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Infinitesimal Lyapunov functions, invariant cone families and stochastic properties of smooth dyanmical systems

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Abstract. We establish general criteria for ergodicity and Bernoulliness for volume-preserving diffeormorphisms and flows on compact manifolds. We prove that every ergodic component with non-zero Lyapunov exponents of a contact flow is Bernoulli. As an application of our general results, we construct on every compact 3-dimensional manifold a C^{∞} Riemannian metric whose geodesic flow is Bernoulli.

1. *Introduction*

This paper represents a completed, revised and expanded version of the 1988 preprint 'Invariant cone families and stochastic properties of smooth dynamical systems' by the first author. The current version was written during his visit to IHES at Bures-sur-Yvette in May–June 1991, whose support and hospitality are readily acknowledged.

Our primary goal is to establish verifiable criteria for ergodicity and stronger stochastic properties, specifically the Bernoulli property, for several important classes of smooth dynamical systems with absolutely continuous invariant measure. The first part of the paper which includes §2 and the first part of §3 is primarily expository. It is needed both to provide a conceptual framework and to establish convenient notations for the original results which are contained in the rest of the paper.

We consider, in particular, symplectic diffeomorphisms of compact symplectic manifolds and geodesic flows on compact Riemannian manifolds and, more generally, contact flows on compact contact manifolds. The most widely applicable general known method of proving ergodicity and other stochastic properties for smooth dynamical systems is to deduce it from a certain kind of asymptotic hyperbolicity for infinitesimal

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families of orbits. This method goes back to the seminal works of Hopf **[H]** and Anosov **[A]** who showed how ergodicity (and in Anosov's case stronger stochastic properties) can be obtained from uniform hyperbolicity. The method was later extended to apply in the much more common situation of non-uniform hyperbolicity. Since the history of the emergence and applications of this method is long and some aspects of it, especially those related to the study of dynamical systems with singularities, are rather involved, we will not try here to give comprehensive historical remarks. Instead we will discuss primarily the contributions of Pesin and Wojtkowski which are crucial for establishing a natural conceptual framework for the subject and on which our work is directly based.

The results of Pesin **[P1]**, **[P2]** play the fundamental role in this area. Pesin shows that a rather weak, at least very non-uniform, kind of hyperbolicity, namely non-vanishing of Lyapunov characteristic exponents, produces ergodic and Bernoulli components of positive measure. In §3 below, we present appropriately adapted versions of some of his results (cf, Theorems 3.2, 3.5 and Corollary 3.1). In the continuous time case, according to Pesin, every ergodic component with non-zero characteristic exponents is either Bernoulli or admits a measurable eigenfunction. We prove (Theorem 3.6) that for a contact flow the first alternative always takes place. In order to build effective criteria for ergodicity upon these results, one needs to append the Pesin theory on both sides, i.e. to find verifiable methods for checking the non-vanishing of Lyapunov exponents and for a better understanding of the nature of ergodic components which are in general described by Pesin theory in a rather indirect way.

The first task was very effectively accomplished by Wojtkowski in **[W]**. He showed that the existence of a family of cones in the tangent bundle, which is mapped into itself by the linearized dynamical system, is in a number of cases sufficient for the non-vanishing of the exponents. Certainly Wojtkowski was not the first one to associate cone families with hyperbolicity. The importance of his work lies in the general and purely qualitative character of the cone conditions he uses. In fact, Wojtkowski's results do not depend on the smooth structure of the system; they deal with linear extensions of measure-preserving transformations and flows. It turns out that Wojtkowski's results can be put into a more general and more convenient framework. This task is accomplished in §2. The notion of infinitesimal Lyapunov function which we introduce helps to clarify the conditions under which the existence of an invariant cone family guarantees non-vanishing of all Lyapunov exponents. Our approach is a development of that by Lewowicz **[L1]**, **[L2]** and Markarian **[Ma]**. In particular, Theorem 2.1 is a generalization of Theorem 1 of **[Ma]**.

Passing from the ergodic components of positive measure given by Pesin's theorems to actual ergodicity requires some assumptions about 'uniformity' of the non-uniformly hyperbolic structure. Pesin's own strategy for doing that, which he applied to geodesic flows on surfaces without focal points **[P3]**, used monotonicity and convexity properties for the Jacobi fields and included the construction of a global, i.e. defined everywhere outside of a fixed exceptional set and not just almost everywhere, expanding foliation whose leaves include local expanding manifolds as open sets. A similar approach was used in the first author's work on Bernoulli diffeomorphism on surfaces **[K]** and related later work on smooth (Gerber and Katok **[GK]**) and real-analytic (Gerber **[G]**) models of pseudo-Anosov maps. Such procedures involve first producing a global invariant plane field inside the cone field and then integrating it. Those steps usually required *ad hoc* arguments, often long and delicate, based on special structures of the examples under consideration.

The main technical advance which allows us to bypass the subtleties of the construction of a global foliation is an observation that a continuous version of the same condition (existence of an infinitesimal Lyapunov function or an invariant cone family) which guarantees non-vanishing of the Lyapunov exponents allows one to extend almost every local stable and unstable manifold so that it reaches uniform size without too much wiggling (cf, §5). Let us point out that the two-dimensional case can be treated separately by a method suggested by Burns and Gerber [**BG1**] which does not extend to the multi-dimensional case. After the extension of the stable and unstable manifolds is achieved, a relatively standard application of methods of Pesin theory leads to the conclusion that ergodic components are essentially open sets. A somewhat stronger version of the same condition which guarantees the uniform transversality of stable and unstable manifolds almost everywhere, then allows it to be concluded that the ergodic components must contain a connected component of the open set carrying the invariant cone family (§6).

The results of this paper (Theorem 2.1, Theorems 4.1 and 4.2) provide a unified and simplified treatment of the ergodicity and strong stochastic behavior for all known cases of smooth invertible conservative dynamical systems for which some sort of non-uniformly hyperbolic behavior has been found. They also provide a framework for finding new examples of systems with ergodic and Bernoulli behavior. As an interesting application we construct in §7 a C^∞ Riemannian metric on every three-dimensional compact manifold with Bernoulli geodesic flow. The construction appeared as a result of discussions between the first author and Michael Anderson. Further development in this direction appeared in the joint work of Marlies Gerber and the second author [**BG3**]. They constructed Riemannian metrics with Bernoulli geodesic flows on every smooth manifold which is a product of factors of dimension less than or equal to three.

Similar methods can be applied to dynamical systems with singularities. The main results of Pesin's work were extended in [**KS**] to a fairly general axiomatically defined class of systems with singularities which includes billiard systems and other interesting physical models. It seems that in order to obtain openness of ergodic components it is necessary to impose extra more geometric assumptions on the singularities of the system, in addition to assuming the existence of an infinitesimal Lyapunov function. The key ideas for overcoming the influence of singularities were suggested by Bunimovich and Sinai [**BS**] and developed in a systematic way by Chernov and Sinai [**CS**]. Based on their method, Krámli, Simányi and Szász made important progress in the famous problem of the hard sphere gas [**KSS1**], [**KSS2**]. Liverani and Wojtkowski combined the general approach developed in [**W**] and the earlier version of the present paper with the Chernov–Sinai method and proved criteria for symplectic systems with singularities to have stochastic behavior [**LW**]. In the non-singular case their result is essentially the same as Corollary 4.1 of rhe present paper.

2. *Cocycles over dynamical systems, characteristic exponents, Lyapunov functions and cone families*

Let (X, μ) be a Lebesgue probability space, $T : (X, \mu) \to (X, \mu)$ be a measure-preserving transformation and $A : X \to GL(m, \mathbb{R})$ be a measurable map such that

$$\max(\log \|A\|, \log \|A^{-1}\|) \in L^1(X, \mu). \tag{2.1}$$

These data determine a linear extension

$$T^{(A)} : X \times \mathbb{R}^m \to X \times \mathbb{R}^m, \; T^{(A)}(x, v) = (Tx, A(x)v).$$

Let

$$A(x, n) = \begin{cases} A(T^{n-1}x) \cdots A(Tx)A(x) & \text{for } n > 0 \\ A^{-1}(T^{-n}x) \cdots A^{-1}(T^{-1}x) & \text{for } n > 0. \end{cases} \tag{2.2}$$

Obviously, $(T^{(A)})^n(x, v) = (T^n x, A(x, n)v)$. Formula (2.2) determines a $GL(n, \mathbb{R})$-valued cocycle over the $\mathbb{Z}$-action $\{T^n\}_{n \in \mathbb{Z}}$. By a slight abuse of terminology, we will sometimes call the map A itself a cocycle.

The multiplicative ergodic theorem **[O]** asserts that for almost every $x \in X$ the following limits

$$\lim_{n \to \infty} \frac{1}{n} \log \|A(x, n)v\| \stackrel{\text{def}}{=} \chi^+(v, x; T, A) \stackrel{\text{def}}{=} \chi^+(v)$$

and

$$\lim_{n \to \infty} \frac{1}{n} \log \|A(x, -n)v\| \stackrel{\text{def}}{=} \chi^-(v, x; T, A) \stackrel{\text{def}}{=} \chi^-(v)$$

exist for every $v \neq 0$.

Furthermore, there is a $T^{(A)}$-invariant measurable decomposition defined for almost every $x \in X$,

$$\mathbb{R}^m = \bigoplus_{i=1}^{k(x)} E_x^i \tag{2.3}$$

such that $\chi^{\pm}(v) = \pm\lambda_i(x)$ for every $v \in E_x^i \setminus \{0\}$, where $\lambda_1(x) < \lambda_2(x) < \cdots < \lambda_{k(x)}(x)$. The T-invariant functions $\lambda_i(x)$ are called the *Lyapunov characteristic exponents* of the extension $T^{(A)}$. The dimension of the space E_x^i is called the *multiplicity of the exponent* $\lambda_i(x)$. If the transformation T is ergodic with respect to μ, the Lyapunov characteristic exponents and their multiplicities are independent of x.

Let Q be a continuous real-valued function in $\mathbb{R}^m$ which is homogeneous of degree one and takes both positive and negative values. We will call the set

$$C^+(Q) \stackrel{\text{def}}{=} Q^{-1}((0, \infty)) \cup \{0\}$$

the *positive cone associated to Q* or simply the *positive cone of Q*. Similarly,

$$C^-(Q) \stackrel{\text{def}}{=} Q^{-1}((-\infty, 0)) \cup \{0\}$$

is the *negative cone associated to Q* or the *negative cone of Q*. We will call the *positive* (resp. *negative*) *rank* of Q and denote by $r^+(Q)$ (resp. $r^-(Q)$) the maximal dimension of a linear subspace $L \subset \mathbb{R}^m$ such that $L \subset C^+(Q)$ (resp. $L \subset C^-(Q)$). Obviously,

$r^+(Q) + r^-(Q) \leq m$. Our assumption implies that $r^+(Q) \geq 1$ and $r^-(Q) \geq 1$. We will call the function Q *complete* if

$$r^+(Q) + r^-(Q) = m.$$

The prime examples of functions of this sort are

$$Q(v) = \operatorname{sign} K(v, v) \cdot |K(v, v)|^{1/2}, \tag{2.4}$$

where $K(v, v)$ is a non-degenerate indefinite quadratic form. The positive and negative rank of such a Q are equal correspondingly to the positive and negative indices of inertia, i.e. the number of positive and negative eigenvalues for the quadratic form K. The function Q defined by (2.4) is complete.

More generally, if λ is a positive real number and F is a real function on $\mathbb{R}^m$ which is homogeneous of degree λ and takes both positive and negative values, one can define a homogeneous function Q of degree one by

$$Q(v) = \operatorname{sign} F(v) \cdot |F(v)|^{1/\lambda}. \tag{2.5}$$

Then one would mean by the positive and negative cone, positive and negative rank and completeness of F the corresponding properties of Q.

The notions of positive and negative rank and completeness can be defined in a somewhat more general context. Let C be an open cone in $\mathbb{R}^m$, i.e. a homogeneous subset $C \subset \mathbb{R}^m$ such that $C \setminus \{0\}$ is open. The *rank* of C, $r(C)$, is defined as the maximal dimension of a linear subspace $L \subset \mathbb{R}^m$ which is contained in C. The *complementary cone* $\widehat{C}$ to C is defined by

$$\widehat{C} = (\mathbb{R}^n \setminus \operatorname{Clos} C) \cup \{0\}.$$

Obviously the complementary cone to $\widehat{C}$ is C.

A pair of complementary cones C, $\widehat{C}$ is called *complete* if $r(C) + r(\widehat{C}) = m$.

We will call a real-valued measurable function Q on $x \times \mathbb{R}^m$ a *Lyapunov function for the extension* $T^{(A)}$ (or simply *for the cocycle* A) if

(i) For almost every $x \in X$ the function Q_x on $\mathbb{R}^m$ defined by $Q_x(\cdot) = Q(x, \cdot)$ is continuous, homogeneous of degree one and takes both positive and negative values.

(ii) The positive rank $r^+(Q_x)$ and the negative rank $r^-(Q_x)$ are constant almost everywhere and Q_x is complete for almost every x.

(iii) For almost every $x \in X$

$$Q_{T_x}(A(x)v) \geq Q_x(v) \quad \text{for all } v \in \mathbb{R}^m.$$

If the inequality in (iii) is strict for every $v \neq 0$, we will call Q a *strict Lyapunov function* for $T^{(A)}$. The notion which is both useful and flexible lies in between the Lyapunov and the strict Lyapunov property.

Definition 2.1. A real-valued measurable function Q on $X \times \mathbb{R}^m$ is called *an eventually strict Lyapunov function* for $T^{(A)}$ if it satisfies conditions (i)–(iii) above and the following condition:

(iv) For almost every $x \in X$ there exists $n = n(x) > 0$ such that for all $v \in \mathbb{R}^m \setminus \{0\}$

$$Q_{T^n(x)}(A(x,n)v) > Q_x(v)$$

and

$$Q_{T^{-n}(x)}(A(x,-n)v) < Q_x(v).$$

Condition (ii) allows one to define the positive and negative rank $r^+(Q)$ and $r^-(Q)$ of a Lyapunov function as the common values of $r^+(Q_x)$ and $r^-(Q_x)$ respectively for almost every x.

The notion of eventually strict Lyapunov function gives a convenient and concise way to formulate a generalization of some results of Wojtkowski from [**W**]. For Wojtkowski's results in their original form see Proposition 2.1 and Corollary 2.2 below.

THEOREM 2.1. *If a cocycle $A : X \to GL(n,\mathbb{R})$ satisfies (2.1) and the extension $T^{(A)}$ possesses an eventually strict Lyapunov function Q, then $T^{(A)}$ has almost everywhere exactly $r^+(Q)$ positive Lyapunov characteristic exponents and $r^-(Q)$ negative ones. For almost every x one has $E_x^+ \subset C^+(Q_x)$ and $E_x^- \subset C^-(Q_x)$.*

This theorem was proved by Markarian [**Ma**, Theorem 1] in the case when Q is obtained from a quadratic form by formula (2.4).

Proof. First, let us consider the decomposition of T into ergodic components. Both condition (2.1) and the existence of an eventually strict Lyapunov function are inherited by almost every ergodic component of T. On the other hand, the conclusion of the theorem would hold for T if it held for almost every ergodic component of T. Thus we may assume without loss of generality that T is ergodic.

Secondly, in order to establish the conclusion of the theorem, it is sufficient to show that for almost every $x \in X$ there exist subspaces D_x^+ and D_x^- of $\mathbb{R}$ of dimension $r^+(Q)$ and $r^-(Q)$ respectively, such that for all integers n (both positive and negative)

$$A(x,n)D_x^{\pm} \subset C^{\pm}(Q_{T^n x}) \tag{2.6}$$

and for all non-zero $v \in D_x^{\pm}$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|A(x,\mp n)v\| < 0. \tag{2.7}$$

In fact one then has $D_x^{\pm} = E_x^{\pm}$ for almost every $x \in X$.

We shall prove the existence of the spaces D_x^+. The argument for D_x^- is completely similar, with T^{-1} replacing T and the cones $C^-(Q_x)$ playing the role of $C^+(Q_x)$.

Let $\overline{C}_x^+$ be the closure of the cone $C^+(Q_x)$. According to our assumption, it contains a subspace of dimension $r^+(Q)$. For $n = 1, 2 \ldots$, let

$$C_{n,x}^+ = A(T^{-n}x, n)\overline{C}_{T^{-n}x}^+.$$

By condition (iii) from the definition of a Lyapunov function, the sequence $\{C_{n,x}^+\}$ is nested, i.e. $C_{1,x}^+ \supset C_{2,x}^+ \supset \ldots$; obviously each set $C_{n,x}^+$ still contains a subspace of dimension $r^+(Q)$. Using compactness of the intersection of $C^+_{n,x}$ with the unit sphere, we deduce that the intersection

$$C^+_{\infty,x} = \bigcap_{n=1}^{\infty} C^+_{n,x}$$

also contains a subspace of dimension $r^+(Q)$. From the construction of the set $C^+_{n,x}$ and from conditions (iii) and (iv), we see that for almost every $x \in X$, any $v \in C^+_{\infty,x}$ and any integer n

$$A(x,n)v \subset C^+(Q_{T^n x}).$$

Thus if we take as D^+_x any $r^+(Q)$-dimensional space lying inside $C^+_{\infty,x}$ condition (2.6) will be satisfied. In particular,

$$C^+_{\infty,x} \subset C^+(Q_x),$$

so that the function Q_x is positive on $C^+_{\infty,x}$. Since the intersection of the set $C^+_{\infty,x}$ with the unit sphere is compact, the function $Q_x(v)/\|v\|$ has a positive lower bound $q(x)$ on the set $C^+_{\infty,x}$. On the other hand, since Q_x is a continuous homogenous function of degree one, the function $Q_x(v)/\|v\|$ has an upper bound $s(x)$. Thus there is a set of positive measure $E \subset X$ and positive constants c_1, c_2 such that for all $x \in E$ and all $v \in C^+_{\infty,x}$

$$c_1\|v\| \le Q_x(v) \le c_2\|v\|. \tag{2.8}$$

By ergodicity of T, almost every $x \in X$ has infinitely many positive and negative iterates in the set E. If we replaced T by the induced map $T_E : E \to E$ and the extension $T^{(A)}$ by the corresponding induced extension on $E \times \mathbb{R}^n$, the assumptions of the theorem would still hold. On the other hand the assertions hold for T if they hold for T_E. Thus we may assume without loss of generality that (2.8) holds.

If $x \in X$ and n is a positive integer, let

$$\rho_n(x) = \sup_{v \in C^+_{\infty,x}\setminus\{0\}} \frac{Q_{T^{-n}x}(A(x,-n)v)}{Q_x(v)} \tag{2.9}$$

and

$$L(x,n) = \log \rho_n(x).$$

Since $A(x)C^+_{\infty,x} = C^+_{\infty,Tx}$, it follows that

$$\rho_{m+n}(x) \le \rho_n(x) \cdot \rho_m(T^{-n}x).$$

Therefore $L(x,n)$ is a sub-additive cocycle over T^{-1}.

Condition (iii) implies that $\rho_n(x) \le 1$ for almost every $x \in X$. From condition (iv) and the compactness of the intersection of the set $C^+_{\infty,x}$ with the unit sphere, it follows that for almost every $x \in X$ there exists $n(x)$ such that $\rho_{n(x)}(x) < 1$. Hence $\int_X L(x,n)d\mu < 0$ for all large enough n. Since we assumed that T is ergodic, the subadditive ergodic theorem implies that for almost every $x \in X$

$$\lim_{n\to\infty} \frac{L(x,n)}{n} = \lim_{n\to\infty} \int_X L(x,n)d\mu < 0. \tag{2.10}$$

By (2.8) and (2.9), any $v \in C^+_{\infty,x}$ satisfies

$$\|A(x,-n)v\| \le c_1^{-1} Q_{T^{-n}x}(A(x,-n)v) \le c_1^{-1} Q_x(v)\rho_n(x) \le c_2 c_1^{-1}\rho_n(x)\|v\|. \tag{2.11}$$

By taking logarithms, passing to the limit in (2.11) and using (2.10), we obtain for any non-zero $v \in C^+_{\infty,x}$ (and hence for any non-zero $v \in D^+_x$)

$$\limsup_{n\to\infty} \frac{\log \|A(x,-n)v\|}{n} \le \lim_{n\to\infty} \int_X L(x,n)d\mu < 0,$$

thus verifying (2.7). □

Lyapunov functions are intimately related to the invariant families of cones studied by Wojtkowski and other authors. For a Lyapunov function Q, let

$$C_x = C^+(Q_x).$$

Of course, C_x is a cone in $\mathbb{R}^m$. Condition (ii) implies that the pair $(C_x, \widehat{C}_x)$ is complete†. Condition (iii) implies

$$A(x)C_x \subset C_{Tx}, \ A^{-1}(x)\,\widehat{C}_x \subset \widehat{C}_{T^{-1}x}, \tag{2.12}$$

and (iv) means that for almost every $x \in X$ there exists $n = n(x)$ such that

$$\mathrm{Clos}(A(x,n)C_x) \subset C_{T^n x} \quad \text{and} \quad \mathrm{Clos}(A(x,-n)C_x) \subset C_{T^{-n}x}. \tag{2.13}$$

Definition 2.2. Let $C = \{C_x\}$, $x \in X$ be a measurable family of cones in $\mathbb{R}^m$. Assume that for almost every x the pair $(C_x, \widehat{C}_x)$ is complete and properties (2.12) and (2.13) are satisfied. Then the family C is called *an eventually strictly invariant family of cones* for the extension $T^{(A)}$ (or just for the cocycle A).

Thus the existence of an eventually strict Lyapunov function for $T^{(A)}$ implies the existence of an eventually strictly invariant family of cones. Conversely, if C is an eventually strictly invariant family of cones, it is not difficult to see that there is some eventually strict Lyapunov fuction Q such that $C_x = C^+(Q_x)$‡. But if we begin with a homogeneous function Q and find that the cone field $C^+(Q_x)$ is eventually strictly invariant, we cannot expect Q to be an eventually strict Lyapunov funtion. For certain interesting classes of cocycles and cones, however, this does occur. The most important case for applications involves cocyles with values in the symplectic group $\mathrm{Sp}(2m, \mathbb{R})$ $m = 1, 2, \ldots$ and the so-called symplectic cones which are defined later. For the sake of clarity, we will precede the discussion of this situation by that of the special case $m = 1$, i.e. we will consider $\mathbb{R}^2$ extensions and $SL(2, \mathbb{R})$ cocycles. For this case, we will present an explicit and very elementary proof.

Let us call a cone $C \subset \mathbb{R}^m$ *connected* if its projection to the projective space $\mathbb{R}P(n-1)$ is a connected set. A connected cone in $\mathbb{R}^2$ is simply the union of two opposite sectors formed by two different straight lines intersecting at the origin plus the origin itself. By a linear coordinate change such a cone can always be reduced to the following standard cone

$$S = \{(u, v) \in \mathbb{R}^2 : uv > 0\} \cup \{(0,0)\}. \tag{2.14}$$

† Note that the complementary cone $\widehat{C}_x$ is not always equal to $C^-(Q_x)$. This happens exactly when arbitrarily close to each v such that $Q_x(v) = 0$ one can find v' such that $Q_x(v') > 0$.

‡ We thank Marlies Gerber for this remark.

THEOREM 2.2. *If an $SL(2,\mathbb{R})$ cocycle possesses an eventually strictly invariant family of connected cones $C = \{C_x\}_{x\in X}$ then it has an eventually strict Lyapunov function Q of the form (2.4) such that the zero set of the function Q_x coincides with the boundary of the cone C_x.*

Proof. First, assume that $C_x = S$ for almost every $x \in X$. Then if

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix},$$

(2.12) implies that $a(x), b(x), c(x), d(x)$ are non-negative numbers. Since $A(x) \in SL(2,\mathbb{Z})$ we have $1 = a(x)d(x) - b(x)c(x)$. On the other hand, let $K(u,v) = uv$ and assume that $(u,v) \in S$. Then $uv > 0$ and

$$\begin{aligned} K(A(x)(u,v)) &= (a(x)d(x) + b(x)c(x))uv + a(x)c(x)u^2 + b(x)d(x)v^2 \\ &\geq (a(x)d(x) + b(x)c(x))uv \geq (a(x)d(x) - b(x)c(x))uv = K(u,v). \end{aligned} \tag{2.15}$$

Applying a similar argument to the iterate

$$A(x,n) = (\, a(x,n) \quad b(x,n)c(x,n) \quad d(x,n)\,),$$

we deduce from (2.13) that for $n = n(x)$ we have $b(x,n) > 0$ and $c(x,n) > 0$, which immediately implies using (2.15) that $K(A(x,n)(u,v)) > K(u,v)$.

In the case of an arbitrary family of connected cones, let us introduce a coordinate change $L : X \to SL(2,\mathbb{R})$ which takes two lines bounding the cone C_x into the coordinate axis. Then $L(x)C_x = S$. For the cocycle B, $B(x) = L(Tx)A(x)L^{-1}(x)$, the constant family of cones S is eventually strictly invariant and hence, by the previous argument, the function $Q_0(x,u,v) = \operatorname{sign}(uv)\cdot|uv|^{1/2}$ is an eventually strict Lyapunov function. Hence for the original cocycle A, the function $Q(x,u,v) = Q_0(L(x)(u,v))$ has the same properties. □

Let us proceed to the general symplectic case. We denote by ω the standard symplectic form in $\mathbb{R}^{2m}$,

$$\omega(x,y) = \sum_{i=1}^{m}(x_i y_{m+i} - y_i x_{m+i}),$$

and by K the following non-degenerate quadratic form of signature zero:

$$K(x) = \sum_{i=1}^{m} x_i x_{m+i}.$$

The cone

$$S = \{x \in \mathbb{R}^{2m} : K(x) > 0\} \cup \{0\}$$

will be called the *standard symplectic cone*. The image of the standard symplectic cone under an invertible linear symplectic map will be called a *symplectic cone*. Wojtkowski suggested the following elegant coordinate-free description of symplectic cones and the corresponding quadratic forms [**LW**]. It is possible that this description has been known in symplectic geometry before, although we were not able to find an appropriate source.

Let L_1, L_2 be two transversal Lagrangian subspaces in an $2m$-dimensional symplectic space (H, ω), i.e. complementary m-dimensional subspaces on which the symplectic form ω vanishes. Then for any $v \in H$ there is a unique decomposition

$$v = v_1 + v_2, \quad v_i \in L_i, \ i = 1, 2.$$

Let

$$K_{L_1,L_2}(v) = \omega(v_1, v_2) \quad \text{and} \quad C_{L_1,L_2} = K^{-1}_{L_1,L_2}((0, \infty)) \cup \{0\}. \tag{2.16}$$

Then C_{L_1,L_2} is a symplectic cone and K_{L_1,L_2} is a corresponding quadratic form.

It is easy to see (e.g. by a direct calculation in the standard case) that for a given symplectic cone C in a symplectic space there are exactly two isolated Lagrangian subspaces L_1, L_2 which belong to the boundary of C and that either $C = C_{L_1,L_2}$ or $C = C_{L_2,L_1}$. Thus the cone C canonically determines the form K: we have

$$K(C) = K_{L_1,L_2} \text{ or } K_{L_2,L_1}, \tag{2.17}$$

according to which form is positive on C.

For example, the standard cone S is C_{L_1,L_2}, where

$$L_1 = \{(x, 0) : x \in \mathbb{R}^m\} \quad \text{and} \quad L_2 = \{(0, x) : x \in \mathbb{R}^m\}.$$

The following statement is a reformulation of Proposition 5.1 from Wojtkowski's paper [W] in coordinate-free terms.

PROPOSITION 2.1. *Let H, H' be two $2m$-dimensional symplectic spaces. Let $L_1, L_2 \subset H$, $L'_1, L'_2 \subset H'$ be pairs of transversal Lagrangian subspaces and $A : H \to H'$ be a symplectic linear transformation such that $A\, C_{L_1,L_2} \subset C_{L'_1,L'_2}$. Then for all $v \in H$*

$$K_{L'_1,L'_2}(Av) \geq K_{L_1,L_2}(v).$$

Furthermore, if

$$A(\operatorname{Clos} C_{L_1,L_2}) \subset C_{L'_1,L'_2},$$

then for any norm in H there exists $\varepsilon > 0$ such that

$$K_{L'_1,L'_2}(Av) \geq K_{L_1,L_2}(v) + \varepsilon \parallel v \parallel.$$

The last proposition immediately implies the following relation between invariant cone families and Lyapunov functions.

COROLLARY 2.1. *Let $A : X \to Sp(2m, \mathbb{R})$ be a cocycle over a measure preserving transformation $T : (X, \mu) \to (X, \mu)$. If A has an eventually strictly invariant family of symplectic cones $C = \{C_x\}$, $x \in X$, then it also has an eventually strict Lyapunov function Q, where Q_x has the form (2.4) with the quadratic form K_x of signature zero. Furthermore, the zero set of the function Q_x coincides with the boundary of the cone C_x.*

Combining Corollary 2.1 with Theorem 2.1 we immediately obtain

COROLLARY 2.2. *If a cocycle $A : X \to Sp(2m, \mathbb{R})$ satisfies (2.1) and has an eventually strictly invariant family of symplectic cones, then the extension $T^{(A)}$ has m positive and m negative characteristic exponents.*

Now we will very briefly mention how to obtain counterparts to the results from this section for continuous-time dynamical systems.

Let $\{T_t\}_{t\in\mathbb{R}}$ be a measure-preserving flow on the Lebesgue measure space (X, μ). A *matrix cocycle* over the flow is a measurable map $A : X \times \mathbb{R} \to GL(m, \mathbb{R})$ such that

$$A(x, t_1 + t_2) = A(T_{t_1}x, t_2)A(x, t_1) \quad \text{for a.e. } x \in X \text{ and all } t_1, t_2 \in \mathbb{R}.$$

The *linear extension* $\{T_t^{(A)}\}_{t\in\mathbb{R}}$ of the flow determined by the cocycle A is defined as follows:

$$T_t^{(A)}(x, v) = (T_t x, A(x, t)v).$$

Condition (2.1) becomes $\sup_{1\le t\le 1} \|A(\cdot, t)\| \in L^1(X, \mu)$. The definition of Lyapunov characteristic exponents, the multiplicative ergodic theorem, the decomposition (2.3) and all definitions and results concerning Lyapunov functions and cone families for linear extensions are completely similar to the discrete time case.

3. *Survey of Pesin Theory; the Bernoulli property for contact flows*

3.1. Now let us consider a $C^{1+\varepsilon}$ ($\varepsilon > 0$) diffeomorphism f of a compact m-dimensional differentiable manifold M, preserving a Borel probability measure μ. The differential $Df : TM \to TM$ is a linear extension of f to the tangent bundle TM. Although topologically the tangent bundle may not be the direct product of M and R^m, this is always true up to a set of measure zero. Moreover, one can fix a Riemannian metric on M and assume that the norm of vectors in TM generated by that metric corresponds to the norm in the direct product. Thus the Lyapunov characteristic exponents exist almost everywhere and define a decomposition of T_xM similar to (2.3). When it does not cause confusion we will use the same notations as in §2. Let

$$E_x^+ = \bigoplus_{i:\lambda_i(x)>0} E_x^i \quad \text{and} \quad E_x^- = \bigoplus_{i:\lambda_i(x)<0} E_x^i. \tag{3.1}$$

The subspaces E_x^- and E_x^+ are called the *stable and unstable subspaces* at the point x. The first important result of Pesin theory is a kind of unique integrability of the families of stable and unstable subspaces. Let $\dim E_x^- = s(x)$ and $\dim E_x^+ = u(x)$.

THEOREM 3.1. *There exists a set Λ, $\mu(M \setminus \Lambda) = 0$, such that for every $x \in \Lambda$ there are C^1 submanifolds W_x^s and W_x^u with the following properties:*

(a) $W_x^s \cap W_x^u = \{x\}$.

(b) *W_x^s and W_x^u are embedded diffeomorphic images of closed balls of dimensions $s(x)$ and $u(x)$ respectively. Moreover, such diffeomorphisms can be effected by the exponential maps from certain neighborhoods of the origin in the spaces E_x^- and E_x^+.*

(c) *$T_xW_x^s = E_x^-$ and $T_x^u = E_x^+$.*

(d) *$fW_x^s \subset W_{f(x)}^s$ and $fW_x^u \supset W_{f(x)}^u$.*

(e) *If $x, y \in \Lambda$ and $y \in W_x^s$, then for an open neighborhood U of y we have $U \cap W_y^s = U \cap W_x^s$; similarly for W_x^u and W_y^u.*

(f) *If $x \in \Lambda$, then the distance between $f^n(x)$ and $f^n(y)$ goes to 0 exponentially as $n \to \infty$ if and only if $f^n(y) \in W_{f^n(x)}^s$ for some n. The same is true as $n \to -\infty$ if and only if $f^n(y) \in W_{f^n(x)}^u$ for some n.*

(g) *The submanifolds W_x^s and W_x^u depend on x in a measurable way.*

The manifold W_x^s is called the *local stable* or *contracting manifold* of the point x; similarly, W_x^u is called the *local unstable* or *expanding manifold* of x. Since the local stable manifolds for f are at the same time the local unstable for f^{-1}, every result about stable manifolds implies a similar statement about unstable ones. Accordingly, we will sometimes omit one of those parallel statements.

Let us assume that W_x^s and W_x^u are considered as elements of the disjoint union of spaces of C^1 embeddings of the standard k-dimensional ball into M for $k = 0, 1, \cdots, m$. We can define the *size* of a local manifold in the following way. According to 3.1.b the exponential map at the point x with respect to the Riemannian metric induced on W_x^s establishes a diffeomorphism between a subset D of E_x^- and W_x^s. The size of W_x^s is equal to the radius of the maximal ball about the origin which is contained in D.

By Luzin's theorem, one can find for every $\varepsilon > 0$ a closed set $\Lambda_\varepsilon \subset \Lambda$ such that $\mu(M \setminus \Lambda_\varepsilon) < \varepsilon$ and the maps $x \to W_x^s$ and $x \to W_x^u$ are uniformly continuous on Λ_ε. By throwing away a set of measure 0, we can assume that $\Lambda = \bigcup_{\varepsilon>0} \Lambda_\varepsilon$. In particular, there is a positive lower bound $\sigma(\varepsilon)$ for the sizes of local stable and unstable manifolds for the points of the set Λ_ε. For $x \in \Lambda_\varepsilon$ and for any positive number $\delta < \sigma(\varepsilon)$ we define the *δ-truncated stable manifold* of x, $W_x^{s,\delta} \subset W_x^s$, as the image of the δ-ball about the origin in E_x^- under the exponential map.

Let

$$\Lambda_\varepsilon^{k,\ell} = \{x \in \Lambda_\varepsilon : \dim W_x^s = k,\ \dim W_x^u = \ell\}.$$

Pick a point $x \in \Lambda_\varepsilon^{k,\ell}$ and consider two small $(m-k)$-dimensional transversals T_1 and T_2 to the local stable manifold W_x^s. For every point $y \in \Lambda_\varepsilon^{k,\ell}$ sufficiently close to x, the local stable manifold W_y^x intersects each of the two transversals at exactly one point. Correspondence between these intersection points defines a continuous map between certain subsets of the transversals. Let us denote the domain of this map by $D_{\varepsilon,T_1,T_2}^{k,\ell}$, its range by $R_{\varepsilon,T_1,T_2}^{k,\ell}$ and the map itself by $H_{\varepsilon,T_1,T_2}^{k,\ell}$. We will usually suppress the dependence on k and ℓ in our notations. A completely similar construction can be carried out for local unstable manifolds. If $k+\ell = m$, then local unstable manifolds can be used as transversals to the stable ones and vice versa.

Let us call a measure on M *absolutely continuous* if its restriction to any coordinate neighborhood is absolutely continuous with respect to the Lebesgue measure in that neighborhood. The following result of Pesin plays the central role in the study of ergodic properties of smooth dynamical systems via Lyapunov characteristic exponents. It establishes the property which is usually called absolute continuity of families of local stable manifolds. Let ξ be a partition of M into open subsets of local stable manifolds of points from $\Lambda_\varepsilon^{k,\ell}$ and $M \setminus \bigcup W_x$. Let μ_x^s be the conditional measure induced by μ on the element of ξ which contains $x \in \Lambda_\varepsilon^{k,\ell}$.

THEOREM 3.2. *Suppose that the measure μ is absolutely continuous and that $x \in \Lambda_\varepsilon^{k,\ell}$ is a Lebesgue density point of the set $\Lambda_\varepsilon^{k,\ell}$. For any two transversals T_1, T_2 to the local stable manifold W_x^s, which are sufficiently close to each other, the sets $D_{\varepsilon,T_1,T_2}^{k,\ell}$ and $R_{\varepsilon,T_1,T_2}^{k,\ell}$ have positive $(m-k)$-dimensional Lebesgue measure and the map $H_{\varepsilon,T_1,T_2}^{k,\ell}$ is* *absolutely continuous with respect to that measure. Furthermore, for almost every x in* $\Lambda_\varepsilon^{k,\ell}$ *the conditional measure* μ_x^s *is absolutely continuous and its density with respect to the measure* λ_s *induced by a Riemannian metric is bounded between two positive constants which depend only on x and the Riemannian metric.*

Non-vanishing of all characteristic exponents is sufficient for a kind of local ergodicity.

THEOREM 3.3. *If, under the assumptions of Theorem 3.2, we have* $k + \ell = m$*, then there exists* $\delta = \delta(\varepsilon) > 0$ *such that for each* $x \in M$ *almost every point of the set*

$$\bigcup_{y \in \Lambda_\varepsilon^{k,\ell}:\, \mathrm{dist}(x,y)<\delta} (W_y^s \cup W_y^u)$$

belongs to the same ergodic component E of f. In particular, almost every point of M lies in an ergodic component of positive measure.

Local stable manifolds can be extended or 'saturated' in a natural fashion. If $x \in M$ and $n > 0$, let us denote the manifold $f^{-n}W_{f^n x}^s$ by $W_{x,n}^s$. By Theorem 3.1(d) if $n > n'$, then $W_{x,n}^s \supset W_{x,n'}^s$. Now we can define the *global stable manifold* of x, $\widetilde{W}_x^s = \bigcup_{n=1}^\infty W_{x,n}^s$, which in general is not an embedded submanifold of M. The manifolds $W_{x,n}^u$ and $\widetilde{W}_x^u$ are defined similarly.

COROLLARY 3.1. *In the assertion of Theorem 3.3 the manifolds* W_y^s *and* W_y^u *can be replaced by* $W_{y,n}^s$ *or* $\widetilde{W}_{y,n}^s$ *and* $W_{y,n}^s$ *or* $\widetilde{W}_y^u$ *correspondingly.*

Pesin analyzes ergodic properties of diffeomorphisms with non-vanishing Lyapunov characteristic exponents in great detail. His results in that direction can be summarized in the following way.

THEOREM 3.4. *Let E be an ergodic component for f which has positive measure and non-zero Lyapunov characteristic exponents. Then E is a union of disjoint measurable sets* $E_1, \ldots, E_n = E_0$ *such that* $fE_k = E_{k+1}$, $k = 0, \ldots, N-1$, *and* f^N *restricted to each set* E_k *is a Bernoulli map. Furthermore, the sets described in Theorem 3.3 and Corollary 3.1 belong to the same* E_k.

The sets E_k from the theorem are uniquely defined up to a set of measure zero. We will call these sets *Bernoulli components* for f.

Theorem 3.1 and 3.2 remain true for $C^{1+\varepsilon}$ flows with appropriate modifications. A major but obvious difference for the case of smooth flows is the presence of an invariant one-dimensional distribution determined by the direction of the flow. If the invariant measure μ vanishes on the set of the fixed points of the flow, which we will always assume, this distribution contributes a zero Lyapunov exponent for the flow of differentials. Theorem 3.3 and Corollary 3.1 are extended in a natural way to $C^{1+\varepsilon}$ flows for which the zero exponent has multiplicity one. The counterpart of Theorem 3.4 looks as follows.

THEOREM 3.5. *Let E be an ergodic component of positive measure for a* $C^{1+\varepsilon}$ *flow on a compact manifold which preserves an absolutely continuous measure. Then either the flow on the set E is a Bernoulli flow or it possesses a non-constant eigenfunction. In the latter case, the flow E is isomorphic to a constant-time suspension over a Bernoulli map.*

For the original proofs of Theorems 3.1–3.5 see [**P1**] and [**P2**]. Pesin's proofs are basically sound but some of them, especially in the absolute continuity part (Theorem 3.2), contain numerous minor gaps and errors. Proofs following very closely the line of Pesin's argument but with the gaps filled and errors corrected can be found in [**KS**]. However, the presentation there is rather heavy, not surprisingly, most of all again in the part concerning absolute continuity. An extra source of heavy notation in [**KS**] is the need to generalize Pesin's theory to systems with singularities. A more conceptual and lucid presentation of Pesin's theory is forthcoming in [**KM1**]; the shorter version [**KM2**] may serve as a complement to the present account. Another account has recently appeared in [**PS**].

3.2. There is a significant special case in which only the former alternative in Theorem 3.5 is possible.

Let us assume that M is a compact manifold of odd dimension $2m+1$. A *contact form* on M is C^1 differential 1-form α such that the $(2m+1)$-form $\alpha \wedge (d\alpha)^m$ is non-zero at every point. The kernel of α is a codimension 1 distribution on M. The restriction of the 2-form $d\alpha$ to $\ker \alpha$ determines a symplectic structure there.

There is a unique vector field X on M such that $d\alpha(X, Y) = 0$ for all vector fields Y and $\alpha(X) = 1$. The flow $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ defined by X is called the *contact flow on* M. It preserves the contact form α. Conversely, any flow on M that preserves α is a constant speed reparametrization of ϕ. The contact flow preserves the distribution $\operatorname{Ker}\alpha$, the symplectic structure there and the measure μ on M determined by the volume form $\alpha \wedge (d\alpha)^m$.

The following result constitutes a useful new addition to Pesin theory.

THEOREM 3.6. *Let M be a contact manifold as above. Let E be an ergodic component of the contact flow ϕ which has positive measure and non-zero Lyapunov exponents except in the flow direction. Then the flow on E is Bernoulli.*

Proof. By Theorem 3.5 it suffices to show that any eigenfunction on E is μ-a.e. constant. Recall that $f : E \to \mathbb{C}$ is called an *eigenfunction* if f is measurable and there is $\lambda \in \mathbb{R}$ such that

$$f(\phi_t x) = e^{i\lambda t} f(x) \quad \text{for almost all } x \in M \text{ and } t \in \mathbb{R}. \tag{3.2}$$

We shall show that if f is an eigenfunction and $\Delta > 0$, we can choose, for μ-a.e. $x \in E$, a number $\Delta(x)$ such that

$$0 < |\Delta(x)| < \Delta \quad \text{and} \quad f(\phi_{\Delta(x)}x) = f(x). \tag{3.3}$$

If follows from (3.3) and the eigenfunction property (3.2) that f is a.e. constant. □

First we apply a version of the classical Hopf argument to f. In the flow case the local stable and unstable manifolds are denoted by W_x^{ss} and W_x^{su} correspondingly where an extra s stands for "strong". Acoordingly the families of those manifolds are denoted by W^{ss} and W^{su}. Those families are integrable with the orbit foliation. Resulting integral manifolds are denoted by W_x^{os} and W_x^{ou}. Let μ_{ss} and μ_{su} be the conditional measures induced by μ on the leaves of W^{ss} and W^{su}.

LEMMA 3.1. *Let $\mathcal{G}$ be the set of $x \in \Lambda$ such that $f(y) = f(x)$ for μ_{ss}-a.e. $y \in W_x^{ss}$ and $f(y') = f(x)$ for μ_{su}-a.e. $y' \in W_x^{su}$. Then $\mu(M \setminus \mathcal{G}) = 0$.*

Proof. We may assume that when we applied Luzin's theorem to choose the closed set Λ_ε, we also arranged for f to be continuous on Λ_ε. Let Λ_ε^* be the set of x in Λ_ε for which $\{t \in \mathbb{R} : \phi_t x \in \Lambda_\varepsilon\}$ has upper density $> 1/2$ as $t \to \infty$ and as $t \to -\infty$. Since $\mu(M \setminus \Lambda_\varepsilon) \to 0$, we see that $\mu(\Lambda_\varepsilon \setminus \Lambda_\varepsilon^*) \to 0$ as $\varepsilon \to 0$. Observe also that if $x \in \Lambda_\delta^*$ and $y \in W_x^{ss} \cap \Lambda_\varepsilon^*$ for some $\varepsilon \le \delta$, then there are arbitrarily large t for which both $\phi_t x$ and $\phi_t y$ are in Λ_ε. We see from (3.2) and the uniform continuity of f on Λ_ε^* (remember that the sets Λ_ε^* are compact !) that $f(x) = f(y)$. If $x \in \Lambda_\delta^*$ and $y' \in W_x^{su} \cap \Lambda_\varepsilon^*$ for some $\varepsilon \le \delta$, we see by a similar argument that $f(x) = f(y')$.

Now consider a fixed $\delta > 0$. Since $\bigcup_{\epsilon \le \delta} \Lambda_\varepsilon^*$ has full μ-measure, it follows from the absolute continuity of W^{ss} that for μ-a.e. $x \in \Lambda_\varepsilon^*$ the sets $W_x^{ss} \cap (\bigcup_{\epsilon \le \delta} \Lambda_\varepsilon^*)$ and $W_x^{su} \cap (\bigcup_{\epsilon \le \delta} \Lambda_\varepsilon^*)$ have full μ_{ss}-measure in W_x^{ss} and full μ_{su}-measure in W_x^{su} respectively. It follows that for each $\delta > 0$, μ-a.e. $x \in \Lambda_\delta^*$ has desired properties. □

Let λ_{ss} and λ_{su} be the Riemannian measure on W_x^{ss} and W_x^{su} respectively. It follows from the version of Theorem 3.2 for flows that for μ-a.e. $x \in \Lambda_\varepsilon^*$, the measures μ_{ss} and μ_{su} are absolutely continuous with respect to λ_{ss} and λ_{su}, and there is a constant $c = c(x)$ such that $d\mu_{ss}/d\lambda_{ss} < c$ everywhere in W^{ss} and $d\mu_{su}/d\lambda_{su} < c$ everywhere in W^{su}. By deleting a set of measure 0, we may assume that every $x \in \mathcal{G}$ has these properties. Now let $\mathcal{G}_\varepsilon = \{x \in \mathcal{G} \cap \Lambda_\varepsilon \cap E : x$ is a μ_{ss}-density point of $\mathcal{G} \cap \Lambda_\varepsilon \cap E \cap W_x^{ss}$ and a μ_{su}-density point of $\mathcal{G} \cap \Lambda_\varepsilon \cap E \cap W_x^{su}\}$. Since $\bigcup_{\varepsilon > 0} \mathcal{G}_\varepsilon$ has full measure, it suffices to show that if $x \in \mathcal{G}_\varepsilon$ for some $\varepsilon > 0$, then x has property (3.3).

Choose a Riemannian metric on a neighborhood of x so that E_x^- and E_x^+ are orthogonal and $\exp_x$ maps neighborhoods of the origin in E_x^- and E_x^+ diffeomorphically onto neighborhoods of x in W^{ss} and W^{su}. For a small $\eta > 0$, let

$$S = \{\exp_x w : w \in E_x^- \oplus E_x^+ \text{ and } \|w\| < \eta\}$$

and let

$$N = \{\phi_t y : y \in S \text{ and } |t| < \eta\}.$$

If $y \in N \cap \Lambda_\varepsilon$, let $W^*(y)$ be the connected component of y in the set $W_y^* \cap N$ for $* = ss,\ su,\ os,\ ou$. We may assume that 2η is less than the length of any closed orbit of ϕ and η is small enough so that S and all sets of the form $W^{ss}(y) \cap S$ or $W^{su}(y) \cap S$ have the property that any two points are connected by a unique geodesic.

We may also assume that if $y, y' \in \Lambda_\varepsilon$, $y = \exp_x v$ with $v \in E_x^-$, and $\|v\| < \eta/2$ and $y' = \exp_x v'$ with $v' \in E_x^+$ and $\|v'\| < \eta/2$, then each of the sets

$$W^{su}(y) \cap W^{os}(y') \quad \text{and} \quad W^{ou}(y) \cap W^{ss}(y')$$

consists of a single point which lies in N. Denote these points by z and z' respectively. Define $\Delta(y, y')$ so that

$$z' = \phi_{\Delta(y,y')} z$$

and the curve $\Gamma_0(s) = \phi_{s\Delta(y,y')} z$, $0 \le s \le 1$, lies in N. Observe that if both y and y' are in $\mathcal{G}_\varepsilon$, we have $f(x) = f(y) = f(z)$ and

$$f(\phi_{\Delta(y,y')} x) = f(\phi_{\Delta(y,y')} y) = f(\phi_{\Delta(y,y')} z) = f(z') = f(y') = f(x)$$

Thus x has property (3.3) if we can choose y in $W^{ss}(x) \cap \mathcal{G}_\varepsilon$ and y' in $W^{su}(x) \cap \mathcal{G}_\varepsilon$ so that $\Delta(y, y')$ is non-zero but as small as we wish. The next two lemmas show that this is possible. Despite its formulation in dynamical terms, the first lemma essentially belongs to symplectic geometry.

LEMMA 3.2. *$\Delta(y, y') = d\alpha(v, v') + o(\|v\|^2 + \|v'\|^2)$ as $y \to x$ in $W^{ss}(x) \cap \Lambda_\varepsilon$ and $y' \to x$ in $W^{su}(x) \cap \Lambda_\varepsilon$.*

Proof. Let $\pi : N \to S$ be the projection along the orbits of ϕ. Let $\widehat{z} = \pi z = \pi z'$. Let γ_1 be the geodesic in $W^{ss}(y')$ with $\gamma_1(0) = \widehat{z}$ and $\gamma_1(1) = y'$. Let γ_2 be the geodesic in $W^{su}(x) \cap S$ with $\gamma_2(0) = y'$ and $\gamma_2(1) = x$. Let γ_3 be the geodesic in $W^{ss}(x) \cap S$ with $\gamma_3(0) = x$ and $\gamma_3(1) = y$. Let γ_4 be the geodesic in $W^{su}(y) \cap S$ with $\gamma_4(0) = y$ and $\gamma_4(1) = \widehat{z}$. Finally let Σ be the surface in S formed by the geodesics joining $\gamma_1(s)$ to $\gamma_2(1-s)$ and $\gamma_3(s)$ to $\gamma_4(1-s)$ for $0 \le s \le 1$. The precise construction of Σ is not important. What matters is that one can see from the convergence of $W^{ss}(y)$ to $W^{ss}(x)$ and of $W^{su}(y)$ to $W^{su}(x)$ in the C^1-topology that

$$\int_\Sigma d\alpha = d\alpha(v, v') + o(\|v\|^2 + \|v'\|^2). \tag{3.4}$$

Recall that Γ_0 is the curve with $\Gamma_0(s) = \phi_{s\Delta(y,y')}z$, $0 \le s \le 1$. Let Γ be the curve in N such that starts at z', is tangential to $\operatorname{Ker}\alpha$ and has $\pi \circ \Gamma = \gamma$. Then Γ ends at z and

$$\int_{\Gamma_0 * \Gamma} \alpha = \int_{\Gamma_0} \alpha = \Delta(y, y') + o(\|v\|^2 + \|v'\|^2), \tag{3.5}$$

because the vector field X which generates ϕ satisfies $\alpha(X) = 1$. Now observe that $\Gamma_0 * \Gamma$ and $\partial\Sigma$ are closed curves that bound a surface which is tangent to the vector field X and $d\alpha$ vanishes on any 2-plane containing X. Using this and Stokes theorem, we obtain

$$\int_\Sigma d\alpha = \int_{\Gamma_0 * \Gamma} \alpha,$$

which together with (3.4) and (3.5) completes the proof. □

LEMMA 3.3. *There is $c_0 > 0$ such that for any small enough $\delta > 0$ there are $v \in E_x^-$ and $v' \in E_x^+$ such that*

(1) $\delta/2 < \|v\|, \|v'\| < \delta$;

(2) $\exp_x v \in \mathcal{G}_\varepsilon$ *and* $\exp_x v' \in \mathcal{G}_\varepsilon$;

(3) $|d\alpha(v, v')| > c_0\delta^2$.

Proof. Let $A_\delta = \{(v, v') \in E_x^- \oplus E_x^+ : \delta/2 < \|v\|, \|v'\| < \delta\}$ and $B_\delta = \{(v, v') \in A_\delta : \|v\| < \delta$ and $\|v'\| < \delta\}$. Let $C_\delta = \{(v, v') \in B_\delta : \exp_x v \in \mathcal{G}_\varepsilon$ and $\exp_x v' \in \mathcal{G}_\varepsilon\}$. Let λ_- and λ_+ be the Lebesgue measures on E_x^- and E_x^+ and λ their product. Since the pullbacks of λ_{ss} and λ_{su} by $\exp_x$ are equivalent to λ_- and λ_+ respectively, and x is a density point for both μ_{ss} and μ_{su}, we see that

$$\frac{\lambda(C_\delta)}{\lambda(B_\delta)} \to 1 \quad \text{as} \quad \delta \to 0. \tag{3.6}$$

Now choose $c_0 > 0$ so that there exists $(v, v') \in A_1$ such $|d\alpha(v, v')| > c_0$. Let $D_\delta = \{(v, v') \in A_\delta : |d\alpha(v, v')| > c_0\delta^2\}$. Then $D_\delta = \{(\delta v, \delta v') : (v, v') \in D_1\}$, and so for all δ

$$\frac{\lambda(D_\delta)}{\lambda(B_\delta)} = \frac{\lambda(D_1)}{\lambda(B_1)} > 0. \tag{3.7}$$

The lemma follows from (3.6) and (3.7). □

4. *Ergodicity and the Bernoulli property for systems with infinitesimal Lyapunov functions: formulation of results*

Various notions of Lyapunov functions and invariant cone families discussed in §2 in the context of linear extensions of measure preserving transformations have natural continuous analogues. We will begin with appropriate general definitions and then adapt them to the specific situation of diffeomorphisms (or smooth flows) of compact manifolds and their differentials.

Let X be a compact metrizable space and B a locally trivial $\mathbb{R}^n$-bundle over X whose fiber B_x is equipped with an inner product that varies continuously with x. Let $f : X \to X$ be a homeomorphism and $\widehat{f} : B \to B$ a linear extension of f. Since, unlike in the measurable situation discussed in §2, the bundle B may be non-trivial globally, the extension $\widehat{f}$ can not in general be determined by a $GL(n, \mathbb{R})$ cocycle over f. However, it is often convenient to cover X by a finite system of neighborhoods over which the bundle trivializes and to represent $\widehat{f}$ locally in matrix form.

Let $U \subset X$ be an open subset and B_U the restriction of the bundle B to U.

Definition 4.1. A continuous real-valued function Q defined on B_U is called a *continuous Lyapunov function* for $\widehat{f}$ if

(i) For every $x \in U$ the function $Q_x = Q(x, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree one and takes both positive and negative values.

(ii) There exist continuous distributions $D_x^+ \subset C^+(Q_x)$ and $D_x^- \subset C^-(Q_x)$ such that $B_x = D_x^+ + D_x^-$ for all $x \in U$. In particular, $r^+(Q_x) = \dim D_x^+$ and $r^-(Q_x) = \dim D_x^-$ are constant on U.

(iii) If $x \in U$, $n \geq 0$ and $f^n x \in U$, then for all $v \in B_x$

$$Q_{f^n x}(\widehat{f}^n v) \geq Q_x(v).$$

Now assume that f has a invariant Borel measure μ that is positive on open sets. A continuous Lyapunov function for $\widehat{f}$ will be called *eventually strict* if

(iv) For μ-almost every $x \in U$ there exist $k = k(x) > 0$ and $\ell = \ell(x) > 0$ such that $f^k x \in U$, $f^{-\ell} x \in U$ and for all $v \in B_x \setminus \{0\}$

$$Q_{f^k x}(\widehat{f}^k v) > Q_x(v) \quad \text{and} \quad Q_{f^{-\ell} x}(\widehat{f}^{-\ell} v) < Q_x(v).$$

A continuous Lyapunov function for $\widehat{f}$ will be called *eventually uniform* if

(v) There exists $\varepsilon > 0$ such that for μ-almost every $x \in U$ there are $k = k(x) > 0$ and $\ell = \ell(x) > 0$ such that for all $v \in B_x$

$$Q_{f^k x}(\widehat{f}^k v) \geq Q_x(v) + \varepsilon\|v\| \quad \text{and} \quad Q_{f^{-\ell} x}(\widehat{f}^{-\ell} v) \leq Q_x(v) - \varepsilon\|v\|.$$

Continuous eventually strict Lyapunov functions are the direct counterpart of the eventually strict Lyapunov functions in the measurable situation (Definition 2.1). They will play the same role for local ergodicity (openness of the ergodic components almost everywhere) as the latter play for the non-vanishing of the Lyapunov exponents. The somewhat stronger notion of continuous eventually uniform Lyapunov function is needed to ensure global ergodicity on connected components of the set U.

In order to introduce topological counterparts for the notions of invariant cone families, we need to define a topology on the space of cones. For open cones in $\mathbb{R}^n$ a convenient one is the Hausdorff topology on the intersection of the *complement* of the cone with the unit sphere. This topology allows one to define a *continuous family of cones* in a locally trivial bundle.

Definition 4.2. A family of cones $\mathcal{C} = \{C_x\}_{x\in U}$ is called a *continuous eventually strictly invariant family of cones* on U for the extension $\widehat{f}$ if it is continuous and

(i) There exist continuous families of subspaces $D_x^+ \subset C_x$ and $D_x^- \subset \widehat{C}_x$ such that $D_x^+ + D_x^- = B_x$.

(ii) If $x \in U$, $n > 0$ and $f^n x \in U$, then $\widehat{f}^n C_{f^{-n}x} \subset C_x$.

(iii) For μ-almost every $x \in U$ there exist $k = k(x) > 0$ and $\ell = \ell(x) > 0$ such that $f^{-k}x \in U$, $f^{\ell}x \in U$ and

$$\widehat{f}^k(\operatorname{Clos} C_{f^{-k}x}) \subset C_x \quad \text{and} \quad \widehat{f}^{-\ell}(\operatorname{Clos} \widehat{C}_{f^{\ell}x}) \subset \widehat{C}_x.$$

One defines for $\varepsilon > 0$ the ε-interior of a cone C as the cone whose intersection with the unit sphere S is the ε-interior of $S \cap C$, i.e. $\{p \in S \cap C; \operatorname{dist}_S(p, \partial\, S \cap C) > \varepsilon\}$.

We will call a continuous family of cones $\mathcal{C}$ *eventually uniformly invariant* if (iii) in Definition 4.2 is replaced by

(iv) There is $\varepsilon > 0$ such that for μ-almost every $x \in U$ there exist $k = k(x) > 0$ and $\ell = \ell(x) > 0$ such that $f^{-k}x \in U$, $f^{\ell}(x) \in U$ and

$$\widehat{f}^k(C_{f^{-k}x}) \subset \operatorname{Int}_\varepsilon C_x \quad \text{and} \quad \widehat{f}^{-\ell}(\widehat{C}_{f^{\ell}x}) \subset \operatorname{Int}_\varepsilon \widehat{C}_x.$$

All of the above definitions can be translated almost verbatim to the case of a continuous flow on a compact metrizable space.

Let us now consider the special case when the compact metrizable space is actually a smooth manifold M, the map f is a diffeomorphism, the bundle B is the tangent bundle TM and the extension $\widehat{f}$ is the differential Df. For the sake of future references it is convenient to give special names for the above-defined notions in this case.

A continuous eventually strict (resp. uniform) Lyapunov function will be called an *infinitesimal eventually strict (uniform) Lyapunov function* over U. Similarly a continuous eventually strict (uniform) family of cones will be called an *infinitesimal eventually strict (uniform) family of cones*.

For the flow case instead of the tangent bundle TM we will consider the vector-bundle $TM|_E$ where E is the one-dimensional subbundle of TM generated by the vector-field which determines the flow. The notions of infinitesimal eventually strict (uniform) Lyapunov function and an infinitesimal strictly (uniformly) invariant family of cones are defined accordingly.

The following theorem represents the main general criterion of ergodicity based on the notion of an infinitesimal Lyapunov function.

THEOREM 4.1. *Let f be a $C^{1+\varepsilon}$ ($\varepsilon > 0$) diffeomorphism of a compact manifold M which preserves an absolutely continuous invariant measure positive on open sets. Let $U \subset M$ be an open set.*

(i) *Assume that f possesses an infinitesimal eventually strict Lyapunov function Q over U. Then almost every ergodic component of f on the invariant set $U_f = \bigcup_{n\in\mathbb{Z}} f^n U$ is open up a set of measure zero.*

(ii) *If f possesses an infinitesimal eventually uniform Lyapunov function Q over U, then every connected component of the set U_f belongs to one ergodic component for f. If U_f is connected then f restricted to U_f is Bernoulli.*

Theorem 4.1 is proved in §5 and 6.

The analogous theorem holds for a flow except that, in general, one cannot say anything about the Bernoulli property. In the case of a contact flow, however, one can combine this result with Theorem 3.6 to obtain

THEOREM 4.2. *Let $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ be a $C^{1+\varepsilon}$ ($\varepsilon > 0$) contact flow on a compact manifold M. Assume that there is an open set U on which the flow ϕ has an infinitesimal eventually uniform Lyapunov function. Then every connected component of $\bigcup_{t\in\mathbb{R}} \phi_t U$ belongs to one Bernoulli component of ϕ.*

Let us consider now the case of a symplectic diffeomorphism f of a symplectic manifold (M^{2m}, Ω) where Ω is a closed non-degenerate two form. The $2m$-form Ω^m determines an invariant absolutely continuous measure on M^{2m} which is sometimes called Liouville measure. The differential $Df : TM^{2m} \to TM^{2m}$ can be viewed as a symplectic cocycle in the sense of §2.

There is a natural topological counterpart of Corollary 2.1 which follows from the fact that the correspondence $C \to K(C)$ defined by (2.11) is continuous from the cone topology to the C^0 topology for homogeneous functions. For the reader's convenience we formulate this statement explicitly.

PROPOSITION 4.1. *Let B be a symplectic locally trivial linear bundle over a metrizable compact space X and let $\widehat{f} : B \to B$ be a symplectic linear extension of a homeomorphism $f : X \to X$. Assume that $\widehat{f}$ has a continuous eventually strictly invariant (resp. eventually uniformly invariant) family of symplectic cones $C = \{C_x\}_{x\in U}$. Then $\widehat{f}$ also admits an eventually strict (resp. eventually uniform) Lyapunov function Q over x, where Q_x has the form (2.4) and $\partial C_x = Q_x^{-1}(0)$.*

This proposition together with Theorem 4.1 immediately implies a criterion for ergodicity of symplectic diffeomorphisms in terms of invariant families of symplectic cones.

COROLLARY 4.1. *Let f be a $C^{1+\varepsilon}$ ($\varepsilon > 0$) symplectic diffeomorphism of a symplectic manifold (M^{2m}, Ω) and let $U \subset M^{2m}$ be an open set.*

(i) *Assume that f admits an infinitesimal eventually strictly invariant family of symplectic cones over U. Then almost every connected component of the set U_f is open up to a set of measure zero.*

(ii) *If f admits an infinitesimal eventually uniform family of symplectic cones over U, then every connected component of the set U_f belongs to one ergodic component of f. If U_f is connected, f restricted to U_f is Bernoulli.*

5. *The non-contraction lemma and the extension of stable and unstable manifolds*

In this section and the next, we assume that M is a compact smooth manifold and $f : M \to M$ is a $C^{1+\varepsilon}(\varepsilon > 0)$ diffeomorphism that preserves a Borel measure μ which is absolutely continuous and has positive density with respect to the Lebesgue measure class. The assumptions on μ mean that the conditional measures μ^s and μ^u of μ on the leaves of the stable and unstable foliations have positive density with respect to the Riemannian measures λ^s and λ^u on those leaves. In particular, a subset of a leaf of $\widetilde{W}^u$ that has full μ^u measure is dense in that leaf. We assume that there is an open set $U \subseteq M$ on which f has a continuous eventually strict infinitesimal Lyapunov function Q. We fix a continuous Riemannian metric on M.

The results in this section are formulated for unstable manifolds; the results for stable manifolds are exactly parallel. There are also analogous results for flows, which we leave to the reader.

The definition of Λ_ε in §3 does not take into account the Lyapunov function Q. In particular, Theorem 2.1 does not tell us that $E_x^+ \subset C^+(Q_x)$ and $E_x^- \subset C^-(Q_x)$ for all $x \in \Lambda_\varepsilon$. For this reason, we introduce

$$V_\varepsilon = \{x \in U \cap \Lambda_\varepsilon : Q_s(v) \geq \varepsilon\|v\| \text{ for all } v \in E_x^+ \text{ and } Q_x(w) \leq -\varepsilon\|w\| \text{ for all } w \in E_x^-\}.$$

Since Q_x and $\|\cdot\|$ are both homogeneous of degree one and vary continuously with x, it is clear that each V_ε is compact. Moreover it follows from Theorem 2.1 that $\bigcup_{\varepsilon>0} V_\varepsilon$ has full measure in U.

LEMMA 5.1. (The non-contraction lemma.)
Let F be a compact subset of U and $\varepsilon > 0$. Then there exist $\delta = \delta(F, \varepsilon)$ and $c = c(F, \varepsilon)$ such that the following hold.
(i) *The truncated unstable manifold $W_x^{u,\delta}$ is defined for every $x \in V_\varepsilon \cap F$.*
(ii) *If $x \in V_\varepsilon \cap F$, $y \in W_x^{u,\delta}$, $n \geq 0$, $f^n y \in F$ and $v \in T_y W_x^{u,\delta}$, then*

$$\|Df^n v\| \geq c\|v\|.$$

Proof. Recall from the discussion after the definition of Λ_ε in §3 that $W_x^{u,\delta}$ is defined for all $x \in \Lambda_\varepsilon \supseteq V_\varepsilon \cap F$, provided $\delta > 0$ is small enough. By the definition of V_ε,

$$\inf_{x \in V_\varepsilon \cap F} \inf_{v \in E_x^+ \setminus \{0\}} \frac{Q_x(v)}{\|v\|} \geq \varepsilon. \tag{5.1}$$

Note that $V_\varepsilon \cap F$ is compact, Q_x varies continuously with x, and the local unstable manifolds W_x^u vary continuously in the C^1 topology as x varies. We see from these observations and (5.1) that if $\delta > 0$ is small enough, then $W_x^{u,\delta}$ is defined for all

$x \in V_\varepsilon \cap F$ and

$$\inf_{x \in V_\varepsilon \cap F} \inf_{y \in W_x^{u,\delta}} \inf_{v \in T_y W_x^{u,\delta}} \frac{Q_y(v)}{\|v\|} = c_1(F, \varepsilon) > 0. \tag{5.2}$$

On the other hand, the uniform continuity of $x \mapsto Q_x$ on the compact set F implies that there is a constant $c_2(F) > 0$ such that for all $x \in F$ and all $v \in T_x M$

$$Q_x(v) \leq c_2(F)\|v\|. \tag{5.3}$$

Finally, since Q is an infinitesimal Lyapunov function,

$$Q_{f^n y}(Df^n v) \geq Q_y(v), \tag{5.4}$$

whenever $y \in U$, $n \geq 0$, $f^n y \in U$ and $v \in T_y M$. Thus if x, y and v are as in the statement of the Lemma, we see from (5.3), (5.4), and (5.2) that

$$\begin{aligned}\|Df^n v\| &\geq c_2(F)^{-1} Q_{f^n y}(Df^n v)\\ &\geq c_2(F)^{-1} Q_y(v)\\ &\geq c_2(F)^{-1} c_1(F, \varepsilon)\|v\|.\end{aligned}$$

□

If W is a C^1 submanifold of M, let g_W be the Riemannian metric induced on W by the metric that we fixed earlier. Define the size $\sigma_x(W)$ of W at a point $x \in W$ to be the maximum radius of an open ball about 0 in $T_x W$ on which the exponential map for g_W is defined and is a diffeomorphism.

COROLLARY 5.1. *Let F be a compact subset of U $F \subset \operatorname{Int} F'$ $\varepsilon > 0$. Then there is $r = r(\varepsilon) > 0$ such that if $y \in F$ and $f^{-n} y \in V_\varepsilon \cap F$ for some $n > 0$, then $\sigma_y(\widetilde{W}_y^u) \geq r$.*

Proof. Choose a compact subset F' of U such that $F \subset \operatorname{Int} F'$. Define c' and δ' by applying Lemma 5.1 to F'. Let $\eta = \inf_{x \in F} \operatorname{dist}(x, \partial F')$. If $y' \in W_{f^{-n} y}^{u,\delta'}$ and $f^n y' \in F'$, then the derivative Df^n contracts vectors in $T_{y'} W_{f^{-n} y}^u$ by at most the factor c'. Hence $f^n(W_{f^{-n} y}^{u,\delta})$ contains a ball whose radius in the induced metric of $\widetilde{W}_y^u$ is at least $\min(\eta, c'\delta')$. □

LEMMA 5.2. *Let F be a compact subset of U. Then there is $R = R(F) > 0$ such that $\sigma_x(\widetilde{W}_x^u) \geq R$ for almost every $x \in F \cap \Lambda$.*

Proof. Choose a compact subset F' of U such that $F \subset \operatorname{Int} F'$. It suffices to show that for almost every $x \in \Lambda \cap F$, there is $r = r(x) > 0$ such that $\sigma_y(\widetilde{W}_y^u) \geq r$ for all y in a dense subset of $\widetilde{W}_x^u \cap F'$. Then we can take $R = \inf_{x \in F} \operatorname{dist}(x, \partial F')$.

Let E be an ergodic component such that $\mu(E \cap F') > 0$. Choose $\varepsilon > 0$ so that $\mu(E \cap F' \cap V_\varepsilon) > 0$. Since f is ergodic on E, μ-a.e. $y \in E$ has the property that there is $n > 0$ with $f^{-n} y \in F' \cap V_\varepsilon$. By Corollary 5.1, there is $r > 0$ such that $\sigma_y(\widetilde{W}_y^u) \geq r$ for μ-a.e. $y \in E \cap F'$. We now see from Corollary 3.1 and Theorem 3.2 that for μ-a.e. $x \in E \cap F'$, the set of y in $\widetilde{W}_x^u \cap F'$ with $\sigma_y(\widetilde{W}_y^u) \geq r$ has full μ^u-measure. By our hypothesis on μ, a subset of $\widetilde{W}_x^u \cap F$ with full μ^u-measure is dense in $\widetilde{W}_x^u \cap F'$.

The lemma now follows, because, as is easily seen from Theorem 3.3, the union of the ergodic components E such that $\mu(E \cap F) > 0$ has full measure in F. □

So far we have seen that the existence of an eventually strict infinitesimal Lyapunov function Q implies that unstable manifolds are typically reasonably large. Now we study how Q controls the direction of the unstable manifolds. Let $C_x^+ = C^+(Q_x)$. We use the Riemannian metric on M to define the ε-interior $\mathrm{Int}_\varepsilon C_x^+$ of this cone. Recall from Theorem 2.1 that for almost every $x \in U \cap \Lambda$ we have

$$T_x \widetilde{W}_x^u = E_x^+ \subset C_x^+. \tag{5.5}$$

For typical x, this relationship extends to the whole of $\widetilde{W}_x^u$.

LEMMA 5.3. *Almost every $x \in U \cap \Lambda$ has the property that*

$$T_y \widetilde{W}_x^u \subset C_y^+ \quad \textit{for all} \quad y \in \widetilde{W}_x^u \cap U.$$

Proof. We may assume that (5.5) holds at x and that $x \in \Lambda_\varepsilon$ for some $\varepsilon > 0$. By the Poincaré Recurrence Theorem, we may also assume that there is a sequence $n_k \to \infty$ such that $f^{-n_k}x \in \Lambda_\varepsilon \cap U$ for all k and $f^{-n_k}x \to x$ as $k \to \infty$. It follows from (5.5), the uniform continuity on Λ_ε of the local unstable manifolds W^u (with respect to the C^1 topology) and the continuity of the cone family C^+ on U that there is $\delta > 0$ such that if $x' \in \Lambda_\varepsilon$, $\mathrm{dist}(x, x') < \delta$ and $y' \in W_{x'}^{u,\delta}$, then $T_{y'} W_{x'}^u \subset C_{y'}^+$. Hence if $y \in W_x^u$, we have

$$T_{f^{-n_k}y} \widetilde{W}_{f^{-n_k}x}^u \subset C_{f^{-n_k}y}^+$$

for large k. Applying $Df^{n_k}x$ gives us

$$T_y \widetilde{W}_y^u \subset Df^{n_k} C_{f^{-n_k}y}^+ \subset C_y^+.$$

□

The next Lemma is crucial in §6. Together with its analogue for stable manifolds, it implies a locally uniform transversality of typical stable and unstable manifolds.

LEMMA 5.4. *For almost every $z \in U$ there are $\theta = \theta(z) > 0$ and a neighbourhood N of z such that, for almost every $x \in N$, we have*

$$T_y \widetilde{W}_x^u \subseteq \mathrm{Int}_\theta C_y^+ \quad \textit{for all} \quad y \in \widetilde{W}_x^u \cap N.$$

Proof. Since Q is eventually strict, we may assume that there is $l > 0$ such that $f^{-l}x \in U$ and $Df^l(C_{f^{-l}x}^+) \subset C_x^+$. By continuity, we can choose a neighbourhood $N \subseteq U$ of x and $\theta > 0$ such that $f^{-l}N \subset U$ and for all $y \in N$

$$Df^l(C_{f^{-l}y}^+) \subset \mathrm{Int}_{2\theta} C_y^+.$$

For almost every $y \in N$ we have $T_{f^{-l}y} \widetilde{W}_{f^{-l}y}^u \subset C_{f^{-l}x}^+$, which implies that

$$T_y \widetilde{W}_y^u \subset Df^l(C_{f^{-l}y}^+) \subset \mathrm{Int}_{2\theta} C_y^+. \tag{5.6}$$

It follows from absolute continuity (Corollary 3.1) that for almost every $x \in U \cap \Lambda$ property (5.6) holds for μ-a.e. $y \in \widetilde{W}_x^u \cap N$. Since a subset of $\widetilde{W}_x^u \cap N$ with full μ-measure is dense, we see that

$$T_y \widetilde{W}_x^u \subset \mathrm{Int}_\theta C_y^+ \quad \text{for all} \quad y \in \widetilde{W}_x^u \cap N.$$

□

If the infinitesimal Lyapunov function Q is eventually uniformly invariant, there is a global version of Lemma 5.4.

LEMMA 5.5. *If Q is eventually uniform, there is $\theta > 0$ such that for almost every $x \in U$ we have*

$$T_y \widetilde{W}^u_x \subset \mathrm{Int}_\theta\, C^+_y \quad \textit{for all} \quad y \in \widetilde{W}^u_x \cap U.$$

Proof. Almost every $y \in U$ has the properties that $y \in \Lambda$ and there is a sequence $n_k \to \infty$ such that $f^{-n_k} y \in U \cap \Lambda$ and $T_{f^{-n_k}y} \widetilde{W}^u_{f^{-n_k}y} \subset C^+_{f^{-n_k}y}$ for each k. For such y we have

$$T_y \widetilde{W}^u_y \subseteq \bigcap_{k=1}^{\infty} Df^{n_k} C^+_{f^{-n_k}y}. \tag{5.7}$$

Since Q is eventually uniform, there is $\theta > 0$ such that for almost all y the right hand side of (5.7) lies in $\mathrm{Int}_{2\theta}\, C^+_y$. Thus for almost every $y \in U \cap \Lambda$, we have

$$T_y \widetilde{W}^u_y \subset \mathrm{Int}_{2\theta}\, C^+_y.$$

Now one can apply essentially the same absolute continuity argument as at the end of the proof of the previous Lemma. □

6. *Proof of the main theorem*

In this section we use Lemmas 5.2, 5.4, and 5.5 together with their analogues for stable manifolds to prove Theorem 4.1. The corresponding result for flows can be proved in a similar way; this is left to the reader.

The first part of the argument is some simple Euclidean geometry. Let k and k' be positive integers and set $n = k + k'$. Let C and C' be cones in $\mathbb{R}^n$ of rank k and k' respectively. Assume there is $\varepsilon > 0$ such that

$$\sphericalangle(v, v') \geq \varepsilon \quad \text{for all} \quad v \in C \setminus \{0\} \quad \text{and} \quad v' \in C' \setminus \{0\}.$$

Let W and W' be C^1 submanifolds of $\mathbb{R}^n$ with dimension k and k' respectively. Suppose that we have

$$T_x W \subseteq C \quad \text{and} \quad T_{x'} W' \subseteq C'$$

whenever $x \in W$ and $x' \in W'$ and we make the canonical identifications of $T_x\mathbb{R}^n$ and $T_{x'}\mathbb{R}^n$ with $\mathbb{R}^n$. Assume that there are $x_0 \in A$ and $x_0' \in W'$ such that

$$\sigma_{x_0} W \geq 1 \quad \text{and} \quad \sigma_{x_0'} W' \geq 1,$$

where σ denotes the injectivity radius of a submanifold of $\mathbb{R}^n$ defined by the Euclidean metric.

LEMMA 6.1. *If $dist(x_0, x_0') < \sin^2(\varepsilon/2)$, then $W \cap W' \neq \emptyset$.*

Proof. Suppose $x \in W$, $x' \in W'$ and $x \neq x'$. Let $u(x, x')$ be the unit vector that points from x to x', $v(x, x')$ the orthogonal projection of $u(x, x')$ onto $T_x W$, and $\theta(x, x') \in [0, \pi/2]$ the angle between $u(x, x')$ and $T_x W$. Similarly, let $u'(x, x')$ be the unit vector that points from x' to x, $v'(x, x')$ the orthogonal projection of $v'(x, x')$ onto $T_{x'} W'$ and $\theta'(x, x')$ the angle between $u(x, x')$ and $T_{x'} W'$. Our assumptions about W, W', C and C' tell us that

$$\min\big(\theta(x, x'), \theta'(x, x')\big) \leq \pi/2 - \varepsilon/2.$$

Now let $\mathcal{W} = \{(x, x') \in W \times W' : x \neq x'\}$ and define the vector field V on $\mathcal{W}$ by

$$V(x, x') = \big(v(x, x'), v'(x, x')\big).$$

We may assume that $(x_0, x_0') \in \mathcal{W}$. Let (x_t, x_t') be the integral curve of V starting at (x_0, x_0'). This curve is defined until $(x_t, x_t') \to \partial(W \times W')$ or $l(t) = \mathrm{dist}(x_t, x_t') \to 0$. Since the curves x_t and x_t' have at most unit speed, the first possibility cannot occur until $t \geq 1$. On the other hand, $l(0) < \sin^2(\varepsilon/2)$ and

$$\frac{dl}{dt} = -\cos^2\theta\left(x_t, x_t'\right) - \cos^2\theta'\left(x_t, x_t'\right) \leq -\cos^2(\pi/2 - \varepsilon/2) = -\sin^2(\varepsilon/2).$$

Therefore there is $\tau \in (0, 1)$ such that $l(t) \to 0$ as $t \nearrow \tau$. It is clear that $\lim_{t \nearrow \tau} x_t$ exists and lies in $W \cap W'$. □

Let us now return to the situation described at the beginning of §5.

Definition 6.1. An open subset N of U has the *intersection property* if there is $\delta > 0$ such that both $W_x^{s,\delta}$ and $W_x^{u,\delta}$ are defined for almost every $x \in N$ and $W_y^{u,\delta} \cap W_z^{s,\delta} \neq \emptyset$ for $\mu \times \mu$ almost every $(y, z) \in N \times N$.

We emphasize that we are requiring only that $W_y^{u,\delta}$ and $W_z^{s,\delta}$ intersect somewhere, and not that they intersect in N. It is a straightforward exercise to prove the following Lemma using Lemmas 5.2, 5.4, 5.5 and 6.1.

LEMMA 6.2. (i) *Under the assumptions of part (i) of Theorem 4.1, almost every $x \in U$ has a neighborhood $N(x)$ with the intersection property.*

(ii) *Under the assumptions of part (ii) of Theorem 4.1, every $x \in U$ has a neighborhood $N(x)$ with the intersection property.*

Now we need a version of the Hopf argument.

LEMMA 6.3. *Let N be an open subset of U that has the intersection property and let φ be an L^1 function that is invariant under f. Then φ is almost everywhere constant on N.*

Proof. For a bounded measurable function $g : M \to \mathbb{R}$, let

$$\widehat{g}^+(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} g(f^k(x)) \quad \text{and} \quad \widehat{g}^-(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} g(f^{-k}(x)).$$

Then $\widehat{g}^+$ and $\widehat{g}^-$ are f-invariant and, by Birkhoff Ergodic Theorem, equal on a set G of full measure. Since $\{\widehat{g}^+ : g \text{ is continuous}\}$ is L^1-dense in the space of L^1 invariant functions, it suffices to prove that $\widehat{g}^+$ is constant when g is continuous. The continuity of g and the contraction of the stable and unstable manifolds as $t \to \infty$ and $t \to -\infty$ respectively imply that $\widehat{g}^+$ is constant on $\widetilde{W}_x^s$ and $\widehat{g}^-$ is contant on $\widetilde{W}_x^u$ for each $x \in \Lambda$.

Since $G \cap \Lambda$ has full measure in M, it follows from absolute continuity that we can choose $x_0 \in N$ so that $G \cap \widetilde{W}_{x_0}^u$ has full μ^u-measure in $\widetilde{W}_{x_0}^u$. The intersection property implies that $\widetilde{W}_x^s \cap \widetilde{W}_{x_0}^u \neq \emptyset$ for almost every $x \in N$. On the other hand, since μ^u-a.e. point of $\widetilde{W}_{x_0}^u$ is in G, the union of the $\widetilde{W}^s$ leaves that intersect $\widetilde{W}_{x_0}^u$ in points that are not in G has measure 0. We see that, for almost $x \in N$, there is a point $y \in \widetilde{W}^s_x \cap \widetilde{W}^u_{x_0} \cap G$. We have

$$\widehat{g}^+(x) = \widehat{g}^+(y) = \widehat{g}^-(x) = \widehat{g}^-(x_0).$$

Thus $\widehat{g}^+(x) = \widehat{g}^-(x_0)$ for almost every $x \in N$.

It follows immediately from Lemma 6.3 that each neighbourhood $N(x)$ in Lemma 6.2 lies modulo a set of measure 0 in a single ergodic component of f. This proves the ergodicity statements in both parts (i) and (ii) of Theorem (4.1). The claims about the Bernouilli property in (ii) follow from Theorem 3.4. The proof of Theorem 4.1 is complete. □

7. *Riemannian metrics with Bernoulli geodesic flows on compact manifolds of dimension 3*

We shall construct on any compact 3-dimensional manifold M a C^∞ Riemannian metric whose geodesic flow is Bernoulli.

The geometric basis of the construction is the fact that M contains a knot K such that $M \setminus K$ admits a hyperbolic structure, i.e. a complete Riemannian metric of finite volume and constant curvature -1. For orientable M this was proved by Myers [**My**], using Thurston's theorem on the existence of hyperbolic structures [**Th**, Theorem 1.2], [**Mo**, Theorem B]. Myers constructs his knot in the following way. Start with a triangulation of M and consider the dual 1-skeleton S of its second barycentric subdivision. Then S is a graph in which four edges meet at each vertex. One now replaces the vertices of S by by so called 'true lover's tangles'. This operation removes the vertex and joins each of the four edges that met at the vertex with one of the others. Applying the operation to all vertices converts S into a link. Myers shows that the complement of any link obtained in this manner has a hyperbolic structure. Since an even number of edges meet at each vertex, it is always possible to choose the edge joinings so that one obtains a knot.

For non-orientable M, consider the double cover D, and let $\tau : D \to D$ be the covering transformation. Myers' construction can be performed in a τ-equivariant way to obtain a τ-invariant link L in D, such that $D \setminus L$ has a hyperbolic structure and L projects to a knot K in M. By the Mostow rigidity theorem, there is a map $\sigma : D \setminus L \to D \setminus L$ that is homotopic to τ and is an isometry of the hyperbolic structure. Note that $\sigma^2 = id$, because σ^2 is an isometry of the hyperbolic structure and is homotopic to $\tau^2 = id$. Since $D \setminus L$ has a hyperbolic structure, $\pi_1(D \setminus L)$ has trivial center, and hence $D \setminus L$ is neither the torus T^3 nor a Seifert fibration. It now follows from a theorem of Tollefson† [**To**] that there is a homeomorphism h of $D \setminus L$ isotopic to the identity such that $\sigma = h^{-1} \circ \tau \circ h$. Hence $(D \setminus L)/\sigma$ and $(D \setminus L)/\tau$ are homeomorphic, and indeed diffeomorphic, since we are in dimension 3. Thus $M \setminus K$ admits a hyperbolic structure.

Now consider $M \setminus K$ with its hyperbolic structure. It has one end, which is a cusp that is a warped product $F \times_{e^{-t}} [0, \infty)$, where F is a compact flat surface. Let $\widehat{M}$ be the compact Riemannian manifold with boundary obtained from $M \setminus K$ by cutting off the cusp along the horospherical surface $F_1 = F \times \{1\}$ and changing the warping function to $f : [0, 1] \to \mathbb{R}$ with the properties:

† We thank Jean-Pierre Otal for drawing Tollefson's result to our attention.

- $f(t) = e^{-t}$ for t near 0.
- f is positive and strictly convex.
- All derivatives of f vanish when $t = 1$.

Then Int $\widehat{M}$ has negative curvature, $\partial\widehat{M} = F_1$ is totally geodesic, and one will obtain a smooth Riemannian manifold by gluing $F_1 \times [0, 1]$ onto $\widehat{M}$.

We now attach to $\widehat{M}$ along F_1 a Riemannian manifold P that is locally the product of an interval and the disc D^2 with a certain Riemannian metric g_0. We choose g_0 so that it is radially symmetric, ∂D^2 is a closed geodesic, and the curvature is a non-negative, non-increasing function of distance from the center and vanishes in a neighborhood of the boundary. To construct P, we form the Riemannian product of (D^2, g_0) with an interval $[0, \ell]$ and then use an isometry ψ of (D^2, g_0) to identify the two ends. With appropriate choices of g_0, ℓ and ψ, there will be an isometry $\varphi : \partial P \to F_1$ and the manifold obtained by using φ to attach P to $\widehat{M}$ will be diffeomorphic to M. Our requirements on g_0 and f ensure that we obtain a C^∞ Riemannian metric.

Now consider the geodesic flow g^t of this metric on the unit tangent bundle T^1M. Let $\pi : T^1M \to M$ be the projection. If $u \in T^1M$, let $\gamma_u(t) = \pi(g^t u)$, and let Y_ξ be the Jacobi field along the geodesic γ_u with

$$Y_\xi(t) = d\pi g^t \xi \quad \text{for all} \quad t.$$

Then $T_u T^1 M = \{\xi \in T_u TM : \langle u, Y'_\xi(0)\rangle = 0\}$. Recall that g^t is a contact flow. The contact form α_u on $T_u T^1 M$ is defined by

$$\alpha_u(\xi) = \langle u, Y_\xi(0)\rangle.$$

We shall prove that g^t is Bernoulli by constructing an infinitesimal eventually uniformly invariant Lyapunov function Q for g^t acting on the restriction to a suitable open set U of the bundle TT^1M/E, where E is the one dimensional subbundle tangential to the flow. We identify the fibre over u of TT^1M/E with

$$\ker \alpha_u = \{\xi \in T_u TM : \langle Y_\xi(0), u\rangle = 0 = \langle Y'_\xi(0), u\rangle\}.$$

Note that the only geodesics of (D^2, g_0) that do not intersect ∂D^2 transversally are closed geodesics in the flat region near ∂D^2 that are parallel to ∂D^2; see [**BG2**, Proposition 2.4]. It follows easily that almost every geodesic of M enters Int $\widehat{M}$. Since $\partial\widehat{M}$ is totally geodesic and $\widehat{M}$ has non-positive curvature, these is $\delta > 0$ such that every geodesic in Int $\widehat{M}$ contains a point with distance at least 2δ from $\partial\widehat{M}$. We define

$$U = \{u \in T^1\widehat{M} : \text{ dist } (\pi v, \partial\widehat{M}) > \delta\}.$$

Let $B = \cup_{u \in U} \ker \alpha_u$, define $Q_0 : B \to \mathbb{R}$ by

$$Q_0(\xi) = \langle Y_\xi(0), Y'_\xi(0)\rangle$$

and set

$$Q(\xi) = \text{ sign } (Q_0(\xi))|Q_0(\xi)|^{1/2}.$$

It is obvious that Q is homogeneous of degree 1 and takes both positive and and negative values on $\ker \alpha_u$ for each $u \in U$. Condition (ii) of Definition 4.1 holds because we take

$$D_u^+ = \{\xi \in \ker \alpha_u : Y_\xi(0) = Y'_\xi(0)\}$$

and
$$D_u^- = \{\xi \in \ker\alpha_u : Y_\xi(0) = -Y'_\xi(0)\}.$$

We now verify condition (iii) of Definition 4.1. Suppose that $u \in U, \tau \geq 0, g^\tau u \in U$ and $\xi \in \ker\alpha_u$. Then
$$Q_0(dg^\tau\xi) = \langle Y_\xi(\tau), Y'_\xi(\tau)\rangle.$$

We need two lemmas.

LEMMA 7.1. *Suppose that Y is a Jacobi field along a geodesic γ in a Riemannian manifold and $\langle R(Y(t), \dot\gamma(t))\dot\gamma(t), Y(t)\rangle \leq 0$ for all t. Then $\langle Y(t), Y'(t)\rangle$ is non-decreasing.*

Proof. $\langle Y, Y'\rangle' = \langle Y', Y'\rangle + \langle Y, Y'\rangle = \langle Y', Y'\rangle - \langle R(Y, \dot\gamma)\dot\gamma, Y\rangle \geq 0$. □

LEMMA 7.2. *Let γ be a geodesic in (D^2, g_0) such that $\gamma(t_1) \in \partial D^2, \gamma(t_2) \in \partial D^2$ and $\gamma(t) \in$ Int D^2 for $t_1 < t < t_2$. Let Y be a Jacobi field along γ. Then*
$$\langle Y, Y'\rangle(t_2) \geq \langle Y, Y'\rangle(t_1).$$

Proof. Let $Y = (Y_\top, Y_\perp)$ be the decomposition of Y into components tangential and perpendicular to $\dot\gamma$. Then $\langle Y, Y'\rangle = \langle Y_\top, Y'_\top\rangle + \langle Y_\perp, Y'_\perp\rangle$ and the tangential term is non-decreasing by Lemma 7.1. Let N be a continuous field of unit normals to γ. Then $Y_\perp(t) = y(t)N(t)$, where $y(t)$ is a solution of the scalar Jacobi equation
$$y'(t) + K(\gamma(t))y(t) = 0. \tag{7.1}$$

There are constants a, b, c, and d such that
$$\begin{pmatrix} y(t_2) \\ y'(t_2) \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y(t_1) \\ y'(t_1) \end{bmatrix}$$
for all solutions $y(t)$ of (7.1); it follows from the argument in Lemma 2.5 in **[BG2]** that there is a solution z of (7.1) with $z(t_1) = 1, z'(t_1) = 0, z(t_2) = -1, z'(t_2) = 0$. Thus $a = -1$ and $c = 0$. Since the Wronskian $y'(t)z(t) - z'(t)y(t)$ of y with z is constant, we see that $y'(t_2) = -y'(t_1)$ for all solutions y of (7.1), and so $d = -1$. It follows from Proposition 2.7 of **[BG2]** that if $y(t_1) = 0$ and $y'(t_1) = 1$, then $y(t_2)y'(t_2) \geq 0$. Hence $b \leq 0$ and
$$\langle Y_\perp(t_2), Y'_\perp(t_2)\rangle = \langle Y_\perp(t_1), Y'_\perp(t_1)\rangle - b\langle Y'_\perp(t_1), Y'_\perp(t_1)\rangle \geq \langle Y_\perp(t_1), Y'_\perp(t_1)\rangle.$$
□

Choose a sequence of times $0 = t_0 < t_1 < \cdots < t_n = \tau$ such that in each interval $[t_i, ti + 1]$ either

(i) $\gamma|_{[t_i, t_{i+1}]}$ lies in $\widehat{M}$, or

(ii) $\gamma|_{[t_i, t_{i+1}]}$ is a maximal geodesic in P.

In case (i) we see from Lemma 7.1 that $\langle Y_\xi(t_{i+1}), Y'_\xi(t_{i+1})\rangle \geq \langle Y_\xi(t_i), Y'_\xi(t_i)\rangle$. In case (ii) we use the fact that P splits locally as the Riemannian product of (D^2, g) and an interval. Let Y_D and Y_I be the projections of Y_ξ onto the D^2 and interval directions. Then we can apply Lemma 7.2 to Y_D and Lemma 7.1 to Y_I, yielding
$$\begin{aligned}
\langle Y_\xi(t_{i+1}), Y'_\xi(t_{i+1})\rangle &= \langle Y_D(t_{i+1}), Y'_D(t_{i+1})\rangle + \langle Y_I(t_{i+1}), Y'_I(t_{i+1})\rangle \\
&\geq \langle Y_D(t_i), Y'_D(t_i)\rangle + \langle Y_I(t_{i+1}), Y'_I(t_{i+1})\rangle \\
&= \langle Y_\xi(t_i), Y'_\xi(t_i)\rangle.
\end{aligned}$$

It now follows that $\langle Y_\xi(\tau), Y'_\xi(\tau)\rangle \geq \langle Y_\xi(0), Y'_\xi(0)\rangle$, and consequently $Q(dg^\tau\xi) \geq Q(\xi)$.

Thus Q is an infinitesimal Lyapunov function. It remains to verify that Q is eventually uniform. To do this, observe that, since $\operatorname{Int}\widehat{M}$ has negative curvature, there is $\eta > 0$ such that $0 < \eta < 1$ and for all $(u, t) \in \overline{U} \times [-\eta, \eta]$ the sectional curvature of every plane at $\gamma_u(t)$ is less than $-\eta^2$. We see from the proof of Lemma 7.1 that if $u \in \overline{U}$ and $|t| \leq \eta$, then

$$\langle Y_\xi(t), Y'_\xi(t)\rangle' > 0 \quad \text{for all non-zero } \xi \in \ker\alpha_u.$$

Hence for all $u \in \overline{U}$ we have

$$Q(dg^\eta\xi) > Q(\xi) > Q(dg^{-\eta}\xi) \quad \text{for all non-zero } \ \xi \in \ker\alpha_u.$$

It follows using the homogeneity of Q and a compactness argument that there is $\varepsilon > 0$ such that for all $u \in U$ and all $\xi \in \ker\alpha_u$ we have

$$Q(dg^\eta\xi) \geq Q(\xi) + \varepsilon\|\xi\|$$

and

$$Q(dg^{-\eta}\xi) \leq Q(\xi) - \varepsilon\|\xi\|.$$

Thus Q is an infinitesimal eventually uniform Lyapunov function. It follows from Theorem 4.2 that g^t is Bernoulli.

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