DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **6**, Number **1**, January **2000** pp. **61–88**

CONSTRUCTION OF WEAKLY MIXING DIFFEOMORPHISMS PRESERVING MEASURABLE RIEMANNIAN METRIC AND SMOOTH MEASURE

Roland Gunesch and Anatole B. Katok

Department of Mathematics
Pennsylvania State University
University Park, PA 16802, U.S.A.

Abstract. We describe in detail a construction of weakly mixing C^∞ diffeomorphisms preserving a smooth measure and a measurable Riemannian metric as well as $\mathbb{Z}^k$ actions with similar properties. We construct those as a perturbation of elements of a nontrivial non-transitive circle action. Our construction works on all compact manifolds admitting a nontrivial circle action.

It is shown in the appendix that a Riemannian metric preserved by a weakly mixing diffeomorphism can not be square integrable.

1. **Relation between differentiable and measurable structure for smooth dynamical systems: a brief overview.** Smooth ergodic theory studies measurable (or measure-theoretic, or ergodic) properties of differentiable dynamical systems with respect to natural invariant measures. (The word *smooth* will mean C^∞ unless explicitly stated otherwise). Such measures include smooth and, more generally, absolutely continuous measures such as Liouville measure for Hamiltonian and Lagrangian systems or Haar measure for homogeneous systems, their limits such as SRB measures, invariant measures for uniquely ergodic systems, measures of maximal entropy on invariant locally maximal sets, and so on. There is a number of situations where a remarkable correspondence appears between the differentiable dynamical structure and properties of invariant measures. We will follow the general scheme of classifying representative behavior of smooth dynamical systems as elliptic, parabolic hyperbolic and partially hyperbolic elaborated in [HK].

One can divide positive results on interrelations between measurable and differentiable structures into two kinds which are not mutually exclusive:

(i) Measurable structure determines differentiable structure completely or to a large extent (rigidity);

(ii) measurable structure (and sometimes also topological or even differentiable structure) within certain classes of systems (such as perturbations of a given one) and on certain parts of phase space conforms to a certain set of standard models (stability).

1.1. **Rigidity.** Rigidity phenomena appear for systems with elliptic and parabolic behavior and for hyperbolic smooth actions of higher rank abelian groups.

A very primitive but archetypal result of this kind asserts that any two metrically conjugate (i.e. isomorphic as measure-preserving transformations) topologically transitive translations or linear flows on a torus are differentiably conjugate (in fact,

1991 *Mathematics Subject Classification.* 58F11 (primary), 28D05, 57R50, 53C99 (secondary).
Key words and phrases. Conjugation-approximation method, almost isometries, weakly mixing diffeomorphisms, partition rearrangements, measure-preserving exponential map.

conjugate via an algebraic isomorphism). We will call this phenomenon *rigidity of measurable orbit structure* within a particular class of systems.

More interesting instances of rigidity of measurable orbit structure appear in systems with parabolic behavior. These include unipotent affine maps on tori, homogeneous flows on nilmanifolds, some other classes of group extensions of toral translations, and, most remarkably, unipotent homogeneous maps and flows on semisimple Lie groups. A prototype result of the last kind is rigidity of measurable orbit structure for horocycle flows on surfaces of constant negative curvature [R1] and their time changes [R2].

An example of both rigidity and stability appears for diffeomorphisms of the circle with Diophantine rotation number: Measurable structure with respect to its unique invariant measure determines rotation number, and by [Y] any such diffeomorphism is differentiably conjugate to a rotation. A simpler manifestation of the same phenomenon is rigidity of smooth time changes for Diophantine translations on a torus.

When one moves from classical dynamical systems (i.e. actions of $\mathbb{Z}$ and $\mathbb{R}$) to actions of higher-rank abelian groups, rigidity of measurable orbit structure appears for very natural algebraic actions such as $\mathbb{Z}^k$-actions by automorphisms of a torus or Weyl chamber flows [KS, KKS, KaK].

Rigidity is also prevalent among actions of semisimple Lie groups all of whose simple factors have rank greater than one as well as lattices in such groups. (See [FK] for a detailed discussion and references.)

1.2. **Stability.** Stability in various forms appears for hyperbolic and stably ergodic partially hyperbolic systems as well as for elliptic systems with Diophantine behavior.

The prototype result of the first kind is that any Gibbs measure (equilibrium state) with Hölder potential for a restriction of a diffeomorphism on a locally maximal hyperbolic set is Bernoulli [B]. This situation includes absolutely continuous invariant measures for Anosov flows, SRB measures for hyperbolic attractors and measures of maximal entropy on locally maximal hyperbolic sets. The same is true for absolutely continuous and SRB measures in the non-uniformly hyperbolic case, i.e. when all Lyapunov characteristic exponents are different from zero [P, L]. Bernoulli behavior extends to hyperbolic flows, unless they are suspensions, and to many partially hyperbolic systems. Notice that Bernoulliness implies extreme flexibility of measure-theoretic conjugacies, something completely opposite to rigidity of measurable orbit structure.

Stability in elliptic systems with Diophantine behavior in dimension higher than one for diffeomorphisms and higher than two for flows appears in the local form: For example, any perturbation of a translation T on a torus with a Diophantine translation vector which is topologically conjugate to T (and hence is metrically conjugate with respect to its unique invariant measure) is in fact differentiably conjugate to T. The main theme of KAM theory can be interpreted as establishing stability on a large part of the phase space of an integrable Hamiltonian system which is filled with Diophantine tori. Let us point out though that away from the low-dimensional cases rigidity and/or stability of measurable Diophantine behavior remains a widely open question.

2. **Liouvillian behavior and absence of rigidity or stability.** The mechanism leading to absence of both rigidity and stability which is best understood is abnormally fast periodic approximation; the prototype model for such behavior

is given by a rotation of the circle by an irrational angle extremely well approximable by rational multiples of π, or, more generally, by a Liouvillian translation on a torus, i.e. a translation T_γ such that the coordinates of the translation vector $\gamma = (\gamma_1, \dots, \gamma_n)$ are simultaneously very well approximable by rational numbers with the same denominator.

More generally, we say that a diffeomorphism $f : M \to M$ of a compact differentiable manifold is *Liouvellian* if for a certain sequence of integers $n_k \to \infty$ the iterates f^{n_k} converge to identity together with their derivatives faster than any polynomial in the C^r-topology for any r. Notice that the orbit closure of an aperiodic Liouvillian diffeomorphism in the space of diffeomorphisms is a perfect set and hence is uncountable. Baire category theorem implies that a dense G_δ subset of the orbit closure of a Liouvillean diffeomorphism consists of Liouvillean diffeomorphisms. Since orbit closure of a diffeomorphism in an appropriate space of maps is contained in its centralizer, the C^∞ centralizer of any Liouvellian diffeomorphism is uncountable and in particular contains an nontrivial copy of $\mathbb{Z}^k$ for any positive integer k. Since both ergodicity and weak mixing can be described by observing behavior of a countable dense collection of sets or functions along a subsequence of iterates (see Lemma 4.22 for the weak mixing property) another standard application of Baire category theorem produces the following observation.

Proposition 2.1. *In the orbit closure of a volume–preserving ergodic (corr. weakly mixing) Liouvillean diffeomorphism a dense G_δ subset consists of ergodic (corr. weakly mixing) Liouvillean diffeomorphisms.*

Corollary 2.2. *Any volume–preserving ergodic (corr. weakly mixing) Liouvillean diffeomorphism can be included into a $\mathbb{Z}^k$ action for any k by ergodic (corr. weakly mixing) diffeomorphisms which belongs to its orbit closure.*

Two basic phenomena related with Liouvillian behavior are the following:

(i) Measurable structure of a Liouvellian diffeomorphism with respect to a smooth invariant measure may be very diverse;

(ii) measurable orbit structure of a Liouvellian diffeomorphism does not determine continuous or differentiable structure even if the former is simple, i.e. a rotation by a certain Liouvillian number.

Given a volume preserving diffeomorphism f, we will call a volume-preserving diffeomorphism g *a nonstandard smooth realization of* f if g is metrically conjugate to f but not differentiably conjugate to it.

2.1. **Isometric extensions and time changes.**

The easiest way to produce Liovillian diffeomorphisms and to observe these phenomena is to look either at S^1-extensions of a Liouvillian rotation, i.e. transformations of T^2 of the form

$$F(x, y) = (x + \alpha, y + \phi(x)) \qquad (\text{mod } 1), \tag{2.1}$$

where α is a Liouvillian number and ϕ a smooth function, or at a time change for the linear flow on the two-dimensional torus with Liouvillian slope.

For an S^1-extension of an appropriate Liouvillian rotation even with a real-analytic function ϕ the spectrum of the associated unitary operator in L^2 may be mixed, or the map may be uniquely ergodic and metrically conjugate to a translation but not topologically conjugate (e.g. topologically weak mixing), or topologically but not smoothly conjugate to a translation, or minimal but not uniquely ergodic.

Similarly the time change may be weakly mixing or may be ergodic with mixed spectrum, or may be metrically but not topologically conjugate to the linear flow or, again, may be topologically but not smoothly conjugate to a linear flow.

2.2. **Conjugation-approximation construction.** A powerful and flexible approach which produces Liouvillian diffeomorphisms with diverse and often "exotic" properties is based on the construction introduced in [AK]. This construction involves consecutive perturbations of elements of a given smooth action of S^1 via differentiable conjugations which diverge but are chosen in such a way that resulting diffeomorphisms converge.

More specifically, let M be a manifold, with the action $\phi = \{\phi_t\},\ t \in \mathbb{R}/\mathbb{Z}$ of the circle. The maps are constructed as limits of conjugates of periodic maps from the action ϕ but conjugating map diverge in an often dramatic but controlled way. So we have $f = \lim_{n\to\infty} f_n$ where

$$f_n = H_n \circ \phi_{\alpha_n} \circ H_n^{-1},\ \ H_{n+1} = H_n \circ h_n,$$

Here $\alpha_n = \frac{p_n}{q_n}$ and $p_n,\ q_n$ are relatively prime integers. Furthermore,

$$h_n \circ \phi_{1/q_n} = \phi_{1/q_n} \circ h_n$$

and $\alpha_{n+1} = \alpha_n + 1/l_n k_n q_n^2$.

At $(n+1)$-st inductive step the correction to conjugacy h_n and the number k_n are constructed first in order to make the orbits of the periodic flow $H_{n+1} \circ \phi \circ H_{n+1}^{-1}$ imitate the desired properties with a certain precision and k_n is chosen to make the discrete orbits of

$$H_{n+1} \circ \phi_{1/k_n q_n} \circ H_{n+1}^{-1}$$

approximate the continuous orbits of $H_{n+1} \circ \phi \circ H_{n+1}^{-1}$. The properties range from ergodicity or minimality in more basic versions, to weak mixing or the complete orbit structure of a map from a specific family in more sophisticated versions of the construction. Then finally, l_n is chosen large enough to provide closeness of f_{n+1} to f_n in the C^∞ topology.

Among original applications of that method are non-standard smooth realizations of some Liouvillian rotations and toral translations on manifolds other than tori and whose dimension is different from the number of frequencies in the spectrum. Furthermore, there are examples of volume-preserving diffeomorphisms metrically isomorphic to certain standard nonsmooth models such as certain translations on infinite-dimensional tori. There are also examples of weakly mixing transformations.

There are many other applications of this method and its potential is far from having been exhausted. In this paper we restrict ourselves to a specific application and carry it out in great detail.

3. **Invariant Riemannian metric, discrete spectrum and isometry.** The property of being "essentially an isometry" is most closely associated with elliptic behavior. Its versions illustrate the interplay of different structures in smooth ergodic theory quite well. Since these properties often appear in the context of actions of more general groups than $\mathbb{Z}$ or $\mathbb{R}$ we will assume that Γ is a locally compact second countable group acting by diffeomorphisms of a compact manifold M. Thus we will consider the following properties:

(D)(Differentiable) Γ preserves a smooth Riemannian metric on M (and hence a smooth volume form generated by the metric);

(C)(Continuous of Topological) Γ preserves a metric on M;

(IM)(Infinitesimal measurable) Γ preserves an absolutely continuous probability measure and a measurable Riemannian metric on M;

(GM)(Global measurable) Γ preserves an absolutely continuous probability measure μ and the induced group of unitary operators in $L^2(M,\mu)$ has *discrete spectrum*, i.e. $L^2(M,\mu)$ splits into orthogonal sum of finite–dimensional Γ invariant subspaces.

Property *(D)* implies all others. In fact, in this case the closure of Γ in the group of diffeomorphisms of M is a compact Lie group and every orbit closure is diffeomorphic to a homogeneous space of G with Γ acting by translations.

Property *(C)* does not imply existence of an absolutely continuous invariant measure but it implies that the action has discrete spectrum with respect to *any* Borel probability invariant measure.

Already in the simplest case $M = S^1$, the circle, and $\Gamma = \mathbb{Z}$, for Liouvillian rotation numbers various non–equivalences appear: property *(C)* always holds by Denjoy Theorem, while if the conjugacy with a rotation is singular none of the other holds. Furthermore if the conjugacy is absolutely continuous but not smooth *(IM)* and *(GM)* hold but not *(D)*. In this case however, preservation of an absolutely continuous measure implies both *(IM)* and *(GM)*.

In the rest of this paper we discuss relationships between the properties *(IM)* and *(GM)* as well as existence of actions satisfying one of these properties but not satisfying the stronger property *(D)*.

3.1. **The main result.**

Theorem 3.1. *On any compact C^∞-manifold of dimension $m \geq 2$ admitting a nontrivial C^∞ circle action there exists a weakly mixing C^∞ Liovillean diffeomorphism that preserves a C^∞ measure and a measurable Riemannian metric.*

Since any diffeomorphism in the orbit closure of a diffeomorphism preserving a measurable Riemannian metric also preserves this metric we immediately deduce from Theorem 3.1 and Corollary 2.2 the following stronger statement.

Corollary 3.2. *On any compact C^∞-manifold of dimension $m \geq 2$ admitting a nontrivial C^∞ circle action there exists a $\mathbb{Z}^k$ action for any positive integer k by weakly mixing C^∞ Liouvillean diffeomorphisms preserving a C^∞ measure and a measurable Riemannian metric.*

The idea of the construction is the following: We create the diffeomorphism f as the limit of C^∞-diffeomorphisms f_n by an appropriately specified version of the conjugation-approximation construction. To do so, we proceed as follows: We show how to exhaust the manifold up to a set of arbitrarily small measure by similar "almost hypercubes" with positive distance. As n increases, the number of hypercubes increases and the area not covered converges to zero as quickly as we wish. In the general "conjugation-approximation" construction, each f_n is a measure-preserving diffeomorphism constructed from certain maps which are not explicitly specified but only required to satisfy certain conditions. In this paper, we construct those maps explicitly. Doing so enables us to equip the maps in the construction with the additional structure of being locally very close to an isometry. We show how enough of this structure gets preserved when we pass to the limit

$n \to \infty$, so that $f = \lim_{n\to\infty} f_n$ still preserves a Riemannian metric. Weak mixing is guaranteed by a combinatorial arrnagement of the hypercubes involved so that a certain iterates of the approximating (and hence limit) diffeomorphism mixes elements of a certain partition (see Lemma 4.22 and these partitions become finer and finer as n increases.

The detailed proof of Theorem 3.1 is given in the next section.

3.2. **Regularity of the invariant Riemannian metric.** It is natural to ask how regular an invariant Riemannian metric ought to be in order to guarantee discrete spectrum.

A simple general observation is that essential boundedness of such a metric from above and below is sufficient. For, given an invariant Riemannian metric which is essentially bounded we may define a bounded invariant *Finsler* metric by defining the norm of a tangent vector v as the essential upper limit of norms of vectors converging to v. Such a Finsler metric then defines an invariant distance via the usual process of defining curve length and taking infimum. Hence the map has property *(C)* and consequently discrete spectrum with respect to any invariant measure.

Moreover, a result of A. Furman shows that if a C^2- action of a countable group preserves a metric with L^2 distortion (i.e. both the norm and its inverse are L^2 functions) then f has discrete spectrum with respect to a smooth invariant measure. The proof of this fact is contained in the Appendix.

3.3. **Related results.** Existence of a measurable invariant metric for a diffeomorphism (or, more generally, a smooth group action) on an n-dimensional manifold is equivalent to existence of measurable cohomology between the derivative cocycle (with respect to any measurable trivialization of the tangent bundle) and a cocycle with values in the orthogonal group $SO(n, \mathbb{R})$. Another equivalent formulation is existence of an invariant measure for the projectivized derivative extension of the action which is absolutely continuous in the fibers. One may naturally ask what would be ergodic properties of the projectivized derivative extension with respect to such a measure. There are two extreme possibilities:

(i) projectivized derivative extension is ergodic, and

(ii) the derivative cocycle is cohomologous to identity.

In case (ii) the projectivized derivative extension is as non–ergodic as possible: it is isomorphic to the direct product of the action in the base with the trivial action in the fibers so that each ergodic component intersects almost every fiber in a single point.

Construction presented in Section 4 in fact realizes case (ii). As it turns out by modifying our construction case (i) as well as intermediate situations may also be achieved.

One can also go in the opposite direction and produce non-standard smooth realizations of certain Liouvillean rotations of the circle and translations of the torus which do not preserve any any measurable Riemannian metric thus showing that *(GM)* does not imply *(IM)* either.

Detailed constructions and proofs will appear in a separate paper.

3.4. **Actions of non–abelian groups.** One of the motivations for considering groups of diffeomorphisms preserving a measurable Riemannian metric came from Margulis–Zimmer rigidity theory. Zimmer's cocycle superrigidity theorem [Z1, Theorem 5.2.5] applied to the case of the derivative cocycle implies that this possibility is essentially the only alternative to the genuine "rigidity" of the derivative cocycle, namely cohomology with a "constant coefficient" cocycle. A natural question arises whether one can make conclusion about global differentiable or at least measurable structure of the action. Zimmer provides partial answers in [Z2, Z3]. In the former paper an extra condition (which is not immediately checkable) is given which is sufficient for existence of a smooth invariant Riemannian metric. The latter contains an elegant general result: for groups satisfying Kazhdan property (T) our property *(IM)* implies discrete spectrum, i.e. *(GM)*.

3.5. **Open problems.** Beyond methods based on super–rigidity and property (T) on one hand, and the conjugation–approximation construction on the other, very little is known about group actions satisfying property *(IM)*. For brevity we will call such actions simply *(IM) actions*.

Let us formulate several interesting open problems. We will not discuss here questions related to "rigid " group actions except for pointing out that the central question arising from superrigidity in this context is still open. Namely it is not known whether any smooth *(IM)* action of a Kazhdan (property (T)) group (or, more specifically, a lattice in simple Lie group of rank greater than one) actually preserves a smooth Riemannian metric.

Problem 3.3. *Does every compact manifold admit an (IM) diffeomorphism?*

The conjugation approximation construction is the key ingredient in the original proof that every compact manifold admits a volume–preserving ergodic diffeomorphism. The essential dynamical part of the proof is a construction of an ergodic diffeomorphism of the closed ball which fixes every point of the boundary and is very "flat" near it. At a certain stage of that construction a suspension flow over a Liouvillian diffeomorphism produces by a conjugation–approximation method is subjected to a time change which vanishes at the boundary. At this step of the construction a possibility of preservation of a measurable invariant metric is lost.

Problem 3.4. *Does there exist a faithful smooth (IM) action of the free group with $m \geq 2$ generators on a compact manifold which does not satisfy (D) (or (C))?*

Since free groups can be embedded in many ways into orthogonal groups there are many actions of free groups by isometries which may serve as starting point of an inductive construction similar to conjugation–approximation. The difficulty lies in approximation the free group in a way similar to approximation of $\mathbb{Z}$ by finite group in the conjugation–approximation method.

Problem 3.5. *Suppose Γ is a nilpotent countable group which does not have an abelian subgroup of finite index. Does there exist a faithful smooth (IM) action of any such group on a compact manifold?*

The difficulty here is that for any such group Γ the image of any homomorphism to an orthogonal group has an abelian subgroup of finite index. Hence Γ cannot act faithfully by isometries of a Riemannian manifold and any construction would have to be of a non–perturbative nature.

The following conjecture is one of a few plausible general results concerning *(IM)* actions

Conjecture 3.6. *Any smooth (IM) action of locally compact second countable group (or just $\mathbb{Z}$) preserving a* rigid geometric structure [Gr] *satisfies property (D).*

Several natural questions are related to the regularity of a measurable invariant Riemannian metric sufficient for discreteness of the spectrum. Here is a small sample.

Conjecture 3.7. *Any smooth volume preserving action of a second countable locally compact topological group on a compact manifold which preserves a measurable Riemannian metric with L^1 distortion has discrete spectrum, i.e satisfies (GM).*

Problem 3.8. *Suppose a volume preserving diffeomorphism of a compact manifold preserves an L^2 (or L^1) Riemannian metric which is almost everywhere positive (the inverse to the norm may not be integrable). Does it have discrete spectrum?*

Finally there are questions concerning *real–analytic* actions. It is worth pointing out that implementation of the conjugation–approximation construction in the real–analytic category meets with great difficulties. So far the only successes are R. Perez-Marco's work in dimension one and very recent results of the second author concerning perturbations of homogeneous actions. However there is hope that other methods may work at least is special cases.

Problem 3.9. *Does there exist a real–analytic volume–preserving weak mixing (IM) diffeomorphism?*

4. Proof of the main result.

4.1. Review of the conjugation-approximation construction. We construct the weakly mixing C^∞-diffeomorphism $f = \lim_{n\to\infty} f_n$ described in theorem 3.1, namely

$$f_n := H_n^{-1} S_{b_{n-1}} H_n f_{n-1} = H_n^{-1} S_{\alpha_n} H_n, \qquad H_n := h_n \circ \cdots \circ h_1,$$

where

$$\alpha_n =: \frac{p_n}{q_n}, \qquad \alpha_{n+1} := \alpha_n + \beta_n, \qquad \beta_n := \frac{1}{q_n^2 k_n l_n}.$$

We will describe the choice of the parameters (namely q_n and k_n) and also explicitly construct the maps h_n and the invariant metric.

4.2. Action and factor space. Let M be an m-dimensional compact smooth manifold admitting a smooth circle action $S = (S_\alpha)_{\alpha\in\mathbb{R}/\mathbb{Z}}$ under which a smooth probability measure μ is invariant. Let $\tau(x)$ denote the smallest period of x, i.e.,

$$\tau(x) := \inf\{t > 0 : S_\tau(x) = x\}.$$

By compactness τ is bounded, and we can assume without loss of generality that $\max_{x\in M} \tau(x) = 1$. Define

$$M_1 := \{x \in M : \tau(x) = 1)\}, \qquad M_0 := M \setminus M_1.$$

Lemma 4.1. *τ is lower semicontinuous, i.e. if $\lim_{n\to\infty} x_n = x$ and $\tau(x_n) \le a$ then $\tau(x) \le a$.*

Proof. Clearly $\tau(x) \in \{0\} \cup \{\frac{1}{i} : i \in \mathbb{N}\}$. If $\lim_{n\to\infty} x_n = x$ and $\tau(x_n) \le a$ then either there is a infinite subsequence along which τ equals $b \le a$, or $\lim_{n\to\infty} \tau(x_n) = 0$. In the first case, the conclusion $\tau(x) = b \le a$ simply follows from continuity of the action. In the second case, if $b := \tau(x)/2 > 0$ then $d(x, S_b x) > \varepsilon$ for some $\varepsilon > 0$. By smoothness of the action and compactness, $\frac{d}{dt} S_t(y)$ is bounded by some constant C. Thus $d(x_n, S_b x_n) \le \tau(x_n) C$. So for n sufficiently large we have $d(x_n, S_b x_n) < \varepsilon$. Thus $d(x, S_b x) \le \varepsilon$, contradicting $\tau(x) > 0$. Therefore $\lim_{n\to\infty} \tau(x_n) = 0$ implies $\tau(x) = 0$. ∎

Lemma 4.2. *M_1 is open and of full measure.*

Proof. If $x_i \to x$ and $x_i \in M_0$ then $\tau(x_i) \leq \frac{1}{2}$, thus $\tau(x) \leq \frac{1}{2}$ by semicontinuity of τ. Therefore M_0 is closed and M_1 open. See [AK] for a reference that $\mu(M_0) = 0$. □

Lemma 4.3. [AK] *M_1 is connected.*

From openness of M_1 follows that the measure of the ε-neighborhood of M_0 decreases to zero as $\varepsilon \to 0$ (because for $p \in M_1$ there is γ such that $B_\gamma p \subset M_1$, thus $p \notin B_\varepsilon(M_0)$ for $\varepsilon < \gamma$). Therefore we can cover M up to arbitrarily small measure by balls of uniform radius ε so that all balls are uniformly bounded away from M_0.

Let N be the factor space M/S.

Remark 4.4. N carries a natural measure ν given by $\nu := \pi_*\mu$, i.e. $\nu(U) = \mu(\pi^{-1}(U))$, where $\pi : M \to N$ is the natural projection. Moreover, $N_1 := \pi(M_1)$ is an open subset of N. The closed set $N_0 := \pi(M_0)$ obviously satisfies $\nu(N_0) = 0$.

4.3. **Smooth trivialization on full measure.** There exists a S-invariant set $M_2 \subset M_1$ such that the following is true: Let $M_0' := M \setminus M_2$. Then $\mu(M_0') = 0$, the set M_2 is diffeomorphic to a disc D^m, the factor space M_2/S can be identified with a submanifold $N_2 \approx D^{m-1}$, and for any $\delta > 0$ there is an S-invariant compact subset $M_\delta \subset M_2$ diffeomorphic to $\bar{D}^{m-1} \times S^1$ and of measure at least $1 - \delta$. Of course, δ will always be small, so there is no danger of confusion between M_1 and M_δ etc. The factor M_δ/S can be identified with a subset $N_\delta \subset M_\delta$ diffeomorphic to $\bar{D}^{m-1}$. N_2 can be approximated by compact sets N_δ diffeomorphic to $\bar{D}^{m-1}$ in such a way that $\nu(N_\delta) > 1 - \delta$.

Lemma 4.5. *Under the above assumptions, there exists a smooth Riemannian metric on N_2 whose induced volume coincides with ν.*

Proof. Let g be any smooth Riemannian metric on N_2 with volume form $d\text{vol}_g$. Let θ be the density function of ν with respect to g, i.e.,

$$\frac{d\nu}{d\text{vol}_g} = \theta.$$

Since multiplying the metric by a multiplies the volume form by $a^{\dim N}$, the metric $g' := \theta^{1/\dim N} \cdot g$ satisfies

$$\frac{d\nu}{d\text{vol}_{g'}} \equiv 1,$$

as desired.

A smooth Riemannian metric induces a natural volume form. Integration gives a natural volume measure. More interestingly, any smooth measure arises this way: □

Proposition 4.6. *There exists a Riemannian metric g on M_2 such that:*

- *The volume measure of g coincides with μ.*
- *N_2 is orthogonal to the fibers of the action, i.e., $g(v, \frac{d}{dt}S_t(x)|_{t=0}) = 0$ for all $v \in T_xN_2$.*
- *For every $\delta > 0$, the measure $\nu = \pi_*\mu$ is smooth with positive density on N_δ.*

Proof. Choose a smooth metric g_{N_2} on N_2. By the previous lemma (4.5) we can assume that its volume equals $\nu|_{N_2}$. Denote the standard metric on S^1 by g_S and its volume (which is just length) by λ. Define the metric g on M_2 to be the product metric of g_{N_2} and g_S. Since g is a product metric, its volume form is the product of the volume form of g_{N_δ} (which is ν) and the volume form of g_S (which is l). Since all S-fibers of M_2 have the same length, we see that $\mathrm{vol}_g = \mu = \nu \times \lambda$ on M_2. Thus the density of ν is positive. Compactness of N_δ gives uniform boundedness. □

Remark 4.7. Thus M_δ with the metric g is (smoothly) isometric to $N_\delta \times S^1$ with the product metric of a smooth metric g_{N_δ} and the canonical (arclength-) metric on the circle S^1. The measure μ coincides with the volume in the metric g.

4.4. **Constructing hypersquares.**

Definition 4.8. For a diffeomorphism f defined on a compact subset U of a smooth Riemannian manifold we define the *deviation from being an isometry* by

$$\mathrm{dev}_U(f) := \max_{v \in TU, ||v||=1} |\log ||df \cdot v|| \,| .$$

This quantity has the following properties:

- $\mathrm{dev}_U(f) \geq 0$.
- $\mathrm{dev}_U(f) = 0$ if and only if f is a smooth isometry of U.
- $\mathrm{dev}_U(f) = \mathrm{dev}_{f(U)}(f^{-1})$.
- $\mathrm{dev}_U(\tilde{f} \circ f) \leq \mathrm{dev}_{f(U)}(\tilde{f}) + \mathrm{dev}_U(f)$.

The exponential map of a Riemannian manifold is not measure-preserving (although it is "infinitesimally measure-preserving"). We now define a measure-preserving diffeomorphism which is arbitrarily close to the exponential map and has arbitrarily small deviation from being an isometry:

Lemma 4.9. *Let (M, g) be a Riemannian manifold, $p \in M$. For all $\varepsilon > 0$ there exists $\rho > 0$ such that for any hypercube $\bar{W} \subset B_\rho(0) \subset T_pM$ there exists $C \in (1-\varepsilon, 1+\varepsilon)$ and a diffeomorphism*

$$e : C \cdot \bar{W} \to \exp(\bar{W})$$

which is measure-preserving (i.e. the pullback via e of the volume form induced by g equals the Euclidean volume form) and which satisfies dev$(e) < \varepsilon$. Here $C \cdot \bar{W}$ denotes homothetic scaling of $\bar{W}$ (about its center) with factor C.

Proof. Since the derivative of the exponential map $\exp_p$ at $0 \in T_pM$ is the identity, the exponential map preserves the length element up to terms of second order in the distance to p. Thus it preserves volume up to second order terms is $d(p, .)$. For any bounded $U \subset T_pM$ we see that

$$\frac{\mathrm{vol}_g(\exp_p(aU))}{\mathrm{vol}_{\mathrm{Eucl}}(aU)} \to 1$$

as $a \to 0$. So for ρ small enough, the number

$$C := \left(\frac{\mathrm{vol}_g(\exp_p(\bar{W}))}{\mathrm{vol}_{\mathrm{Eucl}}(\bar{W})}\right)^{1/\dim M}$$

lies in the interval $(1-\varepsilon, 1+\varepsilon)$.

Since $d\exp$ is arbitrarily close to the identity for ρ small enough, we can find ρ so that $\mathrm{dev}_{B_\rho}(\exp_p) < \varepsilon/2$.

Denote the density function of the pullback of $\mathrm{vol}_g|_{\exp_p(\bar{W})}$ by θ, i.e.

$$\theta := \frac{d(\exp_p^* \mathrm{vol}_g)}{d\mathrm{vol}_{\mathrm{Eucl}}}.$$

Since exp is volume-preserving up to second order terms, it follows that $\theta - 1 \in (-\varepsilon, \varepsilon)$ for ρ small enough.

Choosing a diffeomorphism $A : C\bar{W} \to C\bar{W}$ so that $e(x) := \exp(\frac{1}{C} \cdot A(x))$ is measure-preserving and satisfies $\mathrm{dev}_{C\bar{W}}(A) < \varepsilon/2$, we have proved the claim.

The existence of A can be seen as follows: Choose a gradient vector field V on $C\bar{W}$ so that Φ_V^1 (the time-1 map of V) satisfies $(\Phi_V^1)^*(\theta \cdot \mathrm{vol}_{\mathrm{Eucl}}) = \mathrm{vol}_{\mathrm{Eucl}}$. Such a vector field can be constructed by planting positive divergence where $\theta > 1$, thus "repelling mass" there, and negative divergence where $\theta < 1$, thus "attracting mass" there, and solving the corresponding Laplace equation. Let $A := (\Phi_V^{-1})$. Since $V \to 0$ as $\rho \to 0$, we see that $\mathrm{dev}(h) < \varepsilon/2$ for ρ small enough. Thus $\mathrm{dev}(e) \leq \mathrm{dev}(h) + \mathrm{dev}(\exp) < \varepsilon$.

Another, more explicit, construction of A is to average the mass density θ subsequently in all directions: Let $\theta_0 := \theta$, and for $i \in \{1, \dots, m\}$ let θ_i be the density obtained after applying $A_i \circ \dots \circ A_1$, where we define the smooth map

$$A_i : x_i \mapsto \int^{x_i} \theta_{i-1}(x_1, \cdots, x_{i-1}, t, x_{i+1}, \dots, x_m)dt, \quad x_k \mapsto x_k \, \forall k \neq i.$$

Since θ_i does not depend on x_i, the smooth map $A := A_m \circ \dots \circ A_1$ satisfies the requirements. □

Corollary 4.10. *Let (M, g) be a compact Riemannian manifold. For all $\varepsilon > 0$ there exists $\rho > 0$ such that for all $p \in M$ and any hypercube $\bar{W} \subset B_\rho(0) \subset T_pM$ there exists $C \in (1-\varepsilon, 1+\varepsilon)$ and a smooth diffeomorphism $e : C \cdot \bar{W} \to exp(\bar{W})$ which is measure-preserving and which satisfies $dev(e) < \varepsilon$.*

Definition 4.11. We call a collection of subsets of (M, μ) a *partition mod δ (of M)* if the complement of their union has measure at most δ. In similar spirit, we say that a property is true *mod* δ if the set where it is false has measure at most δ. For example, two sets are equal *mod* δ if their symmetric difference is at most δ.

Proposition 4.12. *(Cutting N_2 into hypersquares.) For all $\delta > 0, \varepsilon > 0, \gamma > 0$ there exists a finite partition $\mathcal{Q}$ of N_2 up to a set of measure less than δ, a number $\sigma > 0$ and a family $(f_{QQ'})_{Q,Q' \in \mathcal{Q}}$ of C^∞-diffeomorphisms $f_{QQ'}\colon Q \to Q'$ so that for all r all $Q, Q' \in \mathcal{Q}$:*

- *Q is a topological ball in N_δ.*
- *$diam(Q) < \gamma$.*
- *$dist(Q, Q') > \sigma$.*
- *$\mu(Q) = \mu(Q')$.*
- *$f_{QQ'}$ is measure-preserving.*
- *$dev_Q(f_{QQ'}) < \varepsilon$.*
- *$f_{Q'Q} = f_{QQ'}^{-1}$.*

Remark. We use the terminology "hypersquares" to denote objects of full dimension on N and "hypercubes" for those on M. The dimension of the hypercubes is m, that of the hypersquares is $m-1$.

Proof. We start by choosing a small enough number ρ. Precise conditions on its size will be described later.

Pick δ' so small that $\nu(N_2 \setminus N_{\delta'}) < \delta/100$. Cover $N_{\delta'}$ by a finite collection of balls of radius ρ, denoted by $(B_\rho(x_i))_{i \in J}$, $J \subset \mathbb{N}$. Define

$$U_i := B_\rho(x_i) \setminus \bigcup_{j<i} U_j.$$

At each point x_i, $i \in J$, choose a linear isomorphism $L_i : \mathbb{R}^{m-1} \to T_{x_i}N$. Choose a small $\gamma' > 0$ (the requirements for smallness of γ' will also be given later).

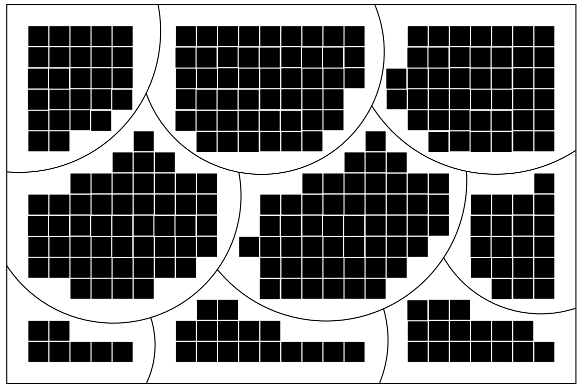

FIGURE 4.1. We can cut N_δ into hypersquares (black) of the same measure which are almost isometric to Euclidean ones. (The circles correspond to the sets U_i.)

Subdivide $\mathbb{R}^{m-1}$ into hypersquares of side length γ'. Since the exponential map is a local diffeomorphism, if ρ was chosen small enough, the hypersquares in the ρ-ball in $\mathbb{R}^{m-1}$ are mapped diffeomorphically into N_2 via $\exp_{x_i} \circ L_i$. Discarding those images that do not lie completely in U_i, we get a collection of images of hypersquares for each $i \in J$. Denote the collection of all these topological balls by $(Q'_i)_{i\in I}$ with some finite index set I. Since, by choosing γ' small enough, each U_i can be exhausted this way up to a set of arbitrarily small measure, we can assume that $\nu(N_2 \setminus \bigcup_{i\in I} Q'_i) < \delta/30$. We separate the Q'_i by shrinking the corresponding hypersquare in $\mathbb{R}^{m-1}$ homothetically (around its center) by a factor $1 - \frac{\delta}{30(m-1)}$. This gives a modified collection $(Q''_i)_{i\in I}$. This modified collection can be assumed to satisfy $\nu(N_2 \setminus \bigcup_{i\in I} Q''_i) < \delta/10$.

Without loss of generality ρ was chosen small enough that $\nu(Q''_i)/\max_{i\in I}\nu(Q''_i) \in [1-\delta/10, 1]$. Thus we can find numbers $\lambda_i \in [1-\delta/10, 1]$ for $i \in I$ such that if we do another scaling of the hypersquare in $\mathbb{R}^{m-1}$ corresponding to Q''_i by λ_i, we get a topological ball Q_i such that $\nu(Q_i) = \nu(Q_j)$ for all $i, j \in I$, and the Q_i are still a positively separated family. Since we can assume $\gamma' < \gamma/\sqrt{m-1}$, the diameter of all Q_i is less then γ. Thus we have produced the collection $\mathcal{Q}$ satisfying the first four properties in the statement.

Now we will construct the maps $f_{QQ'}$. For $Q \in \mathcal{Q}$, we write L_Q for the L_i corresponding to Q. Let e_Q be the map derived from the exponential map corresponding to Q, as explained in lemma 4.9. (More explicitly, if $i(Q)$ is the index such that $Q \subset U_{i(Q)}$, then e_Q corresponds to $\exp_{x_{i(Q)}}$ and L_Q corresponds to $L_{i(Q)}$.) Similarly we define $L_{Q'}$ and $e_{Q'}$.

Obviously the hypersquare $L_Q \circ e_Q^{-1}(Q) \subset \mathbb{R}^{m-1}$ can be mapped to $L_{Q'} \circ e_{Q'}^{-1}(Q') \subset \mathbb{R}^{m-1}$ by some translation $A_{QQ'}$. Define the map $f_{QQ'} : Q \to Q'$ by

$$f_{QQ'} := \mathrm{e}_{Q'} \circ L_{Q'} \circ A_{QQ'} \circ L_Q^{-1} \circ \mathrm{e}_Q^{-1}.$$

If ρ is small enough then $\mathrm{dev}(e_Q), \mathrm{dev}(e_{Q'}) < \varepsilon/2$. Of course, $\mathrm{dev}(A_{QQ'}) = \mathrm{dev}(L_Q) = \mathrm{dev}(L_{Q'}) = 0$ since those maps are smooth isometries. Thus

$$\mathrm{dev}_Q(f_{QQ'}) < \varepsilon.$$

□

4.5. **Geometric partitions.**

Definition 4.13. For $i \in \mathbb{Z}$, $q \in \mathbb{N}$ let $D_{i,q} \subset M_2$ be the rectangle

$$D_{i,q} := N_2 \times \left[\frac{i}{q}, \frac{i+1}{q}\right),$$

where we understand coordinates on S^1 to be taken modulo 1. With this notation we have $S_{l/q} D_{i,q} = D_{i+l,q}$. For $i = 0, \dots, q-1$ the $D_{i,q}$ are disjoint isometric "slices" whose collection partitions M_2. Define the partition

$$Y_n := \{D_{i,q_n} : i \in \{0, \dots, q_n - 1\}\}.$$

It is clear that $Y_{n+1} > Y_n$, i.e., every element of Y_n is union of elements of Y_{n+1}. Now we define the partition

$$X_n := H_n^{-1} Y_n.$$

Let $\delta > 0$ be given. Let $\mathcal{Q}$ be the partition mod $\delta/4$ of $N_{\delta/4}$ defined in Proposition 4.12. Denote the number of its elements by j. Let $\sigma' := \frac{\delta}{2m}$. For $q, k \in \mathbb{N}$, where we can assume without loss of generality that k is a multiple of j, we define the following partitions:

$$\mathcal{I} := \mathcal{I}_c \setminus B_{\sigma'}(\partial I_c \cup M_0) \qquad \text{where} \quad \mathcal{I}_c := \{D_{i,q}\}_{0 \le i < q};$$

$$\mathcal{I}' := \mathcal{I}'_c \setminus B_{\sigma'}(\partial I'_c \cup M_0) \qquad \text{where} \quad \mathcal{I}'_c := \{D_{i,kq}\}_{0 \le i < kq};$$

$$\mathcal{I}^* := \mathcal{I}^*_c \setminus B_{\sigma'}(\partial I^*_c \cup M_0) \qquad \text{where} \quad \mathcal{I}^*_c := \{D_{ij,kq/j}\}_{0 \le i < qk/j}.$$

For σ' small enough, $\mathcal{I}$, $\mathcal{I}'$ and $\mathcal{I}^*$ are partitions mod $\delta/2$ of S^1, consisting of q, kq and kq/j elements, respectively. They satisfy $\mathcal{I}' > \mathcal{I}^* > \mathcal{I} > S^1$. Now define

$$\mathcal{P} := \{N_2 \times I\}_{I \in \mathcal{I}};$$

$$\mathcal{P}' := \{N_2 \times I'\}_{I' \in \mathcal{I}'}.$$

These two are partitions mod $\delta/2$ of M with $\mathcal{P}' > \mathcal{P} > M_2$, consisting of q and of qk elements, respectively. Finally let

$$\mathcal{W} := \{Q \times I^*\}_{Q \in \mathcal{Q}, I^* \in \mathcal{I}^*};$$

$$\mathcal{W}' := \{Q \times I'\}_{Q \in \mathcal{Q}, I' \in \mathcal{I}'}.$$

These two are partitions of M_2 mod δ, consisting of qk and jqk elements, respectively. They satisfy $\mathcal{W}' > \mathcal{W} > M_2$, $\mathcal{W} > [P]$, $\mathcal{W}' > [P']$.

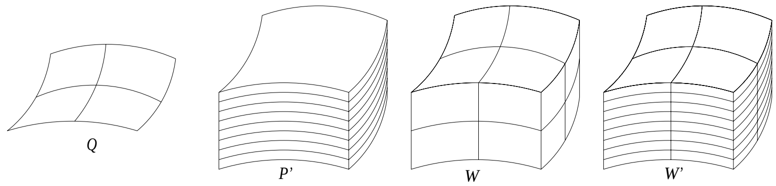

FIGURE 4.2. Schematic illustration of the partition $\mathcal{Q}$, the partitions $\mathcal{P}'$ and $\mathcal{W}$, and the common refinement $\mathcal{W}'$.

4.6. **Permuting building blocks of a partition.** We start by defining, for arbitrary $W, W' \in \mathcal{W}$, a map F which exchanges W with W' and is the identity on any other hypercube in $\mathcal{W}$.

Choose an open neighborhood V of $W \cup W'$, diffeomorphic to a ball, not intersecting any hypercubes in $\mathcal{W}$ other than W and W'. We can make $\mu(V \setminus (W \cup W'))$ arbitrarily small by a suitable choice of V.

Let D_r be the open disc of radius r around the origin in $\mathbb{R}^2$. For numbers $r_1 < r_2$ (whose difference is going to be very small) we let $b_i := \sqrt{\frac{\pi}{8}} r_i$ and define the cylinder $C_i := D_{r_i} \times (-b_i, b_i)^{m-2} \subset \mathbb{R}^m$ for $i = 1, 2$.

(The geometric motivation is the following: A slice of C_i in the (x_1, x_2)-plane has area πr_i^2, and the length of C_i in any x_k-direction equals $2b_i$ for all $k > 2$. If $\pi r_i^2 = 2 \cdot (2b_i)^2$ then we can fit two hypercubes $[-b_i, b_i]^m \subset \mathbb{R}^m$ disjointly into C_i by a map that preserves all x_k-coordinates for $k > 2$.)

Define $\Omega : C_1 \to C_1$ by

$$\Omega := (-\mathrm{id}_{D_{r_1}}) \times \mathrm{id}_{\mathbb{R}^{m-2}}.$$

If $(2b_1)^m > \mu(W) = \mu(W')$ then we can find a volume-preserving diffeomorphism

$$E : V \to \mathbb{R}^m,\ E(W \cup W') \subset C_1 \subset C_2 \subset E(V)$$

satisfying

$$\Omega \circ E|_W = E \circ f_{WW'}.$$

This can be done by first defining E on W such that $E|_W(W) \subset \{(x_1, \ldots, x_m) \in C_1 : x_1 < 0\}$, then extending it to W' by defining $E|_{W'} := \Omega \circ E|_W \circ f_{WW'}^{-1}$ and finally extending it to V smoothly.

By construction, the diagram

$$\begin{array}{ccc} W & \xrightarrow{f_{WW'}} & W' \\ E \downarrow & & \downarrow E \\ \mathbb{R}^m & \xrightarrow{\Omega} & \mathbb{R}^m \end{array}$$

commutes, where the map corresponding to the left vertical arrow is $f_{WW'}$ and the map corresponding to the right arrow is Ω.

The map E is so far only defined on V. We extend it to a C^∞-diffeomorphism $N_2 \to N_2$ as follows:

Choose C^∞-functions $f : [0, \infty) \to [0, 1]$, $f|_{[0,r_1]} \equiv 1$, $f|_{[r_2,\infty)} \equiv 0$ and $g : \mathbb{R} \to [0, 1]$, $g|_{[-b_1,b_1]} \equiv 1$, $g|_{\mathbb{R}\setminus(-b_2,b_2)} \equiv 0$.

Define a map $\rho : \mathbb{R}^m \to \mathbb{R}^m$ by

$$\rho(r, \theta, x_3, \ldots, x_m) := (r, \theta + \pi \cdot f(r) \cdot \prod_{i=3}^{m} g(x_i), x_3, \ldots, x_m),$$

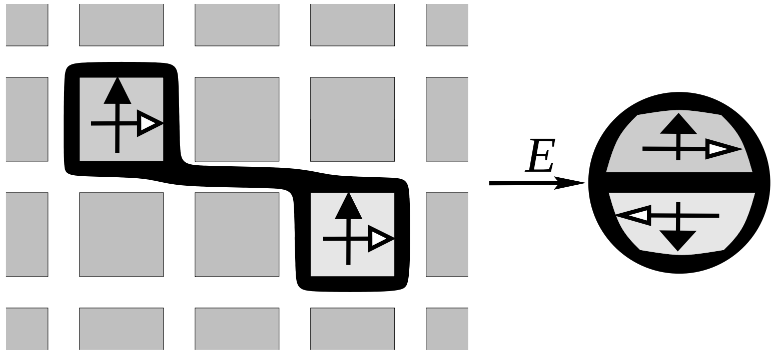

FIGURE 4.3. W and W' can be mapped almost isometrically into a cylinder. Rotation of this cylinder exchanges them and leaves all other pieces unchanged.

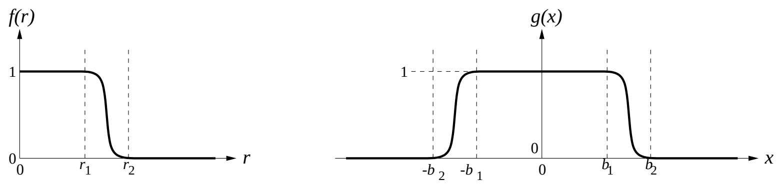

FIGURE 4.4. The cutoff functions f and g ensure that just the inner cylinder gets rotated.

where (r, θ) are the polar coordinates of points in $\mathbb{R}^2$. This is a C^∞-diffeomorphism which rotates C_1 by π and which is the identity outside C_2. The rotation only involves the first two coordinates and leaves the others constant.

Extend the map E in any bijective manner to all of M. Define the map G by $G := E^{-1} \circ \rho \circ E$. Then G is a C^∞-diffeomorphism exchanging W with W' which is the identity outside C_2 and thus on any other element of $\mathcal{W}$.

Remark 4.14. In dimension two, the map G can be realized as the time-1 map of a Hamiltonian flow: Let Ψ_H^t be the time-t map of the Hamiltonian flow

$$\frac{dz}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \operatorname{grad} H|_{z(t)}$$

on $\mathbb{R}^2$ obtained from the Hamiltonian function $H(r, \theta) := 2\pi \int_0^r f(R) \cdot R\, dR$ which is equal to πr^2 inside B_{r_1} and constant outside B_{r_2}, thus it generates rotation with uniform speed on B_{r_1} and no motion outside B_{r_2}. It a simple verification that if $\tilde{V}$ is any subset of N_2 and F is any area-preserving map $U \to \mathbb{R}^2$ with $F(\tilde{V}) \supset B_{r_2}$ then the flow $\Psi^t := E^{-1} \circ (\Psi_H^t) \circ E$ on N_2 is a Hamiltonian flow with Hamiltonian $H' := H \circ F$.

Analogously to proposition 4.12 we get the following statement:

Theorem 4.15. *(Cutting M_2 into hypercubes.) For all $\delta > 0$, $\varepsilon > 0$, $\gamma > 0$ there exists a finite partition $\mathcal{W}$ of M_2 up to a set of measure less than δ, a number $\sigma > 0$ and a family $(f_{WW'})_{W,W' \in \mathcal{W}}$ of C^∞-diffeomorphisms $f_{WW'}\colon W \to W'$ so that for all $W, W' \in \mathcal{W}$:*

- *W is a topological ball in M_δ.*
- *diam$(W) < \gamma$.*

- *dist*$(W, W') > \sigma$.
- $\mu(W) = \mu(W')$.
- $f_{WW'}$ *is measure-preserving.*
- $f_{WW'}$ *is the identity on any* $W'' \in \mathcal{W} \setminus \{W, W'\}$.
- $dev_W(f_{WW'}) < \varepsilon$.
- $f_{W'W} = f_{WW'}^{-1}$.

Proof. Since $(M_\delta, g) = (N_\delta, g') \times (S^1, l)$, we have

$$\exp_{(x,s)}(A \times I) = \exp_x(A) \times \exp_s(I)$$

for all $x \in N_\delta$, $s \in S^1$, $A \subset \mathbb{R}^{m-1}$, $I \subset \mathbb{R}$. (Here exp is taken in M_δ, N_δ and S^1, respectively.) Moreover, for any W we have $W = \exp_{x_{k(W)}} \bar{W}$ for some $\bar{W} \subset \mathbb{R}^m$; for all sets $B \subset \mathbb{R}^k$, $B' \subset \mathbb{R}^{m-k}$ with $B \times B' \subset \bar{W}$ (and thus $\exp_{x_{k(W)}}(B \times B') \subset W$) we have $\exp_{x_{k(W)}}(B \times B') = \exp_{x_{k(W)}}(B) \times \exp_{x_{k(W)}}(B')$. Thus the statement follows immediately from proposition 4.12 by choosing a partition $\mathcal{Q}$ of $N_{\delta/4}$ with elements of diameter less than $\gamma/2$ and omitting a set of measure less than $\delta/4$, and taking the product with a partition of the circle omitting a set of measure less than $\delta/2$ whose elements have length less than $\gamma/2$ and are positively separated. □

4.7. **Changing shape of partitions.** Define the arrangement function

$$a_n : \{0, \dots, k_n - 1\} \to \{0, \dots, q_n\},$$

$$a_n(i) := \begin{cases} \left\lceil \frac{iq_n}{2k_n} - \frac{1}{2} \right\rceil & \text{for } i \text{ odd,} \\ 0 & \text{for } i \text{ even.} \end{cases}$$

In other words, $a_n(i)$ is the integer closest to $\frac{iq_n}{2k_n}$.

Define

$$R^{(n)} = \bigcup_{i=0}^{k_n-1} D_{k_n a_n(i)+i, k_n q_n} = \bigcup_{i=0}^{k_n-1} S_{a_n(i)/q_n} D_{i, k_n q_n}.$$

The meaning of this is that we cut the slice D_{0,q_n} into k_n slicelets $D_{i,k_n q_n} \in \mathcal{P}'$, $i \in \{0, \dots, k_n - 1\}$ and move the i-th slicelet $D_{i,k_n q_n}$ from D_{0,q_n} (in which it is contained) into D_{i,q_n} via rotation by $a_n(i)/q_n$. This makes

$$X_n := \left\{ S_{i/q_n} R^{(n)} \right\}_{i=0}^{q_n - 1}$$

a partition of M_2.

Theorem 4.16. *For all* $\delta > 0$ *and* $q, k \in \mathbb{N}$ *we can find a* C^∞*-diffeomorphism* $h : M \to M$ *which maps* $\mathcal{W}$ *to* $\mathcal{P}'$ *mod* δ *and which satisfies dev*$(h) < \varepsilon$ *mod* δ.

Proof. $\mathcal{P}'$ and $\mathcal{W}$ have a common subpartition $\mathcal{W}'$. By theorem 4.15 we see that we can arbitrarily permute $\mathcal{W}'$ (mod δ) via a smooth map with the desired properties. Thus we can rearrange $\mathcal{P}'$ into $\mathcal{W}$ (mod δ).

Since all partitions have positive distance σ to M_0, we can extend h to all of M by defining it to be the identity on the $\sigma/2$-neighborhood of M_0 and choosing a smooth continuation in between. □

Now we are able to arbitrarily rearrange the partitions. In particular, choosing $\mathcal{P}'$ and $\mathcal{W}$ so that $\mathcal{P}' > Y_n$, $\mathcal{W} > X_n$, we have proved:

Corollary 4.17. *For all $\delta_n > 0$, $\varepsilon_n > 0$ we can map the partition X_n to the partition Y_n mod δ_n with a map h_n satisfying $dev_{\bigcup W \in \mathcal{W}'}(h_n) < \varepsilon_n$ (i.e. an almost isometry on a set whose complement has measure at most δ_n). Moreover, h_n can be chosen to satisfy*

$$h_n S_{\alpha_{n-1}} = S_{\alpha_{n-1}} h_n.$$

Proof. All that is left to show is the commutation relation: Once we have specified h_n on $D_{0,q_n} \cap \mathcal{W}'$, we can simply extend it by $h_n|_{D_{i,q_n}} = S_{i/q_n} \circ h_n|_{D_{0,q_n}} \circ S_{-i/q_n}$. □

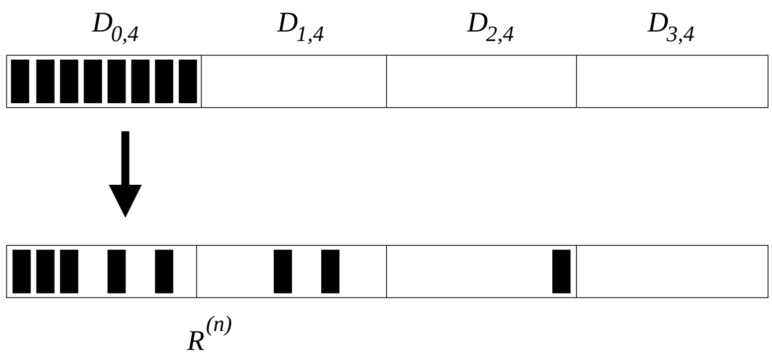

FIGURE 4.5. The map h_n maps the slice D_{0,q_n} to $R^{(n)}$ and thus the partition $\mathcal{P}'$ to the partition $\mathcal{W}$. Here $q_n = 4$, $k_n = 8$ (for illustration purposes; actual values will be much larger).

Remark 4.18. Recall that $\mathcal{W}$ refines X_n. By taking a suitable subpartition, we can guarantee that h_n not only maps X_n to Y_n, but also that the pieces of $\mathcal{W}$ (which obviously are arbitrarily small) remain arbitrarily small after applying h_n. This can be done by simply choosing $\mathcal{W}'$ (the common refinement of $\mathcal{W}$ and $\mathcal{P}'$) to be fine enough. Note also that since $\mathcal{W}$ is only a partition mod δ_n, we can talk about smallness of images of these pieces without having to cut off parts of small measure again.

The map h_n will be chosen to map X_n to Y_n as described. This completes the construction of the diffeomorphism. We proceed to show that it preserves a measurable Riemannian metric and is weakly mixing.

4.8. **The invariant Riemannian metric.** Recall that we have defined $f := \lim_{n\to\infty} f_n$ where $f_n = H_n^{-1} S_{\alpha_n} H_n$, $H_n = h_n \circ h_{n-1} \circ \cdots \circ h_1$. We let g_0 be the product metric on M_2 (with respect to which any M_δ is isometric to $N_\delta \times S^1$) and define

$$g_n := H_n^* g_0 = h_1^* \circ \cdot \circ h_n^* g_0.$$

(Here we use the notation $f^*\omega$ for the pullback of the multilinear map ω via the map f, i.e., for $p \in M, v \in T_pM$ we have $(f^*\omega)|_p(v) = \omega|_{f(p)}(df \cdot v)$. In our notation, $\omega|_p(v)$ means the the form ω at the point p evaluated on the vector v.)

Each g_n is the pullback of a smooth metric on M_2 via a C^∞-diffeomorphism and thus a smooth metric. Moreover, g_n is f_n-invariant since

$$S_{\alpha_n}^* g_0 = g_0$$

and thus

$$f_n^* g_n = H_n^* S_{\alpha_n}^* (H_n^{-1})^* \cdot g_n = H_n^* S_{\alpha_n}^* (H_n^{-1})^* \cdot H_n^* \cdot g_0 = g_n.$$

Now we claim that

$$g_\infty := \lim_{n\to\infty} g_n$$

exists μ-a.e. and is a f-invariant Riemannian metric:

Lemma 4.19. *The sequence $(g_n)_{n\in\mathbb{N}}$ converges μ-a.e. to a limit g_∞.*

Proof. Our construction allows us to choose, for any value of $\delta > 0$ and $\varepsilon > 0$, the map h_n so that $h_n^* g_0$ is ε_n-close to g_0 up to a set of measure δ_n. Thus on a set of measure at least $1 - \sum_{n\geq N} \delta_n$ we have, for all $n > N$,

$$d(g_{n+1}, g_n) = d(H_n^* h_{n+1}^* g_0, H_n^* g_0) \leq ||H_n^*|| \cdot d(h_{n+1}^* g_0, g_0) < \varepsilon_{n+1}$$

□

if h_{n+1} is chosen so that $d(h_{n+1}^* g_0, g_0)$ is small enough. Thus $(g_n|_p)_{n>N}$ is a Cauchy sequence for a set whose measure approaches 1 as $N \to \infty$, therefore it converges to a limit g_∞ on a set of full measure.

Lemma 4.20. *The limit g_∞ is a measurable Riemannian metric.*

Proof. Since g_∞ is the limit of positive definite quadratic forms, it is obviously a nonnegative definite quadratic form. On $T_1 M \otimes T_1 M$ minus a set of measure at most $\sum_{k\geq n} \delta_k$, the form g_∞ is $\sum_{k>n} \varepsilon_k$-close to g_n, which is positive definite. By choosing ε_k, $k > n$, small enough (this depends on g_n), we can guarantee that g_∞ is positive definite up to a set of measure at most $\sum_{k\geq n} \delta_k$. Since this is true for all n, it follows that g_∞ is positive definite on a set of full measure. □

Lemma 4.21. *The metric g_∞ is f-invariant, i.e. $f^* g_\infty = g_\infty$ almost everywhere.*

Proof. We know that the sequence $(g_n)_{n\in\mathbb{N}}$ converges in the C^0-topology pointwise almost everywhere. By Egoroff's theorem, for any $\delta > 0$ we can find a set $C_\delta \subset M$ such that the convergence is uniform on C_δ and $\mu(M \setminus C_\delta) < \delta$.

We know that $\tilde{f}_n := f_n^{-1} \circ f$ converges in C^∞ to the identity as $n \to \infty$, thus uniformly by compactness. Smoothness of f implies $f^* g_\infty = f^* \lim_{n\to\infty} g_n = \lim_{n\to\infty} f^* g_n$. By uniform convergence, $f^* g_\infty = \lim_{n\to\infty} \tilde{f}_n^* f_n^* g_n = \lim_{n\to\infty} \tilde{f}_n^* g_n = g_\infty$ on C_δ. This is true on all sets C_δ with $\delta > 0$, thus on $\bigcup_{\delta>0} C_\delta$, which is a set of full measure, the equation $f^* g_\infty = g_\infty$ holds. □

4.9. **Proof of the weak mixing property.** Almost all arguments in this section are contained in various parts of [AK]; for the convenience of the reader, we present them here with explicit calculations and in one contiguous piece.

Recall that a sequence of finite partitions ξ_n of a metric space is called *exhaustive* if for any measurable set A there exists a sequence of sets A_n composed of elements of ξ_n such that $\mu(A \triangle A_n) \to 0$ as $n \to 0$.

Lemma 4.22. *The map $T : M \to M$ is weakly mixing if and only if there exists an exhaustive sequence of partitions ξ_n of M and a sequence of integers $(m_n)_{n\in N}$ with the property that $m_n \to \infty$ and*

$$\sum_{c,c'\in\xi_n} |\mu(c \cap T^{m_n} c') - \mu(c)\mu(c')| \to 0 \ \text{ as } \ n \to \infty.$$

The proof of this lemma can be found in [AK], pp. 20-21.

Remark 4.23. Since we are able to choose the pieces of the partition $\mathcal{W}$ in the $n+1$-th step to be so small that they remain sufficiently small when applying the conjugating map H_n (see remark 4.18), we can ensure that the diameter of elements of $X_{n,n+1}$ converges to zero as $n \to \infty$.

Measuring overlap after rotation

Define the overlap number $Q_{n,k}$ to be the number of strips in $\mathcal{P}'$ that are in the intersection of $S_{1/q_n k_n} R^{(n)}$ with $S_{k/q_n} R^{(n)}$ (the k-th element of the partition $Y_{n,n+1}$). We can formalize this as follows:

Definition 4.24.

$$Q_{n,k} = k_n q_n \cdot \mu \left(S_{1/q_n k_n} R^{(n)} \cap S_{k/q_n} R^{(n)} \right). \tag{4.1}$$

Thus the intersection of the k-th copy of $R^{(n)}$ with an appropriate iterate of $R^{(n)}$ under a map conjugate to f_n consists of $Q_{n,k}$ elements in $\mathcal{P}'$.

Let $l_n := k_n/q_n$. Since we choose k_n after q_n and our construction only mandates lower bounds for k_n, we can assume that $2q_n|k_n$.

Lemma 4.25. *(Intersection counting.) The preceding construction guarantees that*

$$\begin{aligned} Q_{n,k} &= l_n && \text{for } k \neq 0, q_n/2, \\ Q_{n,k} &\in [0, 2l_n] && \text{for } k = 0, q_n/2. \end{aligned}$$

Proof. Recall that we have labeled the substrips of the strip D_{0,q_n} in the canonical way, i.e., $D_{i,k_n q_n} = S_{1/k_n q_n} D_{0,k_n q_n}$. The substrips of $D_{0,k_n q_n}$ (namely $D_{i,k_n q_n}$, $0 \leq i < k_n$) are mapped to the substrips of $R^{(n)}$ via the map h_{n+1}. Among these substrips, all even-numbered sectors are left in D_{0,q_n}, where they already were before application of the map h_{n+1}. Since we rotate via $S_{1/k_n q_n}$ by exactly one substrip and then map it to D_{0,q_n} via S_{k/q_n}, the measure of the intersection in formula 4.1 is exactly the number of substrips mapped to D_{k,q_n} plus the number of substrips mapped to D_{-k,q_n}. For $0 < k < q_n/2$, the first number equals l_n (because that is the total number of such substrips in D_{k,q_n}) and the second number is zero (because there are no such substrips in D_{-k,q_n}. Similarly, for $q_n/2 < k < q_n$, the first number is zero and the second number equals l_n. For the two remaining values of k, there remain $2l_n$ strips to be distributed, showing the second condition. □

Approximating partitions

For a partition Y of M and a set $S \subset M$ we write $S|Y$ if S is union of elements in Y (in other words, if $\{S, M \setminus S\} < Y$).

Recall that $Y_n < Y_{n+1}$ for all n. Define the map $C_n : \{S : S|Y_n\} \to \{S : S|Y_{n+1}\}$ by $C_n(S) := \bigcup\{c \in Y_{n+1} : c \cap S \neq \emptyset\}$ for $S|Y_n$. Observe that the condition $c \cap S \neq \emptyset$ is equivalent to $c \subset S$ for $S|Y_n$, $c \in Y_{n+1}$.

We define $\tilde{P}^n_{n+1} : \{S : S|Y_n\} \to \{S : S|Y_{n,n+1}\}$ by $\tilde{P}^n_{n+1}(D_{i,q_n}) := C_n(D_{i,q_n})$. This also serves to define $Y_{n,n+1} := \{\tilde{P}^n_{n+1} D_{i,q_n}\}_{i=0}^{q_n-1}$.

Obviously $Y_{n,n+1}$ is increasing in n, since the map C_m for any m maps a partition to another partition with the same number of elements, and increasing n means starting with a finer partition.

Define $X_{n,n+1} := H^{-1}_{n+1} Y_{n,n+1}$.

Let $P^n_{n+1} := H^{-1}_{n+1} \tilde{P}^n_{n+1} H_{n.}$

Fact 4.26. *The partition X_n is e'_n-approximated by the partition $X_{n,n+1}$, that is,*

$$\sum_{c \in X_n} c \triangle P^n_{n+1} c \leq \varepsilon'_n.$$

Proof. Calculation. □

Lemma 4.27. *The maps f_i and f_{i+1} are $\frac{2}{l_i q_i^2}$-close on X_i in the sense that*

$$\sum_{d \in X_i} \mu(f_i d \triangle f_{i+1} d) < \frac{2}{l_i q_i^2}.$$

Proof.

$$\begin{aligned}
\mu\left(f_i H_i^{-1} D_{l,q_i} \triangle f_{i+1} H_i^{-1} D_{l,q_i}\right) &= \mu(H_i f_i H_i^{-1} D_{l,q_i} \triangle H_i f_{i+1} H_i^{-1} D_{l,q_i}) \\
&= \mu(S_{\alpha_i} D_{l,q_i} \triangle h_{i+1} S_{\beta_{i+1}} h_{i+1}^{-1} D_{l,q_i}) \\
&= \mu(S_{\alpha_i} D_{l,q_i} \triangle S_{a_i} h_{i+1} S_{\beta_{i+1}} h_{i+1}^{-1} D_{l,q_i}) \\
&= \mu(D_{l,q_i} \triangle h_{i+1} S_{\beta_{i+1}} h_{i+1}^{-1} D_{l,q_i}) \\
&= \mu(h_{i+1}^{-1} D_{l,q_i} \triangle S_{\beta_{i+1}} h_{i+1}^{-1} D_{l,q_i})
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{d \in X_i} \mu(f_i d \triangle f_{i+1} d) &\leq \sum_{l=0}^{q_n-1} \mu\left(f_i H_i^{-1} D_{l,q_i} \triangle f_{i+1} H_i^{-1} D_{l,q_i}\right) \\
&= \sum_{l=0}^{q_n-1} \mu\left(h_{i+1}^{-1} D_{l,q_i} \triangle S_{\beta_{i+1}} h_{i+1}^{-1} D_{l,q_i}\right) \\
&\leq \sum_{l=0}^{q_n-1} \mu\left(S_{l/q_i} R^{(i)} \triangle S_{\beta_{i+1}} S_{l/q_i} R^{(i)}\right) + 2e_i \\
&\leq \sum_{l=0}^{q_n-1} \mu\left(R^{(i)} \triangle S_{\beta_{i+1}} R^{(i)}\right) + 2e_i \\
&\leq 2\beta_{i+1} k_i,
\end{aligned}$$

since $R^{(i)}$ consists of k_i strips. By definition, $\beta_{i+1} < \beta_i = \frac{1}{k_i l_i q_i^2}$. Thus $2\beta_{i+1} k_i < \frac{2}{l_i q_i^2}$, proving the claim. □

Proposition 4.28. *With respect to the partition $X_{n,n+1}$, the maps f and f_n are close in the sense that*

$$\sum_{c \in X_{n,n+1}} \mu(fc \triangle f_n c) \leq 2\varepsilon'_{n-1} + 2 \sum_{i \geq n-1} \varepsilon'_i + \sum_{i \geq n-1} \frac{1}{l_i q_i^2}.$$

Proof. For $c \in X_{i,i+1}$ we set $d := (P_i^{i-1})^{-1} c$. Then $d \in X_i$. Using this, we find

$$\begin{aligned}
&\sum_{c \in X_{n,n+1}} \mu(fc \triangle f_n c) \\
&\leq \sum_{i \geq n-1} \sum_{c \in X_{n-1,n}} \mu(f_i c \triangle f_{i-1} c) \\
&\leq \sum_{i \geq n-1} \sum_{c \in X_{i-1,i}} \mu(f_i c \triangle f_{i-1} c) \qquad \text{(since } i > n \text{ implies } X_{i,i+1} > X_{n,n+1}\text{)} \\
&\leq \sum_{i \geq n-1} \sum_{c \in X_{i-1,i}} \mu(f_i P_i^{i-1} d \triangle f_{i-1} P_i^{i-1} d) \\
&\leq \sum_{i \geq n-1} \sum_{c \in X_{i-1,i}} \left(\mu(f_i P_i^{i-1} d \triangle f_i d) + \mu(f_i d \triangle f_{i-1} d) + \mu(f_{i-1} d \triangle f_{i-1} P_i^{i-1} d)\right) \\
&\leq 2\varepsilon'_{n-1} + 2 \sum_{i \geq n-1} \varepsilon'_i + \sum_{i \geq n-1} \frac{1}{l_i q_i^2}
\end{aligned}$$

where the last inequality in this calculation follows from lemma 4.27. □

Lemma 4.29. *(Iteration lemma.)*

$$\sum_{c\in X_{n,n+1}} \mu(f^m c \triangle f_i^m c) \le m \cdot \sum_{c\in X_{n+1}} \mu(fc\triangle f_i c).$$

Proof. Since $X_{n,n+1} < X_{n+1}$, the left hand side is $\le \sum_{c\in X_{n+1}} \mu(f^m c\triangle f_k^m c)$. Moreover,

$$\begin{aligned}
\mu(\{x : f^m x \ne f_k^m x\}) &\le \sum_{i=1}^{m} \mu(\{x : f^i x \ne f_k^i x,\ f^{i-1}x = f_k^{i-1}x\}) \\
&\le \sum_{i=1}^{m} \mu(\{x : f^i x \ne f_k f^{i-1} x\}) \\
&= m\cdot \mu(\{x : fx \ne f_k x\})
\end{aligned}$$

since f preserves μ. This completes the proof. □

Conclusion 4.30. Suitable iterates of the maps f and f_n are also close with respect to the partition $X_{n,n+1}$ in the sense that the term

$$\sum_{c,c'\in X_{n,n+1}} |\mu(f^{m_n} c\cap c') - \mu(cf_{n+1}^{m_n} \cap c')|$$

converges to 0 as $n\to\infty$.

Proof.

$$\begin{aligned}
&\sum_{c,c'\in X_{n,n+1}} |\mu(f^{m_n} c\cap c') - \mu(cf_{n+1}^{m_n} \cap c')| \\
&\le \sum_{c,c'\in X_{n,n+1}} \mu((f^{m_n} c\triangle f_{n+1}^{m_n} c)\cap c') \\
&\le \sum_{c\in X_{n+1}} \mu(f^{m_n} c\triangle f_{n+1}^{m_n} c) \qquad \text{(since } X_{n+1} > X_{n,n+1}) \\
&\le m_n \cdot \sum_{c\in X_{n+1}} \mu(fc\triangle f_{n+1} c) \qquad \text{(by the iteration lemma)} \\
&\le m_n \cdot \left(\sum_{c\in X_{n+1}} \mu(fc\triangle fP_\infty^{n+1} c) + \sum_{c\in X_{n+1}} \mu(fP_\infty^{n+1} c\triangle f_{n+1} P_\infty c) \right. \\
&\qquad \left. + \sum_{c\in X_{n+1}} \mu(f_{n+1} P_\infty c\triangle f_{n+1} c) \right) \\
&\le m_n \cdot \left(2\sum_{c\in X_{n+1}} \mu(c\triangle P_\infty^{n+1} c) + \sum_{c\in X_{n+1}} \mu(fP_\infty^{n+1} c\triangle f_{n+1} P_\infty^{n+1} c) \right) \\
&\le m_n \cdot \left(\varepsilon'_{n+1} + \sum_{c\in Z_{n+1}} \mu(fc\triangle f_{n+1} c) \right) \\
&\le m_n \cdot \left(\varepsilon'_{n+1} + 4\sum_{i\ge n} \varepsilon'_i + 2\sum_{i\ge n} \frac{1}{l_i q_i^2} \right)
\end{aligned}$$

by proposition 4.28.

Now we use the fact that $m_n \le q_{n+1}$. For a suitable choice of parameters, the last term in the above equations converges to zero. □

Algorithm 4.31. *Choose $m_n \in \{0, \dots, q_{n+1} - 1\}$ so that $m_n p_{n+1} \equiv l_n q_n$ (in $\mathbb{Z}/q_{n+1}\mathbb{Z}$).*

Proposition 4.32. *For any sequence $(\varepsilon'_n)_{n\in\mathbb{N}}$ we can choose the parameters in the construction so that $\sum_{c,c'\in X_{n,n+1}} |\mu(f_{n+1}^{m_n} c \cap c') - \mu(c)\mu(c')|$ converges to zero as $n \to \infty$.*

Proof. By algorithm 4.31, we have

$$m_n \alpha_{n+1} = m_n p_{n+1}/q_{n+1} = l_n q_n/q_{n+1} = \frac{l_n q_n}{k_n l_n q_n^2} = \frac{1}{k_n q_n}.$$

This yields

$$f_{n+1}^{m_n} = H_{n+1}(S_{\alpha_{n+1}})^{m_n} H_{n+1} = H_{n+1} S_{m_n \alpha_{n+1}} H_{n+1} = H_{n+1} S_{\frac{1}{k_n q_n}} H_{n+1}.$$

Therefore

$$\begin{aligned}
\mu(f_{n+1}^{m_n} c \cap c') &= \mu(H_n f_{n+1}^{m_n} H_n^{-1} c \cap c') \\
&= \mu\left(S_{1/k_n q_n} S_{j/q_n} R^{(n)} \cap S_{k+j/q_n} R^{(n)}\right) \\
&= \mu\left(S_{1/k_n q_n} R^{(n)} \cap S_{k/q_n} R^{(n)}\right) \\
&= \frac{Q_{n,k}}{k_n q_n}.
\end{aligned}$$

If $c, c' \in X_{n,n+1}$ then we can write $c = S_{j/q_n} R^{(n)}$, $c' = S_{k+j/q_n} R^{(n)}$ for suitable integers $j, k \in \{0, \dots, q_n\}$. Here we call the second variable $k + j$ instead of k to make the next calculation easier. Namely, we get the estimate

$$\begin{aligned}
\sum_{c,c'\in X_{n,n+1}} |\mu(f_{n+1}^{m_n} c \cap c') - \mu(c)\mu(c')| &= \sum_{j=0}^{q_n-1} \sum_{k=0}^{q_n-1} \left| \frac{Q_{n,k}}{k_n q_n} - q_n^{-2} \right| \\
&= q_n \sum_{k=0}^{q_n-1} \left| \frac{Q_{n,k}}{k_n q_n} - q_n^{-2} \right|.
\end{aligned}$$

By the intersection counting lemma 4.25, this term equals

$$q_n \cdot \left(\left| \frac{Q_{n,0}}{k_n q_n} - q_n^{-2} \right| + \left| \frac{Q_{n,q_n/2}}{k_n q_n} - q_n^{-2} \right| \right) \le 2 q_n \left| \frac{k_n/2q_n}{k_n q_n} - q_n^{-2} \right| = \frac{1}{q_n},$$

which clearly converges to zero as $n \to \infty$. □

Theorem 4.33. *For any sequence $(\varepsilon'_n)_{n\in\mathbb{N}}$ we can choose the parameters in the construction so that $\sum_{c,c'\in X_{n,n+1}} |\mu(f^{m_n} c \cap c') - \mu(c)\mu(c')|$ converges to zero as $n \to \infty$.*

Proof.

$$\begin{aligned}\sum_{c,c'\in X_{n,n+1}} &|\mu(f_n^{m_n}c\cap c')-\mu(c)\mu(c')| \\ \le &\sum_{c,c'\in X_{n,n+1}} |\mu(f_{n+1}^{m_n}c\cap c')-\mu(c)\mu(c')| \\ &+\sum_{c,c'\in X_{n,n+1}} |\mu(f^{m_n}c\cap c')-\mu(cf_{n+1}^{m_n}\cap c')|.\end{aligned}$$

By proposition 4.32, the first summand converges to zero as $n\to\infty$. By conclusion 4.30, so does the second. □

Conclusion 4.34. f is weakly mixing.

REFERENCES

[AK] D.V. Anosov and A. Katok, *New examples in smooth ergodic theory. Ergodic diffeomorphisms,* Trans. Moscow Math. Soc., vol. 23 (1970).

[B] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer Lecture Notes in Mathematics, vol. 470 (1975)

[FK] R. Feres and A. Katok, *Ergodic theory and dynamics of G-spaces (with a special emphasis on rigidity phenomena),* Preprint (1999), to appear in "Handbook of dynamical systems", vol. 1, Elsevier (2000).

[Gr] M. Gromov, *Rigid transformation groups* in *Géometrié différentielle*, ed. D. Bernard and Y. Choquet-Bruhat, Travaux en cours, vol. 33, 65–139, Hermann, 1988.

[HK] B. Hasselblatt and A. Katok, *Dynamical systems. Principal structures.* Preprint, 1999, to appear in "Handbook in Dynamical Systems", vol. 1, Elsevier (2000).

[KaK] B. Kalinin and A. Katok, *Rigidity of invariant measures, factors and joinings for actions of higher-rank abelian groups*, Preprint (1999).

[KKS] A. Katok, S. Katok, and K. Schmidt, *Rigidity of measurable structure for algebraic actions of higher-rank abelian groups,* Preprint (1999).

[KS] A. Katok and R. Spatzier, *Invariant measures for higher rank hyperbolic Abelian actions.* Erg. Theory and Dynam. Systems, vol. 16 (1996), 751-778.

[L] F. Ledrappier, *Propriétés ergodiques des mesures de Sinaï*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques vol. 59 (1984) 163–188.

[P] Ya. B. Pesin, *Characteristic Lyapunov exponents and smooth ergodic theory*, Russian Math Surveys, vol. 32 (1977), 55–114.

[R1] M. Ratner, *Rigidity of horocycle flows*, Ann. of Math. vol. 115 (1982), 597-614.

[R2] M. Ratner, *Rigidity of time changes for horocycle flows*, Acta Math. vol. 156 (1986), 1-32.

[Y] J.-C. Yoccoz, *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition Diophantienne*, Ann. Sci. École Norm. Sup. vol. 17 (1984) 333–361.

[Z1] R. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, 1984.

[Z2] R. Zimmer, *Lattices in semisimple groups and invariant geometric structures on compact manifolds*, Discrete groups in geometry and analysis (New Haven, Conn., 1984), 152–210, Progr. Math., 67, Birkhäuser Boston, Boston, Mass., 1987.

[Z3] R. Zimmer, *Spectrum, entropy and geometric structures for smooth actions of Kazhdan groups* Isr. Jour. Math., vol. 75 (1991), 65-80.

Received for publication Oct. 1999.

E-mail address: gunesch@math.psu.edu, katok_a@math.psu.edu