INVARIANT MEASURES OF FLOWS ON ORIENTED SURFACES

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1. Let M be an orientable surface of genus $p \geq 1$, v a vector field on M of class C^1. We denote by $\{S_t^v\}$ the flow generated by the vector field v, by $I(v)$ the set of zeros of v, that is, the stationary points of the flow $\{S_t^v\}$, and by $\Omega(v)$ the set of non-wandering points of $\{S_t^v\}$. Everywhere in this note, unless otherwise stipulated, it will be assumed that the following two conditions are satisfied.

1.1. The set $I(v)$ contains only nondegenerate saddle points and consequently consists of precisely $2p - 2$ points.

1.2. $\Omega(v) = M$.

From results of A. G. Maĭer ([1], for the correction of an error in the proof see [2]) it follows that the surface M can be represented as a union of domains $M_1, \cdots, M_k$ with pairwise disjoint interiors, such that the boundaries of these domains consist of separatrices of fixed points and either the interior of a domain M_i is filled out with closed trajectories or any semitrajectory different from a separatrix, lying interior to the domain, is everywhere dense in the domain.

The assertions listed below are most interesting when $\{S_t^v\}$ is topologically transitive.

A Borel measure μ on M will be called a *nontrivial invariant measure for* $\{S_t^v\}$ if it is invariant relative to $\{S_t^v\}$, the measure of any trajectory of $\{S_t^v\}$ is equal to zero and $\mu(M \backslash U) < \infty$ for any neighborhood U of $I(v)$.

Proposition 1. *The flow* $\{S_t^v\}$ *has a nontrivial invariant measure which is positive on any open set.*

2. Let $\gamma: [0; 1] \to M$ be a path of class C^1 on M. We construct a mapping γ' of the oriented square into M by putting $\gamma'(s, t) = S_t^v \gamma(s)$, $s, t \in [0, 1]$. If μ is a nontrivial invariant measure for $\{S_t^v\}$, then the limit

$$\lambda_\mu^v(\gamma) = \lim_{t \to 0} \frac{1}{t} \int_{[0,1] \times [0,t]} \mathrm{sgn}\, J(\gamma^{-1})\, d(\gamma')^* \mu$$

exists finitely ($J(\gamma')$ is the Jacobian of γ'), which, by analogy with the case of smooth measures, we will call the *flux* of μ on translation of v through γ.

We extend the function λ_μ^v linearly to the space $S(M)$ of smooth (class C^1) 1-chains on M with real coefficients.

Proposition 2. λ_μ^ν *is a cocycle, that is,* $\lambda_\mu^\nu(\gamma) = 0$ *for any cycle* γ *homologous to zero.*

We denote the cohomology class of the cocycle λ_μ^ν by $\overline{\lambda}_\mu^\nu$. The Poincaré duality operator $\pi: H^1(M; \mathbf{R}) \to H_1(M; \mathbf{R})$ takes $\overline{\lambda}_\mu^\nu$ to an element of the homology group $H_1(M; \mathbf{R})$, which is called the rotation class of $\{S_t^\nu\}$ relative to μ (see [3], [4]).[1] We will allow a certain freedom in terminology and also call $\overline{\lambda}_\mu^\nu$ the rotation class.

Let μ_1, μ_2 be nontrivial invariant measures of $\{S_t^\nu\}$.

Proposition 3. *If* $\{S_t^\nu\}$ *has no closed trajectories and* $\overline{\lambda}_{\mu_1}^\nu = \overline{\lambda}_{\mu_2}^\nu$, *then* $\mu_1 = \mu_2$.

Proposition 4. $\overline{\lambda}_{\mu_1}^\nu(\pi\overline{\lambda}_{\mu_2}^\nu) = 0$.

Proposition 3 is easily deduced from Proposition 2 and the topological transitivity of $\{S_t^\nu\}$ in the domains $M_1, \cdots, M_k$ (see §1).

Let $\alpha \in H^1(M; \mathbf{R})$ and $\beta \in H_1(M; \mathbf{R})$. The value $\alpha(\beta)$ is equal to the index of the intersection $\pi\alpha \cdot \beta$. Therefore Proposition 4 may be formulated as follows:

The index of the intersection of the rotation classes of any two nontrivial invariant measures of $\{S_t^\nu\}$ *is equal to zero.*

Choose a neighborhood U of $I(\nu)$ such that $\mu(M\backslash U) > 0$ for any nontrivial invariant measure μ. We call μ normalized if $\mu(M\backslash U) = 1$.

Propositions 3 and 4 imply

Theorem 1. *If* $\{S_t^\nu\}$ *has no closed trajectories, then it has no more than* p *different normalized ergodic nontrivial invariant measures.*

3. We denote by $K(\nu)$ the cone in $H^1(M; \mathbf{R})$ which is generated by the rotation classes of all ergodic nontrivial invariant measures of $\{S_t^\nu\}$. This cone is equivariant under homeomorphisms: if the homeomorphism $\phi: M \to M$ takes trajectories of $\{S_t^{\nu 1}\}$ to trajectories of $\{S_t^{\nu 2}\}$, then $\phi^*K(\nu_2) = K(\nu_1)$. If M is a torus ($p = 1$), then it is well known that $K(\nu)$ consists of a unique pencil and completely characterizes the topological type of $\{S_t^\nu\}$. For $p \geq 2$ it is necessary to add to the values of the flux of the measure translatable by the vector field through closed curves the values of the flux of the measure through paths connecting fixed points.

The linear functional λ_μ^ν, on the relative homology group $H_1(M, I(\nu); \mathbf{R})$, generated by the cocycle λ_μ^ν is called the *fundamental class* of $\{S_t^\nu\}$ relative to the nontrivial invariant measure μ. The fundamental classes of ergodic nontrivial invariant measures of $\{S_t^\nu\}$ generate a cone $\hat{K}(\nu)$ in the $(4p - 3)$-dimensional space $(H_1(M, I(\nu); \mathbf{R}))^*$, which is also equivariant under homeomorphisms. The uses of the notion of fundamental class are shown in Theorems 2 and 3.

We denote by $\Gamma^1(Tm)$ the space of C^1 vector fields on M.

Theorem 2. *Let* μ *be a nontrivial invariant measure for* $\{S_t^\nu\}$, *positive on any open set. There exists a neighborhood* V *of the vector field* v *in the space* $\Gamma^1(TM)$ *with*

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[1] If μ is infinite, then the definition of rotation class by asymptotic cycles [3] is not equivalent to that mentioned here.

the following properties: if $v' \in V$, $I(v') = I(v)$ and $\{S_t^{v'}\}$ has a nontrivial invariant measure μ' such that $\hat{\lambda}_{\mu'}^{v'} = \hat{\lambda}_{\mu}^{v}$, then the flow $\{S_t^{v'}\}$ is topologically conjugate to the flow $\{S_t^v\}$.

It is likely that a similar assertion is true not for nearby flows alone.

Conjecture 1. *Let $I(v) = I(v')$ and $\hat{K}(v') = \hat{K}(v)$. Then the flows $\{S_t^v\}$ and $\{S_t^{v'}\}$ are topologically conjugate. If the flows $\{S_t^v\}$ and $\{S_t^{v'}\}$ are topologically transitive, then for their topological conjugacy instead of $\hat{K}(v') = \hat{K}(v)$ it is sufficient to require the existence of nontrivial invariant measures μ and μ' such that $\hat{\lambda}_{\mu}^{v} = \hat{\lambda}_{\mu'}^{v'}$.*

4. Let ω be a nonsingular 2-form of class C^∞ on M. We denote by μ_ω the measure generated by ω, by $\Gamma^\infty(TM, \omega)$ the space of C^∞ vector fields on M preserving ω, $\hat{\lambda}_\omega^v = \hat{\lambda}_{\mu_\omega}^v$. Vector fields v, v' of class C^∞ are called C^∞-equivalent if there exists a C^∞-diffeomorphism $\phi: M \to M$ taking trajectories of $\{S_t^{v'}\}$ to trajectories of $\{S_t^v\}$.

Theorem 3. *Let $v \in \Gamma^\infty(TM, \omega)$ and let $\{S_t^v\}$ satisfy condition 1.1. There is a neighborhood V of v in $\Gamma^\infty(TM, \omega)$ such that any vector field $v' \in V$ for which $I(v') = I(v)$ and $\hat{\lambda}_\omega^{v'} = \hat{\lambda}_\omega^v$ is C^∞-equivalent to v.*

Let I be a finite subset of M consisting of $2p - 2$ points. Let $\Gamma^\infty(TM, I, \omega) = \{v \in \Gamma^\infty(TM, \omega): v(x) = 0 \; \forall \; x \in I\}$, and let $\Gamma_1(I, \omega)$ the subset of $\Gamma^\infty(TM, I, \omega)$ (clearly open) consisting of vector fields satisfying condition 1.1 and consequently different from zero outside I.

Proposition 5. *If a vector field v satisfies conditions 1.1 and 1.2, then there is a vector field $v' \in \Gamma_1(I, \omega)$ such that the flows $\{S_t^v\}$ and $\{S_t^{v'}\}$ are topologically equivalent.*

Denote by $\Gamma_1'(I, \omega)$ the subset of $\Gamma_1(I, \omega)$ consisting of vector fields having closed trajectories homologous to zero, and by $\Gamma_2(I, \omega)$ the complement of $\Gamma_1(I, \omega)$ in the closure of $\Gamma_1'(I, \omega)$.

Proposition 6. *Let $v \in \Gamma_1(I, \omega) \cap \partial\Gamma_2(I, \omega)$. The intersection of the boundary $\partial\Gamma_2(I, \omega)$ with a sufficiently small neighborhood of the vector field v belongs to the union of a finite number of hyperplanes in $\Gamma^\infty(TM, I, \omega)$.*

Proposition 7. *Let $v, v' \in \Gamma_2(I, \omega)$, and let $f: M \to M$ be a diffeomorphism of class C^1, equal to the identity on I, taking trajectories of $\{S_t^{v'}\}$ to trajectories of $\{S_t^v\}$ (f_* is the automorphism of $H_1(M, I; \mathbf{R})$ induced by f).*
Then $\hat{\lambda}_\omega^v = c\hat{\lambda}_\omega^{v'} \circ f_*$, where c is some scalar.

In other words for vector fields from $\Gamma_2(I, \omega)$ the ray $\{t\hat{\lambda}_\omega^v: t \geq 0\} \subset (H_1(M, I; \mathbf{R}))^*$ is equivariant under diffeomorphisms.

Conjecture 2. *If $v, v' \in \Gamma_2(I, \omega)$, $\hat{\lambda}_\omega^v = \hat{\lambda}_\omega^{v'}$, then the vector fields v and v' are C^∞-equivalent.*

Remark. The structure of all the spaces of vector fields described does not depend on the choice of ω and I, since for any nonsingular 2-forms ω_1, ω_2 of class C^∞ with integral 1 and any sets $I_1, I_2 \subset M$ each consisting of $2p - 2$ points we may construct

a C^∞-diffeomorphism $f: M \to M$ such that $f(I_1) = I_2$ and $f^* \omega_2 = \omega_1$; see [5].

5. The mapping $\Gamma_2(I, \omega) \to (H_1(M, I; \mathbf{R}))^*: v \mapsto \hat{\lambda}_\omega^v$ is the restriction to $\Gamma_2(I, \omega)$ of a linear mapping $\Gamma^\infty(TM, I, \omega) \to (H_1(M, I; \mathbf{R}))^*$. It follows from (1) that a vector field $v \in \Gamma_2(I, \omega)$ is either topologically transitive or has closed trajectories not cohomologous to zero or has separatrices proceeding from one fixed point to another. In the second and third cases there exists an integral relation between the values of $\hat{\lambda}_\omega^v$ on the elements of any basis of the integral cohomology group $H_1(M, I; \mathbf{Z}) \subset H_1(M, I; \mathbf{R})$. So we have proved

Proposition 8. *It is possible to find a countable set of hyperplanes in $\Gamma^\infty(TM, I, \omega)$ such that, for any vector field $v \in \Gamma_2(I, \omega)$ not belonging to the intersection of $\Gamma_2(I, \omega)$ with one of these hyperplanes, the flow $\{S_t^v\}$ is topologically transitive.*

If M is a torus, then topological transitivity is equivalent to the uniqueness of the invariant measure. For $p \geq 2$ this is not so (see §6.1 below). However, uniqueness of the nontrivial invariant measure is nevertheless a typical property in $\Gamma_2(I, \omega)$. We denote by $\Gamma_3(I, \omega)$ the subset of $\Gamma_2(I, \omega)$ consisting of those vector fields v for which μ_ω is unique, up to multiplication by a constant, nontrivial invariant measure of $\{S_t^v\}$. For $v \in \Gamma_3(I, \omega)$ it is obvious that $\{S_t^v\}$ is ergodic with respect to the measure μ_ω.

Theorem 4. *The set $\Gamma_3(I, \omega)$ is a subset of second Baire category in $\Gamma_2(I, \omega)$.*

Conjecture 3. *There exists a set $A \subset (H_1(M, I; \mathbf{R}))^*$ of Lebesgue measure zero such that $v \in \Gamma_3(I, \omega)$ if $\hat{\lambda}_\omega^v \notin A$.*

Let σ be the Riemannian metric on M, ω_σ the 2-form associated with σ. A vector field v of class C^∞ is called *harmonic* with respect to the metric σ if $v \lrcorner \omega_\sigma$ is a harmonic 1-form. We denote by $J_\sigma: TM \to TM$ the operator acting in each tangent space $T_x M, x \in M$, as a rotation through $\pi/2$ in a positive direction. It is not difficult to show that v is a vector field harmonic with respect to σ if and only if $v \in \Gamma^\infty(TM, \omega_\sigma)$ and $v \circ J_\sigma \in \Gamma^\infty(TM, \omega_\sigma)$.

Proposition 9. *Let $v \in \Gamma_2(I, \omega)$ and let the flow $\{S_t^v\}$ be topologically transitive. There exists a C^∞-Riemannian metric on M with respect to which the vector field v is harmonic.*

Proposition 9 is used in the proof of Theorem 4 and in addition may turn out to be of use in the proof or disproof of Conjectures 2 and 3.

6. **Examples.**

6.1. *The number of invariant measures.* The estimate given by Theorem 1 is attained. In fact it is no more difficult to construct a flow for which M decomposes into p invariant domains such that a unique normalized nontrivial invariant measure is concentrated in each domain. Examples of such kind are in A. G. Maĭer [1], although there it is not a question of invariant measures. More interesting from the point of view of the circle of problems considered by us is the following assertion proved by a student of Moscow State University, E. A. Sataev.

For any $k \leq p$ *there exists a topologically transitive flow satisfying condition* 1.1 *and having exactly* k *normalized ergodic nontrivial invariant measures.*

6.2. *Finite and infinite measures.* Let $p = 2$. The possibilities are already apparent in this case.

For topologically transitive flows of class C^∞ all five possible situations are realized. Namely, the flow may have a unique normalized ergodic nontrivial invariant measure, finite or infinite, or two finite or two infinite or one finite and one infinite normalized ergodic nontrivial invariant measures. Here one of these finite measures may be chosen to be μ_ω.

The stimulus prompting the author to study the circle of problems considered in this paper was a discussion with S. H. Aranson and V. Z. Grines of the invariant proposed by them for flows on surfaces—the homotopic rotation class (see [2]). In particular the construction at the basis of the examples of §6, for $p = 2$, appeared for the first time in answer to a question put by Aranson and Grines of whether it is possible for a trajectory of a topologically transitive flow to exist without defining a limit direction in $H_1(M; \mathbf{R})$. One of the possible ways of proving Conjectures 1 and 2 consists of a detailed investigation of the connection between the fundamental class and the homotopic rotation class, which completely characterizes the topological type of a flow but is defined nonconstructively.

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