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# Nonuniform Hyperbolicity and Structure o Smooth Dynamical Systems

# §1. Introduction

A substantial part of recent progress in the theory of smooth dynamical systems is based on better and more systematic understanding, than before, of the role played by "hyperbolic" behavior and more specifically by nonuniform hyperbolicity and Lyapunov characteristic exponents. One and probably the most important aspect of this development concerns ergodic properties of smooth dynamical systems with respect to absolutely continuous invariant measures or other measures naturally connected with the smooth structure. The main work in that area in the last decade was done by Pesin [10], [11], [12], [13], and is now often referred to as the Pesin theory. Both the methods employed by Pesin and his results are essential for the subsequent development. He discovered the crucial role of nonuniform hyperbolicity and Lyapunov characteristic exponents and using these tools developed an ergodic theory for smooth dynamical systems with respect to an absolutely continuous invariant measure. His results include the celebrated entropy formula which shows that the entropy comes exclusively from the exponential expansion, the description of  $\pi$ -partition and a complete classification of systems with nonzero exponents.

Among the developments that appeared after Pesin's work I would like to point out Mañé's proof of the entropy formula [8], which contains a fundamental simplification of the original approach, the recent works of Ledrappier [6] and Ledrappier and L.-S. Young [7] on the characterization of measures satisfying the entropy formula and a work on ergodic theory of geodesic flows on manifolds of nonpositive curvature by BalImann and Brin [1]. The lack of space does not allow me to discuss here an extensive work by various authors on absolutely continuous invariant measures for one-dimensional maps and on various special, primarily 2-dimensional examples, including both conservative transformations and maps with nonuniformly hyperbolic attractors.

In this talk I am going to discuss another aspect of the development based on the concept of nonuniform hyperbolicity, namely, how certain global "exponential" properties of a dynamical system produce certain types of orbits including the abundance of periodic orbits and large hyperbolic sets. The structure of a dynamical system on a locally maximal hyperbolic set is well understood. It includes such ingredients as stable and unstable manifolds, local product structure, shadowing property, closing lemma, local stability of the set, density of periodic orbits among the recurrent orbits, Markov partitions, existence and uniqueness of measure with maximal entropy on basic sets, the uniform distribution of periodic orbits according to that measure and the growth of the number of periodic orbits with the exponential rate given by the topological entropy. Thus, the existence of an infinite locally maximal hyperbolic set for a given dynamical system provides considerable information about the orbit structure of the system and all effects obtained that way persist under small perturbations of the system.

Most of the results discussed below are contained in my papers [2], [3], [4], [5], although in several cases I will formulate theorems in slightly stronger or more general form than they were written.

Before proceeding to a more technical discussion let me outline the strategy of the approach. We begin with a certain "global" property which indicates that some kind of exponential growth is present. Here are some examples of global exponential properties.

(i) Positive topological entropy, i.e., the exponential growth rate of the number of different orbits distinguishable with an arbitrary fine buf fixed precision.

(ii) Exponential behavior of the iterates of the map  $f_*$  induced by a diffeomorphism  $f: M \to M$  on the fundamental group  $\pi_1(M)$ , i.e., the exponential growth of the word-length norm of the iterates  $f_*^n \gamma$  for all (or some)  $\gamma \in \pi_1(M) \setminus \{id\}$ .

(iii) Similar exponential behavior of the maps induced on homology groups.

(iv) Exponential growth of the volume of a ball on the universal covering of a compact Riemann manifold M. This property appears when the dynamical system under consideration is the geodesic flow generated by the metric.

(v) In the same situation as in (iv), the exponential growth of the fundamental group  $\pi_1(M)$  is another exponential type property.

We will derive from a global property the existence of invariant measures for the dynamical system such that orbits typical with respect to such a measure possess a weaker type of hyperbolicity than the orbits belonging to a hyperbolic set. The linearized system along such an orbit allows an exponential dychotomy but the coefficients in front of the exponential terms may oscillate as the initial point moves along the orbit. This is the reason for calling those orbits nonuniformly hyperbolic. However, in our case the oscillations of the coefficients are not too big, they are essentially subexponential. The existence of many such regular nonuniformly hyperbolic orbits follows from Oseledec's Multiplicative Ergodic Theorem [9]. A neighborhood of a regular nonuniformly hyperbolic orbit possesses certain properties similar to a neighborhood of a hyperbolic set. Using proper variations of closing and shadowing arguments one can catch many orbits which never leave a (noninvariant) neighborhood with uniform hyperbolic estimates and thus possess a uniform hyperbolic structure. This construction may be supplemented with the estimates on the number of different orbits found and on the quality of hyperbolic estimates along those orbits.

Let us discuss the last notion in detail. Let x be a hyperbolic periodic point of period n. The degree of hyperbolicity of x is measured by the number

$$m(x) = \frac{1}{n} \min_{\lambda \in \text{sp} Df_{x}^{n}} |\log |\lambda||.$$
(1)

Our standard set-up in the discrete time case is to consider a diffeomorphic embedding  $f: U \to M$  of an open neighborhood U of a compact invariant set  $\Gamma$ ; here M is an ambient smooth manifold. Let for an open set  $V \supset \Gamma$ and for  $\chi > 0$ ,  $n \in \mathbb{Z}_+$ ,  $P_{n,\chi}^{V}(f)$  be the number of hyperbolic points  $w \in V$ of period n with  $m(w) \ge \chi$ .

Furthermore, let

$$p_{\mathbf{x}}^{\mathbf{V}}(f) = \varlimsup_{n \to \infty} \frac{\log P_{n,\mathbf{x}}^{\mathbf{V}}(f)}{n}$$

and

$$p_{\mathbf{x}}^{\Gamma}(f) = \inf_{\mathbf{y} \mathrel{\triangleright} r} p_{\mathbf{x}}^{\mathbf{y}}(f).$$

If  $\Gamma = M$  we will write  $p_{\chi}(f)$  instead of  $p_{\chi}^{\Gamma}(f)$ .

Similar definitions can be made for a continuous time dynamical system in a similar set-up. In the definition of m(x), the eigenvalue 1 corresponding to the direction of the vector field should be excluded; instead of periodic points, periodic orbits would be counted; instead of orbits of a fixed period, one should count all orbits of period  $\leq T$ .

#### § 2. Main results and applications

We will assume the standard set-up described above. All maps and flows are assumed of class  $O^{1+\delta}$  for some  $\delta > 0$ . In the continuous time case we also assume that the flow does not have fixed points on  $\Gamma$  (added in proof: I have recently been able to remove this assumption). In both cases,  $h_{\Gamma}$  will denote the topological entropy of the dynamical system restricted to  $\Gamma$ . We assume  $h_{\Gamma} > 0$ .

THEOREM 1. Let  $f: U \rightarrow M$  and dim M = 2. Then for every  $\varepsilon > 0$ 

$$p_{h_{\Gamma}-s}^{\Gamma}(f) \geqslant h_{\Gamma}.$$

THEOREM 2. Let  $f_i: U \to M$  be a flow and dim M = 3. Then for every  $\varepsilon > 0$ 

$$p_{h_{\Gamma}-s}^{\Gamma}(f) \ge h_{\Gamma}.$$

THEOREM 3. Under the assumptions of Theorem 1, for every  $\varepsilon > 0$  and every open set  $V \supset \Gamma$  there exists an invariant locally maximal hyperbolic set  $\Lambda_{\varepsilon} \subset V$  such that  $f|_{\Lambda_{\varepsilon}}$  is topologically conjugate to a subshift of finite type and

$$h(f|_{A_{\bullet}}) > h_{\Gamma} - \varepsilon.$$

THEOREM 4. Under the assumptions of Theorem 2, for every  $\varepsilon > 0$ and every open set  $V \supset \Gamma$  there exists an invariant locally maximal hyperbolic set  $\Lambda_s \subset V$  such that  $f_t|_{\Lambda_s}$  is topologically conjugate to a suspension over a subshift of finite type and

$$h(f_t|_{\Lambda_s}) > h_{\Gamma} - \varepsilon.$$

COROLLARY 1. The topological entropy h(f) of any  $C^{1+\delta}$  diffeomorphism  $f: M \rightarrow M$  is upper-semicontinuous as a function of f in  $C^0$  topology.

*Proof.* Follows immediately from Theorem 3 applied to  $\Gamma = M$  and from the topological stability of hyperbolic sets.

THEOREM 5. Let  $f: M \rightarrow M$  be an area-preserving diffeomorphism of a compact surface. Then f has a hyperbolic periodic point iff

$$\overline{\lim_{n\to\infty}} \frac{\log \|Df^n\|}{n} > 0.$$
<sup>(2)</sup>

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Here we assume that a Riemannian metric is fixed on M so that  $||Df|| = \max$  $\sup \|Dfv\|/\|v\|$ . However, the quantity in the left-hand  $x \in M$   $v \in T_x \overline{M} \setminus \{0\}$ 

part of (2) does not depend on the choice of Riemannian metric.

All results stated above about the existence of many periodic points and nontrivial invariant sets depend on smoothness. M. Rees [14] constructed an example of a minimal homeomorphism of the 2-torus with positive topological entropy. It is not clear, however, whether the  $C^{1+\delta}$  assumption can be replaced by  $C^1$ .

The next group of results deals with the situations where the existence of many periodic orbits has been established by topological or variational methods. Such methods, however, usually say nothing about the hyperbolicity of those orbits. By applying the above-stated theorems one can ensure the existence of many hyperbolic orbits.

Let  $f: T^2 \to T^2$  be a diffeomorphism of the two-dimensional torus which acts on the first homology group hyperbolically. This action is determined by an integer matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  such that det  $A = \pm 1$  and  $|\mathrm{tr} A| > 2$ . Let  $\lambda$  be the eigenvalue of A of absolute value greater than 1. and  $\alpha = \log |\lambda|$ . Then  $h(f) \ge \alpha$ .

COROLLARY 2. For every  $\varepsilon > 0$ 

$$p_{a-s}(f) \ge a$$
.

#### If, in addition, f is an Anosov diffeomorphism then

 $p_{\alpha}(f) \geq \alpha$ .

Let M be a compact surface of genus greater than one and f:  $M \rightarrow M$ be a diffeomorphism homotopic to a pseudo-Anosov map  $f_0$ . Then  $h(f) \ge a$ where  $\alpha = h(f_0)$  and  $\alpha$  is also equal to the exponential growth rate of the word-length norm for the iterates  $f_*^n \gamma$  where  $\gamma$  is an arbitrary element of  $\pi_1(M)$  different from identity. Nielsen's theorem implies that the exponential growth rate of the number of periodic orbits for f is  $\geq \alpha$ .

COROLLARY 3. For every  $\varepsilon > 0$ 

 $p_{a-s}(f) \ge a$ .

The next example is more interesting. Let  $\sigma$  be a Riemannian metric of class  $C^{2+\delta}$  on a compact surface M with negative Euler characteristic E such that the total area of M is equal to v. Let  $\varphi_t^{\sigma}$  be the geodesic flow generated by that metric. The exponential growth rate  $p_{\sigma,x}$  for the num-

ber of hyperbolic closed geodesics with the positive Lyapunov exponent  $\geq \chi$  coincides with what we denote by  $p_{\chi}(\varphi_i^{\sigma})$ . Let  $K(E, v) = (-2\pi E/v)^{1/2}$ . If  $\sigma$  is a metric of constant negative curvature then this number represents the common value of the topological entropy, entropy with respect to Liouville (smooth) measure and the positive Lyapunov exponent along any orbit.

THEOREM 6 [4].

$$p_{\sigma,K(E,v)} \geq K(E,v)$$

and this inequality is strict unless  $\sigma$  is a metric of constant negative curvature. Moreover, for every metric of nonconstant curvature there exists  $\varepsilon_{\sigma} > 0$  such that

$$p_{\sigma,K(E,v)+s_{\sigma}} > K(E,v).$$

Thus, any metric of nonconstant curvature has more closed geodesics with stronger hyperbolic properties than any metric of constant curvature on the same surface with the same total area.

This theorem follows from Theorem 2 and an entropy estimate. The metric  $\sigma$  is conformally equivalent to a metric  $\sigma_0$  of constant negative curvature and the same total area. Let  $\rho$  be the conformal coefficient. Its average is equal to one. Therefore, the average of  $\rho^{1/2}$ , which we will denote by  $\rho_{\sigma}$ , is less than 1 unless  $\rho \equiv 1$ .

Let  $h_{\sigma}$  be the topological entropy of the geodesic flow  $\varphi_i^{\sigma}$ . Here is the desired entropy estimate.

THEOREM 7[4].

$$h_{\sigma} \geqslant \varrho_{\sigma}^{-1} K(E, v).$$

## § 3. Hyperbolic measures

Let  $\mu$  be a Borel probability measure supported by  $\Gamma$  and invariant and ergodic with respect to a map or a flow under consideration. Let  $\chi_1^{\mu} < \chi_2^{\mu} < \ldots < \chi_r^{\mu}$  be the Lyapunov characteristic exponents of the dynamical system with respect to  $\mu$ . The multiplicative ergodic theorem implies that for  $\mu$ -almost every point  $x \in \Gamma$  there exists a measurable invariant decomposition of the tangent space  $T_x M = E_1(x) \oplus \ldots \oplus E_r(x)$  such that for  $v \in E_i(x)$ 

$$\lim_{t\to\pm\infty}\frac{\log\|Df_tv\|}{t}=\pm\chi_i^{\mu}.$$

By the ergodicity, dim  $E_i(x)$  must be constant  $\mu$ -almost everywhere. We will denote this dimension by  $k_i^{\mu}$  and call it the *multiplicity* of the exponent  $\chi_i^{\mu}$ .

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DEFINITION 1. A measure  $\mu$  is called hyperbolic if

(i) in the discrete time case, all  $\chi_i^{\mu}$  are different from 0,

(ii) in the continuous time case, the zero exponent has multiplicity one.

Sometimes we will also call a nonergodic invariant measure hyperbolic if almost all its ergodic components are hyperbolic measures.

For a hyperbolic invariant measure  $\mu$  let

$$m(\mu) = \min_{i: \chi_i^{\mu} \neq 0} (\chi_i^{\mu}).$$

This definition agrees with (1) for a measure concentrated on a single hyperbolic periodic orbit. Naturally,  $m(\mu)$  characterizes the minimal rate of exponential behavior typical for the system.

THEOREM 8. Let  $\mu$  be an invariant ergodic hyperbolic measure for a map or a flow. Let  $x \in \operatorname{supp} \mu$ . Then for any  $\delta > 0$ , any neighborhoods  $V \ni x$  and  $W \supset \operatorname{supp} \mu$ , and any collection of continuous functions  $\varphi_1, \ldots, \varphi_k$  there exists a hyperbolic periodic point  $z \in V$  such that the orbit of z is contained in W and

$$m(z) > m(\mu) - \delta.$$

Moreover, in the diffeomorphism case

$$\left|(\operatorname{per} z)^{-1} \sum_{k=0}^{\operatorname{per} z - 1} \varphi_i(f^k z) - \int \varphi_i d\mu \right| < \delta$$

for i = 1, ..., k. A similar property holds for flows.

The last statement means that the orbit of the point z is almost uniformly distributed with respect to  $\mu$ .

Theorem 5 follows easily from Theorem 8 since (2) implies the existence of an *f*-invariant measure whose largest exponent is positive and the preservation of area ensures that the second exponent for that measure is negative. Another corollary is "weak stability" of hyperbolic measures in  $C^1$  topology.

COROLLARY 4. Let  $\mu$  be an invariant ergodic hyperbolic measure for a diffeomorphism f or a flow  $f_t$ . If  $f_n$  converges to f (correspondingly  $f_t^{(n)}$  converges to  $f_t$ ) in  $C^1$  topology, then  $f_n$  ( $f_t^{(n)}$ ) has an invariant hyperbolic measure  $\mu_n$  such that  $\mu_n$  converges to  $\mu$  weakly.

THEOREM 9. If, under the assumptions of Theorem 8,  $\mu$  is not concentrated on a single periodic orbit, then z has a transversal homoclinic orbit. COROLLARY 5. If a diffeomorphism or a flow has a hyperbolic ergodic invariant measure whose support is an infinite set then its topological entropy is positive.

THEOREM 10. Under the assumptions of Theorem 8, let  $\text{supp } \mu = \Gamma$ and  $h_{\mu}(f)$  (corr.  $h_{\mu}(f_i)$ ) be equal to h > 0. Then for any  $\varepsilon > 0$ 

$$p_{h-s}^{\Gamma}(f) \ge h$$
 (corr.  $p_{h-s}^{\Gamma}(f_i) \ge h$ ).

Theorems 1 and 2 follow easily from Theorem 10, variational principle, and Ruelle's entropy inequality [15].

THEOREM 11. Under the assumptions as in the previous theorem, there exists an f-invariant, locally maximal hyperbolic set  $\Lambda_s$  such that the restriction  $f|_{\Lambda_s}$  is topologically conjugate to a subshift of finite type and

$$h(f|_{\Lambda_{\bullet}}) > h_{\mu}(f) - \varepsilon.$$

Moreover, any orbit on  $\Lambda_s$  is almost uniformly distributed with respect to  $\mu$  (cf. Theorem 8).

Theorem 11 and its counterpart for flows which we do not formulate explicitly imply Theorems 3 and 4 in the same fashion as Theorem 10 implies Theorems 1 and 2.

It also allows us to strengthen weak stability of Corollary 4 to "entropy stability".

COROLLARY 6. Under the assumptions of Corollary 4, the sequence of measures  $\mu_n$  can be chosen with the additional property  $h_{\mu_n}(f_n) \rightarrow h_{\mu}(f)$  (corr.  $h_{\mu_n}(f_i^{(n)}) \rightarrow h_{\mu}(f_i)$ ).

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