DIFFERENTIABILITY, RIGIDITY
AND GODBILLON-VEY CLASSES FOR ANOSOV FLOWS
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CONTENTS

1. Introduction

The central geometric objects associated with an Anosov dynamical system on a compact manifold are the invariant stable and unstable foliations. While each stable and unstable manifold is as smooth as the system itself, the foliations that they form are believed to have only a moderate degree of regularity for most systems. In this paper we will analyze the exact degree of regularity of codimension-one stable and unstable foliations for low dimensional systems. Our main results relate the regularity of these foliations to cohomology classes associated to the system: the *Anosov class*, a new invariant of the flow which we introduce in this paper, and the *Godbillon-Vey class* of the weak-stable foliation, which we show is a well-defined invariant of the system. The Anosov class has a remarkable application: When this cohomology " obstacle to regularity " vanishes,

the foliations must be infinitely smooth. This is the rigidity phenomenon of the title. This implies that the system is algebraic by results of Avez [4] or Ghys [17]. The Godbillon-Vey invariant of the flow has two applications: We show that there are continuous families of topologically conjugate, codimension-one foliations whose Godbillon-Vey invariants vary continuously (and are not constant!). The Godbillon-Vey invariant of the flow characterizes the geodesic flows for metrics of negative curvature as the flows with maximal value for this invariant, among the geodesic flows for metrics of negative curvature on closed surfaces. We use this to give a new proof that the harmonic measure at infinity for metrics of variable negative curvature on surfaces is totally singular.

Both of the cohomology invariants introduced in this paper are parameters on the space of low-dimensional, volume-preserving Anosov systems, for which the 0-set is parametrized by Teichmuller spaces. The authors conjecture that for other values of our cohomology invariants, the system is determined up to smooth equivalence by a finite set of auxillary, Teichmuller-like parameters.

It is useful to first recall the three classical results about regularity of weak-stable and weak-unstable foliations, before describing in more detail the program of this paper. Anosov showed that these foliations are always α-Hölder [1] for some α depending on the rates of expansion and contraction of the system. Hirsch and Pugh proved that the foliations are C^1 for low-dimensional, area-preserving systems [30], and with Shub they studied the relation between regularity and the *global* maximal and minimal rates of expansion and contraction of the system [51]. On the other hand, it was known to Anosov that the foliations need not be C^2. More specifically, he showed that for an area-preserving C^3-Anosov diffeomorphism of the two-dimensional torus, every periodic orbit carries an effectively calculable obstruction to the foliations being C^2 at the point (Chapter 24, [2]). Geometrically, Anosov's obstruction represents the hyperbolic twist of the Poincaré return map of the system at the point, which is equal to the first obstruction to the local Sternberg linearization of the map.

The general program of this paper is to obtain a priori estimates of the degree of regularity for the weak-stable and weak-unstable foliations for a volume-preserving Anosov system of codimension-one. This is based on the authors' study of the local obstacles discovered by Anosov for the two-torus. We settle this problem completely for the cases described. There is great interest in obtaining similar results for higher-codimension systems, but this is a topic of further research.

Our first main result is that the weak-stable and weak-unstable foliations are always C^1, and the modulus of continuity for their first derivatives is in the class $\Omega(t) = O(t.|\log(t)|)$. In addition, we prove that the first transverse derivative of the foliations belongs to the " Zygmund class ", which in particular implies the stated modulus of continuity, but is in fact stronger. This exact degree of regularity is usually associated with the regularity theory of singular integral operators (cf. [44, 59]).

Our second main result is that if *one* of the foliations of a system is C^1 with the transverse modulus of continuity $\omega(t) = o(t.|\log(t)|)$, and the system is C^∞, then both

foliations must be C^∞. This leap in the regularity from degree less than C^2, to C^∞, is one of the more surprising discoveries of this study, and so far remains a phenomenon confined to codimension-one systems.

The definition of the Anosov cohomology invariant for a system is based on a technical observation, that the Anosov obstructions to smoothness at periodic orbits represent the periodic data for a 1-cocycle over the system. The cohomology class of this cocycle is the Anosov invariant of the system. Exceeding the degree of regularity $\Omega(t) = O(t.|\log(t)|)$ for the first transverse derivative of the weak-unstable foliation is equivalent to the vanishing of the Anosov periodic data, which by a celebrated theorem of Livshitz (cf. Theorem 2.1 below, or [45]) implies that the Anosov cocycle is a C^1-coboundary. This fact sets the stage for the remarkable leap in smoothness to C^∞.

The methods of this paper address the regularity of the *weak*-stable and -unstable foliations. In order to treat the regularity of the *strong*-stable and -unstable foliations for flows, we require the existence of a smooth, flow-invariant 1-form dual to the flow. The strong foliations are then as regular as the weak foliations. (A well-known example of Plante [57] shows that without this requirement, the strong foliations need not even be C^1.) For example, such a form always exists for geodesic flows of compact Riemannian surfaces of negative curvature. A natural question is whether, for a given volume-preserving Anosov flow on a 3-dimensional manifold, there exists a smooth time-change such that the new flow preserves a smooth dual 1-form. In a sequel to this paper [41], the second author shows that there is another cohomology invariant associated to the Anosov flow, whose vanishing is equivalent to the existence of such a form. Moreover, the obstruction class for the smooth transverse form is the first of a sequence of cohomology invariants for general Anosov flows. The second in the sequence is the Anosov class of this paper. One interpretation of the " leap to C^∞ " described above is that for a smooth, volume-preserving Anosov flow, the vanishing of the first two Anosov cohomology invariants implies the vanishing of all of the invariants.

There are practical applications of the regularity theory for stable and unstable foliations of Anosov flows. The original motivation for the study leading to this paper concerned properties of the Godbillon-Vey classes of codimension-one foliations (cf. [20]). In particular, when the weak-stable foliation of a geodesic flow for a surface with a metric of strictly negative curvature is C^2, Ghys [16] and Mitsumatsu [53] made several observations about the relation between its Godbillon-Vey invariant and dynamics of the flow. By the results and observations above, this foliation is C^2 only for metrics of constant curvature, in which case the Godbillon-Vey invariant reveals no new information. On the other hand, the first author had shown in [31] that all of the secondary class invariants can be defined for foliations whose differentiability is less than C^2, with the exact degree of regularity required depending upon the codimension and the specific secondary class. For example, the Godbillon-Vey class in codimension-one is defined whenever the degree of regularity is at least C^1 with an α-Hölder condition on the first transverse derivative, for $\alpha > 1/2$. Thus, for both the weak-stable and weak-unstable foliations of a volume-

preserving smooth C^3-Anosov flow on a 3-manifold, there are well-defined real-valued Godbillon-Vey invariants.

The third main result of this paper is that the extension of the Godbillon-Vey invariant to the weak-stable foliation of volume-preserving, C^3-Anosov flows is an invariant of the flow, up to appropriate topological conjugacy of the weak-stable foliations.

The value of the Godbillon-Vey invariant for the geodesic flow of a metric of negative curvature on a closed surface is given by an explicit formula in terms of the curvatures of the horocycles of the flow. The formula calculates the *Mitsumatsu Defect* of the flow, which is the deviation from the value of the Godbillon-Vey invariant for metrics of constant negative curvature. This formula has several consequences, suggested by Mitsumatsu [53]. First, the Godbillon-Vey class *characterizes* the flows associated to metrics of constant curvature among the geodesic flows of metrics of strictly negative curvature on surfaces. We thus obtain a result parallel to the second author's rigidity theorem for the entropies of such flows [39].

The formula of Mitsumatsu shows that a C^4-path of metrics with negative curvature on a closed surface will have continuously varying Godbillon-Vey invariants. In particular, such a path from a metric of constant curvature to a metric of non-constant negative curvature will yield a family of Anosov flows with all of their weak-stable foliations topologically conjugate, but whose Godbillon-Vey invariants vary continuously and non-trivially.

The third application of the Godbillon-Vey invariant for Anosov flows is based on its invariance under absolutely continuous conjugacy. We use this property to prove that when the harmonic measure at infinity is absolutely continuous for a metric of negative curvature on a surface, then the metric has constant curvature (Theorem 8 below). A geometric proof of this result was first given in [40].

The remainder of this paper is organized as follows. In section 2, we present the basic facts about Anosov flows that are needed for the results cited above. A key result of this section, Theorem 2.6, characterizes smooth functions on a manifold by the property that their restrictions to a *regular web of foliations* should be uniformly smooth. This technical result was first proved by R. de La Llavé, J. Marco and R. Moriyon [49] in the case of two complementary foliations, and is the key result for the C^∞-regularity theory. We give an alternative proof, based on elementary properties of the Fourier transform. Our method of proof of Theorem 2.6 has been used by R. de La Llavé to extend the theorem to a characterization of analytic functions [48].

Section 3 states in a precise form the results of this paper, gathered together for the reader's convenience.

In section 4, we prove the regularity theorem for the weak-unstable foliations. Section 5 defines the Anosov cocycle, and gives a formula for its values at periodic orbits, the local obstructions to regularity. Finally, in section 6 we prove that vanishing of the Anosov class implies smoothness of the foliations. We also show that our modulus-of-continuity condition on the transverse derivative is best possible.

Section 7 constructs the Godbillon-Vey class for foliations of regularity class $C^{1+\alpha}$ for $\alpha > 1/2$. Section 8 gives an alternative definition of this invariant for the case where the ambient manifold is a circle bundle. In section 9 we derive the formula for the Godbillon-Vey invariant of geodesic flows for metrics of negative curvature, modelled on Mitsumatsu's results. This yields the characterization of the geodesic flow for a metric of constant negative curvature as the geodesic flow with maximal value for the Godbillon-Vey invariant.

We conclude the paper with a short list of open problems. The first version of this manuscript was circulated in May, 1986, and since that time considerable additional progress has been made in the smooth rigidity theory for Anosov systems (cf. [8, 9, 11, 12, 13, 14, 15, 23, 26, 27, 28, 32, 33, 36, 37, 41, 42, 55]). The results of this paper lay out the theory as the authors envision it will develop for higher dimensional systems, though we expect with far greater difficulties involved. There are still several very interesting questions remaining in the codimension-one case; we give four of them in section 10.

The authors have benefitted from conversations with many mathematicians during the development of this work. We especially thank R. de La Llavé for explaining his papers to us, Y. Mitsumatsu for providing an early version of his seminal paper [53], and B. Hasselblatt for numerous suggestions to improve the clarity of exposition incorporated in this draft. The support of the Mathematical Sciences Research Institute in Berkeley for the first author during the early development of this work is gratefully acknowledged, and we thank the California Institute of Technology for its generous hospitality.

2. Anosov Flows - Preliminaries

Let M be a closed Riemannian manifold. Let $f_t : M \to M$ be a C^∞-flow on M generated by the vector field $\xi = \dfrac{d}{dt}(f_t)|_{t=0}$. The flow is called *Anosov* if there is a continuous splitting $TM = E^+ \oplus E^0 \oplus E^-$ with E^0 spanned by ξ and there are positive constants c_1, c_2 and γ such that

(1)
$$\| Df_t(\eta) \| \geq c_1 . e^{t\gamma} . \| \eta \| \quad \text{for } \eta \in E^+ \text{ and } t \geq 0;$$
$$\| Df_t(\eta) \| \leq c_2 . e^{-t\gamma} . \| \eta \| \quad \text{for } \eta \in E^- \text{ and } t \geq 0.$$

The existence of an Anosov property (1) for a flow does not depend upon a particular choice of Riemannian metric on TM, although the constants c_1, c_2 and γ will, in general. The plane field E^- is called the *strong-stable* or *contracting* distribution for the flow, and E^+ is the *strong-unstable* or *expanding* distribution.

The expanding and contracting distributions are in general only Hölder [1], but

Anosov also proved that they are uniquely integrable. The integral manifolds of E^- form the *strong-stable foliation* denoted by $\mathscr{W}^-$, and the integral manifolds of E^+ form the *strong-unstable foliation* denoted by $\mathscr{W}^+$. We define the *weak-stable* distribution to be the subbundle $E^{ws} = E^0 \oplus E^-$ and the *weak-unstable* distribution to be $E^{wu} = E^0 \oplus E^+$. These are also uniquely integrable, with corresponding foliations $\mathscr{F}^{ws}$ and $\mathscr{F}^{wu}$, respectively. Hirsch and Pugh [29] (see also [51]) proved that the individual leaves of these foliations are smooth submanifolds of M, where the degree of smoothness is that of the flow. Moreover, in the smooth topology on immersions, these submanifolds depend continuously both on the flow, and on the ambient point through which the submanifold passes.

A function $F : M \times \mathbf{R} \to \mathbf{R}$ is called a *1-cocycle over the flow* f_t if it satisfies the cocycle law

(2) $F(p, t + s) = F(p, t) + F(f_t(p), s)$ for all $p \in M$ and $t, s \in \mathbf{R}$.

A cocycle F is said to be differentiable, or C^1 along the flow, if the function $F(f_t(p))$ is a C^1 function of t for all $p \in M$. For such a cocycle we define the *infinitesimal generator*, a continuous function on M given by $\varphi = \xi(F)$. The cocycle F is recovered from φ by the integral formula

$$F(p, t) = \int_0^t \varphi(f_s(p)) \, ds.$$

A cocycle F is called a *1-coboundary* if for some measurable function $\Phi : M \to \mathbf{R}$,

(3) $F(p, t) = \Phi(f_t(p)) - \Phi(p)$.

In particular, if F is smooth along the flow, then $\varphi = \Phi'_\xi \equiv d\Phi(\xi)$. The coboundary function Φ may be required to satisfy an additional regularity condition; e.g., continuity, or C^k-differentiability for some $k \geqslant 1$. Note that if Φ is assumed to be continuous, or even just everywhere defined, then for every periodic point $p \in M$ of period t_0, F must satisfy the relation

(4) $F(p, t_0) = \Phi(f_{t_0}(p)) - \Phi(p) = 0$.

The flow f_t is said to be *topologically transitive* if there exists a point $p_0 \in M$ whose orbit under the flow is dense in M. For cocycles over transitive Anosov flows with F possessing some minimal degree of regularity, the vanishing conditions (4) at periodic orbits turn out to be both necessary for the existence of even a measurable coboundary Φ, and sufficient for the existence of a regular coboundary Φ. The following remarkable results on the existence of Hölder solutions of (3) were obtained by A. Livshitz [45, 46] in the early 1970's. The improvement to C^∞-regularity of the solution was established later, first by V. Guillemin and D. Kazhdan [24] for the geodesic flows of surfaces of negative curvature, and then in complete generality by R. de La Llavé, J. Marco and R. Moriyon [49].

Theorem **2.1** *(Livshitz).* — *Let f_t be a topologically transitive Anosov flow generated by the vector field ξ, and let $\varphi : M \to \mathbf{R}$ be a Hölder function. Then the following are equivalent:*

1. $\varphi = \Phi'_\xi$ *for a Hölder function Φ differentiable along the flow*

2. $\varphi = \Phi'_\xi$ *almost everywhere for a measurable function Φ differentiable along the flow*

3. $\int_0^{t_p} \varphi(f_s(p)) \, ds = 0$ *for every periodic point p with period t_p.*

Moreover, the function Φ is unique up to an additive constant. If φ is a C^1-function, then the coboundary Φ is also C^1.

The C^1 regularity of the solution Φ is easy to establish, given the existence of a continuous solution Φ.

Theorem **2.2** *(Cocycle Regularity).* — *Let φ be a C^∞ function on M which satisfies condition (1) of Theorem 2.1 for $\{f_t\}$ a C^∞ Anosov flow. Then the solution Φ is C^∞. Hence, if f_t is, in addition, topologically transitive and any of the conditions (1), (2) or (3) holds for φ, then a C^∞-solution Φ exists.*

The Cocycle Regularity Theorem is one of the main technical tools of our regularity theory, so we give an essentially self-contained proof of it here, which is considerably briefer than the original proof in [49]. For arbitrary $p \in M$, and $q \in \mathscr{W}^-(p)$, the forward orbits of p and q are attracting so that it is easy to show that

$$\Phi(q) - \Phi(p) = \int_0^\infty \{ \varphi(f_s(p)) - \varphi(f_s(q)) \} ds$$

and for $q \in \mathscr{W}^-(p)$ the backward orbits are attracting so that

$$\Phi(q) - \Phi(p) = \int_0^\infty \{ \varphi(f_{-s}(p)) - \varphi(f_{-s}(q)) \} ds.$$

Naturally, we also have

$$\Phi(f_t(p)) - \Phi(p) = \int_0^t \varphi(f_s(p)) \, ds.$$

From these formulas and the Anosov Property (1), we can conclude that Φ is C^1, as the distributions E^-, E^+ and E^0 span the tangent spaces to M. The C^∞ regularity of Φ is based upon Theorem 2.6 below, and observe that the hypotheses on Φ of that theorem are implied by the above formulas. Theorem 2.6 is postponed until after we discuss an important application of Theorems 2.1 and 2.2.

A *transverse invariant 1-form* for the flow f_t is a continuous 1-form on M which satisfies

(5) $\qquad\qquad \tau(\xi) = 1, \quad \text{and} \quad \tau(\eta) = 0 \quad \text{for } \eta \in E^- \oplus E^+.$

When the flow is topologically transitive, there is a unique continuous 1-form, up to a multiplicative constant, which vanishes on the subbundle $E^- \oplus E^+$, so that in the condition (5) we need only require that $\tau(\xi) = 1$ at one point of M. In general, the form τ is only Hölder continuous (see Theorem 4.1 of [57]). If the form τ happens to be C^1, then it carries considerable additional information.

Theorem **2.3.** — *Let f_t be a C^∞ Anosov flow on a compact 3-manifold M with C^1-transverse 1-form τ. Then*:

1. *the form τ is in fact C^∞;*
2. *the invariant 3-form $\tau \wedge d\tau$ is either identically zero, and then the flow is the suspension of an Anosov diffeomorphism of the 2-torus, or $\tau \wedge d\tau$ is nowhere vanishing.*

Remark. — In the second case of the dichotomy described in (2.3.2), the flow is said to be *contact*. Such a flow can be extended to a Hamiltonian flow on $M \times \mathbf{R}$ with a homogeneous Hamiltonian function (cf. Appendix, [3]). An immediate corollary of (2.3.2) is that the 2-form $d\tau$ is either identically zero in the suspension case, or in the contact case is equal to the *invariant transverse flux form* $i(\xi)$ (d vol) and hence is also C^∞. Here, d vol is the smooth invariant volume form for the flow.

Proof. — First we show (2.3.2). The 3-form $\tau \wedge d\tau$ is continuous and flow-invariant. If it is identically zero, then by Plante (Theorem 3.1, [57]; see also Ghys [19]) there exists a compact smooth section for the flow which must be a 2-torus. Obviously, the section can be chosen to be C^∞, from which the first case of (2.3.2) follows.

If $\tau \wedge d\tau$ is not identically zero, then both its positive and negative parts define absolutely continuous, invariant measures for the flow (unless one of them vanishes identically). An application of the Livshitz Theorem, as described in [47], yields that any absolutely continuous invariant measure for a transitive Anosov flow is given by a positive density; i.e., non-vanishing almost everywhere. The Cocycle Regularity Theorem then implies that this density is a C^∞ non-vanishing 3-form on M, which must either be the positive or negative part of $\tau \wedge d\tau$ and therefore equal to this form.

The proof of (2.3.1) is based upon a result from [41]:

Lemma **2.4.** — *There is a smooth Anosov flow $\{ g_t \}$ with generating vector field $\{ \rho.\xi \}$, for a positive smooth function ρ, for which the invariant transverse form α is C^∞, is C^0-close to τ, and satisfies $d\alpha = d\tau$.* □

The C^∞-function ρ is uniquely determined by α from the equation

$$1 = \alpha(\rho.\xi) = \rho.\alpha(\xi).$$

We claim that this identity also suffices to show that τ is C^∞. The identity $d\tau = d\alpha$ and the de Rham Theorem for M imply that there is a closed C^∞ 1-form β on M and a C^1 function H for which $\tau = \alpha + \beta + dH$. We thus deduce that

$$\frac{1}{\rho} = 1 - \beta(\xi) - H'_\xi.$$

Since ρ, α and β are C^∞, this implies that H'_ξ is C^∞. As a C^1 solution H to the equation (2.1.1) exists, the Cocycle Regularity Theorem implies that H is C^∞, and hence that τ is C^∞. This completes part (1) of Theorem 2.3. □

We conclude this background section with a result which implies the Cocycle Regularity Theorem. For the case of two complementary foliations, it was first proven by R. de La Llavé, J. Marco and R. Moriyon (Lemma 2.3, [49]). The theorem below characterizes the C^∞ functions on $\mathbf{R}^n$ by their local restrictions to complementary $C^{0,\infty}$-foliations of $\mathbf{R}^n$ with a transverse regularity hypothesis. Our proof elaborates upon the (unpublished) idea of C. Toll to use a direct Fourier transform approach, and estimate the decay rates of the transforms via a cone method. We include this proof due to its simplicity, which has led to generalizations to the analytic case [48], and because the multifoliation case has proved to be an essential tool in the study of the smooth stability of lattice actions on higher dimensional tori (cf. [33]). Yet another proof for the case of two foliations has been given by J.-L. Journé [34, 35], with a different regularity hypothesis on the foliations and using the method of Taylor series approximations.

Definition **2.5.** — *Let* $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_r$ *be a set of continuous foliations of* $\mathbf{R}^n$, *with the leaves of* $\mathscr{F}_i$ *of dimension* n_i, *where* $n_1 + n_2 + \ldots + n_r = n$. *We say that these are a* C^k-*regular web of foliations of* $\mathbf{R}^n$ *if they satisfy the additional regularity hypotheses:*

1. For each $1 \leqslant i \leqslant r$, *the leaf* $L_i(p)$ *of* $\mathscr{F}_i$ *through* $p \in \mathbf{R}^n$ *is a* C^k *immersed submanifold of* M, *and the immersion depends continuously (in the* C^k-*topology on immersions) on the point* p.

2. The tangential distributions $T\mathscr{F}_i$ *are pairwise transverse, and moreover there is an internal direct sum decomposition* $\mathrm{TM} = T\mathscr{F}_1 \oplus \ldots \oplus T\mathscr{F}_r$.

3. For each $1 \leqslant i \leqslant r$, *there exists an* $\varepsilon > 0$ *so that for each* $p \in \mathbf{R}^n$, *there is a coordinate system* $\Phi_p : (-\varepsilon, \varepsilon)^n \to \mathbf{R}^n$ *satisfying:*

a) $\Phi_p(0, 0) = p$.

b) *For each* i, *let* $\mathbf{x}_i \in (-\varepsilon, \varepsilon)^{n_i}$, *and write* $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ *for a typical point* $\mathbf{x} \in (-\varepsilon, \varepsilon)^n$. *Then holding* $\mathbf{x}_j$ *fixed for* $j \neq i$ *and letting* $\mathbf{x}_i$ *vary, we obtain a local* C^k-*chart in the leaf* $L_i(\Phi_p(\mathbf{x}_1, \ldots, \mathbf{x}_i = 0, \ldots, \mathbf{x}_r))$. *We define*

$$\Phi_p^i(\mathbf{x})(\mathbf{x}_i) = \Phi_p(\mathbf{x}).$$

c) *For each* $1 \leqslant i \leqslant r$, *let* dv_i *denote the* $(n - n_i)$-*volume form on* $\mathbf{R}^{n - n_i}$ *lifted to* $\mathbf{R}^n$ *via the product structure of (b). Then the push-forward density*

$$\omega_i^\Phi = \Phi_{p,*}(dv_i)$$

is a continuous $(n - n_i)$-*form on the image of* Φ_p. *(This condition implies that each foliation* $\mathscr{F}_i$ *is absolutely continuous transversally, with continuous transverse invariant volume form.)*

d) *The continuous (local) form* ω_i^Φ *restricts to a* C^k-*section of the normal density bundle along each leaf of* $\mathscr{F}_i$. *That is, the global form* ω_i^Φ *determined by the local forms is smooth when restricted to the leaves of* $\mathscr{F}_i$.

Let us mention two examples where regular webs of foliations arise naturally from dynamical systems. The stable and unstable foliations of a C^k-Anosov diffeomorphism of a compact manifold, M, are C^k-immersed submanifolds, with the individual leaves depending continuously on the basepoint through which they pass. The tangential dis-

tributions are transverse, as they are identified with the Anosov splitting of TM from condition (1). The transverse regularity hypothesis $(2.5.3c)$, was essentially proven by Anosov, but a detailed proof following Anosov's ideas is given in (Lemma 2.5, [49]). Thus, given any coordinate chart on M, the restriction of the stable and unstable foliations to the chart will satisfy the hypotheses of Definition 2.5. For an Anosov flow, one considers a smooth transversal to the flow in a coordinate chart, and takes for the foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ the restrictions to the transversal of the weak-unstable and weak-stable foliations of the flow.

The second class of examples is provided by a family of n commuting, volume-preserving C^∞ Anosov diffeomorphisms of the n-torus, T^n. For $1 \leqslant i \leqslant n$, we require that each $\mathscr{F}_i$ be one-dimensional, and be the stable manifold for one of the Anosov diffeomorphisms. It then follows that each $\mathscr{F}_i$ is transversally C^1, and hence, restricted to any coordinate chart on T^n, will be a web of 1-dimensional foliations of $\mathbf{R}^n$. This is a basic example in the studies [32, 33, 42].

Theorem 2.6. — *Let* $\mathscr{F}_1, \ldots, \mathscr{F}_r$ *be a regular web of foliations on* $\mathbf{R}^n$. *Suppose that* $f : \mathbf{R}^n \to \mathbf{R}$ *is continuous, and for each* $1 \leqslant i \leqslant r$ *and* $p \in \mathbf{R}^n$, *the restriction of f to the leaf* $L_i(p)$ *of* $\mathscr{F}_i$ *is* C^k, *with the leafwise* C^k-*jet of* $f|_{L_i(p)}$ *depending continuously on the point p. Then f is* C^{k-n-1} *on* $\mathbf{R}^n$.

Proof. — We give the proof for $k = \infty$, and leave the modifications for the case $k < \infty$ to the reader. The conclusion on f is local, so we can assume that f has compact support in a common foliation chart for all of the foliations, which without loss can be assumed centered at the origin. The first step is to make a change of coordinates, for which the appropriate coordinate subplanes through the origin are the leaves through the origin for the foliations. This is possible as the individual leaves of the foliations are C^∞-immersed submanifolds. Using that the tangential distributions $T\mathscr{F}_i$ are continuous fields, we can moreover assume that on the support of f, the above coordinate subplane distributions are C^0-close to $T\mathscr{F}_i$ near the origin. We put these two conclusions in a precise form, but first must introduce notation.

Recall from the definition (2.5) that there is a product decomposition

$$\mathbf{R}^n = \mathbf{R}^{n_1} \oplus \ldots \oplus \mathbf{R}^{n_r};$$

with $x \in \mathbf{R}^n$, let $x = x_1 + \ldots + x_r$ be the corresponding decomposition of vectors. Let us write $x^i = x - x_i$ for the vector obtained from x by setting the i-th component equal to 0. Write $\widetilde{\mathscr{F}}_i$ for the foliation $\mathscr{F}_i$ in these local coordinates. Then we can assume that there exists $\varepsilon > 0$ so that:

1. The leaf of $\widetilde{\mathscr{F}}_i$ through $x = 0$ contains the subdisc

$$\{ x_j = 0 \text{ for } j \neq i, \| x_i \| < \varepsilon \}.$$

2. For each $1 \leqslant i \leqslant r$, there is a function

$$\psi^i : (-\varepsilon, \varepsilon)^n \to \mathbf{R}^{n-n_i} \subset \mathbf{R}^n$$

so that the general leaf of $\widetilde{\mathscr{F}}_i$ through $\mathbf{x}^i$ is locally given by the graph

$$\mathbf{L}_{\mathbf{x}_i} = \{ \, \mathbf{x}_i + \psi^i(\mathbf{x}_i + \mathbf{x}^i) \mid \mathbf{x}_i \in \mathbf{R}^{n_i} \, \}$$

and there is $0 < \delta < r^{-2}$ such that $\| \, \psi^i(\mathbf{x}) \, \| \leqslant \delta$ for all $\mathbf{x} \in \mathbf{R}^n$ with $\| \, \mathbf{x} \, \| < \varepsilon$.

3. For each $\mathbf{x}^i$, the function

$$\psi^i_{\mathbf{x}^i} : (- \varepsilon, \varepsilon)^{n_i} \to \mathbf{R}^{n-n_i}, \quad \text{defined by } \mathbf{x}_i \mapsto \psi^i(\mathbf{x}_i + \mathbf{x}^i)$$

is C^∞, and in the C^∞-topology on maps depends continuously on the parameter $\mathbf{x}^i$. Moreover, there is the uniform estimate

$$\left\| \frac{\partial}{\partial x_j} (\psi^i(\mathbf{x})) \right\| \leqslant \delta \quad \text{for } 1 \leqslant j \leqslant n.$$

4. For each $\mathbf{x}_i$, the mapping

(6) $$\psi^i_{\mathbf{x}_i} : (- \varepsilon, \varepsilon)^{n-n_i} \to \mathbf{R}^{n-n_i}, \quad \mathbf{x}^i \mapsto \psi^i(\mathbf{x}_i + \mathbf{x}^i) \in \mathbf{R}^n \mapsto \mathbf{R}^{n-n_i}$$

is absolutely continuous, with absolutely continuous inverse, and pushes the standard measure $d\mathbf{x}^i$ on $\mathbf{R}^{n-n_i}$ forward to a continuous measure on a neighborhood of $0 \in \mathbf{R}^{n-n_i}$. (This condition is the coordinate form of the hypothesis (2.5.3c).)

Let $f(\mathbf{x})$ denote the function f in the above coordinates about the origin. Introduce the " dual" variable $\xi \in \hat{\mathbf{R}}^n$ to $\mathbf{x}$. As f is continuous with compact support, we can form its Fourier transform $\hat{f}$ on $\hat{\mathbf{R}}^n$, given by the usual formula

(7) $$\hat{f}(\xi) = (2\pi)^{-n/2} \cdot \int_{\mathbf{R}^n} \exp\{ i(\xi \cdot \mathbf{x}) \} f(\mathbf{x}) \, d\mathbf{x}.$$

Lemma **2.7.** — *For each integer $m > 0$, there exist constants $C(m)$, $T(m) > 0$ such that for all $\xi \in \mathbf{R}^n$ with $\| \, \xi \, \| = 1$,*

(8) $$|\hat{f}(t\xi)| < C(m) \cdot t^{-m} \quad \text{for } t > T(m).$$

The estimate (8) implies that for each $s > 0$, the function f belongs to the s-Sobolev space on $\mathbf{R}^n$. As f has compact support, we can then apply the Sobolev lemma to deduce that f is C^∞. Thus, the proof of Theorem 2.6 is reduced to proving Lemma 2.7.

Proof of Lemma **2.7.** — The " cone method " of proof alluded to above is based on the simple observation that for any unit vector $\xi \in \hat{\mathbf{R}}^n$, there exists an index $1 \leqslant i \leqslant r$ such that $r \cdot \| \, \xi_i \, \| \geqslant \| \, \xi \, \|$. This says that ξ lies in a cone centered on the coordinate plane $\mathbf{R}^{n_i} \subset \mathbf{R}^n$. We fix a particular value of i with this property, and make a change of variables for the integral in (7) using the graph presentation of the foliation $\widetilde{\mathscr{F}}_i$. The estimate (8) will then follow from a second change of variables and the technical hypotheses on ψ^i made above.

Fix a unit vector ξ and index $1 \leqslant i \leqslant r$ so that $\xi = \xi_i + \xi^i$, where $r \cdot \| \, \xi_i \, \| \geqslant \| \, \xi \, \|$. Introduce the function

$$\hat{F}(t) = \hat{f}(t\xi).$$

Next, make a change of coordinates in $\mathbf{R}^n$ using the graph-function of $\mathscr{F}_i$,

$$(\mathbf{x}_i, \mathbf{x}^i) = (\mathbf{x}_i, \psi^i(\mathbf{x}_i, \mathbf{v}^i))$$

into the integral (7), and separate out the variable $\mathbf{v}^i$ to obtain

$$(9) \qquad \hat{F}(t) = (2\pi)^{-n/2} \cdot \int_{\mathbf{R}^{n-n_i}} \Phi(t, \mathbf{v}^i) \, d\mathbf{v}^i$$

where we introduce the function

$$(10) \qquad \Phi(t, \mathbf{v}^i) = \int_{\mathbf{R}^{n_i}} \exp\{ it(\xi_i . \mathbf{x}_i + \xi^i . \psi^i(\mathbf{x}_i, \mathbf{v}^i)) \} f(\mathbf{x}_i, \psi^i(\mathbf{x}_i, \mathbf{v}^i)) \mid \wedge \psi^i \mid (\mathbf{x}_i, \mathbf{v}^i) \, d\mathbf{x}_i$$

$\mid \wedge \psi^i \mid d\mathbf{v}^i$ being the image of the standard volume form $d\mathbf{v}^i$ under the map (6). By our hypotheses, the function $\mathbf{x}_i \mapsto \mid \wedge \psi^i \mid (\mathbf{x}_i, \mathbf{v}^i)$ is C^∞ in $\mathbf{x}_i$, and depends continuously on $\mathbf{v}^i$ in the C^∞-topology on maps.

The second step in the proof, and the key idea, is to make a second change of coordinates in the definition of $\Phi(t, \mathbf{v}^i)$, which will yield superpolynomial decay of this function in the variable t, with the estimates uniform in $\mathbf{v}^i$. The idea is to write the equation (10) as a convolution integral along the leaves of $\widetilde{\mathscr{F}}_i$, with an integrand consisting of f restricted to the leaves, and other terms arising from the change of variables, but uniformly smooth due to the regularity hypotheses on the web of foliations. The decay in t is then a consequence of the standard properties of the Fourier transform of smooth functions. The proof of Lemma 2.7 follows, as $\hat{F}(t)$ is obtained from $\Phi(t, \mathbf{v}^i)$ by integrating the second variable over a compact set.

We begin with some linear algebra. Let A be an invertible $n_i \times n_i$-matrix, whose first row is the vector ξ_i considered as an element of $\mathbf{R}^{n_i}$, and whose subsequent rows form an orthonormal basis for the complement to ξ_i. Let B be an $n_i \times (n - n_i)$-matrix whose first row is ξ^i, and has 0 for all other entries. Introduce a new variable

$$(11) \qquad z_i = z_i(\cdot, \mathbf{v}^i) : (-\varepsilon, \varepsilon)^{n_i} \to \mathbf{R}^{n_i}$$

$$z_i(\mathbf{x}_i, \mathbf{v}^i) = \mathbf{x}_i + A^{-1} . B . \psi^i(\mathbf{x}_i, \mathbf{v}^i).$$

The choice of A and B is made to ensure that

$$\xi_i . z_i(\mathbf{x}_i, \mathbf{v}^i) = \xi_i . \mathbf{x}_i + \xi^i . \psi^i(\mathbf{x}_i, \mathbf{v}^i),$$

and give the matrix norm estimate

$$\| A^{-1} . B \| \leqslant \frac{\| \xi^i \|}{\| \xi_i \|} \leqslant r.$$

Thus, by the estimates in conditions (2, 3) above, the function defined by (11) is injective in $\mathbf{x}_i$, and the matrix differential $\dfrac{\partial z_i}{\partial \mathbf{x}_i}$ is invertible, uniformly in $\mathbf{v}^i$. Introduce the inverse function $\mathbf{x}_i = \alpha(z_i, \mathbf{v}^i)$ which is C^∞ in the variable z_i, uniformly in $\mathbf{v}^i$. Substitute this change of variables into (10) to obtain

$$(12) \qquad \Phi(t, \mathbf{v}^i) = \int_{\mathbf{R}^{n_i}} \exp\{ it\xi_i . z_i \} F(z_i, \mathbf{v}^i) \, dz_i$$

where we have as integrand the product of functions

$$(13) \qquad F(\mathbf{z}_i, \mathbf{v}^i) = f(\alpha(\mathbf{z}_i, \mathbf{v}^i), \psi^i(\alpha(\mathbf{z}_i, \mathbf{v}^i))) \mid \wedge \psi^i \mid (\alpha(\mathbf{z}_i, \mathbf{v}^i), \mathbf{v}^i) \left| \frac{\partial \alpha}{\partial \mathbf{z}_i} \right| (\mathbf{z}_i, \mathbf{v}^i).$$

The function F is compactly supported, and all of the terms appearing in (13) are C^{∞} in $\mathbf{z}_i$, uniformly in $\mathbf{v}^i$. Thus, its Fourier transform in $\mathbf{z}_i$ has superpolynomial decay in the transform variable $\hat{\mathbf{z}}_i$, uniformly in $\mathbf{v}^i$, so that $\Phi(t, \mathbf{v}^i)$ has the same property as was to be shown. $\square$

3. Formulation of Results

A continuous function $f: (a, b) \to \mathbf{R}$ is in the *Zygmund class* $\Lambda_*(a, b)$, or just Λ_* when the domain is clear from the context, if $\Lambda_*(f) < \infty$, where the Zygmund norm is given by

$$(14) \qquad \Lambda_* \overset{\text{déf}}{=} \sup_{a < x < b} \limsup_{h \to 0} \frac{|f(x+h) + f(x-h) - 2f(x)|}{|h|}.$$

Zygmund studies this class of functions in his famous treatise [65], and in particular shows (Theorem 3.4) that a function $f \in \Lambda_*(a, b)$ has modulus of continuity

$$\Omega(s) = O(s \mid \log(s) \mid),$$

and therefore is α-Hölder for all $\alpha < 1$. However, f need not be Lipshitz, nor be of bounded variation. The norm (14) is one of a family of such " norms " which arise in the study of singular integral operators (cf. [44, 59]).

The definition of the Zygmund class for functions on $\mathbf{R}^n$ uses the same norm as in (14), but replaces the open interval with an open subset of $\mathbf{R}^n$. For a non-negative integer, k, and real number $0 \leqslant \alpha < 1$, consider the five classes of functions denoted respectively by:

$$(15) \qquad C^{k, \alpha}, C^{k, 1}, C^{k, \Lambda_*}, C^{k, \Omega}, C^{k, \omega}$$

which are k times differentiable on the appropriate open domain, and whose k-th derivatives are respectively of class

$$\alpha\text{-Hölder, } Lipshitz, Zygmund, \Omega(s) = O(s \mid \log(s) \mid), \omega(s) = o(s \mid \log(s) \mid).$$

A vector subbundle $E \subset TM$ of the tangent bundle of a manifold M is said to be in one of the classes (15) if E is locally spanned by vector fields whose coordinate expressions with respect to a local C^{∞}-framing of TM are in the appropriate class.

Theorem **3.1.** *(Regularity).* — *Let $\{f_t\}$ be a volume-preserving, C^3-Anosov flow on a compact Riemannian 3-manifold* M.

1. *The weak-stable distribution E^{ws} and the weak-unstable distribution E^{wu} are of class C^{1, Λ_*}.*

2. *If $\{f_t\}$ admits a C^1-transverse invariant 1-form, τ, then the strong-stable distribution E^- and the strong-unstable distribution E^+ are also of class C^{1, Λ_*}.*

3

Part (2) of Theorem 3.1 is an immediate consequence of part (1) and Theorem 2.3. The proof of part (1) is given in section 4.

Anosov observed (Lemma 24.1, [2]) that for an area-preserving, C^3-Anosov diffeomorphism $F : T^n \to T^n$ of the 2-torus, if either the stable or the unstable distribution of F is C^2, then at each periodic orbit of F there exists a differential relation of third order which F must satisfy. The next three theorems put this observation into a systematic framework.

Let $\{ f_t \}$ be a volume-preserving, C^3-Anosov flow on the 3-manifold M. For each periodic point $p \in M$ of period t_p, let $\Psi_p : (- \varepsilon, \varepsilon)^2 \to M$ be a C^3-adapted transversal to the flow, as defined in section 4 below. (These always exist for the suspension of a C^3-toral automorphism; in general, the Anosov flow must be C^4 as in Proposition 4.2.) The Poincaré return map of the flow, for the transversal Ψ_p, is defined on an open subset $(0, 0) \in T_p C (- \varepsilon, \varepsilon)^2$. The return map is written in coordinates as

$$F(x,y) = \begin{cases} \mu x + \varphi(x,y) \\ \mu^{-1} y + \psi(x,y) \end{cases}.$$

We adopt the notation that the partial derivatives with respect to the coordinates x and y are denoted by the corresponding subscripts. At the periodic orbit p, we define:

$$(16) \qquad A_f(p, t_p) = (1/2) \mu\psi_{xvv}(0, 0) = (- 1/2) \mu^{-1} \varphi_{yxx}(0, 0).$$

Theorem **3.2** *(Local Vanishing).* — *Let* $\{ f_t \}$ *be a volume-preserving, C^4-Anosov flow on a closed 3-manifold* M. *Let* p *be a periodic orbit of period* $t_p > 0$. *Suppose that one of the following three conditions holds:*

1. the first transverse derivative of E^{ws} *or* E^{wu} *has modulus of continuity* $\omega(s) = o(s.| \log(s) |)$ *at* p;
2. either E^{ws} *or* E^{wu} *has a measurable transverse second derivative almost everywhere on* M;
3. there is a measurable second derivative for the local angle function between E^{ws} *and* E^{wu} *almost everywhere on* M.

Then $A_f(p, t_p) = 0$.

The proof of the local vanishing theorem above is given in section 5, and is based on a detailed study of how the third jet of the Poincaré map dictates the regularity of the unstable foliations at periodic orbits.

In section 5 we define the *Anosov cocycle* A_f^Ψ over the flow $\{ f_t \}$. For a periodic point $p \in M$ of period t_p, the special values of this cocycle are given by (16). Let $H^1(\{ f_t \}; R)$ denote the group of C^1-cocycles over the flow $\{ f_t \}$ modulo the C^1-coboundaries.

Theorem **3.3** *(Anosov Class).* — *Let* $\{ f_t \}$ *be a volume-preserving, C^4-Anosov flow on a closed 3-manifold* M.

1. The cohomology class $A_f \in H^1(\{ f_t \}, R)$ *of the cocycle* A_f^Ψ *is independent of the choice of adapted transverse coordinates* Ψ *for the flow.*

2. If $\{f_t\}$ and $\{\widetilde{f_t}\}$ are two volume-preserving, C^6-Anosov flows on M, *and* Θ *is a* C^1-*diffeomorphism of* M *conjugating the two flows, then* $\Theta^* \widetilde{A}_{\widetilde{f}} = A_f$.

It is well-known that when two Anosov flows, $\{f_t\}$ and $\{\widetilde{f_t}\}$, on a compact manifold M are topologically conjugate by a homeomorphism F, then there exists an $\alpha > 0$ such that F is α-Hölder. We can thus compare their Anosov classes, A_f and $\widetilde{A}_{\widetilde{f}}$, among the Hölder 1-cocycles over the flow $\{f_t\}$. By the Livshitz Theorem 2.1, the cohomology classes of these flows are determined by their values at periodic orbits. We fix the flow $\{f_t\}$, and a basic problem is to find a characterization of the Anosov flows topologically conjugate to $\{f_t\}$, modulo C^1-conjugacy, with the same Anosov class. The next result solves this problem for the case when the Anosov class vanishes.

Theorem **3.4** *(Smooth Rigidity)*. — *Let* $\{f_t\}$ *be a volume-preserving,* C^k-Anosov *flow on a closed 3-manifold* M, *for* $k \geqslant 5$. *The Anosov class* A_f *vanishes if and only if the distributions* E^{wu} *and* E^{ws} *are* C^{k-3}. *For* $k = \infty$, *the distributions are* C^∞.

The proof of Theorem 3.4 begins with the study of the Anosov cocycle in section 5, and is concluded in section 6.

Corollary **3.5**. — *Let* $\{f_t\}$ *be as in Theorem 3.4. The local obstructions* $A_f(p, t_p)$ *of* (16) *vanish for every periodic orbit of the flow if and only if the weak-unstable and the weak-stable distributions are* C^{k-2}. *In particular, if either distribution is of class* $C^{1,\omega}$, *then both distributions are* C^{k-2}.

Proof. — If either distribution is C^2, then the local obstructions vanish by Theorem 3.2, and by the Livshitz Theorem the Anosov cocycle is a coboundary. Conversely, by the Livshitz Theorem, if the local obstacles vanish, then the Anosov class vanishes and by Theorem 3.4 the distributions are C^{k-2}. $\square$

Corollary **3.6**. — *Let* F : $T^2 \to T^2$ *be an area-preserving,* C^k-Anosov *diffeomorphism of the 2-torus, for* $k \geqslant 5$. *Then there is the dichotomy*:

1. for some periodic point $p \in T^2$ *of* F, *one of the stable or unstable distributions* E^- *or* E^+ *is not* $C^{1,\omega}$ *at* p;

2. the diffeomorphism F *is* C^{k-3}-*conjugate to a linear Anosov automorphism of* T^2.

In particular, if the Anosov class of the flow obtained by suspending F *vanishes, then* F *is* C^{k-3} *conjugate to a linear automorphism.*

Proof. — The suspension of the diffeomorphism F produces a flow $\{f_t^F\}$ on the closed 3-manifold

$$M = \frac{T^2 \times \mathbf{R}}{(x, r) \sim (F(x), r + 1)}.$$

The adapted system of local transversals for this flow can be chosen to lie in the submanifolds $T^2 \times \{r\}$, for $r \in [0, 1]$, so that the local Anosov obstructions for the flow are

calculated in terms of F at periodic points. We then apply the Livshitz Theorem to this flow to conclude that either some local obstacle is not zero, and hence by Theorem 3.2 one of the distributions E^- or E^+ is not $C^{1,\omega}$ at this periodic point, or the Anosov class of the flow vanishes and hence by Theorem 3.4 the distributions are C^{k-3}. The flow $\{f_t^F\}$ has a C^k-transverse 1-form, so the strong foliations are of the same class, and hence the stable and unstable foliations of F are C^{k-3}. The proof of Theorem 10.2 by Avez [4] then implies that F is C^{k-3}-conjugate to an Anosov linear automorphism. □

Corollary **3.7.** — *Let g denote a* C^∞ *Riemannian metric on a closed 2-dimensional orientable manifold* S, *such that the geodesic flow* $\{f_t^g\}$ *on the unit tangent bundle* $M = T^1 S$ *is Anosov. Then the local obstructions* $A_f(p, t_p)$ *for the flow vanish at all periodic orbits if and only if the metric g has constant negative curvature.*

Remark. — A metric with strictly negative curvature has a contact Anosov geodesic flow [2]. However, there are weaker hypotheses which will guarantee that the flow is Anosov, as discussed by Eberlein [10].

Proof. — A metric of constant negative curvature has analytic weak-stable and weak-unstable foliations, so that all of the local obstacles vanish. Conversely, note that the Louiville measure on M is flow invariant. Thus, if the local obstacles vanish, then the horocycle foliations of the geodesic flow are C^∞ by Theorem 3.4. It then follows by the work of Ghys [17] that the flow is C^∞-conjugate to a flow for a metric of constant negative curvature. This implies that the metric and topological entropies of the flow $\{f_t^g\}$ must agree, so by the metric entropy rigidity theorem of Katok [39] the metric g must have constant curvature. Alternately, the Godbillon-Vey invariant of the weak-stable foliation of the flow must assume the maximal possible value, as the foliation is C^∞-conjugate to a Roussarie foliation, hence by Corollary 9.3 below the metric g has constant curvature. □

The second part of this paper centers on the Godbillon-Vey class of codimension-one foliations. This is a degree 3 cohomology class, traditionally defined for transversally C^2 foliations (cf. [6, 22]). The foliations arising from Anosov dynamical systems are conjecturally rarely C^2, so it is of significance that an extension of the invariant to foliations of differentiability class less than C^2 can be given. The next four results describe the properties of this extended class.

Let us first recall two definitions from the theory of foliations. A codimension-q foliation $\mathcal{F}$ on a manifold M is said to be of class $C^{k,\alpha}$ if its tangential distribution $T\mathcal{F}$ is defined as the annihilating subspace of a q-form, θ, on M where in local coordinates θ is of differentiability class $C^{k,\alpha}$. The standard equivalence relation for cohomology invariants of foliations is *concordance*. We say that two codimension-q $C^{k,\alpha}$-foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ on M are $C^{k,\alpha}$-concordant if there is a codimension-q $C^{k,\alpha}$-foliation $\mathcal{F}$ on the product $M \times [0,1]$, such that the restriction $\mathcal{F} \mid M \times \{i\}$ is $C^{k,\alpha}$-conjugate to $\mathcal{F}_i$ for $i = 0, 1$.

Theorem **3.8** *(Dynamical Godbillon-Vey).* — *Let $\mathscr{F}$ be a codimension-one $C^{1,\alpha}$-foliation of a closed orientable 3-manifold M, for $\alpha > 1/2$. Then there is a Godbillon-Vey class,*

$$GV(\mathscr{F}) \in H^3(M; \mathbf{R})$$

which extends the usual Godbillon-Vey class for C^2-foliations, and satisfies:

1. the cohomology class $GV(\mathscr{F})$ depends continuously on the defining 1-form θ for $\mathscr{F}$, in the $C^{1,\alpha}$-topology on 1-forms;

2. if $\mathscr{F}_0$ is $C^{1,\alpha}$-concordant to $\mathscr{F}_1$, then $GV(\mathscr{F}_0) = GV(\mathscr{F}_1)$;

3. let $\Theta : M_0 \to M_1$ be a diffeomorphism conjugating two $C^{1,\alpha}$-foliations, $\mathscr{F}_0$ to $\mathscr{F}_1$. If either

* *Θ is $C^{1,\beta}$ for $\alpha + \beta > 1$, or*

* *Θ and Θ^{-1} are transversally Lipshitz for $\alpha = 1$,*

then $GV(\mathscr{F}_0) = \Theta^* GV(\mathscr{F}_1)$.

Remarks.

* T. Tsuboi has shown that it is not possible to define a " Godbillon-Vey " class for C^1-foliations which is invariant under C^1-concordance [64]. Moreover, Tsuboi has constructed a family of examples of codimension-one C^2-foliations on a 3-manifold such that their Godbillon-Vey invariants do not depend continuously on the foliations in the $C^{1,\alpha}$-topology for $\alpha < 1/2$.

* The definition of the Godbillon-Vey invariant given in section 7 is based on distribution theory. An alternative construction of our extension is given in section 8, based on Thurston's " area functional " approach. The methods introduced in section 8 are the basis for Tsuboi's more general extension of the Godbillon-Vey class, which incorporates both the invariant of Theorem 3.8 and the piecewise-C^2 extension defined by Ghys (cf. [18, 63]).

* The C^1-invariance of the Godbillon-Vey class for C^2-foliations was first proved by G. Raby [58]. However, Ghys and Tsuboi [21] proved that in the C^2 case, the result is weak, as the C^1 conjugacy Θ between the two foliations must actually be C^2 on the support of the cohomology class $GV(\mathscr{F}_1)$. The results (3.8.3) are new, and part of a more general study of the C^1-invariance of the secondary cohomology invariants of foliations [31].

Define the stable and unstable Godbillon-Vey invariants of a volume-preserving C^3-Anosov flow $\{f_t\}$ on a closed, oriented 3-manifold M, to be the real numbers:

$$(17) \qquad \begin{aligned} gv^s(\{f_t\}) &= \langle\, GV(\mathscr{F}^{ws}), [M] \,\rangle \\ gv^u(\{f_t\}) &= \langle\, GV(\mathscr{F}^{wu}), [M] \,\rangle \end{aligned}$$

where the Godbillon-Vey classes of the weak-stable and weak-unstable foliations are well-defined by Theorems 3.1 and 3.8, and we pair these classes with the fundamental class [M] of the 3-manifold.

Corollary **3.9.** — *Let* $\{f_{\lambda,t} \mid 0 \leqslant \lambda \leqslant 1\}$ *be a* 1-*parameter family of volume-preserving* C^3-*Anosov flows on a closed* 3-*manifold* M, *which vary* C^3 *in the parameter* λ. *Then both the stable and unstable Godbillon-Vey invariants of the flows vary continuously with* λ.

Proof. — The weak-stable and weak-unstable foliations of the family of flows depend $C^{1,\alpha}$ on the parameter λ, for any $\alpha < 1$ (cf. Lemma 3.8 of [52], or see [43]). The corresponding Godbillon-Vey classes then vary continuously with the parameter by Theorem 3.8.1. □

The geodesic flow of a metric of negative curvature on a closed surface is on a circle bundle over the surface, and the weak-stable foliation is transverse to the fibers of this bundle. In this context, there is a technical, but significant strengthening of the conclusion (3.8.3) above. The proof of the following is given at the end of section 7.

Theorem **3.10** *(Absolute-Continuity Invariance).* — *Let* $\mathcal{F}$ *and* $\widetilde{\mathcal{F}}$ *be codimension-one,* $C^{1,\alpha}$-*foliations on closed oriented* 3-*manifolds* M *and* $\widetilde{M}$, *respectively, for* $\alpha > 1/2$. *Suppose that* $\mathcal{F}$ *is the weak-stable foliation of a volume-preserving* C^3-*Anosov flow on* M, *and that there exists a homeomorphism* $\Theta : M \rightarrow \widetilde{M}$ *conjugating* $\mathcal{F}$ *to* $\widetilde{\mathcal{F}}$ *with* Θ *transversally absolutely continuous. Then* $GV(\mathcal{F}) = \pm \Theta^* GV(\widetilde{\mathcal{F}})$ *according to whether* Θ *is orientation preserving or reversing.*

The last result in this development is the *Formula of Mitsumatsu* for the value of the Godbillon-Vey invariant of a geodesic flow of a closed surface. It is necessary to introduce a few technical properties of these flows before we can state the result.

Let $\{f_t(g)\}$ be the geodesic flow on $M = T^1 \Sigma$ for a smooth metric g of strictly negative curvature on an oriented surface Σ. Give TM the natural Riemannian metric induced by the metric on $T\Sigma$ on its bundle of orthonormal frames, and let d vol denote the Riemannian volume form on M. The $\pi/2$-rotation on M smoothly conjugates the weak-stable to the weak-unstable foliation of the flow, so the two invariants defined by (17) coincide, and we denote their common value by $gv(g)$. For a metric of *constant* negative curvature, the weak-stable foliation on M is called the *Roussarie* foliation, after the paper [Ro] where the Godbillon-Vey invariant for this foliation was calculated to be $4\pi^2 \chi(S)$. The integer $\chi(\Sigma)$ is the Euler characteristic of the surface Σ. Mitsumatsu [53] calculated the value of $gv(g)$ for all metrics of negative curvature for which the weak-stable foliation is of class C^2. As seen above, this restriction forces the metric to be of constant curvature. However, the formula he derived continues to be defined for the non-constant curvature case, and a modification of his proof yields a calculation of $gv(g)$ in the more general case.

Let $\partial/\partial\varphi$ denote the unit tangent vector field on M tangent to the fibers, whose time t flow is rotation by t radians in $T^1 \Sigma$. Introduce the positive, global C^1-solution $H = H^+ : M \rightarrow \mathbf{R}$ to the Riccati equation

$$(18) \qquad \xi_g(H) + H^2 + k(g) \circ \pi = 0,$$

where $k(g) : \Sigma \to \mathbf{R}$ is the Gaussian curvature function of the metric g. We define the *Mitsumatsu Defect*

$$(19) \qquad \mathrm{De} f(g) = 3 . \int_{\mathbf{M}} \left(\frac{\partial \mathrm{H}}{\partial \varphi} \right)^2 . d \, \mathrm{vol}.$$

Theorem **3.11** *(Mitsumatsu Formula).* — *Let g be a C^4 metric with strictly negative curvature on a closed surface Σ. Then the weak-stable (and weak-unstable) foliation of the geodesic flow has a well-defined Godbillon-Vey invariant, given by the formula*

$$gv(g) = 4\pi^2 \chi(\Sigma) - \mathrm{De} f(g).$$

Moreover, $\mathrm{De} f(g)$ is non-negative and equal to zero if and only if g has constant curvature.

Remark. — There is a striking resemblance between the formula (19) and the Pesin Formula [56] for the metric entropy $h(g)$ of the geodesic flow $\{ f_t(g) \}$ with respect to the Liouville measure d vol on M:

$$(20) \qquad h(g) = \int_{\mathbf{M}} \mathrm{H} . d \, \mathrm{vol}.$$

Recall that the entropy of an Anosov flow is a measure of the growth rate of the lengths of the closed orbits of the flow [38]. Comparing the formulas (19) and (20) suggests that the term $\mathrm{De} f(g)$ should be viewed as a type of " mean variation " of the distribution of the closed orbits.

We conclude the section on results with the two applications of the Formula of Mitsumatsu mentioned in the Introduction.

Corollary **3.12** *(Topological Non-invariance of Godbillon-Vey).* — *Let M be the unit tangent bundle to a closed oriented surface Σ with negative Euler characteristic. There exists a continuous family of codimension-one $\mathrm{C}^{1, \Lambda*}$-foliations on M parametrized by the space of metrics of negative curvature,*

$$\{ \mathscr{F}_g \mid g \ \text{a metric on } \Sigma \text{ of strictly negative curvature} \},$$

such that the Godbillon-Vey invariants $gv(\mathscr{F}_g)$ vary continuously and non-trivially in the C^4-topology on metrics. Furthermore, all of the foliations $\mathscr{F}_g$ in this family are Hölder-topologically conjugate, but not topologically conjugate by absolutely-continuous maps.

Corollary **3.13** *(Measure Rigidity).* — *Let g be a C^4 metric of strictly negative curvature on a closed surface Σ. Then the geodesic measure class at infinity coincides with the harmonic measure class of g if and only if g has constant curvature.*

Proof. — The metric g is conformally equivalent to a metric of constant negative curvature, g_0, so there is a positive scalar function $\rho : \Sigma \to \mathbf{R}^+$ such that $g = \rho g_0$. By

the conformal invariance of the Laplacian in dimension two, the harmonic measure classes for g and g_0 coincide. If the geodesic measure is absolutely continuous with respect to the harmonic one, then there exists a continuous, absolutely continuous orbit equivalence h between the geodesic flows $\{f_t(g)\}$ and $\{f_t(g_0)\}$. (For a proof of this folk-lore theorem, see [40].) We then have that h conjugates the weak stable foliation of the metric g to the weak-stable foliation of the metric g_0, and the conjugacy is transversally absolutely continuous. Reversing the rôles of g and g_0 shows that the inverse conjugacy is also absolutely continuous, so by Theorem 3.10, $gv(\{f_t(g)\}) = 4\pi^2$, and hence by Theorem 3.11 g has constant curvature. $\square$

4. Regularity of the Weak-Unstable Foliations

Let $\{f_t\}$ be a C^3-Anosov flow on the closed Riemannian 3-manifold, M, which leaves the Riemannian volume form d vol invariant. In this section, we prove that the weak-unstable foliation of the flow is in the class C^{1, A_*}. We will first show that the transverse derivative has modulus of continuity $\Omega(s) = O(s \mid \log(s) \mid)$, and then observe that a modification of the argument establishes the stronger result that the derivative is in the Zygmund class.

We can assume without loss that the bundles E^+ and E^- are orientable. Let η^+ denote a unit vector field spanning E^+, and η^- denote a vector field spanning E^- so that the triple $\{\xi, \eta^+, \eta^-\}$ is a unit volume frame at each point. Define the *local multipliers of the flow*, $\lambda^+(p, t)$ and $\lambda^-(p, t)$, by the equation

(21)
$$Df_t(\eta^+(p)) = \lambda^+(p, t) \cdot \eta^+(f_t(p))$$
$$Df_t(\eta^-(p)) = \lambda^-(p, t) \cdot \eta^-(f_t(p)).$$

The flow invariance of the volume form implies that the multipliers satisfy $\lambda^+ . \lambda^- = 1$, and we say that the local expansion and contraction multipliers of the flow are equal. The work of Hirsch and Pugh [30] then implies that the foliations $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$ are C^1, and the transverse derivative is α-Hölder for some $\alpha < 1$.

There are three steps in the proof. We first introduce *adapted transverse coordinates* for the flow, based on the C^1-foliations $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$. The vector field η^+ above need not even be C^1, so we replace it with a unit vector field e^+ which is C^1, and the pair $\{\xi, e^+\}$ still spans E^{wu}. The Hölder function Ω is used to define a norm on the set of C^2-vector fields near to e^+. We essentially prove that there is a compact set in this norm which is invariant under the projectivized transverse action of the flow. More precisely, we show that any C^2-vector field v which is C^1-close to e^+ is exponentially attracted to e^+ by the forward iterates of the flow, yielding a sequence of vector fields which are Cauchy in the Ω-norm. This implies that e^+ is $C^{1, \Omega}$, which proves that E^{wu} is of class $C^{1, \Omega}$.

For each $p \in M$, let L_p^{wu} and L_p^{ws} denote the weak-unstable and weak-stable manifolds through p.

Definition **4.1** *(Adapted Transverse Coordinates).* — Let $\{f_t\}$ be a volume-preserving C^k-*Anosov flow on the closed* 3-*manifold* M, *for* $k \geqslant 2$. C^k-*adapted transverse coordinates for the flow consists of a* C^1-*map*

$$\Psi : M \times (-\varepsilon, \varepsilon)^2 \to M$$

for some $\varepsilon > 0$, *which satisfies the following conditions.*

    1. For each $p \in M$ *the map*

(22)
$$\begin{aligned} \Psi_p : (-\varepsilon, \varepsilon) &\to M \\ \Psi_p(x,y) &= \Psi(p, x, y) \end{aligned}$$

is a C^k-*diffeomorphism into, with the vectors* $D_{(x,y)} \Psi_p(\partial/\partial x)$ *and* $D_{(x,y)} \Psi_p(\partial/\partial y)$ *uniformly transverse to the flow vector field* $\xi(\Psi_p(x,y))$: *We specify that the angles be everywhere greater than* $\pi/4$, *and orthogonal to the vector* $\xi(p)$ *for* $(x,y) = (0, 0)$.

    2. The maps

(23)
$$\begin{aligned} \Psi_p &: \{ (x, 0) \mid \mid x \mid < \varepsilon \} \to L_p^+ \subset L_p^{wu} \\ \Psi_p &: \{ (0, y) \mid \mid y \mid < \varepsilon \} \to L_p^- \subset L_p^{ws} \end{aligned}$$

are coordinates onto the 1-*dimensional submanifolds* L_p^+ *and* L_p^- *centered at* p, *and depend* C^1 *on the basepoint* p, *when considered as* C^k-*immersions of* $(-\varepsilon, \varepsilon)$ *into* M.

    3. For each $p \in M$, *define* $X_p = \Psi_p((-\varepsilon, \varepsilon)^2)$, *which by* (4.1.1) *above is a uniformly embedded transversal to the flow.*

• *The* C^1-*foliation* W_p^u *of* X_p *defined by the restriction* $\mathscr{F}^{wu} \mid X_p$ *is* C^1-*tangent at* p *to the linear foliation of* X_p *by the coordinate lines parallel to the x-axis, in the coordinates provided by* Ψ_p.

• *The* C^1-*foliation* W_p^s *of* X_p *defined by the restriction* $\mathscr{F}^{ws} \mid X_p$ *is* C^1-*tangent at* p *to the linear foliation of* X_p *by the coordinate lines parallel to the y-axis, in the coordinates provided by* Ψ_p.

    4. Let $d\nu$ *be the restriction of the* 2-*form* $i(\xi)$ $(d\,\mathrm{vol})$ *to* X_p. *Then*

$$\Psi_p^*(d\nu) = dx \wedge dy.$$

Remark. — Adapted transverse coordinates are a cocycle form of the Moser-Sternberg canonical coordinates for a symplectomorphism. We briefly recall the relevant result from their theory (cf. [54, 60]). Let $F = (F_1(x,y), F_2(x,y))$ be a C^∞-local diffeomorphism of an open neighborhood of the origin,

$$F : U \to \mathbf{R}^2, \quad \text{with } F(0, 0) = (0, 0), \text{ and } F^*(dx \wedge dy) = dx \wedge dy$$

such that the eigenvalues of the differential DF are not of modulus 1. Then by Theorem 1 of [54] (cf. also Theorem 9, [60]), there is a C^∞, volume-preserving local change of coordinates about $(0, 0)$ so that the germ of F in the new coordinates $(\tilde{x}, \tilde{y})$ has the form

(24)
$$\tilde{x}^* = \tilde{F}_1 \sim \tilde{x}(\mu + a\tilde{x}\tilde{y} + a_2(\tilde{x}\tilde{y})^2 + \ldots)$$

(25)
$$\tilde{y}^* = \tilde{F}_2 \sim \tilde{y}(\mu^{-1} - a\tilde{x}\tilde{y} + b_2(\tilde{x}\tilde{y})^2 + \ldots).$$

At (0,0), we identify the non-zero first derivatives

$$(26) \qquad \frac{\partial(\widetilde{F}_1)}{\partial \widetilde{x}} = \mu, \qquad \frac{\partial(\widetilde{F}_2)}{\partial \widetilde{y}} = \mu^{-1},$$

the second derivatives of $\widetilde{F}$ vanish, and there are two non-vanishing third derivatives

$$(27) \qquad \frac{\partial^3(\widetilde{F}_1)}{\partial \widetilde{x}\, \partial \widetilde{x}\, \partial \widetilde{y}} = 2a = - \frac{\partial^3(\widetilde{F}_2)}{\partial \widetilde{x}\, \partial \widetilde{y}\, \partial \widetilde{y}}.$$

At a periodic orbit of the flow, the conditions (4.1.1-4) imply that the Poincaré return map of the flow on the transversal X_p in the local coordinates Ψ_p agrees to *second order* with the Moser-Sternberg local canonical form. This is stated explicitly in Lemma 4.7 below, and is a key point in our constructions.

Proposition 4.2. — C^{k-1}-*adapted transverse coordinates exist for a volume preserving* C^k-*Anosov flow on a closed* 3-*manifold, for* $k \geqslant 3$.

Proof. — The foliations $\mathcal{F}^{ws}$ and $\mathcal{F}^{wu}$ are C^1 by the Hirsch-Pugh theory [30], and we assume that the Anosov distributions are orientable, so we can choose unit C^1-vector fields e^+ and e^- on M satisfying for each $p \in$ M:

* $e^+(p) \in E^{wu}(p)$ and $e^-(p) \in E^{ws}(p)$;
* $e^+(p)$ and $e^-(p)$ are orthogonal to $\xi(p)$.

There exist a constant $0 < c < 1$ such that the Riemannian exponential map $\exp : TM \to M \times M$ is a diffeomorphism into, when restricted to a c-tube around the zero section in TM. Thus, there exists a constant $0 < \varepsilon < c$ so that the map

$$(28) \qquad \begin{aligned} &E : M \times (-2\varepsilon, 2\varepsilon)^2 \to M \times M \\ &(p, (a, b)) \mapsto \exp(a.e^+(p) + b.e^-(p)) \end{aligned}$$

satisfies:

* E is a C^1-diffeomorphism into;
* $E_p(a, b) = E(p, (a, b))$ is C^∞ in the variables (a, b);
* $X_p \overset{\text{def}}{=} E_p((\varepsilon, \varepsilon)^2)$ is uniformly transverse to the vector field ξ, with angles bounded below by $\pi/4$.

Recall that L_p^{wu} is the curve through p in the transversal X_p contained in a leaf of the weak-unstable foliation, and L_p^{ws} is the corresponding curve for the weak-stable foliation. The images of these curves under the local coordinates E_p are C^k, and tangent to the x and y-axis, respectively, at the origin $(0, 0)$. Also, the C^k-germs of these image curves depend C^1 on the base point p by an application of the C^1-Section Theorem 3.5 of [51]. Therefore, we can introduce a C^k-change of coordinates on the domain of E_p,

$$(a, b) = h_p(\widetilde{a}, \widetilde{b}) = (\widetilde{a} + \psi_p(\widetilde{b}), \widetilde{b} + \varphi_p(\widetilde{a}))$$

where L_p^{wu} is given by the graph $(\tilde{a}, \varphi_p(\tilde{a}))$, and L_p^{ws} is given by the graph $(\psi_p(\tilde{b}), \tilde{b})$. Note that both φ_p and ψ_p have vanishing first derivative at $(0, 0)$, and their k-jets at $(0, 0)$ depend C^1 on the point p.

Finally, we introduce a C^k-change of coordinates $(\tilde{a}, \tilde{b}) = k_p(\tilde{x}, \tilde{y})$ with

$$k_p(\tilde{x}, 0) = (\tilde{x}, 0), \qquad k_p(0, \tilde{y}) = (0, \tilde{b})$$

so that the composition

$$(29) \qquad \Psi_p(\tilde{x}, \tilde{y}) \overset{\text{def}}{=} E_p \circ h_p \circ k_p(\tilde{x}, \tilde{y})$$

pulls the volume form dv on X_p back to the form $d\tilde{x} \wedge d\tilde{y}$. The pull-back volume form $(E_p \circ h_p)^*(dv)$ depends C^{k-1} on the point $(\tilde{x}, \tilde{y})$, so the coordinate map k_p can be chosen to be C^{k-1}, and depends C^1 on the point p. It is then easy to see that the map (29) defines C^{k-1}-adapted coordinates. $\square$

Let us introduce the spaces of vector fields that we work with:

$$\Gamma_\xi(TM) = \{ v \in \Gamma^k(TM) \mid \langle v(p), \xi(p) \rangle = 0 \ \forall p \in M \},$$

$$\Gamma_\xi^*(TM) = \{ v \in \Gamma_\xi(TM) \mid v(p) \neq 0 \ \forall p \in M \},$$

$$S\Gamma_\xi(TM) = \{ v \in \Gamma_\xi(TM) \mid \| v(p) \| = 1 \ \forall p \in M \}.$$

For $\delta > 0$, define a subset of $S\Gamma_\xi(TM)$,

$$(30) \qquad V(\delta) = \{ v \in S\Gamma_\xi(TM) \mid v = \alpha e^+ + \beta e^-, \text{ for } \alpha, \beta \in C^k(M)$$
$$\text{with } |\beta(p)| < \delta \text{ and } |\nabla\beta(p)| < \delta \ \forall p \in M \}.$$

The closure of this set in the C^1-uniform topology, denoted by $\overline{V(\delta)}$, is a compact set of C^1-vector fields containing e^+. It is clear from the definition that e^+ is the unique element of the intersection of all of the sets $\overline{V(\delta)}$, $\delta > 0$.

The differential Df_t of the flow does not map the set $V(\delta)$ to itself, but does have the property that for any $v \in V(\delta)$, $\langle Df_t(v(p)), e^+(f_t(p)) \rangle > 0$. Let $\pi_\xi : TM \to TM$ be the fiberwise projection map onto the C^k-subbundle of vectors orthogonal to ξ. Define a projectivized form of Df_t:

$$(31) \qquad \begin{array}{l} Pf_t : S\Gamma_\xi(TM) \to S\Gamma_\xi(TM) \\ Pf_t(v)\,(f_t(p)) = \| \pi_\xi(Df_t(v(p))) \|^{-1} \cdot \pi_\xi(Df_t(v(p))). \end{array}$$

Proposition **4.3.** — *1. $Pf_t \circ Pf_s = Pf_{s+t}$ for all $s, t \in \mathbf{R}$.*
2. For $v \in V(\delta)$, $\lim_{t \to \infty} Pf_t(v) = e^+$ uniformly in the C^1-topology.
3. For each $\delta > 0$, there exists $t(\delta) > 0$ such that $Pf_t : V(\delta) \to V(\delta)$ for $t > t(\delta)$.

Proof. — $(4.3.1)$ is immediate as $Df_t(\xi) = \xi$. $(4.3.2)$ is a direct consequence of the Anosov conditions (1), which imply that e^+ is exponentially expanded and e^- is exponentially contracted by Df_t for t large. As the multipliers are equal, a local calculation shows that the expansion and contraction are uniform in the C^1-topology. (See

the C^r-Section Theorem 3.5 of [51] for the details of the proof.) Define a 1-cocycle over the flow by the rule

$$(32) \qquad \pi_\xi(Df_t(e^+)\,(f_t(p))) = \mu^+(p, t)\, e^+(f_t(p)), \quad \text{for } \mu^+ \in C^k(M \times \mathbf{R}).$$

Similarly, we define $\mu^-(p, t)$ for e^-. The Anosov condition implies that

$$\lim_{t \to \infty} \mu^+(p, t) = \infty, \qquad \lim_{t \to \infty} \mu^-(p, t) = 0,$$

uniformly in p. Together these imply (4.3.3). $\square$

Let us fix $p \in M$, with L_p^- the " stable " curve through p defined by (4.1.2), and y the local coordinate for this curve. Given a vector field $v \in \Gamma_\xi(TM)$, its restriction to X_p is projected onto TX_p, and then expressed in the local coordinates Ψ_p as $(v_1(x, y), v_2(x, y))$. Denote the restriction of this vector field on $(-\varepsilon, \varepsilon)^2$ to the y-axis by $(v_1(y), v_2(y))$. This construction depends upon the basepoint p; we emphasize this fact by using the notation $v_p(y) = (v_{p,1}(y), v_{p,2}(y))$. Let D_y denote the derivative with respect to the y-coordinate.

Our local " norm " based on the function Ω is defined for $v \in V(\delta)$ when $\delta < 1$. The definition requires locally rescaling the vector field. For each $p \in M$, let $\tilde{v}_p$ denote the local vector field at p along the stable manifolds L_p^- obtained from v_p by pointwise scaling so that in coordinates we have $\tilde{v}_p(y) = (1, a_p(y))$. For each $\varepsilon_0, k > 0$, introduce the set

$$(33) \qquad V(\delta; \varepsilon_0; k) = \{ v \in V(\delta) \mid \forall p \in M, \; \forall y \quad \text{with } \varepsilon_0 \leqslant |y| \leqslant \varepsilon,$$

$$(34) \qquad | D_y(a_p)\,(y) - D_y(a_p)\,(0) | \leqslant k.|y|.| \log(|y|) | \}.$$

The condition (34) is closed in the C^1-topology for fixed $\delta < 1$, so each set $V(\delta; \varepsilon_0; k)$ is C^1-precompact.

Our approach to the regularity of $\mathscr{F}^{wu}$ is based on the following technical result. The interested reader can skip ahead to Corollary 4.8 to see how it implies $C^{1, \Omega}$-regularity.

Proposition **4.4.** — *For* $0 < \delta < 1$, *there exist constants* $k, T > 0$ *and* $0 < \varepsilon_0 < \varepsilon$, *so that for all positive integers* n,

$$Pf_{nT} : V(\delta) \to V(\delta; \varepsilon_0^n; k).$$

Proof. — The proof is based on induction. We start with an elementary observation.

Lemma **4.5.** — *Given* $0 < \delta < 1$ *and* $0 < \varepsilon_0 < 1$, *there exists* $k > 1$ *so that* (34) *holds for all* $v \in V(\delta)$. *That is,* $V(\delta) \subset V(\delta; \varepsilon_0; k)$.

Proof. — The uniform continuity in $v \in V(\delta)$ of $v_{1,p}(y)$ and $D_y(v_p)(y)$ on the set $(p, y) \in M \times \{ |y| \leqslant \varepsilon \}$ implies there is a maximum of the left-hand-side of the expression (34) on the set $V(\delta)$. Fixing $\varepsilon_0 > 0$, we can then choose $k > 0$ so that (34) holds uniformly on $V(\delta)$. $\square$

Fix $0 < \delta < 1$, then choose $T > t(\delta) > 0$ so large that the function μ defined in (32) satisfies $\mu(p, t) > 4$ for all $p \in M$, $t > T$. Fix $p \in M$ and set $p' = f_T(p)$. Let $F : X_p \to X_{p'}$ denote the Poincaré return map for the flow. That is, there is a C^k function $\tau : X_p \to \mathbf{R}$ with $\tau(p) = 0$ and, for $q \in X_p$, $f_{t + \tau(q)}(q) \in X_{p'}$. Then $F(q) = f_{t + \tau(q)}(q)$.

Note that the function $\mu(p, T)^{-1} < 1/4$ is the exponent of contraction for the local maps F with respect to the unit vector fields e^- along the curves L_p^-. The vector field e^- need not be tangent to this curve, but by the hypothesis (4.1.1) on adapted coordinates, it makes an angle at most $\pi/4$ with the tangent line to the curve. Thus, for all $p \in M$ and $q \in L_p^-$, the projection of $e^-(q)$ to the line $T_q L_p^- \subset T_q X_p$ has length at least $1/\sqrt{2}$, with a similar bound on the converse projection. It then follows from our choice of T that F is a strict contraction with exponent less than $1/2$ from the curve L_p^- into the curve $L_{p'}^-$.

Introduce the coordinate y along L_p^- and z along $L_{p'}^-$. Write $(u, z) = F(x, y)$ and more specifically, $z = F(0, y)$. For $v \in V(\delta)$, let $(1, b_{p'}(z))$ denote the local rescaling of v_T at p'. The heart of the proof of Proposition 4.4 is the next result, which follows from explicit local calculations.

Proposition **4.6.** — *There exist* $0 < \varepsilon_1 < \varepsilon$, $\delta > 0$, *and* $k > 1$ *such that for all* $p \in M$ *and* $|y| < \varepsilon_1$, *if* $v \in V(\delta)$ *satisfies the estimate*

$$(35) \qquad | D_y(a_p) (y) - D_y(a_p) (0) | \leqslant k. |y|. |\log(|y|)|,$$

then

$$(36) \qquad | D_z(b_{p'}) (z) - D_z(b_{p'}) (0) | < k. |z|. |\log(|z|)|.$$

Proof. — Expand the local coordinate form of F into the " Moser local form ",

$$(u, z) = F(x, y) = (\mu x + \varphi(x, y), \mu^{-1} y + \psi(x, y)),$$

where $\mu = \mu(p, T)$ as defined in (32), and $\varphi_x(0, 0) = \psi_y(0, 0) = 0$. (Recall that subscripts denote the respective partial derivatives.)

Lemma **4.7.** — *The local forms* φ *and* ψ *satisfy the differential identities*:

1. $(\mu + \varphi_x) (\mu^{-1} + \psi_y) - (\varphi_y) (\psi_x) = 1$ *at all points* (x, y);
2. $\varphi_x(0, 0) = \varphi_y(0, 0) = 0$; $\psi_x(0, 0) = \psi_y(0, 0) = 0$;
3. $\varphi_{xy}(0, 0) = \varphi_{xx}(0, 0) = \varphi_{yy}(0, 0) = 0$; $\psi_{xy}(0, 0) = \psi_{yy}(0, 0) = \psi_{xx} = 0$;
4. *if* φ *and* ψ *are* C^3 *in a neighborhood of* $(0, 0)$, *then* $\varphi_{yyy}(0, 0) = 0 = \psi_{xxx}(0, 0)$ *and* $\mu^{-1} \varphi_{yxx}(0, 0) + \mu \psi_{xyy}(0, 0) = 0$.

Proof. — The 2-form $i(\xi) (d \, \mathrm{vol})$ on M is a transverse invariant volume form for the flow, as $d \, \mathrm{vol}$ is flow invariant. Therefore the restrictions $d\nu$ to transversals are invariant under the Poincaré maps $F = F_p$. The local coordinates are chosen so that $\Psi_p^*(d\nu) = dx \wedge dy$, so in coordinates $F^*(du \wedge dz) = dx \wedge dy$, which implies (4.7.1). The manifolds L_p^+ and L_p^- are invariant under the map F, which in coordinates is equivalent to $\varphi(0, y) = 0 = \psi(x, 0)$ for all (x, y). Differentiating this relation yields (4.7.2). Pro-

perty (4.1.3) implies that the vector field $DF_{(x,y)}(\partial/\partial x) = (\mu + \varphi_x, \psi_x)$ is C^1-tangent to the vector field $\partial/\partial u$ at the origin $(0, 0)$. This yields $\varphi_{xy}(0, 0) = \psi_{xy}(0, 0) = 0$. Differentiating equation (4.7.1) and using that these mixed partials vanish yields $\varphi_{xx}(0, 0) = \psi_{yy}(0, 0) = 0$. Finally, differentiating (4.7.1) twice, with respect to x and y, and using (4.7.2) and (4.7.3) yields the equality (4.7.4). $\square$

Fix an $\varepsilon_0 < \varepsilon$, and consider $v \in V(\delta)$. We obtain an estimate for $w_{p'} \stackrel{\text{def}}{=} v_{\mathrm{T},\,p'} = Pf_{\mathrm{T}}(v)_{p'}$ in terms of the functions φ and ψ, and the local coordinates of v_p. Let $\tilde{v}$ denote the local vector field at p obtained from v_p by pointwise scaling so that in coordinates we have $\tilde{v}(0, y) = (1, a(y))$. Then apply DF to $\tilde{v}$ and rescale to obtain a vector field $\tilde{w}$ at p' which in local coordinates has the form $\tilde{w}(0, z) = (1, b(z))$. Elementary calculation then gives the following differential expression for $b(z)$ in terms of $a(y)$ and the partial derivatives of φ and ψ:

$$(37) \qquad b(z) = \frac{(\mu^{-1} + \psi_y) \cdot a + \psi_x}{(\mu + \varphi_x) + \varphi_y \cdot a}\,(y(z)).$$

Let us expand all of the terms in equation (37) into their second order expansions: the first order partials of φ and ψ vanish uniformly at $(0, 0)$; the coordinate expression $z = \mu^{-1} y + (1/2)\,\psi_{yy}(0)\,y^2 + o(|y|^2)$ inverts to give the expression

$$y = \mu z - (1/2)\,\psi_{yy}(0)\,\mu^3\,z^2 + o(|z|^2);$$

the function $a(y)$ is C^1, so we can write it as $a(y) = a_0 + a_1 y + A(y)$, where A is a C^1-function with $A(0) = 0$ and vanishing first derivative $A'(y)$ at 0. Expanding (37) in these second order terms then yields the simple estimate

$$(38) \qquad D_z(b)\,(z) - D_z(b)\,(0) = \mu^{-1}\,A'(\mu z) + O(|z|).$$

Our hypothesis (35) translates into the estimate:

$$(39) \qquad |A(y)| \leqslant k \cdot |y| \cdot |\log(|y|)|.$$

Expand $b(z) = b_0 + b_1 z + B(z)$ with $B(0) = B'(0) = 0$; then, combining the previous two estimates and substituting in the second order expansion for $y = y(z)$, we obtain

$$(40) \qquad |B(z) - B(0)| \leqslant k \cdot (|z| + O(|z|^2)) \cdot |\log(|z|) + \log(\mu) + O(|z|)|.$$

Now require $\varepsilon_1 > 0$ to be sufficiently small so that $-\log(|z|) \gg \log(\mu)$ for $|z| < \varepsilon_1$. This can be chosen to hold uniformly in p. The right hand side of (40) is then estimated by

$$(41) \qquad |B(z) - B(0)| \leqslant k \cdot |z| \cdot |\log(|z|)| - k \cdot \log(\mu)\,(|z| + o(|z|)) + O(|z|)$$

The error term, $O(|z|)$, which arises from the second order terms in the expansions of the terms in (34), is uniform in p and independent of k. Thus, for a suitably large choice of k and $\varepsilon_1 > 0$ small as indicated above, we have that

$$-\{k \cdot \log(\mu)\,(|z| + o(|z|))\} + O(|z|) < 0$$

uniformly in $p \in M$ and $|z| < \varepsilon_1$. Thus $|B(z) - B(0)| < k \cdot |z| \cdot |\log(|z|)|$, which was to be shown. $\square$

It remains to deduce Proposition 4.4 from Proposition 4.6. Choose $\varepsilon_1 > 0$ and $0 < \delta < 1$ so that Proposition 4.6 is satisfied. Choose $0 < \varepsilon_0 < \varepsilon_1$ such that for all $p \in M$, $F_p(0, \varepsilon_1) > \varepsilon_0$, and similarly $F_p(0, -\varepsilon_1) < -\varepsilon_0$. (This condition has a very simple geometric interpretation: we require that the Poincaré maps F_p send the segments of the curves in L_p^- corresponding to these coordinate intervals, to curves in L_p^- which overlap with them. Thus, the union of the forward images of the segments $\Psi_p(\{(0, y) \mid \varepsilon_0 < |y| < \varepsilon_1\})$ covers all of a deleted neighborhood of $p \in L_p^-$.) Let $k > 0$ be given by Lemma 4.5 for this value of ε_0.

Fix $v \in V(\delta)$ and $n \geqslant 1$. We must show that $Pf_{nT}(v) \in V(\delta; \varepsilon_0^n; k)$. Fix $p \in M$ and y so that $\varepsilon_0^n < |y| < \varepsilon$. By our choice of ε_0, there exists an integer $0 < m \leqslant n$ such that for some y'' with $\varepsilon_0 < |y''| < \varepsilon_1$, we have $F^m(0, y'') = (0, y)$. Now observe that we can apply Proposition 4.6 iteratively to the vector field v and the points $f_{iT}(p)$ for $0 \leqslant i < m$, as the map F is a strict contraction on the sets $L_{f_{iT}(p)}^-$. As the estimate (35) holds for v, we conclude that (36) holds for $(0, z) = F^m(0, y'') = (0, y)$ as was to be shown. $\square$

Corollary **4.8.** — *The vector field* e^+ *is in the class* $C^{1,\Omega}$.

Proof. — Choose $\delta, k, T > 0$ as in Proposition 4.4. For $v \in V(\delta)$, the vector fields $v_n = Pf_{nT}(v)$ converge in the uniform C^1-topology to e^+ by Proposition 4.3.2, hence the local rescaled vector fields $\tilde{v}_{nT, p}$ converge uniformly to the rescaled field $\tilde{e}_p^+$. Then we have, for $p \in M$ and $|y| < \varepsilon_1$,

$$(42) \qquad |D_y(\tilde{e}_p^+)(y) - D_y(\tilde{e}_p^+)(0)| = \lim_{n \to \infty} |D_y(\tilde{v}_{nT, p})(y) - D_y(\tilde{v}_{nT, p})(0)|$$

$$\leqslant k.|y|.|\log(|y|)|.$$

since we can apply the estimate (34) for arbitrary $\varepsilon_0 > 0$ as $nT \to \infty$. This shows that the local fields $\tilde{e}_p^+$ are $C^{1,\Omega}$ in the y-coordinate. The vector field e^+ is locally obtained from the local field $\tilde{e}_p^+$ by dividing by its length, so it will also be $C^{1,\Omega}$ along the stable manifold L_p^-. The vector field e^+ is known to be C^k along the unstable manifold L_p^{wu}, so that e^+ is $C^{1,\Omega}$ uniformly in p. $\square$

A variant of the proof of Proposition 4.6 yields the estimate needed to establish that Zygmund regularity of the vector field e^+. The key result is obtained by using the full strength of the second order expansion of (37).

Proposition **4.9.** — *There exist* $0 < \varepsilon_2 < \varepsilon$, $\delta > 0$ *and a monotone decreasing function* $C(k) > 0$ *of* $k > 0$ *such that for all* $p \in M$ *and* $|y| < \varepsilon_2$, *if* $v \in V(\delta)$ *satisfies the estimate (in the notation of Proposition 4.6),*

$$(43) \qquad |D_y(a_p)(y) + D_y(a_p)(-y) - 2D_y(a_p)(0)| \leqslant k|y|,$$

then,

$$(44) \qquad |D_z(b_{p'})(z) + D_z(b_{p'})(-z) - 2D_z(b_{p'})(0)| < k|z|(1 + |z| C(k)).$$

Proof. — The second order expansion of the terms appearing in (37) yields the following estimate of the left-hand side of (44)

(45) $| D_z(b_{p'}) (z) + D_z(b_{p'}) (-z) - 2D_z(b_{p'}) (0) |$

$$= \mu^{-1} | A'(y(z)) + A'(y(-z)) - 2A'(0) | + O(| z |^2)$$

$$\leqslant k\{ | z | + \mu^2 \, \psi_{yy}(0)| z |^2/2 \} + O(| z |^2)$$

$$\leqslant k | z | (1 + | z | C(k)),$$

where $C(k)$ is chosen so that

(46) $C(k) \geqslant \mu^2 \, \psi_{yy}(0)/2 + O(| z |^2)/k | z |^2$

uniformly in $p \in M$. Note that the second order error term $O(| z |^2)$ is independent of k, so we can choose $C(k)$ to be uniformly decreasing. □

The inductive estimate (44) is used to prove the analogue of Proposition 4.4.

Proposition 4.10. — *For $0 < \delta < 1$, there exist constants $K > 0$ and $0 < \varepsilon_3 < 1$ so that, for all positive integers n and for all $v \in V(\delta)$, the vector field $v_{nT} = Pf_{nt}(v)$ satisfies the local estimates*

(47) $| D_y(a_{nT, p})(y) + D_y(a_{nT, p})(y) - 2D_y(a_{nT, p})(0) | \leqslant K | y |; \ 2^{-n}\varepsilon_3 < | y | < \varepsilon_3.$

Proof. — Fix $0 < \delta < 1$. Recall that T was chosen so that the local expansion constants $\mu(p, T) > 4$, so we can choose $\varepsilon_3 > 0$ such that at every point $p \in M$, the local coordinate expansion of the Poincaré map F for the flow f_T satisfies $| \mu^{-1} + \psi_y(y) | < 1/2$ for all $| y | < \varepsilon_3$.

Select ε_4 with $0 < \varepsilon_4 < \varepsilon_3$ so that the forward images of the segments with coordinates $\{ \varepsilon_4 < | y | < \varepsilon_3 \}$ under the iterates F^n cover the deleted stable manifolds, as in the proof of Proposition 4.4. For example, choose $\varepsilon_4 < \inf\{ | \mu^{-1} + \psi_y(y) | . \varepsilon_3 : | y | < \varepsilon_3 \}$. Then choose a constant K_0 so that there is the uniform estimate for all $v \in V(\delta)$ and $p \in M$:

$$| D_y(a_p) (y) + D_y(a_y) (-y) - 2D_y(a_p) (0) | \leqslant K_0 \, \varepsilon_4 < K_0 . | y |$$

$$\text{for } \varepsilon_4 < | y | < \varepsilon_3.$$

Set

$$K_n = K_0 . \prod_{m-1}^{n} (1 + 2^{-m} C(K_0) \, \varepsilon_3),$$

$$K = \lim_{n \to \infty} K_n.$$

Let y satisfy $2^{-n} \varepsilon_3 < | y | < \varepsilon_3$. Then for some $0 \leqslant m \leqslant n$, there is y'' with $\varepsilon_4 < | y'' | < \varepsilon_3$ so that $y = F^m(y'')$. Note that $d_j = | F^j(y'') | < 2^{-j} \varepsilon_3$, so we can successively apply Proposition 4.9 and the estimate (44) with $k = K_j$ and

$$C(k) = C(K_j) < C(K_0)$$

to the vector field $v_{(n-j)T} \in V(\delta)$ to obtain

(48) $| D_z(a_{nT, p'}) (z) + D_z(a_{nT, p'}) (-z) - 2D_z(a_{nT, p'}) (0) | < K_m | z | \leqslant K | z |$

which finishes the proof of Proposition 4.10. □

We conclude this section with the proof of Theorem 3.1.1. The vector field e^+ is the limit in the uniform C^1-topology of vector fields v_{nT}, so we obtain from Proposition 4.10 a uniform estimate of the local renormalized form of e^+ along the stable curves L_p^-. This implies that e^+ is of class C^{1,Λ_*} along each curve L_p^-, uniformly in p. The field e^+ is C^k on the submanifolds L_p^{wu}, so we obtain the desired estimate locally about p, uniform in p. $\square$

5. Anosov Cocycle and Local Obstructions to Regularity

As noted in the Introduction, Anosov found obstructions to the stable and unstable foliations being C^2 for a volume-preserving C^3 Anosov diffeomorphism of T^2. Similar obstructions can be found for flows via selecting cross-sections to the flow. One of the original observations which led to this work was the realization that the Anosov obstructions represent the periodic data for a 1-cocycle which arises from the induced action on the 2-jets transverse to the flow.

In this section, we construct the Anosov cocycle A_f^Ψ for a volume-preserving, C^4-Anosov flow, and show that its cohomology class, A_f, is a flow invariant. Geometrically, the value of this cocycle is the hyperbolic twist of the Poincaré map in the adapted transverse coordinates to the flow; at periodic orbits this is the first obstruction to linearizing the Poincaré return map.

Anosov's observation can also be generalized in a different direction. Namely, if the value of the cocycle along a periodic orbit through $p \in M$ does not vanish, then the semi-norm $\| \ \|_p^\Omega$ of $D_y(e^+)$, associated to the modulus of continuity $\Omega(s) = s\,|\log(s)|$, does not vanish at p. This shows that the assertion of Theorem 3.1.1 cannot be improved by replacing Ω with any other modulus of continuity.

We assume that the flow is at least C^3 and admits C^3-adapted coordinates. For example, these always exist for a C^4-flow by Proposition 4.2. For $0 < \delta < 1$ and $v \in V(\delta)$, adopt the convention introduced in the last section that $\tilde{v}_p$ denotes the rescaled vector field defined in a neighborhood of $p \in M$, which is given in adapted transverse coordinates along the stable curve L_p^- by an expression $(1, a_p(y))$. Let $\tilde{v}_{t,f_t(p)}$ denote the rescaled image of $\tilde{v}$ under the Poincaré map $F_t : X_p \to X_{f_t(p)}$, with local form $(1, a_{t,f_t(p)}(z))$.

Lemma **5.1.** — *Let $\tilde{v}$ have local form $(1, a_p)$ with second order expansion*

$$a_p(y) = a_0 + a_1 y + y\alpha_p(y) + o(|y|^2),$$

where $\alpha_p(y)$ is continuous and vanishes at 0. Then the rescaled vector field $(1, a_{t,f_t(p)}(z))$ has second order expansion for z near to 0 given, for $\mu = \mu(p, t)$, by

$$(49) \qquad a_{t,f_t(p)}(z) = \mu^{-2} a_0 + \mu^{-1} a_1 z + \mu^{-1} z\alpha_p(\mu z) + A_f^\Psi(p, t)\, z^2$$
$$+ a_0\, B_f^\Psi(p, t)\, z^2 + o(|z|^2),$$

where $A_f^\Psi(p, t) = (1/2)\,\mu\psi_{xvv}(0, 0)$ *and* $B_f^\Psi(p, t) = (1/2)\,\{\,\mu\psi_{vvv}(0, 0) - \mu^{-1}\varphi_{xvv}(0, 0)\,\}.$

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Proof. — Write each of the functions of y appearing in (37) in their second order expansions, then use Lemma 4.7 to simplify the resulting quotient, noting that we have $y = \mu z + o(|z|^2)$. ☐

We can apply Lemma 5.1 to the vector field e^+, which is characterized by the property that $a_0 = 0$ for every $p \in M$. Equation (49) simplifies to

(50) $\qquad a_{1, f_t(p)}(z) = \mu^{-1} a_1 z + \mu^{-1} z\alpha_p(\mu z) + A_f^\Psi(p, t) z^2 + o(|z|^2).$

Isolating the second order component of (50) yields

(51) $\qquad \alpha_{f_t(p)}(z) = \mu^{-1} \alpha_p(\mu z) + A_f^\Psi(p, t) z + o(|z|),$

which shows that A_f^Ψ measures the translational contribution by the Poincaré map to the " α "-term of e^+. We develop this remark later in the section to obtain obstacles to e^+ being $C^{1,\omega}$ if A_f^Ψ does not vanish. Let us first make a few basic observations about the coefficient A_f^Ψ.

Corollary 5.2 *(Anosov Obstacles).* — *One has* $A_f^\Psi(p, t_p) = 0$ *if* E^{wu} *is* C^2 *at the periodic point* p *for the flow.*

Proof. — Assume that the vector field e^+ has a second order expansion with $\alpha(y) = a_2 y + o(|y|)$ at a periodic point p. Choose $v = e^+$ and $t = t_p$ in Lemma 5.1, then note that (51) reduces to $a_2 = a_2 + A_f^\Psi(p, t_p)$. ☐

Proposition 5.3 *(Anosov Class).* — *Let* $k \geqslant 3$.

1. Let Ψ *be a choice of* C^k-*adapted coordinates. Then* $A_f^\Psi : M \times \mathbf{R} \to \mathbf{R}$ *is a* C^1-*cocycle over the flow.*

2. The C^1-*cohomology class of* A_f^Ψ *is independent of the choice of Riemannian metric on* TM *and* C^k-*adapted coordinates.*

Proof. — (1) The C^k-immersions Ψ_p depend C^1 on the basepoint p by (4.1.2), so the third jet of the return map of f_t will depend C^1 on p in these adapted transverse coordinates. Hence the expression $A_f^\Psi(p, t) = (1/2) \mu(p, t) \psi_{zw}$ depends C^1 on p also. The cocycle law for A_f^Ψ is the identity

$$A_f^\Psi(p, t + s) = A_f^\Psi(p, t) + A_f^\Psi(f_t(p), s)$$

which follows from explicit calculation, using the chain rule and the identities of Lemma 4.7.

(2) Let Ψ and $\widetilde{\Psi}$ be two choices of adapted coordinates for the flow $\{f_t\}$, possibly with respect to different Riemannian metrics on TM. Then at each $p \in M$, there is a local C^3-diffeomorphism $(\tilde{x}, \tilde{y}) = T_p(x, y)$ such that, for some C^3-function $\sigma : (-\varepsilon, \varepsilon)^2 \to \mathbf{R}$ with $\sigma(0, 0) = 0$,

$$f_{\sigma(x, y)} \circ \Psi_p(x, y) = \widetilde{\Psi}_p \circ T_p(x, y).$$

The transverse 2-form $dv = i(\xi)$ $(d\,\mathrm{vol})$ is flow invariant, so the coordinate change maps T_p preserve the volume 2-form $dx \wedge dy$. The stable and unstable manifolds through p correspond to the x and y axes in coordinates, so T_p must also preserve the x and y axes. From this we observe that the identities of Lemma 4.7 also hold for the maps $T_p(x,y) = (T_p^1(x,y), T_p^2(x,y))$. (If the metrics are the same for both sets of adapted coordinates, we also have that the partials $T_{p,x}^1(0, 0) = T_{p,y}^2(0, 0) = 1$.)

The local Poincaré maps with respect to the two sets of adapted coordinates are related by

$$(52) \qquad T_{f_t(p)}^{-1} \circ \widetilde{F}_p \circ T_p = F_p.$$

The calculation used to show that A_f^{Ψ} is a cocycle in (5.3.1) above shows more generally that for any local volume-preserving C^3-diffeomorphism T fixing $(0, 0)$, the expression $T_x^1(0, 0)\,T_{xy}^2(0, 0)$ is an additive quantity. That is, under composition of such maps this expression combines linearly. Applying this remark to the equation (52), and letting $\Phi(p) = (1/2)\,T_{p,x}^1(0, 0)\,T_{p,xy}^2(0, 0)$, we obtain

$$(53) \qquad - \Phi(f_t(p)) + A_f^{\widetilde{\Psi}}(p, t) + \Phi(p) = A_f^{\Psi}(p, t).$$

The function $\Phi(p)$ depends C^1 on the basepoint p by the properties of adapted coordinates and the chain rule, so this completes the proof of (5.3.2). □

Corollary 5.4. — *Let $\{f_t\}$ and $\{\widetilde{f}_t\}$ be volume-preserving, C^6-Anosov flows on closed 3-manifolds* M *and* $\widetilde{\mathrm{M}}$, *respectively. Let* $\Theta : \mathrm{M} \to \widetilde{\mathrm{M}}$ *be a C^1-diffeomorphism conjugating the two flows up to a time-shift. Then the induced map* Θ^* *on 1-cocycles identifies the respective Anosov classes:* $\Theta^* \widetilde{A}_{\widetilde{f}} = A_f$.

Proof. — The continuous invariant volume form for a transitive Anosov flow is unique up to a scalar multiple, so Θ must be volume-preserving, and hence Θ is C^3 by an application of the regularity theory in [50], and Theorem 2.6 for $k = 6$ and $n = 2$ the dimension of the transversal to the flow. Let Ψ be adapted coordinates on M, and $\widetilde{\Psi}$ be adapted coordinates on $\widetilde{\mathrm{M}}$. The pull-back Anosov cocycle $\Theta^* \widetilde{A}_{\widetilde{f}}^{\widetilde{\Psi}}$ is clearly the Anosov cocycle over $\{f_t\}$ constructed from the C^3-adapted transverse coordinates $\Theta^{-1} \circ \widetilde{\Psi}$, which by Proposition 5.3 is cohomologous to A_f^{Ψ}. □

Proposition 5.3 and Corollary 5.4 together yield a proof of Theorem 3.3. In the remainder of this section we prove the results that are used to establish Theorem 3.2. We first examine the applications of the Livshitz Theorem 2.1 to the vanishing of the Anosov class.

Proposition 5.5. — *Let $\{f_t\}$ be a volume-preserving, C^4-Anosov flow on the closed 3-manifold* M.

1. For each period orbit $p \in \mathrm{M}$ of period t_p, the Anosov obstacle $A_f(p, t_p) \overset{\mathrm{def}}{=} A_f^{\Psi}(p, t_p)$ is independent of the choice of adapted transverse coordinates.

2. The Anosov class $A_f = 0$ *if and only if* $A_f(p, t_p) = 0$ *for all periodic orbits.*

3. $A_f = 0$ *if either* E^{ws} *or* E^{wu} *has a measurable transverse second derivative on a set of positive Lebesgue measure in* M.

4. $A_f = 0$ *if the local angle function between* E^{ws} *and* E^{wu} *has a measurable second derivative on a set of positive Lebesgue measure in* M.

Proof. — (1) At a periodic point, the equation (53) reduces to $A_f^{\Psi}(p, t_p) = A_f^{\tilde{\Psi}}(p, t_p)$.

(2) If the Anosov class is zero, then the coboundary equation (3) shows that $A_f^{\Psi}(p, t_p) = 0$ at periodic orbits. The converse is Theorem 2.1.3.

(3, 4) The set of points $p \in M$ where the vector field e^+ has a transverse second derivative is measurable and flow invariant. As the flow is ergodic, this set must have either measure zero or full measure. Each of the hypotheses implies that the local functions $\alpha_p(y)$ have a derivative at $y = 0$ for a set p of positive measure, and hence $a_2(p) = \alpha'_p(0)$ exists for a flow invariant set of p with full measure. The equation (53) implies

$$(54) \qquad a_2(f_t(p)) - a_2(p) = A_f^{\Psi}(p, t).$$

Then by the Livshitz Theorem 2.1 there is a C^1-extension of $a_2(p)$ so that equation (54) holds everywhere, and thus $A_f = 0$. □

The last result of this section is a local calculation relating the non-vanishing of $A_f(p, t_p)$ with the modulus of continuity of the field e^+ at p. This will complete the proof of Theorem 3.2. We formulate the result in a general fashion.

Let $F : (-\epsilon, \epsilon)^2 \to \mathbf{R}^2$ be a C^3-embedding for which the local coordinates $F(x, y) = (\mu x + \varphi(x, y), \mu^{-1} y + \psi(x, y))$ satisfy $\mu > 1$, and

$$\varphi(x, 0) = 0 = \psi(0, y),$$

$$\varphi_x(0, 0) = 0 = \psi_y(0, 0),$$

$$(\mu + \varphi_x(x, y))(\mu^{-1} + \psi_y(x, y)) - \varphi_y(x, y)\psi_x(x, y) = 1.$$

These conditions imply that the first and second partial derivatives of φ and ψ vanish at $(0, 0)$ (cf. proof of Lemma 4.7). We can assume that F extends to a hyperbolic, volume-preserving C^3-diffeomorphism of $\mathbf{R}^2$, and so can define stable and unstable invariant vector fields e^- and e^+, which will be tangent to the y and x axes, respectively, near the origin. In a neighborhood of $(0, 0)$ we can assume that e^+ has the coordinate form $e^+(x, y) = (1, a(x, y))$. An easy calculation shows that e^+ is C^1 at $(0, 0)$, so there is an expansion $a(0, y) = a_1 y + \tau(y)$ where $\tau(y) = y\alpha(y)$ and α is a C^1-function away from $y = 0$, and continuous at $y = 0$ with $\alpha(0) = 0$. We define $A_F = (1/2) \mu\psi_{xyy}(0, 0)$.

Define a semi-norm on the vector field e^+ by setting

$$(55) \qquad \| D_y(e^+) \|_0^{\Omega} = \sup_{0 < |y| < \epsilon} \frac{\tau'(y)}{|y||\log(|y|)|}.$$

Theorem **5.6.** — *Let f be a C^3-map as above. Then*

$$(56) \qquad \| D_y(e^+) \|_0^\Omega \geq \frac{| A_F |}{\log(\mu)}.$$

Proof. — Make a volume-preserving, C^3-change of coordinates so that $F(x, 0) = (\mu x, 0)$ and $F(0, y) = (0, \mu^{-1} y)$. The vector field e^+ is projectively invariant under the differential DF, so, for $|y| < \varepsilon$, applying DF and rescaling yields a recursive equation for α, which is a specialized form of (53):

$$(57) \qquad \alpha(\mu^{-1} y) = \mu^{-1} \{ \alpha(y) + A_F y + o(|y|) \}.$$

Iterate the estimate (57) to obtain the general formula

$$(58) \qquad \alpha(\mu^{-n} y) = \mu^{-n} \{ \alpha(y) + n A_F y + E(n, y) \},$$

$$(59) \qquad E(n, y) = \sum_{i=0}^{n-1} \mu^{i+1} o(\mu^{-i} |y|).$$

Fix a point $0 < y < \varepsilon$ and $n > 0$. By the Mean Value Theorem, there is a point z_n with $0 < z_n < \mu^{-n} y$ so that

$$\tau'(z_n) = \frac{\tau(\mu^{-n} y)}{\mu^{-n} y} = \alpha(\mu^{-n} y).$$

We can assume without loss that $\varepsilon < 1/e$ so that the function $s \, | \log(s)|$ is monotone increasing for $0 < s < \varepsilon$, and thus

$$\frac{| \tau'(z_n) |}{z_n | \log(z_n) |} \geq \frac{| \alpha(\mu^{-n} y) |}{(\mu^{-n}) | \log(\mu^{-n} y) |}.$$

By (58), the right-hand-side is equal to

$$(60) \qquad \left| \frac{\alpha(y)}{y(n \log(\mu) - \log(y))} + \frac{n . A_F}{(n \log(\mu) - \log(y))} + \frac{E(n, y)}{y(n \log(\mu) - \log(y))} \right|.$$

By a delicate and fortuitous coincidence, the term $E(n, y)/n$ is seen to tend to zero as $n \to \infty$, so the limit of (60) is $| A_F |/\log(\mu)$, which establishes (56). $\square$

Corollary **5.7.** — *Let $\{ f_t \}$ be a volume-preserving, C^3-Anosov flow on a 3-manifold M, and p a periodic orbit with period t_p. If $A_f(p, t_p) \neq 0$, then E^{wu} is not $C^{1, \omega}$ at p. More precisely, the first transverse derivative at p of the C^1 vector field e^+ has the estimate*

$$(61) \qquad \| D_y(e^+) \|_p^\Omega \geq | A_f(p, t_p) | / \log(\mu(p, t_p)).$$

6. Smooth Rigidity

Smooth rigidity for Anosov flows is the phenomenon that when appropriate cohomology invariants vanish, the weak-unstable and weak-stable foliations are smooth. As discussed in the Introduction, it is then known in some cases that the flow is smoothly conjugate to an algebraic flow. In this section, we show that when the Anosov cocycle

of a volume-preserving C^k-Anosov flow on a 3-manifold is a C^1-coboundary, then the weak-unstable foliation is C^{k-3}. The same conclusion then follows for the weak-stable foliation by the time-reversing symmetry between the stable and unstable foliations, which completes the proof of Theorem 3.4.

The critical aspect of the proof of C^{k-3}-regularity of the weak-unstable foliation is to show that the field e^+ is transversally C^3. We then invoke a standard bootstrap method for hyperbolic systems to deduce that e^+ is transversally C^k, and an application of Theorem 2.6 yields that e^+ is C^{k-3} on M.

Fix C^k-adapted coordinates Ψ for the flow f_t, and let $a : M \to \mathbf{R}$ be a C^1-function which satisfies the coboundary equation

(62) $A_f^\Psi(p, t) = a(f_t(p)) - a(p)$ for all $(p, t) \in M \times \mathbf{R}$.

In section 4 we observed that for a C^k-vector field $v \in V(\delta)$ for $\delta < 1$, the forward images $v_t = Pf_t(v)$ converge exponentially fast to the field e^+ in the uniform C^1-topology. This was codified in the equation (49). The idea of the proof is to use the coboundary relation (62) to deduce that the formal second derivative terms in the Taylor expansions of the local fields $\tilde{v}_t = (1, a_{t, p}(y))$ are bounded independently of t and $p \in M$, from which we deduce that the field is transversally $C^{1,1}$. An application of the uniqueness part of the Livshitz Theorem then implies that the local fields $\tilde{v}_t$ are Cauchy in the uniform C^3-topology on y, from which we deduce that e^+ is transversally C^3. This method is a combination of techniques, similar to those utilized in the Hirsch-Pugh Theory of regularity: We use a compactness condition on the transversal 2-jets and the Livshitz Theorem 2.1.1, 2.1.2 to obtain a formal candidate for the transversal 2-jet of e^+; the uniqueness part of the Livshitz Theorem and properties of absolutely continuous functions on the line then imply that the formal transversal 2-jet is the actual second derivative.

Let us fix a C^k-vector field $v \in V(\delta)$ for some $0 < \delta < 1$, and set $v_t = Pf_t(v)$. Adopt the notation of Lemma 5.1; then the key technical result is the estimate:

Proposition 6.1. — *The absolute value of the term B_f^Ψ in (49) is uniformly bounded on the set* $M \times [0, \infty)$.

Proof. — From the identities (4.7) and (62) we can write

$$B_f^\Psi(p, t) = (1/2)\, \mu \psi_{vv}(0, 0) + A_f^\Psi(p, t)$$

(63) $$= \frac{\psi_{vv}(0, 0)}{2\psi_v(0, 0)} + a(f_t(p)) - a(p)$$

(64) $$= (1/2) \frac{d^2}{dy^2} (\log\{\,\psi_v(0, y)\,\})\, (0) + a(f_t(p)) - a(p).$$

It clearly suffices to give an estimate for the derivative term in (64).

We can assume without loss that $t = NT$ for a positive integer N and T as in the proof of Lemma 4.5. For $0 \leqslant i \leqslant N$, we set $p_i = f_{iT}(p)$, and let y_i denote the local trans-

verse coordinate through p_i. For each i, the Poincaré map of f_T from X_{p_i} to $X_{p_{i+1}}$ is given in local coordinates by $y_{i+1} = \psi_i(0, y_i)$, for C^3-functions ψ_{i, p_i} depending uniformly on p_i and y. Then use (4.7.2) to observe that

$$(65) \qquad \frac{d^2}{dy^2} (\log\{ \psi_v(0, y)\}) (0) = \frac{d^2}{dy^2} (\sum_{i=0}^{N-1} \log\{ \psi_{i, v}(0, \psi^i(0, y))\}) (0)$$

$$(66) \qquad = \sum_{i=0}^{N-1} \frac{\psi_{i, vv}(0, 0)}{\psi_{i, v}(0, 0)} \psi_v^i(0, 0)^2$$

where we use the convenient notation

$$\psi^i(0, y) = y \quad \text{for } i = 0,$$

and $\qquad \psi^i(0, y) = \psi_{i-1} \circ \ldots \circ \psi_0(0, y) \quad \text{for } i > 0.$

The quotient $| \psi_{i, vv}(0, 0)/\psi_{i, v}(0, 0)|$ is uniformly bounded for $p \in M$, and by the choice of T in section 4 there is a uniform estimate $\psi_v^i(0, 0)^2 < 4^{-2i}$. Therefore, the sum in (66) is uniformly convergent in p. The Proposition follows from this. $\square$

Corollary **6.2** *(C^2-Compactness).* — *Fix a C^2-vector field $v \in V(\delta)$ for $0 < \delta < 1$, and let $a_{t, 2}(p)$ denote the coefficient of the quadratic term in the Taylor expansion at $y = 0$ of the local vector field $\tilde{v}_{t, p}$ about p. Then there is a constant $K(v)$, depending only on the vector field v, so that $| a_{t, 2}(p)| < K(v)$ for all $t \geqslant 0$.*

Proof. — The relation (49) yields the formula

$$(67) \qquad a_{t, 2}(p) = a_{0, 2}(f_{-t}(p)) + A_f^Y(f_{-t}(p), t) + a_{0, 0}(f_{-t}(p)) B_f^Y(f_{-t}(p), t),$$

where $| A_f^Y(f_{-t}(p), t)|$ is uniformly bounded by (62), and $| a_{0, 0}(f_{-t}(p)) B_f^Y(f_{-t}(p), t)|$ is bounded by Proposition 6.1. $\square$

Proposition **6.3.** — *The vector field e^+ is transversally C^3.*

Proof. — We first show that e^+ is transversally $C^{1, 1}$, with an integrable second derivative. Corollary 6.2 shows that e^+ is the C^1-limit of C^3-vector fields

$$\{ v_t = Pf_t(v) | t \geqslant 0 \},$$

so that at each $p \in M$ the sequence $\{ \tilde{v}_{t, p}(y) | t \geqslant 0 \}$ has a uniform bound on its second derivative in y at $y = 0$. The choice of adapted transverse coordinates is uniform in p in the C^3-topology on immersions, so there is a uniform change of coordinates between points $q, q' \in X_p$ on the same transversal. (To define the change of coordinates, it is necessary to apply the flow for small time, but there are uniform C^3-estimates on the flow for small time also.) Therefore, we conclude that for fixed $p \in M$, the sequence of local vector fields $\{ \tilde{v}_{t, p}(y)\}$ is bounded in the C^2-topology, uniformly in p and $|y| < \varepsilon/2$. Thus, the limiting local vector field $\tilde{e}_p^+(y) = \lim_{t \to \infty} \tilde{v}_{t, p}(y)$ has a Lipshitz estimate on its first derivative for $|y| < \varepsilon/2$. In the local expansion $\tilde{e}_p^+ = (1, \tilde{a}_p(y))$, this says that the derivative $\tilde{a}_p'(y)$ is Lipshitz, and hence is absolutely continuous. This

implies that a second derivative $\tilde{a}_p''(y)$ exists for almost every $|y| < \varepsilon/2$, and $\tilde{a}_p''(y)$ is integrable to the function $\tilde{a}_p'(y)$.

The existence of a second derivative for $\tilde{e}_p^+(y)$ at y implies the existence of a second derivative for $\tilde{e}_q^+(z)$ at $z = 0$, for q the point on X_p corresponding to y. Thus, for all $p \in M$ and for almost every $q \in X_p$, the vector field e^+ has a transverse second derivative at q. By the Fubini Theorem, the set of points $p \in M$ at which e^+ has a second derivative is a set of positive Lebesgue measure. This set is clearly flow invariant, so by ergodicity must be a set of full measure. Therefore, for almost every $p \in M$, the local expansion $\tilde{e}_p^+(y) = (1, \tilde{a}_p(y))$ has a second derivative at $y = 0$. Denote this second derivative by $2\tilde{a}_2(p)$.

Equation (51) shows that the function $\tilde{a}_2(p)$ is an almost everywhere defined solution of the coboundary equation (62) for A_f^{Ψ}. It is given that there is a C^1-solution, $a(p)$, of this equation, so by the uniqueness of solutions, the function $\tilde{a}_2(p)$ can be extended to a C^1-function defined on all of M. Therefore, for each $p \in M$, the derivative function $\tilde{a}_p'(y)$ is the integral of a C^1-function, and hence $\tilde{a}_p(y)$ is a C^3-function of y. It follows that e^+ is transversally C^3 at every point of M with uniform estimates on its transverse 3-jet. $\square$

The second technical result that we need to establish the regularity of e^+ is a " bootstrap " procedure for concluding that e^+ is transversally C^k. Bootstrapping is a familiar phenomenon in hyperbolic dynamical systems, which is based on expressing the higher derivatives of an invariant line field in terms of an exponentially converging power series, such that each successive derivative converges even faster. Thus, once the bootstrap can be invoked, the invariant field is as smooth as the system. Our use of this principle is very similar to that in (page 568, [49]).

Proposition **6.4** *(Bootstrap).* — *Let* $\{f_t\}$ *be a volume-preserving,* C^k-*Anosov flow on a 3-manifold* M. *Suppose that* e^+ *is transversally* C^n, *with uniform estimates on the transverse n-jets. If* $3 \leqslant n < k$, *then* e^+ *is transversally* C^{n+1}, *with uniform estimates on the transverse* $(n+1)$-*jets.*

Proof. — The method of proof is based on standard techniques, so we will be brief with details, and leave to the reader the often tedious explicit calculations. Fix a point p and let $t = T$ be as in the proof of Lemma 4.5. Expand the local vector field

$$\tilde{e}_p^+(y) = \tilde{a}_2(p)\,y^2 + \ldots + \tilde{a}_n(p)\,y^n + o(|y|^n).$$

For the Poincaré map f_T, we similarly expand the quantities

$$\psi_y(0,y),\ \psi_x(0,y),\ \varphi_x(0,y),\ \varphi_y(0,y)$$

appearing in (37) into their Taylor expansions in the powers $\{y, y^2, \ldots, y^n\}$, and expand the inverse of the function $z = z(y) = \mu(p, T)^{-1}y + \psi(0,y)$ up to order n. Substitute these expansions into (37) to obtain the relation

$$(68) \qquad \tilde{a}_n(f_T(p)) = \mu(p, T)^{n-2}\,\tilde{a}_n(p) + \zeta_n(p, T),$$

where $\zeta_n(p, T)$ is a polynomial in the coefficients of these expansions not involving $\widetilde{a}_n(f_T(p))$. The function $\zeta_n(p, T)$ is seen to be uniformly C^1 in p. Now rewrite (68) as

$$(69) \qquad \widetilde{a}_n(p) = \mu(p, T)^{2-n} \{ \widetilde{a}_n(f_T(p)) + \zeta_n(p, T) \}.$$

Recursively substitute (69) into itself N times to obtain the formula

$$(70) \qquad \widetilde{a}_n(p) = \{ \sum_{i=0}^{N-1} \mu(p, iT + T)^{2-n} \zeta_n(f_{iT}(p), T) \}$$

$$+ \mu(p, NT)^{(2-N)} \{ \widetilde{a}_n(f_{NT-T}(p)) + \zeta_n(f_{NT-T}(p), T) \}.$$

Both $\widetilde{a}_n(q)$ and $\zeta_n(q, T)$ are uniformly bounded functions of $q \in M$, so the sum in (70) is uniformly convergent, with uniform estimates in p. Let N tend to infinity to obtain the closed formula

$$(71) \qquad \widetilde{a}_n(p) = \sum_{i=0}^{\infty} \mu(p, iT + T)^{2-n} \zeta_n(f_{iT}(p), T).$$

The functions appearing in (71) can be restricted to the curves X_p and expressed in the local coordinate y. We abuse notation, and let p represent the local coordinate y in this case. There are then uniform estimates on the first transverse derivatives:

$$(72) \qquad \left| \frac{d\zeta_n(f_{iT}(p))}{dy} \right| \leqslant K_1$$

$$(73) \qquad \left| \frac{d\{ \mu(p, iT + T)^{2-n} \}}{dy} \right| \leqslant K_2 \, \mu(p, iT + T)^{2-n}.$$

Combine (71) with the bounds (72) and (73) to see that $d\{ \widetilde{a}_n(p) \}/dy$ exists everywhere, given by the term-by-term differentiation of (71). We also obtain the uniform estimate

$$\left| \frac{d\{ \widetilde{a}_n(p) \}}{dy} \right| \leqslant \sum_{i=0}^{\infty} K_3 \, \mu(p, iT + T)^{2-n}$$

$$\leqslant K_3 \sum_{i=0}^{\infty} 4^{(2-n)i}.$$

The existence of a transverse derivative for the Taylor coefficients up to order n then implies that e^+ has a transverse Taylor expansion of order $(n + 1)$. $\square$

We can now deduce the Smooth Rigidity Theorem 3.4.

Corollary 6.5. — *Suppose that* $\{f_t\}$ *is a volume-preserving, C^k-Anosov flow on a closed 3-manifold M, for $k \geqslant 5$. If the Anosov class A_f vanishes, then the vector field e^+ is C^{k-3} on M.*

Proof. — The assumption is that we can solve equation (62) for a C^1-function $a(p)$. Then by Proposition 6.3, the vector field e^+ is transversally C^3. The Bootstrap Proposition 6.4 implies that e^+ is transversally C^k. It is known from stable manifold theory (cf. [29]) that the field e^+ is C^k when restricted to the leaves of the unstable foliation.

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Restricting the vector field e^+ to the adapted transversal X_p, we can invoke Theorem 2.6 to conclude that for all $p \in M$, the field $e^+|_{X_p}$ is C^{k-3}. All of our calculations are natural with respect to the flow, so the field e^+ will be C^{k-3} on all of M. □

7. Dynamical Godbillon-Vey Classes

The Godbillon-Vey class for a C^2, codimension-one foliation $\mathscr{F}$ is a cohomology class $GV(\mathscr{F}) \in H^3(M; \mathbf{R})$. Its definition by Godbillon and Vey [22], and calculation in a pivotal example by Roussarie, led to the explosion in the study of secondary invariants in the 1970's. In this section, we refine the definition of this class so that it makes sense for foliations of differentiability class $C^{1,\alpha}$ for all $\alpha > 1/2$. The invariance of the Godbillon-Vey class under diffeomorphisms of low-differentiability is shown, extending the previous work of Raby [58] for C^2-foliations.

The existence of the extended Godbillon-Vey class was observed by the first author in 1984. It raised the question about the regularity of Anosov foliations, whose solution we addressed in the first part of this paper. The degree of regularity is fortuitously compatible with the extended definition of the Godbillon-Vey class. The consequences of this will be examined in section 9, where we develop the formula of Mitsumatsu for the Godbillon-Vey classes of the weak-unstable foliations of geodesic flows of negative curvature. This formula is the non-homogeneous counterpart of the Roussarie calculation, except that, as discussed in the Introduction, the values of the classes now vary continuously and non-trivially with the parameter in the space of all metrics of negative curvature.

For a transversally oriented, codimension-one C^2-foliation $\mathscr{F}$, the Godbillon-Vey class has a well-known elementary definition. Choose a non-vanishing C^2-1-form, θ, whose kernel is the tangential distribution to $\mathscr{F}$. We can solve the equation $d\theta = \eta \wedge \theta$ for a C^1-1-form η, and the Godbillon-Vey class is the cohomology class of the closed continuous 3-form $\eta \wedge d\eta$. When M is a closed oriented 3-manifold, we can also define the Godbillon-Vey invariant

$$(74) \qquad gv(\mathscr{F}) = \int_M \eta \wedge d\eta.$$

The foliation $\mathscr{F}$ is said to be transversally $C^{1,\alpha}$ if its tangential distribution $T\mathscr{F}$ is $C^{1,\alpha}$, and the leaves of $\mathscr{F}$ are a C^1-family of smoothly immersed submanifolds.

Proposition 7.1. — *Let $\mathscr{F}$ be a codimension-one $C^{1,\alpha}$-foliation of a closed oriented 3-manifold M. For $\alpha > 1/2$, there is a natural, well-defined Godbillon-Vey invariant $gv(\mathscr{F})$, which extends the definition* (74) *for $\mathscr{F}$ of class C^2.*

Proof. — The integral (74) is considered as a quadratic form on the 1-form η, and we want to extend its natural domain from the obvious class of C^1-forms, to 1-forms which are distributions transversally. This is accomplished via the Čech definition of

the Godbillon-Vey class. The hypothesis that the form θ is $C^{1,\alpha}$ implies that η is C^{α}, and we use a standard result from harmonic analysis on the line to show that the distributional pairing (74) is defined for such forms.

Let us begin with the result needed from harmonic analysis, based on the Fourier Transform technique. Let $\mathscr{A}^{\alpha}$ be the topological space of α-Hölder continuous, compactly supported functions on the real line **R**, with norm

$$(75) \qquad \|f\|_{\alpha} = \sup_{r \in \mathbf{R}} |f(r)| + \sup_{r \neq s} \frac{|f(s) - f(r)|}{|s - r|^{\alpha}}.$$

For a pair of compactly supported C^1-functions f and g on **R**, define a skew-symmetric bilinear form

$$(76) \qquad I(f, g) = \int_{\mathbf{R}} f(r)\, g'(r)\, dr.$$

Proposition **7.2.** — *For* $\alpha, \beta > 0$ *with* $\alpha + \beta > 1$, (76) *extends to a skew-symmetric, bilinear form*

$$I : \mathscr{A}^{\alpha} \times \mathscr{A}^{\beta} \to \mathbf{R}.$$

Moreover, $I(f, g)$ *is jointly continuous on subspaces of functions with uniformly bounded support.*

Proof. — The Fourier Transform $\hat{f}(\xi)$ of a continuous function $f(r)$ with compact support is defined as

$$\hat{f}(\xi) = \mathscr{F}\mathscr{C}(f)(\xi) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-ir\xi} f(r)\, dr.$$

The operator $\mathscr{F}\mathscr{C}$ induces an isometric isomorphism between the Hilbert spaces $L^2(\mathbf{R}, dr)$ and $L^2(\mathbf{R}, 2\pi\, d\xi)$. The key point in the proof is to identify the range of the α-Hölder functions under $\mathscr{F}\mathscr{C}$. The proof of the next lemma follows Stein (cf. page 139, [59]).

Lemma **7.3.** — *Let* $f \in \mathscr{A}^{\alpha + \varepsilon}$ *for some* $\alpha, \varepsilon > 0$ *with support in the interval* $(-\delta, \delta)$. *Then there is a constant* $K(\alpha) > 0$ *independent of* f *such that*

$$(77) \qquad 2\pi \int_{\mathbf{R}} |\hat{f}(\xi)|^2\, |\xi|^{2\alpha}\, d\xi \leqslant \frac{2\delta}{\varepsilon K(\alpha)} \cdot \|f\|_{\alpha + \varepsilon}^2$$

Proof. — There is a uniform estimate $|f(r + t) - f(r)| \leqslant \|f\|_{\alpha + \varepsilon} |t|^{\alpha + \varepsilon}$, which under the Fourier transform yields

$$(78) \qquad 2\pi \int_{\mathbf{R}} |\hat{f}(\xi)|^2\, |e^{it\xi} - 1|^2\, d\xi = \int_{\mathbf{R}} |f(r + t) - f(r)|^2\, dr$$

$$(79) \qquad \qquad \leqslant 4\delta\, \|f\|_{\alpha + \varepsilon}^2\, |t|^{2\alpha + 2\varepsilon}.$$

Divide the two sides of the estimate (79) by $t^{1+2\alpha}$ and integrate:

$$(80) \qquad \frac{2\delta}{\varepsilon} \, \|f\|_{\alpha+\varepsilon}^2 \geqslant 2\pi \int_{\mathbf{R}} |\hat{f}(\xi)|^2 \int_0^1 \frac{|e^{it\xi} - 1|^2}{t^{1+2\alpha}} \, dt \, d\xi$$

$$(81) \qquad \geqslant 2\pi K(\alpha) \int_{\mathbf{R}} |\hat{f}(\xi)|^2 \, |\xi|^{2\alpha} \, d\xi$$

where we use the substitution $s = t\xi$ to obtain

$$K(\alpha) = \int_0^1 \frac{|e^{is} - 1|^2}{s^{1+2\alpha}} \, ds. \qquad \square$$

Let $f \in \mathscr{A}^\alpha$ and $g \in \mathscr{A}^\beta$ with $\alpha + \beta > 1$. Define

$$(82) \qquad I(f, g) = - 2\pi \int_{\mathbf{R}} \hat{f}(\xi) \, \hat{g}(- \xi) \, \xi \, d\xi.$$

Set $\varepsilon = (\alpha + \beta - 1)/2$, and let $\delta(f) > 0$ be the least number δ such that $(- \delta, \delta)$ contains the support of f, and similarly for g. The Cauchy-Schwartz inequality and Lemma 7.3 yield the estimate

$$| I(f, g)|^2 \leqslant 2\pi \int_{\mathbf{R}} |\hat{f}(\xi)|^2 \, |\xi|^{2\alpha - 2\varepsilon} \, d\xi \cdot \int_{\mathbf{R}} |\hat{g}(\xi)|^2 \, |\xi|^{2\beta - 2\varepsilon} \, d\xi$$

$$(83) \qquad \leqslant \frac{4\delta(f) \, \delta(g)}{\pi \varepsilon^2 \, K(\alpha - \varepsilon) \, K(\beta - \varepsilon)} \, \|f\|_\alpha \, \|g\|_\beta$$

which shows in particular that (82) is well-defined. Definition (82) extends (76) by standard properties of the Fourier Transform. It is obvious from the definition that $I(f, g) = - I(g, f)$. Finally, the estimate (83) shows that if we bound the functions $\delta(f)$ and $\delta(g)$ from above, then the form $I(f, g)$ is jointly continuous in f and g, completing the proof of Proposition 7.2. $\square$

Let v be a smooth unit vector field on M which is everywhere transverse to the foliation $\mathscr{F}$. (If $\mathscr{F}$ is not transversally orientable, then we pass to the appropriate double covering of M.) Define θ to be the 1-form on M which vanishes when restricted to leaves of $\mathscr{F}$, and has $\theta(v) = 1$. We take $\eta = \iota(v) \, d\theta$, and note that $d\theta = \eta \wedge \theta$ follows from the Cartan identity $L(v) = d \circ \iota(v) + \iota(v) \circ d$. The form η is in general only of class C^α, so that the exterior product $\eta \wedge d\eta$ is a distribution. Given a finite set of smooth, non-negative compactly supported functions $\{ \lambda_i \mid 1 \leqslant i \leqslant n \}$ with $\Sigma_{i=1}^n \lambda_i = 1$, we write $\eta_i = \lambda_i . \eta$, and then

$$(84) \qquad \eta \wedge d\eta = \sum_{1 \leqslant i, j \leqslant n} \eta_i \wedge d\eta_j.$$

It will suffice to define the local integrals $\int_U \eta_i \wedge d\eta_j$ for $U \subset M$ an open set containing the support of the integrand.

Definition 7.4 (Foliation Chart). — *A $C^{1,\alpha}$-foliation chart (Φ, U) for $\mathscr{F}$ is an open set $U \subset M$ and a $C^{1,\alpha}$-diffeomorphism $\Phi : (-1, 1)^3 \to U \subset M$ onto U, such that:*

1. for $-1 \leqslant z \leqslant 1$, the restriction $\Phi_z : (-1, 1)^2 \to U$ is onto a connected component of a leaf of $\mathscr{F} \mid U$;

2. the immersions Φ_z are smooth, and their jets depend C^1 on the parameter z;

3. the vector field $\partial/\partial z$ is positively oriented with respect to the transverse field v.

Standard methods of foliation theory show:

Lemma 7.5. — *For a transversally-oriented $C^{1,\alpha}$-foliation $\mathscr{F}$ on a compact 3-manifold M, there exists a finite covering of M by $C^{1,\alpha}$-foliation charts.* □

Let $\{ \lambda_i \mid 1 \leqslant i \leqslant n \}$ be a smooth partition-of-unity subordinate to a fixed covering by $C^{1,\alpha}$-foliation charts, $\{ (\Phi_i, U_i) \mid 1 \leqslant i \leqslant n \}$. The 1-form θ restricted to U_i is expressed in local coordinates as $\theta_i = h_i(x, y, z) \, dz$, where h_i is smooth in (x, y) and is $C^{1,\alpha}$ in the variable z. The restriction of the 1-form η_i to U_i has the local form

$$\eta_i = a_i(x, y, z) \, dx + b_i(x, y, z) \, dy + c_i(x, y, z) \, dz$$

where the coefficients are smooth in x and y, and C^α in z. (The coefficient of dz is ambiguous in this local form, as η is only specified up to exterior product with θ, which is $h_i \, dz$ in local coordinates.)

For a C^2-foliation, we can expand the definition (74) into the sum

$$(85) \qquad gv(\mathscr{F}) = \int_M \sum_{1 \leqslant i, j \leqslant n} \eta_i \wedge d\eta_j = \sum_{1 \leqslant i, j \leqslant n} \int_{U_i} \eta_i \wedge d\eta_j.$$

For a foliation of transversal class $C^{1,\alpha}$, the formal differential $d\eta_j$ has two components, a continuous part when differentiating with respect to x or y, and a distributional part when differentiating with respect to z. This suggests rewriting each of the summands in (85) according to their continuous or distributional character. For a pair of continuous functions $f(z)$ and $g(z)$ with compact support, set $\langle f, g \rangle = \int_{\mathbf{R}} f(z) \, g(z) \, dz$. If f, g are also functions of x, y then the integral will be a function of x, y, which we indicate by $\langle f, g \rangle (x, y)$. The same considerations apply to the skew-symmetric form $I(f, g)$, and in this sense we define

$$AB_{ij} = \iint I(a_i, b_j) \, dx \, dy \qquad\qquad BA_{ij} = \iint I(b_i, a_j) \, dx \, dy$$

$$AC_{ij} = \iint \langle a_i, \partial c_j/\partial y \rangle (x, y) \, dx \, dy \qquad CA_{ij} = \iint \langle c_i, \partial a_j/\partial y \rangle (x, y) \, dx \, dy$$

$$BC_{ij} = \iint \langle b_i, \partial c_j/\partial x \rangle (x, y) \, dx \, dy \qquad CB_{ij} = \iint \langle c_i, \partial b_j/\partial x \rangle (x, y) \, dx \, dy$$

where all double integrals are over the set $\mathbf{R}^2$. We then define the Godbillon-Vey invariant of $\mathscr{F}$ by the formula

$$(86) \qquad gv(\mathscr{F}; \eta) = \sum_{1 \leqslant i, j \leqslant n} \{ - AB_{ij} + BA_{ij} + AC_{ij} - CA_{ij} - BC_{ij} + CB_{ij} \}.$$

There are two sets of choices made in obtaining the definition (86): a choice of transverse vector field v, and hence of 1-forms θ and η; and a choice of foliation charts which cover M. The first ambiguity is equivalent to considering the result of choosing $\widetilde{\eta} = \eta + df + g\theta$ for functions f of transverse class $C^{1,\alpha}$ and g of transverse class C^{α}.

Lemma 7.6. — *One has $gv(\mathscr{F}; \eta) = gv(\mathscr{F}; \widetilde{\eta})$ where $\widetilde{\eta} = \eta + df + g\theta$ for functions f of transverse class $C^{1,\beta}$ and g of transverse class C^{β}, when $\alpha + \beta > 1$. When $\mathscr{F}$ is of class C^2, then we can assume simply that g is smooth on leaves and transversally integrable, and f is transversally absolutely continuous; i.e., df exists almost everywhere, is smooth on leaves and transversally integrable.*

Proof. — It will suffice to consider f and g of class C^2, for the pairings $\langle \cdot, \cdot \rangle$ and $I(\cdot, \cdot)$ are continuous in the appropriate topologies when we substitute the coefficients of the local form

$$\widetilde{\eta}_i = (a_i + \lambda_i f_x)\, dx + (b_i + \lambda_i f_y)\, dy + (c_i + \lambda_i \{f_z + g\})\, dz$$

into (86). Next, write each local form η_j as a limit in the transverse C^{α}-topology

$$\eta_j = \lim_{p \to \infty} \eta_{i,p}$$

of C^1-forms with compact support on U_j. Then calculate

$$(87) \qquad gv(\mathscr{F}; \widetilde{\eta}) = gv(\mathscr{F}; \eta) + \lim_{p \to \infty} \int_M \{\, df + g\theta\,\} \wedge \Big(\sum_{i=1}^{*} d\eta_{j,p}\Big).$$

The form $d\theta = \eta \wedge \theta$ is closed as a distribution, so the product $d\eta \wedge \theta$ is zero as a distribution. This implies that the $g\theta$ contribution to the limit in (87) vanishes. The other component of this limit vanishes by Stokes' Theorem. Note that the above proof only used that df is transversally C^{β}. Similarly, the conclusion for C^2-foliations actually holds when the 1-form η is only transversally Lipshitz, so that its coefficient functions have bounded transverse derivatives almost everywhere. $\square$

The ambiguity introduced by the choice of a covering is easily dealt with. Given two choices of coverings of M by foliation charts, we can assume that the partition-of-unity $\{\lambda_i\}$ is subordinate to both covers, so the task is to prove that the local terms in the sum (86) are independent of the choice of coordinates. We let η_j be a limit of C^1-forms $\eta_{j,p}$ as before; then these terms are actually integrals of a 3-form over the appropriate foliation charts. These integrals are independent of chart, up to C^1-diffeomorphisms, so they agree in any chart chosen, and hence their limits agree. It follows that $gv(\mathscr{F}; \eta)$ is independent of choices, so yields a well-defined invariant $gv(\mathscr{F})$.

It is clear that the formula (86) reduces to (85) when the form η is C^1 as the skew-symmetric product $I(\cdot, \cdot)$ reduces to the ordinary integral when one of the arguments is a C^1-function.

The naturality of $gv(\mathscr{F})$ will be addressed in the next proposition, which will complete the proof of Proposition 7.1. $\square$

Naturality of the Godbillon-Vey invariant is its invariance under orientation-preserving, $C^{1,\alpha}$-diffeomorphisms. This invariant actually has a much stronger invariance property, as given by the next two results (cf. Raby [58]).

Proposition 7.7. — *Let $\mathscr{F}$ and $\widetilde{\mathscr{F}}$ be codimension-one $C^{1,\alpha}$-foliations on closed oriented 3-manifolds M and $\widetilde{M}$, respectively, for $\alpha > 1/2$. Suppose there exits an orientation-preserving, $C^{1,\beta}$-diffeomorphism $\Theta : M \to \widetilde{M}$ conjugating $\mathscr{F}$ to $\widetilde{\mathscr{F}}$ with $\alpha + \beta > 1$. Then $gv(\mathscr{F}) = gv(\widetilde{\mathscr{F}})$.*

Proof. — We can assume that Θ is smooth along leaves by standard smoothing techniques for leafwise diffeomorphisms. Let θ be a transverse 1-form for $\mathscr{F}$, and $\widetilde{\theta}$ the same for $\widetilde{\mathscr{F}}$. Then $\Theta^*(\widetilde{\theta}) = \exp(g)\,\theta$ for a function $g : M \to \mathbf{R}$ which is smooth along leaves of $\mathscr{F}$, but is a priori only transversally C^β. Then define 1-forms η, $\widetilde{\eta}$, $\widehat{\eta}$ by the relations

$$d\theta = \eta \wedge \theta, \qquad d\widetilde{\theta} = \widetilde{\eta} \wedge \widetilde{\theta}, \qquad \Theta^*(\widetilde{\eta}) = \widehat{\eta}.$$

Fix a covering of M by $C^{1,\alpha}$-foliation charts, with notation as above. For a function $f : M \to \mathbf{R}$ which is C^1 along the leaves of $\mathscr{F}$, we define the " leafwise differential " $d_{\mathscr{F}} f$ using the foliation charts: write $f = \Sigma_{i=1}^n f_i$, where each $f_i = \lambda_i . f$. Then in coordinates, $d_{\mathscr{F}} f_i = f_{i,x}\, dx + f_{i,y}\, dy$, and $d_{\mathscr{F}} f$ is obtained by summing up the local differentials. A key property of the leafwise differential is that $df = d_{\mathscr{F}} f + h\theta$ for some function h on M.

We can assume without loss that $\beta < \alpha$. Let $\{\, g_p \mid p = 1, \ldots \}$ be a sequence of smooth forms converging to g in the transverse C^β-topology, with $\{\, h_p \}$ such that $dg_p = d_{\mathscr{F}}\, g_p + h_p\, \theta$. The proof of Lemma 7.6 gives that

$$gv(\mathscr{F};\eta) = gv(\mathscr{F};\eta + dg_p) = gv(\mathscr{F};\eta + d_{\mathscr{F}}\, g_p) \quad \text{for all } p.$$

The evaluation of $gv(\widetilde{\mathscr{F}};\widetilde{\eta})$ in terms of M is facilitated by the next elementary calculation.

Lemma 7.8. — *We have $\widehat{\eta} = \eta + d_{\mathscr{F}}\, g + h\theta$, where the function h on M is C^β.*

Proof. —

$$\widehat{\eta} \wedge \exp(g)\,\theta = \Theta^*(d\widetilde{\theta}) = d\Theta^*(\widetilde{\theta}) = d\{\exp(g)\,\theta\}$$

$$= d_{\mathscr{F}}\, g \wedge \exp(g)\,\theta + \eta \wedge \exp(g)\,\theta \quad !$$

so that $\widehat{\eta} = \eta + d_{\mathscr{F}}\, g + h\theta$ for some function h. However, $\widehat{\eta}$ is transversally C^β and $d_{\mathscr{F}}\, g$ is C^β, so the claim follows. $\square$

The composition of Θ with foliation charts on M yields C^1-foliation charts on $\widetilde{M}$, which are sufficient to calculate $gv(\widetilde{\mathscr{F}};\widetilde{\eta})$ by the remarks following the proof of Lemma 7.6. We let $gv(\mathscr{F};\widehat{\eta})$ denote the evaluation of this invariant by the formula (86)

with entries the coefficients of $\hat{\eta}$. By Lemmas 7.8 and 7.6, and the continuity of the inner products in (86), we have

$$gv(\mathscr{F}; \hat{\eta}) = gv(\mathscr{F}; \eta + d_{\mathscr{F}} g) = \lim_{p \to \infty} gv(\mathscr{F}; \eta + d_{\mathscr{F}} g_p) = gv(\mathscr{F}; \eta). \qquad \square$$

When the foliation is transversally C^2, the natural extension of the proof of Proposition 7.8 yields that $gv(\mathscr{F})$ is invariant under C^1-diffeomorphisms. The above proof then becomes essentially the same as the original proof of Raby. A somewhat stronger conclusion can be shown:

Proposition **7.9.** — *Let $\mathscr{F}$ and $\widetilde{\mathscr{F}}$ be codimension-one $C^{1,1}$-foliations on closed oriented manifolds M and $\widetilde{M}$, respectively. Suppose there exist an orientation-preserving homeomorphism $\Theta : M \to \widetilde{M}$ conjugating $\mathscr{F}$ to $\widetilde{\mathscr{F}}$ such that Θ and Θ^{-1} are transversally Lipshitz, and leafwise C^2 with the 2-jet depending Lipshitz on the transverse parameter. Then $gv(\mathscr{F}) = gv(\widetilde{\mathscr{F}})$.*

Proof. — Let us elaborate on the regularity of Θ. In local coordinates (U_i, Φ_i) on M and $(\widetilde{U}_i, \widetilde{\Phi}_i)$ on $\widetilde{M}$, we write $(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \Theta_i(x, y, z)$. Then, for x_0, y_0 fixed, the function $z \mapsto \Theta_i(x_0, y_0, z)$ is Lipshitz, so has a bounded derivative almost everywhere which integrates back to the function. Moreover, the derivative depends C^1 on the points x, y. Fixing the variable z_0, the function $(x, y) \mapsto \Theta_i(x, y, z_0)$ is C^2, and the 2-jet is a Lipshitz function of z_0. We require that the same conclusions hold on the inverse function Θ^{-1}.

The regularity of $\mathscr{F}$ implies that we can choose a transversally Lipshitz 1-form η satisfying $d\theta = \eta \wedge \theta$, and similarly choose $\widetilde{\eta}$ for $\widetilde{\mathscr{F}}$. The function Θ almost everywhere has a transverse derivative, so the 1-form $\widetilde{\theta}$ pulls-back under Θ to an almost everywhere defined 1-form on M with transversally integrable coefficients, whose kernel contains the tangential distribution to $\mathscr{F}$. Thus, there is a leafwise C^1, measurable function g on M with $\exp(g)$ integrable so that $\Theta^*(\widetilde{\theta}) = \exp(g) \theta$ almost everywhere. Note that $\exp(g)$ is bounded on M by a multiple of the Lipshitz constant for Θ, and similarly $\exp(-g)$ is bounded by a multiple of the Lipshitz constant for Θ^{-1}. Therefore, g is a bounded measurable function on M. We need for the proof only that $|g|$ is integrable, which is clearly implied by it being a bounded function.

The coefficients of $d\widetilde{\eta}$ are integrable functions on $\widetilde{M}$, so the calculation of $gv(\widetilde{\mathscr{F}}; \widetilde{\eta})$ involves ordinary integrals over an open cover for $\widetilde{\mathscr{F}}$. These integrals are invariant under an absolutely continuous change of coordinates, so we can calculate them using the foliation cover of M via the map Θ, given by the expression denoted $gv(\mathscr{F}; \hat{\eta})$ as in the proof of Proposition 7.7, where $\hat{\eta} = \Theta^*(\widetilde{\eta})$. The same method as used in Lemma 7.8 also yields

Lemma **7.10.** — *We have $\hat{\eta} = \eta + d_{\mathscr{F}} g + h\theta$ for a transversally integrable function h.*

By Lemma 7.6, $gv(\mathscr{F}; \eta) = gv(\mathscr{F}; \eta + d_{\mathscr{F}} f + h\theta)$, for arbitrary C^1-functions f and h, and the calculations are continuous in h for the L^1-norm transversally. Choose a

sequence of C^1-functions $\{\, g_p \mid p = 1, \ldots \,\}$ converging to g in the C^1-topology on leaves, and in the L^1-norm transversally. Then as in the proof of Proposition 7.7 we have

$$(88) \qquad gv(\mathscr{F}; \eta) = gv(\mathscr{F}; \eta + d_{\mathscr{F}}\, g_p) = gv(\mathscr{F}; \eta + d_{\mathscr{F}}\, g) = gv(\widetilde{\mathscr{F}}; \hat{\eta}). \qquad \square$$

Propositions 7.1, 7.7 and 7.9 together yield the claims of Theorem 3.8.1 and 3.8.3. The continuous dependence of $gv(\mathscr{F})$ on parameters follows from the continuity of the formula (86) in the C^α-topology. The concordance invariance of the extended God-billon-Vey invariant follows from a standard extension of our definition, via simplicial methods, to a cohomology class which can be defined on foliated 4-manifolds with boundary. However, the proof is omitted as the result is not central to this work. (The interested reader can consult [31] for a detailed treatment.) Let us conclude this section with a proposition which implies Theorem 3.10.

Proposition 7.11. — *Let $\mathscr{F}$ be a codimension-one $C^{1,\alpha}$-foliation on a closed oriented 3-manifold M. Suppose that $\{f_t\}$ is a volume-preserving C^3-Anosov flow on M, such that the leaves of $\mathscr{F}$ are the weak-unstable manifolds of the flow. Then for any codimension-one $C^{1,\alpha}$-foliation $\widetilde{\mathscr{F}}$ on a closed oriented 3-manifold $\widetilde{M}$, if there is a transversally absolutely-continuous, leafwise C^2-homeomorphism $\Theta : M \to \widetilde{M}$ conjugating $\mathscr{F}$ to $\widetilde{\mathscr{F}}$, then $gv(\mathscr{F}) = \pm\, gv(\widetilde{\mathscr{F}})$ with sign according to whether Θ is orientation-preserving or -reversing.*

Proof. — We can assume without loss that Θ is orientation-preserving. Let θ be a transverse $C^{1,\alpha}$ 1-form defining $\mathscr{F}$, and $\widetilde{\theta}$ a defining 1-form for $\widetilde{\mathscr{F}}$ with the same orientation as θ with respect to Θ. The flow $\{f_t\}$ preserves the foliation $\mathscr{F}$, so there is a $C^{1,\alpha}$-function $\varphi : M \times \mathbf{R} \to \mathbf{R}$ such that $f_t^*\,\theta\,|_p = \exp\{\,\varphi(p, t)\}\,\theta\,|_p$. It is clear that φ is C^1-1-cocycle over the flow. The push-forward flow $\{\widetilde{f}_t = \Theta \circ f_t\}$ on $\widetilde{M}$ similarly acts on the transverse form $\widetilde{\theta}$ to define a 1-cocycle $\widetilde{\varphi}$ over $\{\widetilde{f}_t\}$, which we pull back to a Lipshitz 1-cocycle $\hat{\varphi}$ over $\{f_t\}$ which is differentiable along the flow.

The hypothesis that Θ is transversally absolutely continuous implies that there is a measurable function $G : M \to \mathbf{R}$ such that $\Theta^*\,\widetilde{\theta} = G\theta$. The set $X = \{\, p \in M \mid G(p) = 0\,\}$ is flow invariant as the holonomy of $\mathscr{F}$ is at least C^1. The set X cannot be of full measure, as the transverse derivative integrates back to the transverse coordinate of Θ, and Θ is a homeomorphism. Therefore, ergodicity of the flow implies that X has measure zero, and we can define a measurable function g on M such that $G = \exp(g)$.

The function g satisfies the coboundary relation

$$(89) \qquad \hat{\varphi}(p, t) = \varphi(p, t) + g(f_t(p)) - g(p).$$

The difference of cocycles $\hat{\varphi} - \varphi$ is Lipshitz on M, so by Theorem 2.1 the coboundary g is β-Hölder for some $\beta > 0$. By Theorem 3.1, the foliation $\mathscr{F}$ is α-Hölder for all $1 > \alpha > 1 - \beta$, and we conclude from the proof of Proposition 7.7 that $gv(\mathscr{F}) = gv(\widetilde{\mathscr{F}})$. $\square$

8. Godbillon-Vey Classes for Circle Bundles

Let M be a closed 3-manifold which fibers over an oriented surface Σ with circle fibers, and let $\mathscr{F}$ be a codimension-one $C^{1,\alpha}$-foliation of M whose leaves are everywhere transverse to the fibers. The holonomy of such a foliation determines a $C^{1,\alpha}$-action of the fundamental group $\Gamma = \pi_1(\Sigma, x_0)$ of the surface on the S^1-fiber over x_0. In this section we will give an alternate construction of $gv(\mathscr{F})$ based on this holonomy action. The formula we obtain generalizes the well-known " Thurston cocycle " for C^2-actions, and is striking for the intuitive feel it gives to the extension of the Godbillon-Vey invariant. It shows, for example, that the restriction $\alpha > 1/2$ is dictated by considerations of Hausdorff dimension for curves in the plane. The geometric construction of this section is also the original method used to extend the range of definition for the Godbillon-Vey invariant, and has far-ranging generalizations [31].

We assume that the foliation $\mathscr{F}$ is transversally oriented. Fix a basepoint $x_0 \in \Sigma$ and an identification $S^1 \cong \pi^{-1}(x_0)$ of the fiber of $\pi : M \to \Sigma$. The *holonomy homomorphism* (cf. Chapter 5, [7]) of the foliation $\mathscr{F}$ is the representation

$$(90) \qquad h = h_{\mathscr{F}} : \Gamma \to \mathrm{Diff}_+^{1+\alpha}(S^1)$$

into the group of orientation-preserving, $C^{1,\alpha}$-diffeomorphisms of the circle.

The *additive Radon-Nikodyn* 1-cocycle for a C^1-action $h : \Gamma \times S^1 \to S^1$ is defined as the logarithm of the volume expansion:

$$\mu_h : \Gamma \times S^1 \to \mathbf{R}$$
$$\mu_h(\gamma, \theta) = \log(h'(\gamma) \,|_\theta),$$

where $h'(\gamma)$ denotes the derivative with respect to the natural length coordinate on the circle, which we assume to have total length 2π.

Integration over the fiber defines a natural isomorphism of cohomology groups

$$\pi_* : H^3(M; \mathbf{R}) \to H^2(\Sigma; \mathbf{R}) \cong H^2(\Gamma; \mathbf{R})$$

where the latter space is the group cohomology of Γ. Thus, the Godbillon-Vey class $GV(\mathscr{F}) \in H^3(M; \mathbf{R})$ is identified with a group 2-cocycle over Γ.

Definition 8.1. — *The Thurston cocycle c_h for a C^2-action $h : \Gamma \times S^1 \to S^1$ is the group 2-cocycle defined by*

$$(91) \qquad c_h(\gamma_1, \gamma_2) = \int_{S^1} \mu_h(\gamma_2, \theta) \cdot \frac{d\mu_h(\gamma_1 \cdot \gamma_2, \theta)}{d\theta} \, d\theta; \quad \gamma_1, \gamma_2 \in \Gamma.$$

The following is an unpublished result of Thurston (cf. [61, 62]); a detailed proof was published by Brooks in (Appendix to [6]).

Proposition 8.2 *(Thurston Cocycle Formula).* — *Let $\mathscr{F}$ be a C^2-foliation transverse to the fibers of $\pi : M \to \Sigma$ with holonomy homomorphism $h_{\mathscr{F}}$. Then the cohomology class $[c_{h_{\mathscr{F}}}] \in H^2(\Sigma; \mathbf{R})$ equals $\pi_*(GV(\mathscr{F}))$.* $\square$

Let $\mathscr{B}^\alpha$ denote the topological space of α-Hölder continuous functions on the circle, with norm the obvious modification of (75). The method of proof of Proposition 7.2 also establishes the corresponding result for functions on the circle:

Proposition 8.3. — *For $\alpha, \beta > 0$ with $\alpha + \beta > 1$, there is a jointly-continuous skew-symmetric bilinear form*

(92) $$I : \mathscr{B}^\alpha \times \mathscr{B}^\beta \to \mathbf{R}$$

which extends the natural pairing

$$I(f, g) = \int_{S^1} f \cdot dg$$

for C^1-functions f and g. $\square$

Let $h : \Gamma \times S^1 \to S^1$ be an orientation-preserving $C^{1,\alpha}$-action on the circle. For $\alpha > 1/2$, define the *extended Thurston cocycle*

(93) $$c_h^I(\gamma_1, \gamma_2) = I(\mu_h(\gamma_2), \mu_h(\gamma_1 \cdot \gamma_2)).$$

Each function $\mu_h(\gamma)$ belongs to $\mathscr{B}^\alpha$, so the formula is well-defined by Proposition 8.3. The pairing I is invariant under a C^1 change of variable, so the usual proof [6] that c_h^I is a cocycle on $\text{Diff}_+^2(S^1)$ works as well for $\text{Diff}_+^{1+\alpha}(S^1)$.

The fundamental class $[\Sigma] \in H_2(\Sigma; \mathbf{R})$ corresponds to an integral group 2-homology class for Γ, which can be written

$$[\Sigma] = \sum_{i=1}^p n_i \{ \gamma_{1,i} \times \gamma_{2,i} \}$$

for integers n_i and group elements $\{ \gamma_{k,i} \}$. The proof of Proposition 8.2 can be adapted to show:

Proposition 8.4

(94) $$gv(\mathscr{F}) = c_{h,\mathscr{F}}^I([\Sigma]) = \sum_{i=1}^p n_i \, c_{h,\mathscr{F}}^I(\gamma_{1,i}, \gamma_{2,i})$$ $\square$

The purpose of this section is to give an alternative, more geometric definition of the extended cocycle $c_h^I(\gamma_1, \gamma_2)$. This is based on Thurston's " remark " (cf. page 40, [6]) that for C^2-actions, $c_h^I(\gamma_1, \gamma_2)$ is the algebraic area inside the C^1-plane curve

(95) $$BT_h : S^1 \to \mathbf{R}^2$$

$$\theta \mapsto (\mu_h(\gamma_2), \mu_h(\gamma_1 \cdot \gamma_2)).$$

Let us consider the problem in a more general framework. Given functions $f, g : S^1 \to \mathbf{R}$, define $C(f, g) : S^1 \to \mathbf{R}^2$ where $C(f, g)(\theta) = (f(\theta), g(\theta))$. Identify S^1 with the boundary of the unit disk $D^2 = \{(x, y) \mid x^2 + y^2 \leqslant 1 \}$. If f and g are differentiable,

then we can choose a differentiable extension of $C(f, g)$ to $D(f, g) : D^2 \to R^2$. Thurston's remark is based on an application of Stokes' Theorem:

$$\int_{S^1} f \cdot dg = \int_{S^1} C(f, g)^* (x \, dy) = \int_{D^2} D(f, g)^* (dx \wedge dy) \stackrel{\text{def}}{=} A(f, g)$$

where $A(f, g)$ is the " algebraic area " enclosed by the C^1-curve $C(f, g)$. The general problem is then to determine the degree of regularity on functions f and g necessary to define a well-behaved " algebraic area " enclosed by the curve $C(f, g)$.

The first remark is that the value $\alpha = 1/2$ is special for maps of the circle into the plane:

Lemma 8.5. — Let $f, g : S^1 \to R$ be C^α-continuous functions. Then the Hausdorff dimension of the image set $X(f, g) = \{ C(f(\theta), g(\theta)) \mid \theta \in S^1 \}$ is at most the minimum of $\{ 1/\alpha, 2 \}$. □

To define the algebraic area inside of a curve, it is natural to require that the curve have area zero, or Hausdorff dimension less than 2. This corresponds to the values $\alpha > 1/2$. The estimate in Lemma 8.5 is sharp, as it is easy to give examples of functions f and g based on Fourier series expansions with the given α-Hölder condition and Hausdorff dimension as close to the estimate as desired. It was a surprising discovery to find that this condition was also sufficient.

Proposition 8.6. — Let f, g be C^α-functions on the circle. For $\alpha > 1/2$, there is a well-defined algebraic area, $A(f, g)$, enclosed by the curve $C(f, g)$. Moreover, this area is independent of Lipshitz reparametrizations of the circle, and depends continuously on f, g in the α-Hölder norm.

Proof. — Represent the circle S^1 as the interval $[0, 2\pi]$ with the endpoints identified. Let $T = \{ y_1, y_2, \ldots, y_N \} \subset S^1$ be a finite set of points, $N > 1$, given in increasing order, $0 \leqslant y_1 < y_2 < \ldots < y_N < 2\pi$, with $y_{N+1} = y_1$ for notational convenience. Let $f, g : S^1 \to R$ be given C^α-functions for $\alpha > 1/2$. Define piecewise-linear functions $f_T, g_T : S^1 \to R$ by requiring $f_T(y_i) = f(y_i)$ and $g_T(y_i) = g(y_i)$ for $i = 1, \ldots, N$, and f_T, g_T are linear between the points of T. We then obtain a piecewise-linear curve

$$C(f, g, T) = C(f_T, g_T) : S^1 \to R^2.$$

Let $A(f, g, T)$ denote the algebraic area enclosed by $C(f, g, T)$.

Let T_0 denote the given finite set, then define T_{n+1} inductively as the barycentric subdivision of T_n.

Lemma 8.7. — 1. For fixed T_0, the sequence $\{ A(f, g, T_n) \mid n = 0, 1, \ldots \}$ is Cauchy.
2. The limit

$$(96) \qquad A(f, g) = \lim_{n \to \infty} A(f, g, T_n)$$

is independent of the choice of initial set $T \subset S^1$.

Proof. — Let K be a constant such that

$$\max\{\,|f(\theta_1)-f(\theta_2)|,\,|g(\theta_1)-g(\theta_2)|\,\} \leqslant K\,|\,\theta_1-\theta_2\,|^\alpha$$

$$\text{for all } 0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi.$$

For an ordered subset set $Z = \{\,z_1 < \ldots < z_p\,\} \subset S^1$, define

$$\mathrm{mesh}(Z) = \max_{1\leqslant i\leqslant p+1}|\,z_{i+1}-z_i\,|.$$

Given a larger subset Z' containing Z, we order the elements of Z' lexicographically. That is, write

$$Z' = \{\,z_{i,j}\mid 1 \leqslant i \leqslant p;\, 0 \leqslant j \leqslant p_i\,\}$$

for integers p_i depending on i, and so that

$$z_i = z_{i,0} < z_{i,1} < \ldots < z_{i,\nu_i} < z_{i+1}.$$

For notational convenience, set $z_{i,\nu_i+1} = z_{i+1}$. The following is the key estimate:

Lemma 8.8. — *Assume that $p_i \leqslant d$ for all $1 \leqslant i \leqslant p$. (So that there are at most d points of Z' between any two adjacent points of Z.) Then*

$$(97) \qquad |\,A(f,g,Z') - A(f,g,Z)\,| \leqslant p\,dK^2.\mathrm{mesh}(Z)^{2\alpha}.$$

Proof. — Set

$$\mathbf{x}_i = (f(z_i),g(z_i)) \in \mathbf{R}^2 \quad \text{for } 1 \leqslant i \leqslant p,$$

$$\mathbf{x}_{i,j} = (f(z_{i,j}),g(z_{i,j})) \in \mathbf{R}^2 \quad \text{for } 1 \leqslant i \leqslant p;\; 0 \leqslant j \leqslant p_i,$$

$$\overline{x_i\,x_{i+1}} = \text{line segment from } \mathbf{x}_i \text{ to } \mathbf{x}_{i+1},$$

$$\overline{x_{i,j}\,x_{i,j+1}} = \text{line segment from } \mathbf{x}_{i,j} \text{ to } \mathbf{x}_{i,j+1},$$

and let $E(i)$ denote the algebraic area bounded by the segment $\overline{x_i\,x_{i+1}}$ and the polygonal curve joining $\mathbf{x}_{i,0}$ to $\mathbf{x}_{i+1,0}$ via the segments $\overline{x_{i,j}\,x_{i,j+1}}$. (See Figure 1 below.) Then
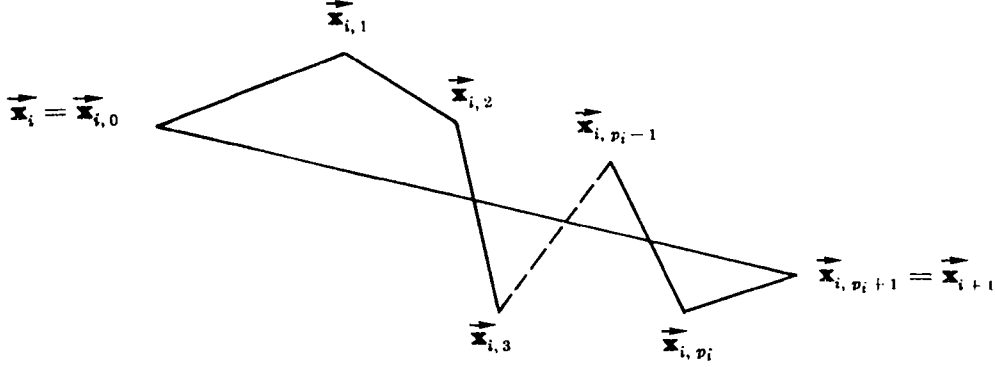
$$(98) \qquad |\,A(f,g,Z') - A(f,g,Z)\,| = |\,\sum_{i=1}^{p} E(i)\,|.$$

Let $\Delta_{i,j} \geqslant 0$ be the *area* of the plane triangle with vertices $\{\,\mathbf{x}_i, \mathbf{x}_{i,j}, \mathbf{x}_{i,j+1}\,\}$. We use the Hölder hypothesis to estimate $\Delta_{i,j}$. First note that $|\,z_i - z_{i,j}\,| \leqslant \mathrm{mesh}(Z)$, so that

$$|f(z_{i,j}) - f(z_i)| \leqslant K.\mathrm{mesh}(Z)^\alpha,$$

$$|g(z_{i,j}) - g(z_i)| \leqslant K.\mathrm{mesh}(Z)^\alpha.$$

This yields the estimates

$$\mathrm{dist}(\mathbf{x}_{i,j},\mathbf{x}_i) \leqslant \sqrt{2}K.\mathrm{mesh}(Z)^\alpha;\quad 1 \leqslant j \leqslant p_i + 1,$$

$$(99) \qquad \Delta_{i,j} \leqslant K^2.\mathrm{mesh}(Z)^{2\alpha};\quad 1 \leqslant i \leqslant p.$$

Fig. 1. — Geometry of E(i)

Combine the estimate (99) with plane geometry to obtain

$$| A(f, g, Z') - A(f, g, Z) | \leqslant \sum_{i=1}^{p} \sum_{j=1}^{p_i} \Delta_{i, j}$$

$$\leqslant \sum_{i=1}^{p} \sum_{j=1}^{p_i} K^2 . \mathrm{mesh}(Z)^{2\alpha}$$

$$\leqslant p \, dK^2 . \mathrm{mesh}(Z)^{2\alpha}. \qquad \square$$

We take $Z = T_n$ and $Z' = T_{n+1}$ in Lemma 8.8, with $d = 2$, $p = 2^n N$ and $\mathrm{mesh}(Z) = 2^{-n} \mathrm{mesh}(T_0)$, to obtain

(100) $| A(f, g, T_{n+1}) - A(f, g, T_n) | \leqslant 2^n N2K^2 . 2^{-2n\alpha} \mathrm{mesh}(T_0)^{2\alpha}$

$$= 2NK^2 . 2^{n(1-2\alpha)} \mathrm{mesh}(T_0)^{2\alpha}.$$

For $\alpha > 1/2$, this last term is summable in n, hence $\{ A(f, g, T_n) \}$ is a Cauchy sequence.

Let T' be another choice of a finite subset of S^1, and introduce the union of the initial choices, $T'' = T \cup T'$. Then there exists integers d, d' so that for all $n > 0$, the number of points in T''_n between any two adjacent points of T_n is bounded by d, and between any two points of T''_n by d'. (We say that T_n and T''_n are *comparable* subdivisions.) Apply Lemma 8.8 for $Z = T_n$ and $Z' = T''_n$ to obtain the estimate

(101) $| A(f, g, T''_n) - A(f, g, T_n) | \leqslant dNK^2 . 2^{n(1-2\alpha)} \mathrm{mesh}(T)^{2\alpha}$

which tends to zero as n tends to infinity. The similar estimate for T'_n yields that the limit (96) is independent of the choice of initial set T.

A Lipshitz change of coordinates Θ for S^1 has a constant K' so that for all $n > 0$, there is an estimate

$$\mathrm{mesh}(\Theta(T_n)) \leqslant K' . \mathrm{mesh}(T_n)$$

and similarly for the inverse. Therefore, we can find a uniform d as above so that T_n and $\Theta(T_n)$ are comparable subdivisions. As above, this implies that the limit of $\{ A(f, g, \Theta(T_n)) \}$ equals the limit of $\{ A(f, g, T_n) \}$, which proves that $A(f, g)$ is an invariant of Lipshitz coordinate changes. $\square$

We now define the *geometric extended cocycle*, c_h^A, for a $C^{1,\alpha}$-group action h by replacing the skew-product I of (92) with the area functional A of (96). Our final result of this section is that this geometric extension of the Godbillon-Vey invariant agrees with the previous analytic extension of the Godbillon-Vey invariant.

Proposition 8.9. — *Let* $h : \Gamma \times S^1 \to S^1$ *be a* $C^{1,\alpha}$-*action by orientation-preserving diffeomorphisms. Then* $c_h^I(\gamma_1, \gamma_2) = c_h^A(\gamma_1, \gamma_2)$ *for all* $\gamma_1, \gamma_2 \in \Gamma$.

Proof. — It suffices to show that the pairings I and A agree on the topological space $\mathscr{B}^\alpha \times \mathscr{B}^\alpha$. The pairings I and A agree on piecewise smooth functions, so for $f, g \in \mathscr{B}^\alpha$ we have that $I(f_{T_n}, g_{T_n}) = A(f_{T_n}, g_{T_n})$ for all $n > 0$. The result then follows from continuity of the pairings and the estimate:

Lemma 8.10. — *Let* $f \in \mathscr{B}^\alpha$ *for* $0 < \alpha < 1$, *and* $Z \subset S^1$ *a finite subset. Then*

$$(102) \qquad \| f - f_Z \|_\alpha \leqslant 4 \, \| f \|_\alpha \cdot \mathrm{mesh}(Z)^\alpha.$$

Proof. — For $0 \leqslant x < y < 2\pi$, we will estimate

$$(103) \qquad |\{ f(y) - f_Z(y) \} - \{ f(x) - f_Z(x) \}| \leqslant |f(y) - f_Z(y)| + |f(x) - f_Z(x)|.$$

Let $z_x \in Z$ be a point closest to x, and $z_y \in Z$ a point closest to y. Then

$$|f(x) - f_Z(x)| \leqslant |\{ f(x) - f_Z(x) \} + \{ f(z_x) - f_Z(z_x) \}|$$
$$\leqslant |f(x) - f(z_x)| + |f_Z(x) - f_Z(z_x)|$$
$$\leqslant \| f \|_\alpha \{ |x - z_x|^\alpha + |z_x' - z_x|^\alpha \}$$
$$\leqslant 2 \, \| f \|_\alpha \cdot \mathrm{mesh}(Z)^\alpha,$$

where $z_x' \in Z$ is the closest point to x in Z such that x lies between z_x and z_x'. A similar estimate for y yields (102). $\square$

9. Mitsumatsu Defect and Rigidity

The geodesic flow for a surface of variable, strictly negative curvature is a very special example of a smooth Anosov system. In this section, we will show that the Godbillon-Vey invariant for the weak-unstable foliations of such a flow is given by a formula whose terms are derivable from the curvature of the metric. This *Formula of Mitsumatsu* has surprising consequences, as originally noted by Mitsumatsu [53]. The formula is straightforward to derive for C^2-foliations, and establishing it for $C^{1,\alpha}$-foliations requires only a (non-trivial) technical modification.

Let (Σ, g) denote a closed orientable surface with strictly negative Gaussian curvature function $k(g) : \Sigma \to \mathbf{R}$ for the Riemannian metric g. Let $\pi : M \to \Sigma$ denote the unit tangent bundle to Σ, with $\{ f_t(g) \}$ the geodesic flow and $\xi = \xi(g)$ the geodesic spray. The group S^1 acts on the fibers of π by the counter-clockwise angle rotation in the tangent bundle, so we can introduce a uniform parameter φ on fibers, so that each fiber has length 2π. Let $w = \partial/\partial\varphi$ be the corresponding unit tangent vector field to the fibers.

The weak-unstable foliation of the flow $\{ f_t(g) \}$, denoted by $\mathscr{F}_g$, is $C^{1,\alpha}$ for all $0 < \alpha < 1$, so in particular has a well-defined Godbillon-Vey invariant $gv(g) = gv(\mathscr{F}_g)$. Note that as the weak-stable and weak-unstable foliations of the flow are conjugate by the $\pi/2$-rotation, there is only one secondary-type invariant for the flow.

The Riccati Equation along the flow (18) has a unique bounded positive solution which extends to a continuous function on M, denoted by $H : M \to \mathbf{R}$. The function H is actually as smooth as the weak-stable and weak-unstable foliations of the flow, so will be smooth along weak-stable and weak-unstable foliations, and transversally C^1. In particular, the Lie derivative wH is continuous.

Proposition **9.1** *(Formula of Mitsumatsu).* — *Let* (Σ, g) *be a closed oriented Riemann surface with strictly negative Gaussian curvature and Euler characteristic* $\chi(\Sigma)$. *Then*

$$(104) \qquad gv(g) = 4\pi^2 . \chi(\Sigma) - 3 . \int_M (wH)^2 \, d\,\mathrm{vol}.$$

Proof. — The idea of Mitsumatsu is to find an explicit 1-form θ defining $\mathscr{F}_g$ expressed in terms of the vector fields ξ, w and function H. Introduce the smooth vector field $\sigma = [w, \xi]$, the Lie commutator of ξ and w. It follows that σ is orthogonal to w and ξ, and the triple $\{ \xi, \sigma, w \}$ form an oriented frame field on M. Let $\{ \xi^*, \sigma^*, w^* \}$ denote the corresponding orthonormal dual framing of the cotangent bundle. Then w^* is the connection 1-form for the metric g, and $\{ \xi^*, \sigma^* \}$ are the Sölder horizontal 1-forms (cf. [5]). The 3-form $d\,\mathrm{vol} = \xi^* \wedge \sigma^* \wedge w^*$ is flow invariant, so gives the Liouville measure for the system.

The connection 1-form satisfies the structure equation

$$dw^* = k(g) \circ \pi . \xi^* \wedge \sigma^*$$

and therefore the vector fields satisfy the structure equations

$$(105) \qquad [w, \xi] = \sigma, \qquad [w, \sigma] = -\xi, \qquad [\xi, \sigma] = -k(g) \circ \pi . w.$$

The dual 1-form ξ^* is the invariant contact form for the flow, so the strong-stable and strong-unstable vector fields are written in terms of the framing by

$$(106) \qquad \begin{aligned} \eta^+ &= \sigma + H^+ \, w \\ \eta^- &= \sigma + H^- \, w \end{aligned}$$

for $C^{1,\Lambda*}$-functions H^+, H^- on M. The weak-unstable distribution E^{wu} is spanned by the pair $\{ \xi, \eta^+ \}$, and the Anosov condition (1) implies that both H^+ and H^- satisfy

the Riccati equation (18). Therefore, $H = H^+$ is the unique bounded positive solution, and H^- is the unique bounded negative solution. Thus,

(107) $\theta = w^* - H\sigma^*$

is a defining 1-form for the distribution E^{wu}. Use the identities (105) to calculate

$$d\theta = dw^* - dH \wedge \sigma^* - h \cdot d\sigma^*$$
$$= - k(g) \circ \pi \cdot \xi^* \wedge \sigma^* - dh \wedge \sigma^* + H \cdot w^* \wedge \xi^*$$
$$= \eta \wedge \theta,$$

where

(108) $\eta = - \iota(w) \, d\theta = (wH) \cdot \sigma^* - H \cdot \xi^*$.

If the 1-form η is C^1, then we can calculate

(109) $\eta \wedge d\eta = \{ - 2(wH)^2 - H^2 + H(w(wH)) \} \cdot d\,\mathrm{vol}$.

The strategy for the case when η is only transversally C^α is to establish an approximate form of equation (109). This will be based on the observation that differentiating the Riccati equation yields

(110) $w(\xi H) + 2H \cdot wH = 0$,

from which we deduce that $wH = - 1/2 \cdot \xi(\log(H))$, which is a C^1-function on M. Therefore, $w(wH)$ is a continuous function.

   Lemma 9.2. — *There exists a sequence of C^2-functions $\{ H_N \mid N = 1, \ldots \}$ on M such that $\{ wH_N \}$ converges uniformly to wH in the C^α-topology for all α, and $\{ w(wH_N) \}$ converges uniformly to $w(wH)$.*

   Proof. — The space of continuous functions on M decomposes into a Fourier series with respect to the fiberwise group action of S^1. Use this decomposition to write $H = \Sigma H_n \circ \pi \cdot e^{in\varphi}$, where each $H_n : \Sigma \to \mathbf{R}$ is a $C^{1,\alpha}$-function (cf. [25]). Choose C^2-functions $\{ H_{n,N} \}$ so that $\{ n^2 H_{n,N} \}$ converges in the $C^{1,\alpha}$-norm to $\{ n^2 H_N \}$, with uniform estimates on Σ, independent of n, and set

$$H_N = \sum_{n=-N}^{N} H_{n,N} \cdot e^{in\varphi}. \quad \square$$

Define 1-forms

$$\theta_N = w^* - H_N \cdot \sigma^*$$
$$\eta_N = - \iota(w) \, d\theta_N$$
$$= (wH_N) \cdot \sigma^* - H_N \cdot \xi^*$$

and calculate

(111) $\eta_N \wedge d\eta_N = \{ - 2(wH_N)^2 - H_N^2 + H_N(w(wH_N)) \} \cdot d\,\mathrm{vol}$.

Mitsumatsu made the observation that the S^1-invariance of d vol and the Gauss-Bonnet Theorem allows rewriting the integral of the formula (111) into the expression (104). The key identity is

$$(112) \qquad 0 = \int_M w(H_N \cdot wH_N) \cdot d\,\mathrm{vol} = \int_M (wH_N)^2 \cdot d\,\mathrm{vol} + \int_M H_N(w(wH_N)) \cdot d\,\mathrm{vol}.$$

The forms $\{\eta_N\}$ converge to η in the C^α-topology, so we use continuity of the pairing I with the identities (111) and (112) to calculate

$$
\begin{aligned}
gv(g) &= \lim_{N\to\infty} \int_M \eta_N \wedge d\eta_N \\
&= \lim_{N\to\infty} \left\{ -\int_M H_N^2 \cdot d\,\mathrm{vol} - 3 \cdot \int_M (wH_N)^2 \cdot d\,\mathrm{vol} \right\} \\
&= -\int_M H^2 \cdot d\,\mathrm{vol} - 3 \cdot \int_M (wH)^2 \cdot d\,\mathrm{vol}.
\end{aligned}
$$

Finally, the Riccati equation and Gauss-Bonnet Theorem imply that

$$
\begin{aligned}
-\int_M H^2 \cdot d\,\mathrm{vol} &= \int_M \xi H \cdot d\,\mathrm{vol} + \int_M k(g) \circ \pi \cdot d\,\mathrm{vol} \\
&= 2\pi \cdot \int_\Sigma k(g) \cdot d\Sigma \\
&= 4\pi^2 \cdot \chi(\Sigma). \qquad \square
\end{aligned}
$$

Corollary **9.3** *(Mitsumatsu)*. — *One has* $gv(g) = 4\pi^2 \cdot \chi(\Sigma)$ *if and only if g has constant negative curvature.*

Proof. — The formula of Mitsumatsu reduces the claim to showing that $wH = 0$ is equivalent to g having constant curvature. For $k(g) = -1$, the function $H^+ = 1$ is a positive bounded solution of the Riccati equation, so $H = 1$ and we have $wH = 0$.

For the converse, first note that the Lie identity $[w, \xi] = \sigma$ implies that

$$(113) \qquad w(\xi H) + \xi(wH) = \sigma H.$$

Combine equations (110) and (113) to conclude that $wH = 0$ is equivalent to $\sigma H = 0$. This last condition implies that H is constant along the flow of the vector field σ, which is ergodic. Thus, H is constant on M and the Riccati equation implies that $k(g) \circ \pi = -H^2$ is constant. $\square$

10. Some Open Problems

The Anosov class A_f of a flow $\{f\}$ is determined by the numbers $A_f(p, t_p)$ for periodic orbits by the Livshitz Theorem. For a geodesic flow, the periodic orbits project to closed geodesics on Σ, and each such orbit is uniquely determined by a non-trivial free homotopy class of closed curves on Σ. The set $\hat{\Gamma}$ is independent of the metric, being equal to the set of conjugacy classes in Γ. Thus, the class $A_{f(g)}$ of a metric g determines a real-valued function $A(g) : \hat{\Gamma} \to \mathbf{R}$, and we say that two metrics have the same Anosov class $A_{f(g)}$ if their corresponding functions $A(g)$ on $\hat{\gamma}$ coincide. Obviously, any diffeomorphism of Σ isotopic to the identity, applied to a metric, does not change the function $A(g)$.

Problem **10.1.** — *Characterize the set of metrics of negative curvature on a closed surface* Σ *with the same Anosov class* $A_{f(g)}$. *More specifically, is this set always finite-dimensional, after factorizing by the action of the group of diffeomorphisms of* Σ *isotopic to the identity?*

The length function $L_g : \hat{\Gamma} \to [0, \infty)$ of a Riemannian surface (Σ, g) assigns to a closed path in Σ the length of the shortest geodesic in the free homotopy class of the path. This function characterizes the isometry class of the metric [8, 55].

Problem **10.2.** — *Find a formula for the Godbillon-Vey invariant* $gv(g)$ *of the metric in terms of the length function* L_g. *For example, can the value of* $gv(g)$ *be derived from the* ζ-*function of* L_g?

The class of Zygmund functions most commonly arises in the study of singular integral operators, and defines a natural norm for these kernal operators (cf. [44, 59]). The conclusion that the weak-stable and weak-unstable foliations of a volume-preserving C^3-Anosov flow on a closed 3-manifold have regularity C^{1, A_*} was a surprise to the authors, which was observed strictly on the basis of obtaining the optimal conclusion from our techniques.

Problem **10.3.** — *Can the* C^{1, A_*}-*regularity of the weak-stable foliation be deduced from an analytic principle? For example, when* $\{f_t(g)\}$ *is a geodesic flow, is there a natural singular integral operator associated to* g, *or possibly to the action of the fundamental group* Γ *at infinity, whose regularity properties imply those of the weak-stable foliation?*

When a volume-preserving, C^∞-Anosov flow on a closed 3-manifold has C^∞ strong-stable and strong-unstable foliations, then the flow is algebraic by Ghys [17]. However, this conclusion is not known if only the weak-stable and weak-unstable foliations are given to be C^∞. The examples of Plante [57] show that an algebraic Anosov flow can be perturbed by a time change, so that it has C^∞ weak-stable and weak-unstable foliations, but the strong-stable and strong-unstable foliations are not even differentiable.

Problem **10.4.** — *Suppose that* $\{f_t\}$ *is a volume-preserving,* C^∞-*Anosov flow on a closed 3-manifold* M. *If the flow has* C^2 *weak-stable and weak-unstable foliations, show that it is possible to make a time-change for the flow, so that the new flow has* C^1 *(and hence* C^∞ *by Theorem* 2.3*) strong-stable and strong-unstable foliations.*

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