{p}(C,D) = 1$ and $\overline{q}(C,D) = 0$ for $G'_\mathbb{R}$. Therefore we get the four equations for e_1, e_2, f_1, and f_2

$$p(e_1+\sqrt{2}e_2,\, f_2+\sqrt{2}f_1) = 1\,, \qquad \overline{p}(e_1-\sqrt{2}e_2,\, f_2-\sqrt{2}f_1) = 1\,,$$
$$q(e_1+\sqrt{2}e_2,\, f_2+\sqrt{2}f_1) = 0\,, \qquad \overline{q}(e_1-\sqrt{2}e_2,\, f_2-\sqrt{2}f_1) = 0\,;$$

the two on the left (respectively, right) correspond to the equations for $G_\mathbb{R}$ (respectively, $G'_\mathbb{R}$). Since the right equations are the conjugates of the left ones, we can rewrite them—similarly to (9.4)—as four equations with rational coefficients. This shows that the image $H = \mathrm{Im}(\psi)$ is the subgroup of $\mathrm{SL}(4n,\mathbb{R})$ whose elements satisfy a certain set of polynomial equations—H is algebraic over $\mathbb{Q}$.

Every element g of $H_\mathbb{Q} = H \cap \mathrm{SL}(4n,\mathbb{Q})$ is diagonalizable because $\psi^{-1}(g)$ decomposes into two blocks $\phi(w) \in G_{\mathbb{Q}[\sqrt{2}]}$ and $\phi'(v) \in G_{\mathbb{Q}[\sqrt{2}]}$. To see this, note that by definitions of e_i and f_i the element $\psi(\phi(w), \phi'(v))$ belongs to $H_\mathbb{Q}$ if and only if

$$A = e_1+\sqrt{2}e_2\,, \quad B = f_1+\sqrt{2}f_2\,, \quad C = e_1-\sqrt{2}e_2\,, \quad D = f_1-\sqrt{2}f_2\,,$$

and e_1, e_2, f_1, and f_2 belong to $\mathrm{Mat}(n,\mathbb{Q})$; this holds similarly for $\mathbb{Z}$ instead of $\mathbb{Q}$. By the construction H is isomorphic to the product $\mathrm{SL}(n,\mathbb{R}) \times \mathrm{SU}(n)$ and is semisimple. From the Borel-Harish-Chandra theorem it follows that $H_\mathbb{Z} = H \cap \mathrm{SL}(4n,\mathbb{Z})$ is a uniform lattice in H. This shows that $\psi^{-1}H_\mathbb{Z}$ is a uniform lattice in $G_\mathbb{R} \times G'_\mathbb{R}$. However, the factor $G'_\mathbb{R}$ is compact and therefore $\pi_1(\psi^{-1}H_\mathbb{Z}) = G_{\mathbb{Z}[\sqrt{2}]}$

is a uniform lattice in $G_{\mathbb{R}}$. The isomorphism φ concludes the proof that Γ is a uniform lattice in $SL(n, \mathbb{R})$.

Let $\mathcal{P}$ be a partition of $\{1, \dots, n\}$. The set $\Gamma \cap D$ forms a uniform lattice in D since one can use $\tau = 3 + 2\sqrt[4]{2} + 2\sqrt{2} + 2\sqrt[4]{8}$ to define diagonal elements of Γ that generate a lattice in D. In the case of the one-dimensional subgroup D_1, it is enough to note that

$$\begin{pmatrix} \tau \, \mathrm{Id}_2 & \\ & \tau^{-2} \end{pmatrix} \in \Gamma \cap D_1 \tag{9.6}$$

and similarly for D_2. Let $SL(P, \mathbb{R})$ be the subgroup of $SL(n, \mathbb{R})$ corresponding to a partition element $P \in \mathcal{P}$, and denote the lattice in $SL(n, \mathbb{R})$ by Γ_n. The subgroup $SL(P_i, \mathbb{R})$ is isomorphic to $SL(|P_i|, \mathbb{R})$; this isomorphism carries the intersection $\Gamma_n \cap SL(P_i, \mathbb{R})$ to $\Gamma_{|P_i|}$. Therefore $\Gamma_n \cap H$ forms a uniform lattice in H for every direct factor H of $G_{\mathcal{P}}$, and $\Gamma_n \cap G_{\mathcal{P}}$ is a uniform lattice in $G_{\mathcal{P}}$, which concludes the proof. □

Let $X_{\mathcal{P}} = G_{\mathcal{P}}/(\Gamma \cap G_{\mathcal{P}})$. Since $\Gamma \cap G_{\mathcal{P}}$ is a lattice, there exists a left invariant probability measure $\mu_{\mathcal{P}}$. Let $M_{\mathcal{P}}$ denote the image of $X_{\mathcal{P}}$ inside X; then $M_{\mathcal{P}}$ is a compact submanifold. Write $\mu_{\mathcal{P}}$ again for the image of the measure.

LEMMA 9.2 *Let $\mathcal{P}$ be a partition, and let $\mu_{\mathcal{P}}$ be as above. The entropy of $\alpha^{\mathbf{t}}$ with respect to $\mu_{\mathcal{P}}$ vanishes if and only if $\alpha^{\mathbf{t}}$ belongs to D. The conditional measure $(\mu_{\mathcal{P}})_x^{(a,b)}$ for a pair of different indices $a \neq b$ is nonatomic* (*and in this case Lebesgue*) *if and only if a and b belong to the same partition element.*

PROOF: Since $(X, \mu_{\mathcal{P}})$ and $(X_{\mathcal{P}}, \mu_{\mathcal{P}})$ are isomorphic as measure spaces, we can calculate the entropy of $\alpha^{\mathbf{t}}$ in $(X_{\mathcal{P}}, \mu_{\mathcal{P}})$. If $\alpha^{\mathbf{t}} \in D$, then $\alpha^{\mathbf{t}}$ only acts as translation in the direction of D and entropy vanishes. If $\alpha^{\mathbf{t}} \notin D$, then $t_a \neq t_b$ for a pair of different indices a and b in the same partition element P_i. Here $\alpha^{\mathbf{t}}$ acts on $SL(P_i, \mathbb{R})/\Gamma_{|P_i|}$ with positive entropy.

If a and b belong to the same partition element, then $\mathrm{Id} + s v_{a,b}$ belongs to $G_{\mathcal{P}}$, and left invariance of $\mu_{\mathcal{P}}$ shows that $(\mu_{\mathcal{P}})_x^{(a,b)}$ is Lebesgue a.e.

Assume a and b belong to different partition elements P_i and P_j. Let $\alpha^{\mathbf{t}} \in D$ be such that $t_a > t_b$. As $F_{a,b}$ is expanded by $\alpha^{\mathbf{t}}$ and the entropy vanishes, the conditional measure $(\mu_{\mathcal{P}})_x^{(a,b)}$ has to be trivial a.e. □

EXAMPLE 9.3 For the partition $\mathcal{P}_1 = \{\{1, 2\}, \{3\}\}$ and the corresponding measure $\mu_1 = \mu_{\mathcal{P}_1}$, we consider the Weyl chamber flow on $SL(3, \mathbb{R})/\Gamma$: The flow along the direction $\mathbf{t} = (1, 1, -2)$ is periodic since the matrix in (9.6) is in the center of $G_{\mathcal{P}_1}$ and in the lattice and therefore acts trivially on $(SL(3, \mathbb{R})/\Gamma, \mu_1)$. The flow along the direction of $\mathbf{t} = (1, -1, 0)$ has positive entropy—there is one expanding (and one contracting) one-dimensional foliation whose conditional measure is nonatomic (Lebesgue). Locally the system is a direct product of an Anosov flow and a periodic flow on the circle.

There are many more invariant measures on $\mathrm{SL}(3,\mathbb{R})/\Gamma_3$. In fact, let ν be any invariant measure on $Y = \mathrm{SL}(2,\mathbb{R})/\Gamma_2$ (with positive entropy or zero entropy for the flow); then $\nu \times m$ defines an invariant measure on $Y \times (D/(\Gamma \cap D))$. The system $G_{\mathcal{P}_1}$ is a finite-to-one factor of $Y \times (D/(\Gamma \cap D))$; let μ be the induced measure on $\mathrm{SL}(3,\mathbb{R})/\Gamma_3$. Again the flow along $\mathbf{t} = (1, 1, -2)$ is periodic and the conditional measure on the one expanding (respectively, contracting) one-dimensional foliation is equal to the conditional measure for ν.

EXAMPLE 9.4 A similar analysis for the partition $\mathcal{P}_2 = \{\{1, 2, 3\}, \{4, 5\}\}$ shows that with respect to the measure $\mu_{\mathcal{P}_2}$, the space $\mathrm{SL}(5,\mathbb{R})/\Gamma_5$ is locally a direct product of $\mathrm{SL}(3,\mathbb{R})/\Gamma_3$, $\mathrm{SL}(2,\mathbb{R})/\Gamma_2$, and a periodic flow on the circle. The entropy vanishes only for the periodic direction.

To construct invariant measures other than Haar measures coming from subgroups, one can again use an ergodic invariant measure ν on $\mathrm{SL}(2,\mathbb{R})/\Gamma_2$ to define a measure μ on $\mathrm{SL}(5,\mathbb{R})/\Gamma_5$. If the measure ν is not a Haar measure on $\mathrm{SL}(2,\mathbb{R})/\Gamma_2$, then the measure μ is invariant under the $\mathrm{SL}(3,\mathbb{R})$ action, α-ergodic but not a Haar measure on $\mathrm{SL}(5,\mathbb{R}/\Gamma_5)$.

We conclude the paper with two simple examples showing that several of the conditions of Theorem 4.1 fail in the nonsplit case or if $G \neq \mathrm{SL}(n,\mathbb{R})$.

EXAMPLE 9.5 Let $G = \mathrm{Sp}(n,\mathbb{R}) \subseteq \mathrm{SL}(2n,\mathbb{R})$ be the symplectic group, i.e., the Lie group of matrices that leave the exterior form

$$x_1 \wedge x_{n+1} + x_2 \wedge x_{n+2} + \cdots + x_n \wedge x_{2n}$$

in $\mathbb{R}^{2n}$ invariant. The subgroup consisting of diagonal matrices of the form

$$\alpha^{\mathbf{t}} = \begin{pmatrix} D_{t_1,\dots,t_n} & \\ & D^{-1}_{t_1,\dots,t_n} \end{pmatrix} \quad \text{where} \quad D_{t_1,\dots,t_n} = \begin{pmatrix} e^{t_1} & & \\ & \ddots & \\ & & e^{t_n} \end{pmatrix}$$

make up a maximal Cartan subgroup in G; the rank of G is n. It is well-known that G is a simple split Lie group. By the Borel-Harish-Chandra theorem the subgroup $\Gamma = \mathrm{SL}(2n,\mathbb{Z}) \cap G$ of integer matrices is a lattice in G. Let $X = G/\Gamma$. For a fixed index $1 \leq j \leq n$ the subgroup G_j of matrices $g \in \mathrm{Sp}(n,\mathbb{R})$ that leave the basis elements e_i for $i \neq j, n+j$ and the subspace $\langle e_j, e_{n+j}\rangle$ invariant is isomorphic to $\mathrm{SL}(2,\mathbb{R})$. Clearly the elements of G_i and G_j for $i \neq j$ commute with each other; we write $G' = \prod_i G_i$ for the generated subgroup of G. Then

$$\Gamma' = \Gamma \cap G' = \prod_i (\Gamma \cap G_i) \quad \text{is a lattice in } G'.$$

Let ν be any ergodic measure on $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ for the geodesic flow so that the flow has positive entropy. The product measure $\mu = \nu \times \cdots \times \nu$ on

$$\prod_{i=1}^{n} G_i/(\Gamma \cap G_i) \cong G'/\Gamma' \subset G/\Gamma$$

is ergodic under the left action α of the Cartan subgroup. Suppose $\mathbf{t} \neq 0$; then it is easy to see from the construction that $\mathrm{h}_\mu(\alpha^{\mathbf{t}}) > 0$.

This gives an example of a simple, split Lie group $G = \mathrm{Sp}(n, \mathbb{R})$, a lattice $\Gamma \subset G$, and an ergodic measure that is not the Haar measure m on G/Γ such that every nontrivial element of the flow has positive entropy. Therefore condition (iv) in Theorem 4.1 does not characterize the Haar measure in general.

EXAMPLE 9.6 Let $Y = \mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ and $X = \mathrm{SL}(n, \mathbb{C})/\mathrm{SL}(n, \mathbb{Z}[i])$. We can consider Y as a subset $Y \subset X$ and the Haar measure m_Y as a measure on X. For any $\alpha^{\mathbf{t}}$ the entropy with respect to m_Y is half the entropy with respect to m_X. In particular, the entropy function is fully positive. For every (two-dimensional) foliation $F_{a,b}$ the conditional measure is nonatomic a.e.; in fact, it is Lebesgue supported on a one-dimensional line.

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