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COCYCLE SUPERRIGIDITY AND RIGIDITY FOR LATTICE ACTIONS ON TORI

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1. INTRODUCTION AND STATEMENT OF RESULTS

This note is part of an ongoing program directed at understanding the actions of lattices in semisimple Lie groups on compact manifolds by diffeomorphisms. A brief account of the history and current state of this program will be given in Section 3. Our main result is the following.

Theorem 1.1. *Suppose Γ is a subgroup of finite index in* $\mathbf{SL}(n, \mathbb{Z})$, $n \geqslant 3$, $M = \mathbb{T}^n$, *and* $\rho: \Gamma \to \mathrm{Diff}(M)$ *is a smooth action such that*

(i) *ρ preserves a non-atomic probability measure μ,*

(ii) *there exists an element $\gamma_0 \in \Gamma$ such that the diffeomorphism $\rho(\gamma_0)$ is Anosov and in addition one of the following three conditions hold:*

(a) *The measure μ is positive on open sets, i.e.* supp $\mu = \mathbb{T}^n$ *and ρ is ergodic with respect to μ.*

(b) *μ is absolutely continuous.*

(c) *Let $\rho_*: \Gamma \to \mathbf{GL}(n, \mathbb{Z})$ denote the homomorphism corresponding to the action on $H_1(M) \simeq \mathbb{Z}^n$. Then $\rho_*(\gamma_0)$ is an irreducible matrix over $\mathbb{Q}$ and either $n \geqslant 4$ or if $n = 3$ the eigenvalues of $\rho_*(\gamma_0)$ are real.*

Then there exists a 1-*cocycle $\alpha: \Gamma \to \mathbb{Q}^n/\mathbb{Z}^n$ (where Γ acts on $\mathbb{Q}^n/\mathbb{Z}^n$ via ρ_*) and a diffeomorphism h of M conjugating ρ to the affine action given by ρ_* and α, i.e. $\rho(\gamma) = h(\rho_*(\gamma) + \alpha(\gamma))h^{-1}$ for every $\gamma \in \Gamma$. In particular, ρ is smoothly conjugate to ρ_* on a subgroup of finite index.*

Here and below we slightly abuse notations by using the same symbol for an integer $n \times n$ matrix and the endomorphism of the n-dimensional torus $\mathbb{T}^n$ induced by that matrix.

At the beginning of the next section we will show that cases (b) and (c) are reduced to (a).

Observe also that the homomorphisms $\Gamma \to \mathbf{GL}(n, \mathbb{Z})$ can be completely classified using the (finite-dimensional) superrigidity theorem of Margulis [30, Theorem 5.1.2]. The consequences for the homomorphism ρ_* corresponding to the action on H_1, given that some element acts by an Anosov diffeomorphism, are worked out in Section 2 of [9]. The precise conclusion is that there exists a matrix $A \in \mathbf{GL}(n, \mathbb{Q})$ and a homomorphism $\iota: \Gamma \to \{\pm I\}$ such that either $\rho_*(\gamma) = \iota(\gamma)A\gamma A^{-1}$ for every $\gamma \in \Gamma$ or $\rho_*(\gamma) = \iota(\gamma)A(\gamma^{-1})^t A^{-1}$ for every $\gamma \in \Gamma$.

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Recall that if Γ is any finitely generated discrete group and G is any topological group whatsoever, we denote by $R(\Gamma, G)$ the space of homomorphisms of Γ into G with the compact/open topology. A homomorphism $\rho_0 \in R(\Gamma, G)$ is said to be *locally rigid* if there exists a neighborhood U of ρ_0 in $R(\Gamma, G)$ such that for every $\rho \in U$ there exists $g \in G$ such that $\rho(\gamma) = g\rho_0(\gamma)g^{-1}$ for every $\gamma \in \Gamma$.

An easy argument (Lemma 2.6, which first appeared in [26]) shows that any C^1 perturbation of a volume-preserving action of a Kazhdan group on a compact manifold must preserve an absolutely continuous probability measure. Furthermore, the standard action of any finite-index subgroup Γ of $\mathbf{SL}(n, \mathbb{Z})$ contains Anosov elements and the property of being an Anosov is C^1 open. These remarks allow to apply case (b) of Theorem 1.1 to obtain the local C^1 rigidity of standard actions. For $n \geqslant 4$ this was proved in [8] but the case $n = 3$ is new. The notion of local rigidity in this statement corresponds to the representations of Γ into the group Diff $\mathbb{T}^n$ of C^∞ diffeomophisms of the n-torus provided with the C^1 topology.

COROLLARY 1.2. *Let* $\Gamma = \mathbf{SL}(n, \mathbb{Z})$ *or any subgroup of finite index,* $n \geqslant 3$. *Then the standard action of* Γ *on* $\mathbb{T}^n$ *is locally* C^1 *rigid.*

Our proof of Theorem 1.1 is based on a result due to Zimmer (extending ideas of Margulis), the "superrigidity theorem for cocycles" [30, Theorems 5.2.5 and 9.4.14]. The cocycle superrigidity theorem yields a homomorphism $\pi : \Gamma \to \mathbf{SL}(n, \mathbb{R})$ and a measurable framing, σ, of the tangent bundle TM with respect to which the derivative $D_x\rho(\gamma)$ is given by $\pi(\gamma)$ for every $\gamma \in \Gamma$ and $x \in M$. The dynamical hypothesis (ii) in Theorem 1.1 makes it possible to conclude that the framing σ, which is *a priori* only measurable, is in fact continuous on supp μ, which by assumption (a) means everywhere. This implies that elements $\gamma \in \Gamma$ with $\pi(\gamma)$ hyperbolic act by Anosov diffeomorphisms, with Lyapunov exponents determined by the eigenvalues of $\pi(\gamma)$. In particular, we are in a position to apply either Theorem 4.12 in [8] or the argument in [7] to conclude that there exists a free abelian subgroup $\mathscr{A}$ of rank $n - 1$ in Γ whose action is smoothly conjugate to the action on homology. Then it is easy to see that ρ_* and π must coincide, and that the continuous linearizing frame σ is the image of a constant frame (i.e. constant with respect to the standard trivialization $TM \simeq \mathbb{T}^n \times \mathbb{R}^n$) under the conjugating diffeomorphism for the action of the abelian subgroup $\mathscr{A}$. Theorem 1.1 follows.

We should probably remark at the outset that the cocycle superrigidity theorem is applicable in much greater generality, in particular, to actions of more general groups and on other compact manifolds. Indeed, the argument we shall present below applies, with very minor modifications, to some additional cases, such as $\Gamma \subset \mathbf{Sp}(n, \mathbb{Z})$ of finite index acting on $M = \mathbb{T}^{2n}$, $n \geqslant 2$. However, we have deliberately restricted the scope of this note in order to present the essential new ideas in the most strightforward possible setting. The issues which arise in extending Theorem 1.1 to more general actions will be addressed elsewhere; here we shall content ourselves with a brief discussion in the final section.

2. PROOFS

In this section we provide the proofs of Theorem 1.1 and its Corollary 1.2. By an elementary argument (cf. Lemmas 2.6 and 2.14 in [9]), it will suffice to establish Theorem 1.1 on any subgroup $\Gamma' \subset \Gamma$ of finite index. In particular, we may assume without loss of generality that the action ρ is orientation-preserving.

First let us show that cases (b) and (c) of the theorem follow from case (a).

Any absolutely continuous invariant measure of an Anosov diffeomorphism is given by a smooth positive density and hence is positive on open sets [14, 15]. Furthermore, any Anosov diffeomorphism is ergodic with respect to such a measure [1] (a more recent exposition of the relevant results appears in Section III.2 of [16]). Thus the diffeomorphism $\rho(\gamma_0)$ and hence the whole action ρ is ergodic with respect to μ.

For case (c) we will assume that μ is ergodic (by taking ergodic components if necessary and noticing that the measures will still be non-atomic) and will show that $\operatorname{supp}\mu = \mathbb{T}^n$. First let us recall that any Anosov diffeomorphism of $\mathbb{T}^n$ is topologically conjugate to the linear map given by the action on the first homology group [17]. The conjugacy is a homeomorphism homotopic to identity and is uniquely determined by the image of the origin which, of course, must be fixed by our diffeomorphism. Thus $\rho(\gamma_0) = h(\rho_*(\gamma_0))h^{-1}$ for some homeomorphism h homotopic to identity. For any hyperbolic linear automorphism of the torus its centralizer in Homeo($\mathbb{T}^n$) coincides with centralizer in the space of affine maps of the torus (cf. e.g. [23]). Furthermore, this centralizer contains a finite index subgroup which belongs to $\mathbf{SL}(n, \mathbb{Z})$, and hence a finite index subgroup C which lies in $\rho_*(\Gamma)$. Since $\rho(\Gamma)$ contains at most one diffeomorphism in each homotopy class we conclude that $\rho(C) = h(\rho_*(C))h^{-1}$. Since any ρ-invariant measure is in particular $\rho(C)$-invariant, we conclude that $\mu = h_*\nu$ for a $\rho_*(C)$-invariant non-atomic measure ν whose support is obviousy an infinite $\rho_*(C)$-invariant closed set. From irreducibility of the matrix $\rho_*(\gamma_0)$ it follows immediately that its eigenvalues are all different (otherwise the characteristic polynomial of the matrix has non-trivial greatest common divisor with its derivative and is hence reducible over the rationals). The rank of the free part of its centralizer is equal to the rank of the group of units in the ring of integers in the number field $\mathbb{Q}(\gamma_0)$. By the Dirichlet unit theorem the latter rank is equal to the number of real eigenvalues of the matrix plus the number of pairs of complex-conjugate eigenvalues minus one. Thus condition (c) implies that the centralizer of γ_0 contains $\mathbb{Z}^2$ and hence the rank of C is $\geqslant 2$. Now we invoke the result by Berend [2, Theorem 2.1] which implies that any infinite $\rho_*(C)$-invariant closed set is the whole torus, hence $\operatorname{supp}\mu = h_*\operatorname{supp}\nu = \mathbb{T}^n$ and case (a) applies.

Now we proceed to the proof of case (a). Let PM denote the principal bundle of n-frames in the tangent bundle TM to $M = \mathbb{T}^n$. As usual, we identify each $\varphi \in P_xM$ with an isomorphism $\varphi : \mathbb{R}^n \to T_xM$. The following proposition summarizes the consequences of the cocycle superrigidity theorem in the context of Theorem 1.1. Although the argument is by now standard, we provide a complete proof for the reader's convenience.

Proposition 2.1. *Suppose that* $\Gamma \subset \mathbf{SL}(n, \mathbb{Z})$, $n \geqslant 3$, $M = \mathbb{T}^n$, *and* $\rho : \Gamma \to \operatorname{Diff}(M)$ *satisfy the hypotheses of Theorem* 1.1 *and in addition, that* ρ *is orientation-preserving. Then there exists a measurable section* $\sigma : M \to PM$ *and a homomorphism* $\pi : \Gamma \to \mathbf{SL}(n, \mathbb{R})$ *such that with respect to the framing* σ, *the derivative* $D_x\rho(\gamma)$ *is given by* $\pm\,\pi(\gamma)$, *i.e. for every* $\gamma \in \Gamma$,

$$D_x\rho(\gamma)\sigma(x) = \pm\,\sigma(\rho(\gamma)x)\pi(\gamma)$$

for almost every $x \in M$. *Moreover, there is a matrix* $A \in \mathbf{GL}(n, \mathbb{R})$ *of determinant* ± 1 *which conjugates the representation* π *to either the identity representation or the involution* $\gamma \mapsto (\gamma^{-1})^t$ *(inverse transpose).*

Proof. Let $\tau : M \to PM$ denote the standard framing on M, i.e. τ corresponds to the constant section $\tau(x) = I$ under the natural identifications $TM \simeq M \times \mathbb{R}^n$, $PM \simeq M \times \mathbf{GL}(n, \mathbb{R})$. Then let $\xi : \Gamma \times M \to \mathbf{GL}(n, \mathbb{R})$ denote the derivative cocycle for the action ρ with respect to the section τ, so that

$$D_x\rho(\gamma)\tau(x) = \tau(\rho(\gamma)x)\xi(\gamma, x)$$

for every $\gamma \in \Gamma$ and $x \in M$. Since Γ acts ergodically with respect to μ the algebraic hull $\mathbf{H}_{\mathbb{R}} \subset \mathbf{GL}(n, \mathbb{R})$ of ξ (or more precisely, its conjugacy class) is well-defined (cf. [30, Section 9.2.]) Let $\tilde{\xi}: \Gamma \times M \to \mathbf{H}_{\mathbb{R}}$ denote a (measurable) cocycle equivalent to ξ, and let $\tilde{\tau}: M \to \mathbf{GL}(n, \mathbb{R})$ denote the Borel map which gives the equivalence between ξ and $\tilde{\xi}$, so that for every $\gamma \in \Gamma$,

$$\tilde{\xi}(\gamma, x)\tilde{\tau}(x) = \tilde{\tau}(\rho(\gamma)x)\xi(\gamma, x)$$

for almost every $x \in M$.

The existence of an ergodic invariant probability measure, together with our assumption that ρ is orientation-preserving, imply that we can take $\mathbf{H} \subset \mathbf{SL}(n, \mathbb{C})$. This follows from splitting the cocycle into the $\mathbf{SL}(n, \mathbb{Z})$ part and $\mathbb{R}_*$ part (the latter given by the determinant) and noticing that the second component is cohomologous to identity since Γ is a Kazhdan group.

Let $\mathbf{H}^0$ denote the (Zariski) connected component of $\mathbf{H}$, and set $S = \mathbf{H}_{\mathbb{R}}/(\mathbf{H}^0) \times M$ with the $\tilde{\xi}$-twisted action $\gamma(\bar{h}, x) = (\tilde{\xi}(\gamma, x)\bar{h}, \rho(\gamma)x)$. Define $\beta: \Gamma \times S \to \mathbf{H}_{\mathbb{R}}, \beta(\gamma, \bar{h}, x) = \tilde{\xi}(\gamma, x)$. Then by [30, 9.2.6], the action of Γ on S is ergodic, and the algebraic hull of the cocycle β is $(\mathbf{H}^0)_{\mathbb{R}}$.

Write $\mathbf{H}^0 = \mathbf{L} \ltimes \mathbf{U}$ where $\mathbf{L}$ is reductive, $\mathbf{U}$ is unipotent, and $\mathbf{L}$ and $\mathbf{U}$ are $\mathbb{R}$-groups. Let $\delta: \Gamma \times S \to \mathbf{L}_{\mathbb{R}}$ denote the cocycle obtained by composing β with the projection onto $\mathbf{L}_{\mathbb{R}}$. Let $p: \mathbf{L}_{\mathbb{R}} \to \mathbf{L}_{\mathbb{R}}/[\mathbf{L}_{\mathbb{R}}, \mathbf{L}_{\mathbb{R}}]$ denote the projection. Then the algebraic hull of $p \circ \delta$ is $\mathbf{L}_{\mathbb{R}}/[\mathbf{L}_{\mathbb{R}}, \mathbf{L}_{\mathbb{R}}]$, which is amenable, so since Γ is a Kazhdan group, $\mathbf{L}_{\mathbb{R}}/[\mathbf{L}_{\mathbb{R}}, \mathbf{L}_{\mathbb{R}}]$ is compact by [30, 9.1.3]. Since $\mathbf{L} = [\mathbf{L}, \mathbf{L}] \cdot Z(\mathbf{L})$, where $Z(\mathbf{L})$ denotes the center, and $[\mathbf{L}, \mathbf{L}] \cap Z(\mathbf{L})$ is finite, it follows that $Z(\mathbf{L})_{\mathbb{R}}$ is also compact.

Write $\mathbf{L}/Z(\mathbf{L})$ as a product of semisimple $\mathbb{R}$-groups, $\mathbf{L}/Z(\mathbf{L}) = \mathbf{L}_1 \times \mathbf{L}_2$, where $(\mathbf{L}_2)_{\mathbb{R}}$ is compact and $(\mathbf{L}_1)_{\mathbb{R}}$ is center-free with no compact factors. Let $q: \mathbf{L}_{\mathbb{R}} \to (\mathbf{L}_1)_{\mathbb{R}}$ denote the projection. Then $q \circ \delta: \Gamma \times S \to (\mathbf{L}_1)_{\mathbb{R}}$ is a cocycle with algebraic hull $(\mathbf{L}_1)_{\mathbb{R}}$, so by the superrigidity theorem for cocycles [30, 9.4.14], there is an $\mathbb{R}$-rational homomorphism $\tilde{\pi}: \mathbf{SL}(n, \mathbb{C}) \to \mathbf{L}_1$ such that $q \circ \delta$ is equivalent to the cocycle $\xi_{\tilde{\pi}|\Gamma}: (\gamma, x) \to \tilde{\pi}(\gamma)$.

Examining the list of representations of $\mathbf{SL}(n, \mathbb{C})$, we see that there are only two possibilities: either $\mathbf{L}_1$ is trivial, or $\mathbf{L}_1 = \mathbf{SL}(n, \mathbb{C})/Z(\mathbf{SL}(n, \mathbb{C})$. In the first instance, we could conclude that the algebraic hull of the derivative cocycle ξ is compact. But it's easy to see that this contradicts hypothesis (ii), the existence of $\gamma_0 \in \Gamma$ with $\rho(\gamma_0)$ Anosov. Thus $\mathbf{L}_1 = \mathbf{L}/Z(\mathbf{L}) = \mathbf{H}^0/Z(\mathbf{H}^0) = \mathbf{H}/Z(\mathbf{H}) = \mathbf{SL}(n, \mathbb{C})/Z(\mathbf{SL}(n, \mathbb{C}))$. Moreover, $\mathbf{H} = \mathbf{SL}(n, \mathbb{C})$ is connected, $S = M$, and there are essentially only two possibilities for the representation $\tilde{\pi}: \mathbf{SL}(n, \mathbb{C}) \to \mathbf{L}_1$. Namely, $\tilde{\pi} = r \circ \pi$, where $\pi: \mathbf{SL}(n, \mathbb{C}) \to \mathbf{SL}(n, \mathbb{C})$ is either the identity map or the involution $\gamma \mapsto (\gamma^{-1})^t$ followed by conjugation by some matrix $A \in r^{-1}((\mathbf{L}_1)_{\mathbb{R}})$, and r denotes the projection $\mathbf{SL}(n, \mathbb{C}) \to \mathbf{L}_1 = \mathbf{SL}(n, \mathbb{C})/Z(\mathbf{SL}(n, \mathbb{C}))$.

Note that $r^{-1}((\mathbf{L}_1)_{\mathbb{R}})$ can be described more concretely:

$$r^{-1}((\mathbf{L}_1)_{\mathbb{R}}) = \{g \in \mathbf{SL}(n, \mathbb{C}) \mid Ad(g) \text{ is an } \mathbb{R}\text{-rational automorphism of } \mathfrak{sl}(n, \mathbb{C})\}.$$

In general, the elements of $r^{-1}((\mathbf{L}_1)_{\mathbb{R}})$ have complex entries. However, it's easy to see that every $A \in r^{-1}((\mathbf{L}_1)_{\mathbb{R}})$ is of the form $A = \lambda A'$, where $\lambda \in \mathbb{C}$ is a scalar (in fact, a root of unity) and $A' \in \mathbf{GL}(n, \mathbb{R})$ with determinant ± 1. In particular, we can replace the conjugating matrix A in the preceding paragraph with a matrix in $\mathbf{GL}(n, \mathbb{R})$, and if $\bar{\eta}: M \to (\mathbf{L}_1)_{\mathbb{R}}$ denotes the Borel map which gives the equivalence between $q \circ \delta$ and $\xi_{\tilde{\pi}|\Gamma}$, $\bar{\eta}$ lifts to a map $\tilde{\eta}$: $M \to \mathbf{GL}(n, \mathbb{R})$. Then $\tilde{\eta}$ reduces $\tilde{\xi}$ to $\pm \pi$, i.e. for every $\gamma \in \Gamma$,

$$\tilde{\xi}(\gamma, x)\tilde{\eta}(x) = \pm \tilde{\eta}(\rho(\gamma)x)\pi(\gamma)$$

for almost every $x \in M$. Then $\sigma: M \to \mathbf{GL}(n, \mathbb{R})$, $\sigma(x) = \tilde{\tau}(x)^{-1}\tilde{\eta}(x)$, or more precisely, the corresponding section $\tau\sigma: M \to PM$, $x \mapsto \tau(x)\sigma(x)$, is the required framing. □

The statement of Proposition 2.1 is perhaps a bit awkward; what the argument actually provides is a Borel map $\bar{\sigma}: M \to \mathbf{PGL}(n, \mathbb{R})$ such that for every $\gamma \in \Gamma$,

$$\bar{\xi}(\gamma, x)\bar{\sigma}(x) = \bar{\sigma}(\rho(\gamma)x)\bar{\pi}(\gamma)$$

for almost every $x \in M$. (Here $\bar{\pi}: \mathbf{SL}(n, \mathbb{C}) \to \mathbf{L}_1$ as above, and $\bar{\xi}: M \to \mathbf{PGL}(n, \mathbb{R})$ is the map corresponding to the derivative cocycle $\xi: M \to \mathbf{GL}(n, \mathbb{R})$. Note that

$$\mathbf{PGL}(n, \mathbb{R}) = \mathbf{GL}(n, \mathbb{R})/Z(\mathbf{GL}(n, \mathbb{R}))$$

is naturally identified with

$$(\mathbf{L}_1)_{\mathbb{R}} = (\mathbf{SL}(s, \mathbb{C})/Z(\mathbf{SL}(n, \mathbb{C}))_{\mathbb{R}};$$

every matrix in $\mathbf{GL}(n, \mathbb{R})$ is a scalar multiple of a matrix in $\mathbf{SL}(n, \mathbb{C})$, so there is a natural inclusion $\mathbf{PGL}(n, \mathbb{R}) \hookrightarrow (\mathbf{L}_1)_{\mathbb{R}}$, and our observation in the preceding paragraph shows that this map is surjective.)

Observe that $\bar{\sigma}$ is unique. For if $\bar{\sigma}: M \to \mathbf{PGL}(n, \mathbb{R})$ is another map with the same property, then with $\varphi: M \to \mathbf{PGL}(n, \mathbb{R})$; $x \mapsto \bar{\sigma}'(x)^{-1}\bar{\sigma}(x)$, for every $\gamma \in \Gamma$, we have $\varphi(\rho(\gamma)x) = \bar{\pi}(\gamma)\varphi(x)\bar{\pi}(\gamma)^{-1}$ for almost every $x \in M$. Then the measure $\varphi_* \mu$ on $\mathbf{PGL}(n, \mathbb{R})$ is invariant under conjugation by $\bar{\pi}(\Gamma)$. But the only $\bar{\pi}(\Gamma)$ invariant probability measure on $\mathbf{PGL}(n, \mathbb{R})$ is the point mass at the identity. Although one can give an explicit, elementary proof in this special case, and the general result is probably well-known, we know of no convenient reference and therefore provide a proof which works in general.

LEMMA 2.2. *Suppose $\mathbf{G} \subset \mathbf{GL}(n, \mathbb{C})$ is a semisimple $\mathbb{R}$-group such that $G = \mathbf{G}_{\mathbb{R}}$ has no compact factors, and $\Gamma \subset G^0$ (connected component in the Hausdorff topology) is a lattice. Then the only probability measures on G which are invariant under conjugation by the elements of Γ are those supported on the center of G.*

Proof. Let μ be a Γ-invariant probability measure on G. By standard arguments (cf. e.g. [29, Section 3]), we may assume without loss of generality that μ is ergodic. Since the action of G on itself by conjugation is algebraic, every conjugacy class is locally closed by [3] (cf. [30, 3.1.3]). Then by [30, 2.1.11], any Γ-ergodic measure must be supported on a single conjugacy class. Thus there exists $g_0 \in G$ such that μ is supported on the conjugacy class $\{gg_0g^{-1} \mid g \in G\}$, which is isomorphic to the quotient variety $G/Z_G(g_0)$ with G acting by left translation. By a theorem of S. G. Dani [4 Corollary 2.6] which generalizes a theorem of C. C. Moore [19] the stabilizer

$$G_\mu = \{g \in G \mid \text{the } g\text{-action on } G/Z_G(g_0) \text{ preserves } \mu\}$$

is (equal to the $\mathbb{R}$-points of) an $\mathbb{R}$-algebraic group, and

$$J_\mu = \{g \in G \mid gx = x, \forall x \in \operatorname{supp} \mu\}$$

is a normal, co-compact subgroup in G_μ. Then by the Borel density theorem, $\Gamma \subset G_\mu$ implies $G_\mu = G$, and since G has no compact factors, $J_\mu \supset G^0$. In other words, each point in supp μ is centralized by G^0, hence supp $\mu \subset Z(G)$. □

The next step is to show that in the presence of an Anosov diffeomorphism, the section $\bar{\sigma}$, which is *a priori* only measurable, must in fact be continuous.

LEMMA 2.3. *Under the hypotheses of Theorem* 1.1 (a) *and the notation of the preceding section, the section $\bar{\sigma}: M \to \mathbf{PGL}(n, \mathbb{R})$ is continuous.*

Proof. Fix $\gamma \in \Gamma$ and decompose $\mathbb{R}^n$ into characteristic subspaces for the action of $\pi(\gamma) \in \mathbf{SL}(n, \mathbb{R})$:

$$\mathbb{R}^n = \oplus W_i, \quad W_i = \{w \neq 0 \in \mathbb{R}^n \mid \lim_{m \to \pm\infty} (1/m) \ln \| \pi(\gamma)^m \| / \|w\| \to \chi_i\} \cup \{0\};$$

$\chi_i \in \mathbb{R}$ are called the *characteristic exponents* for $\pi(\gamma)$. Define measurable distributions $\mathscr{W}_i$ on M, $\mathscr{W}_i(x) = \tau(x)\bar{\sigma}(x)W_i$. Let $\|\cdot\|$ denote the standard (Riemannian) fiber metrics on TM and PM.

LEMMA 2.4. *For μ-almost every $x \in M$,*

$$\mathscr{W}_i(x) - \{0\} = \{w \neq 0 \in T_xM \mid \lim_{m \to \pm\infty} (1/m) \ln \|D_x\rho(\gamma)^m w\| / \|w\| \to \chi_i\} \cup \{0\}.$$

Proof. Given $0 < \varepsilon < 1$, there exists a subset $\mathscr{S} \subset M$ such that $\mu(\mathscr{S}) > 1 - \varepsilon$ and both $\|\sigma(x)\|$ and $\|\sigma(x)^{-1}\|$ are uniformly bounded for $x \in \mathscr{S}$. Then for almost every $x \in \mathscr{S}$, $\rho(\gamma)^{\pm m} x \in \mathscr{S}$ for infinitely many $m \in \mathbb{N}^+$. Thus, for $w \in W_i - \{0\}$ and almost every $x \in \mathscr{S}$, there is a subsequence of m's for which $\lim_{m \to \pm\infty} (1/m) \ln \|D_x\rho(\gamma)^m w(x)\| / \|w(x)\| \to \chi_i$, where $w(x) = \sigma(x)w$.

On the other hand, since the derivative cocycle for $\rho(\gamma)$,

$$\mathbb{Z} \times M \to \mathbf{GL}(n, \mathbb{R}); \quad (m, x) \mapsto \xi(\gamma^m, x) = \tau(\rho(\gamma^m)x)^{-1} D_x\rho(\gamma^m)\tau(x)$$

is clearly integrable, Oseledec's multiplicative ergodic theorem [22] (cf. also [18, Section V.2]) implies that the limits exist for almost every $x \in M$. Thus the assertion holds for almost every $x \in \mathscr{S}$, and Lemma 2.4 follows by taking $\varepsilon \to 0$. □

Recall that there is a $\gamma_0 \in \Gamma$ such that $\rho(\gamma_0)$ is Anosov. Lemma 2.4 implies that $\pi(\gamma_0)$ is hyperbolic (i.e. has no eigenvalues on the unit circle) and that with $W^+ = \bigcup_{|\chi_i| > 1} W_i$ and $W^- = \bigcup_{|\chi_i| < 1} W_i$, where W_i, χ_i denote the characteristic subspaces and exponents for $\pi(\gamma_0)$, $\mathscr{W}^+(x) = \sigma(x)W^+$ ($\mathscr{W}^-(x) = \sigma(x)W^-$) is equal to the unstable (stable) subspace in T_x for $\rho(\gamma_0)$ for almost every $x \in M$. In particular, the *a priori* only measurable distributions $\mathscr{W}^\pm$ are in fact continuous.

We need to recall part of the discussion from [9, Section 3]. Let $\mathbf{G}_k(n, \mathbb{R})$ denote the Grassman variety of k-planes in $\mathbb{R}^n$, $1 \leqslant k \leqslant n - 1$. For each point $p_0 \in \mathbf{G}_k(n, \mathbb{R})$, the map

$$\mathbf{PGL}(n, \mathbb{R}) \to \mathbf{G}_k(n, \mathbb{R}); \quad \bar{g} \mapsto \bar{g}p_0$$

is smooth. The map

$$\mathbf{PGL}(n, \mathbb{R}) \to \mathbf{G}_k(n, \mathbb{R})^\ell = \underbrace{\mathbf{G}_k(n, \mathbb{R}) \times \cdots \times \mathbf{G}_k(n, \mathbb{R})}_{\ell \text{ times}}; \quad \bar{g} \mapsto (\bar{g}p_1, \ldots, \bar{g}p_\ell)$$

is a local embedding in an open neighborhood of $\bar{g}_0$ (via the inverse function theorem) if and only if the intersection of the infinitesimal stabilizers $\bigcap_{i=1,\ldots,\ell} \mathfrak{pgl}(n, \mathbb{R})_{\bar{g}_0^{-1}p_i}$ is zero. In particular, this condition is satisfied provided that the intersection of the stabilizers $\bigcap_{i=1,\ldots,\ell} \mathbf{PGL}(n, \mathbb{C})_{\bar{g}_0^{-1}p_i}$ is trivial.

With γ_0, $W^\pm$, and $\mathscr{W}^\pm$ as above, set $k = \dim W^-$ and $p_1 = W^- \in \mathbf{G}_k(n, \mathbb{R})$. By the Borel density theorem, the intersection

$$\bigcap_{\gamma \in \Gamma} \mathbf{PGL}(n, \mathbb{C})_{\pi(\gamma)p_1} = \bigcap_{\gamma \in \Gamma} \pi(\gamma)(\mathbf{PGL}(n, \mathbb{C})_{p_1})\pi(\gamma)^{-1}$$

is a (proper) normal subgroup of $\mathbf{PGL}(n, \mathbb{C})$, hence is trivial. Thus we can choose matrices

$\gamma_1 = I, \gamma_2, \dots, \gamma_\ell \in \Gamma$ such that with $p_i = \pi(\gamma_i)p_1$, which is the stable subspace for $\pi(\gamma_i\gamma_0\gamma_i^{-1})$, the stabilizer of $p = (p_1, \dots, p_\ell) \in \mathbf{G}_k(n, \mathbb{R})^\ell$ in $\mathbf{PSL}(n, \mathbb{C})$ is trivial. Set U equal to the orbit of p under $\mathbf{PGL}(n, \mathbb{R})$, $U = \mathbf{PGL}(n, \mathbb{R})p \subset \mathbf{G}_k(n, \mathbb{R})^\ell$. Note that by the preceding paragraph, for every $\bar{g}_0 \in \mathbf{PGL}(n, \mathbb{R})$, the map $\bar{g} \mapsto \bar{g}q$ is a local embedding on a neighborhood of $\bar{g}_0$ in $\mathbf{PGL}(n, \mathbb{R})$, where $q = (q_1, \dots, q_\ell) = (\bar{g}_0 p_1, \dots, \bar{g}_0 p_l) = \bar{g}_0 p \in U$.

For $1 \leqslant i \leqslant \ell$, let $q_i: M \to \mathbf{G}_k(n, \mathbb{R})$ denote the map corresponding to the stable distribution for the Anosov diffeomorphism $\rho(\gamma_i\gamma_0\gamma_i^{-1})$ under the identification of TM with $M \times \mathbb{R}^n$ via the standard trivialization. By Lemma 2.4, $q_i = \bar{\sigma} \cdot p_i$. (For example, with the above notation, $q_1(x) = \tau(x)^{-1}\mathscr{W}^-(x)$ for almost every $x \in M$.) Define $q: M \to \mathbf{G}_k(n, \mathbb{R})^\ell$, $q = (q_1, \dots, q_\ell)$. Since each of the maps q_i is continuous (in the Hausdorff topology on $\mathbf{G}_k(n, \mathbb{R})$), it will follow that $\bar{\sigma}: M \to \mathbf{PGL}(n, \mathbb{R})$ coincides with the continuous map which is uniquely determined by $\bar{\sigma}(x)p = q(x)$, provided that this makes sense, i.e. provided that $q(x) \in U$ for every $x \in M$.

Set $S = q^{-1}(U) \subset M$. Since $q = \bar{\sigma}p$, $q(x) \in U$ for almost every $x \in M$. In particular, S is dense in M. By Lemma 2.4, we can fix a representative for the Borel map $\bar{\sigma}$ and delete a subset of measure zero from S to obtain a set $S' \subset S$ such that S' is dense in M and $q(x) = \bar{\sigma}(x)$ and $\bar{\sigma}(x) W^\pm = \tau(x)^{-1}\mathscr{W}^\pm(x)$ for every $x \in S'$.

Now we make use of an elegant trick due to Furstenburg [6]. Suppose that $x_0 \in M - S$ and fix a sequence $x_m \in S'$ with $x_m \to x_0$. As above, for $g \in \mathbf{GL}(n, \mathbb{R})$, we write $\bar{g}$ for its image in $\mathbf{PGL}(n, \mathbb{R})$. Since we are free to multiply by non-zero scalars, we can choose $g_m \in \mathbf{GL}(n, \mathbb{R})$ so that $\|g_m\|$ is bounded and $q(x_m) = \bar{g}_m p$. Then passing to a subsequence, we may assume that $g_m \to g_0 \in \mathbb{R}^{n \times n}$, an $n \times n$ matrix. If g_0 were in $\mathbf{GL}(n, \mathbb{R})$, we would have $q(x_0) = \bar{g}_0 p$ with $\bar{g}_0 \in \mathbf{PGL}(n, \mathbb{R})$, contradicting $x_0 \notin S$. Thus $g_0 \notin \mathbf{GL}(n, \mathbb{R})$, and in particular, $K = \ker(g_0)$ and $V = g_0(\mathbb{R}^n)$ are proper, non-zero subspaces in $\mathbb{R}^n$.

For any subspace $W \subset \mathbb{R}^n$, the set of $g \in \mathbf{SL}(n, \mathbb{C})$ for which $\pi(g)W$ and K are in general position is a non-empty Zariski open set, so by the Borel density theorem, there exists $\gamma \in \Gamma$ such that $\pi(\gamma)W^\pm$ and K are in general position. In other words, replacing γ_0 by $\gamma\gamma_0\gamma^{-1}$, we may assume that $W^\pm$ and K are in general position, i.e.

$$\dim(W^\pm \cap K) = \sup\{\dim W^\pm + \dim K - n, 0\}.$$

Now suppose that W is any subspace of $\mathbb{R}^n$ which is in general position with respect to K and that the sequence $g_m W$ converges in the appropriate Grassman variety, say $g_m W \to W_0$. Then either $\dim W + \dim K \leqslant n$, in which case $W \cap K = (0)$ and $W_0 = \lim g_m W = g_0 W \subset V$, or $\dim W + \dim K > n$, in which case $W + K = \mathbb{R}^n$ and $W_0 = \lim g_m W \supset g_0(\mathbb{R}^n) = V$.

In particular, by compactness of the Grassman varieties, we may again pass to a subsequence and so assume that the sequences $g_m W^\pm$ converge, say $g_m W^\pm \to W_0^\pm$, and there are three possibilities:

(1) $W_0^+, W_0^- \subset V$,
(2) $W_0^+, W_0^- \supset V$, or
(3) $W_0^\pm \subset V \subset W_0^\mp$.

In any case, $W_0^+ \cap W_0^- \neq (0)$. On the other hand, $\bar{g}_m W^\pm = \bar{\sigma}(x_m) W^\pm = \tau(x_m)^{-1}\mathscr{W}^\pm(x_m)$ by construction, hence $\tau(x_0)^{-1}\mathscr{W}^\pm(x_0) = W_0^\pm$ by continuity. In other words, the isomorphism $\tau(x_0): \mathbb{R}^n \to T_{x_0}M$ identifies $W_0^\pm$ with the stable and unstable subspaces for the Anosov diffeomorphism $\rho(\gamma_0)$, which are transversal. This contradiction establishes $S = M$, and completes the proof of Lemma 2.3. □

One immediate consequence of Lemma 2.3 is that the diffeomorphism $\rho(\gamma)$ is Anosov whenever the matrix $\pi(\gamma)$ (equivalently, γ itself) is hyperbolic. Thus we are in a position to apply Theorem 4.12 in [8] to conclude that the action of a suitable abelian subgroup $\mathscr{A}$ in Γ is linear. (In fact, since $\bar{\sigma}$ also determines the periodic data for the action of $\mathscr{A}$, we might just as well apply the regularity theorem in [7].) We summarize this portion of the argument as follows.

LEMMA 2.5. *There exists a free abelian subgroup $\mathscr{A} \subset \Gamma$ of rank $n-1$ (more precisely, $\mathscr{A}$ is a co-compact subgroup in a maximal split Cartan subgroup A of $\mathbf{SL}(n, \mathbb{R})$) acting by Anosov diffeomorphisms and a C^∞ diffeomorphism $h: M \to M$, homotopic to the identity, such that $\rho(\gamma) = h\rho_*(\gamma)h^{-1}$ for every $\gamma \in \mathscr{A}$, where $\rho_*: \mathscr{A} \to \mathbf{SL}(n, \mathbb{Z})$ denotes the map induced by the action on $H_1(M) \simeq \mathbb{Z}^n$.*

Define a smooth section $\tau': M \to PM$, $\tau'(x) = (D_{h^{-1}x}h)\tau(h^{-1}x)$, and set σ' equal to the corresponding map $M \to \mathbf{GL}(n, \mathbb{R})$, $\sigma'(x) = \tau(x)^{-1}\tau'(x)$. Then

$$\xi(\gamma, x)\sigma'(x) = \sigma'(\rho(\gamma)x)\rho_*(\gamma) \quad \text{for every } \gamma \in \mathscr{A} \text{ and } x \in M$$

where, as above, ξ denotes the derivative cocycle for the action ρ with respect to the standard framing τ. As usual, we write $\bar{\sigma}': M \to \mathbf{PGL}(n, \mathbb{R})$ for the map induced by σ'.

Define a continuous map $\psi: M \to \mathbf{PGL}(n, \mathbb{R})$, $\psi(x) = \bar{\sigma}'(x)^{-1}\bar{\sigma}(x)$, and observe that

$$\bar{\rho}_*(\gamma)\psi(x) = \psi(\rho(\gamma)x)\bar{\pi}(\gamma) \quad \text{for every } \gamma \in \mathscr{A} \text{ and } x \in M.$$

Since ψ is bounded, this relation implies that ψ takes values in the set of intertwining automorphisms for the representations $\bar{\rho}_* \mid \mathscr{A}$ and $\bar{\pi} \mid \mathscr{A}$, i.e.

$$\bar{\rho}_*(\gamma)\psi(x) = \psi(x)\bar{\pi}(\gamma) \quad \text{for every } \gamma \in \mathscr{A} \text{ and } x \in M.$$

Thus ψ is constant along $\mathscr{A}$ orbits. But since the restriction $\rho \mid \mathscr{A}$ is conjugate to the linear action $\rho_* \mid \mathscr{A}$, we know in particular that $\rho \mid \mathscr{A}$ is topologically transitive, i.e. some orbit is dense. Thus ψ is constant, and we conclude that $\bar{\sigma}' = \bar{\sigma} \cdot \bar{g}$, where $\bar{g} \in \mathbf{PGL}(n, \mathbb{R})$ intertwines $\bar{\rho}_* \mid \mathscr{A}$ and $\bar{\pi} \mid \mathscr{A}$.

Unraveling the notation, we see that

$$D_x(h^{-1}\rho(\gamma)h)\tau(x) = \pm\,\tau(h^{-1}\rho(\gamma)hx)\rho_*(\gamma) \quad \text{for every } \gamma \in \Gamma \text{ and } x \in M.$$

In other words, $h^{-1}\rho(\gamma)h$ is an affine transformation of the torus, with linear part $\pm\,\rho_*(\gamma)$. Since the actions on H_1 coincide, the sign is always positive.

Now set $\alpha(\gamma) = h^{-1}\rho(\gamma)h(0)$, so that $\rho(\gamma) = h(\rho_*(\gamma) + \alpha(\gamma))h^{-1}$; $\alpha: \Gamma \to \mathbb{T}^n$ is a 1-cocycle, $\alpha \in Z^1(\Gamma, \mathbb{T}^n)$, where Γ acts on the abelian group $\mathbb{T}^n$ via ρ_*. All that remains to complete the proof of Theorem 1.1 is to show that the cocycle α takes values in $\mathbb{Q}^n/\mathbb{Z}^n$. To be more precise, we need to show that α is equivalent to a cocycle with values in $\mathbb{Q}^n/\mathbb{Z}^n$. (We can vary α within its cohomology class by adjusting our choice of origin, i.e. by pre-composing h with a translation.) This is equivalent to the purely geometric assertion that $\rho(\Gamma)$ has a finite orbit.

Recall that $H^1(\Gamma, \mathbb{R}^n) = 0$ (this is established for a general linear representation of Γ in [18]). Thus the short exact sequence of coefficient modules $\mathbb{Z}^n \to \mathbb{R}^n \to \mathbb{T}^n$ gives a long exact sequence

$$0 \to H^1(\Gamma, \mathbb{T}^n) \to H^2(\Gamma, \mathbb{Z}^n) \to H^2(\Gamma, \mathbb{R}^n).$$

Now by the Künneth formula,

$$H^2(\Gamma, \mathbb{R}^n) = H^2(\Gamma, \mathbb{Z}^n \otimes \mathbb{R}) \simeq H^2(\Gamma, \mathbb{Z}^n) \otimes \mathbb{R}$$

so that the kernel of the natural map $H^2(\Gamma, \mathbb{Z}^n) \to H^2(\Gamma, \mathbb{R}^n)$ is precisely the torsion subgroup of $H^2(\Gamma, \mathbb{Z}^n)$.† Combining the two observations, we see that every element of $H^1(\Gamma, \mathbb{T}^n)$ is torsion. In other words, there exist $m \in \mathbb{N}^+$ and $x_0 \in \mathbb{T}^n$ such that

$$m \cdot \alpha(\gamma) = \gamma x_0 - x_0 \quad \text{for every } \gamma \in \Gamma.$$

Fix $x_1 \in \mathbb{T}^n$ such that $m \cdot x_1 = x_0$, and set $h'(x) = h(x - x_1)$, $\alpha'(\gamma) = (h')^{-1}\rho(\gamma)h'(0)$. Then $m \cdot \alpha' \equiv 0$, i.e. α' takes values in the m-division points in $\mathbb{T}^n$. This completes the proof of Theorem 1.1. □

In order obtain Corollary 1.2 from Theorem 1.1, we need the following lemma, which first appeared in [26].

LEMMA 2.6. *Suppose Γ is a discrete Kazhdan group (i.e. Γ satisfies Kazhdan's "Property T"), M is a compact manifold, and $\rho_0 \in R(\Gamma, \mathrm{Diff}^1(M))$ is an action of Γ on M by C^1 diffeomorphisms. Then if ρ_0 preserves an absolutely continuous probability measure μ on M, there is a neighborhood U of ρ_0 in $R(\Gamma, \mathrm{Diff}^1(M))$ such that each $\rho \in U$ preserves an absolutely continuous measure μ_ρ.*

Proof. Let $\delta \in L^1(M)$ denote the invariant density, so that for every $\gamma \in \Gamma$,

$$\delta(\rho(\gamma)x) = |D_x\rho(\gamma)|^{-1}\delta(x)$$

for almost every $x \in M$. In other words, $\delta^{1/2} \in L^2$ is a non-trivial invariant vector under the unitary representation

$$\pi_0 : \Gamma \times L^2 \to L^2; \quad (\pi_0(\gamma)\varphi)(x) = |D_{\rho_0(\gamma)^{-1}x}\rho_0(\gamma)|^{-1/2}\varphi(\rho_0(\gamma)^{-1}x).$$

One formulation of Property T for discrete groups is as follows. Γ is finitely generated, and corresponding to any fixed finite generating set $\{\gamma_1, \ldots, \gamma_m\}$ for Γ, there exists $\varepsilon > 0$ with the following property: if $(\pi, \mathscr{H})$ is any unitary representation of Γ and there exists a unit vector $v \in \mathscr{H}$ with $\|\pi(\gamma_i)v - v\| < \varepsilon$ for $1 \leqslant i \leqslant m$, then π actually has non-trivial invariant vectors. (A unit vector v satisfying the preceding condition is said to be *ε-invariant* with respect to the γ_i.)

It is obvious from the definitions that there exists a neighborhood $U \subset R(\Gamma, \mathrm{Diff}^1(M))$ such that for each $\rho \in U$, the vector $\delta^{1/2} \in L^2$ is ε-invariant under the unitary representation π corresponding to ρ. Thus π fixes some unit vector $\varphi_\rho \in L^2(M)$, and the probability density $|\varphi_\rho|^2$ is invariant under ρ. □

Thus Theorem 1.1 implies that there is a neighborhood $U \subset R(\Gamma, \mathrm{Diff}(M))$ (recall that we consider the group $\mathrm{Diff}(M)$ of C^∞ diffeomorphisms with the C^1 topology) of the standard action (by orientation-preserving automorphisms of $M = \mathbb{T}^n$) such that every $\rho \in U$ is smoothly conjugate to a (rational) affine action with standard linear part. As we have already observed, $H^1(\Gamma, \mathbb{R}^n) = 0$. Thus, by a theorem due to Stowe [27], the set of

†The second author is grateful to M. Raghunathan for pointing this out.

$\rho \in R(\Gamma, \mathrm{Diff}(M))$ with a fixed point (near the origin) also contains a neighborhood of the standard action. Corollary 1.2 follows.

3. CONCLUDING REMARKS

Zimmer outlined a general program directed at understanding smooth actions of lattices in semisimple Lie groups of $\mathbb{R}$-rank $\geqslant 2$ on compact manifolds in his 1986 address to the International Congress of Mathematicians [28]. (He had previously conjectured that the action of $\mathbf{SL}(n, \mathbb{Z})$ on $\mathbb{T}^n$, $n \geqslant 3$, was locally rigid during the conference on ergodic theory, differential geometry, and Lie groups in May, 1984, at the Mathematical Sciences Research Institute.) Lewis established an infinitesimal rigidity result for subgroups of finite index in $\mathbf{SL}(n, \mathbb{Z})$ acting on $\mathbb{T}^n$ for $n \geqslant 7$ in [12]. Hurder obtained rigidity under *continuous deformations* for $n \geqslant 3$ in [7]. His key contribution was the use of the theorem by Stowe [27]. A further developement in that direction which goes beyond actions containing Anosov elements is due to Qian [24].

The first local rigidity result is due to Katok–Lewis for the standard $\mathbf{SL}(n, \mathbb{Z})$ action on $\mathbb{T}^n$, $n \geqslant 4$ in [8]; subsequently they extended the technique to obtain global results (again, for $n \geqslant 4$) in [9]. Their method is based on the recovery of an invariant rational structure using a combination of hyperbolicity and Stowe's theorem.

We recall that the main result (Corollary 2.14) in [9] yields the same conclusion as Theorem 1.1, above, but under the alternative hypotheses that ρ has a finite orbit and with some special restrictions on the element γ_0 such that $\rho(\gamma_0)$ is Anosov. (The hyperbolic matrix γ_0 is required to preserve a non-trivial rational product structure on $\mathbb{Q}^n$, which is equivalent to the requirement that $\rho(\gamma_0)$ preserve a non-trivial decomposition of some finite cover of $\mathbb{T}^n$ as a product of compact subtori. It is this condition that leads to the restriction $n \geqslant 4$.) Note that the two results are parallel, in that the existence of a finite orbit is equivalent to the existence of an invariant atomic measure.

On the other hand, as we indicated at the outset, the cocycle superrigidity theorem is applicable in much greater generality than we have discussed above, and we believe that it can be used to overcome some essential limitations of the techniques in [7–9] which are based on existence of an invariant rational structure. In particular, this approach is applicable in the absence of finite orbits. Very briefly, we will try to describe the main obstacles to extending our technique to more general actions (e.g. to obtain local rigidity for other non-isometric algebraic examples) as well as some approaches to overcome those obstacles.

First, there is the general problem of determining the algebraic hull of the derivative cocycle. It is not at all clear how to use the superrigidity theorem to obtain useful geometric information about the action unless this group is semisimple. In the case that the lattice Γ is cocompact, Zimmer shows in [31] that the algebraic hull is reductive with compact center. We might hope that this is generally the case, but this has not, as yet, been established. The theorem provides definitive information analogous to that summarized in Proposition 2.1, (at least directly) only in very special cases; essentially when the dimension n of the compact manifold M coincides with the minimum n for which there exist representations $\Gamma \to \mathbf{SL}(n, \mathbb{R})$ whose image has non-compact closure. R. Feres succeeded in extending this approach under somewhat more general hypotheses, cf. [5].

Secondly, the requirement that there exists $\gamma_0 \in \Gamma$ such that $\rho(\gamma_0)$ is Anosov only makes sense in special cases, and must be systematically replaced by more general hypotheses. A first successful step in the partially hyperbolic case has been recently made by Nitica and

Török [20, 21] who developed powerful new analytical techniques allowing to classify cocycles with values in the diffeomorphism groups. Note that the examples described in [9, Section 4] make it clear that, at least in the most general setting, we cannot hope to eliminate the dynamical hypotheses entirely.

Finally, the argument (based on Lemma 2.5) for passing from a continuous linearizing frame to a smooth conjugacy needs to be generalized. A relatively straightforward extension to the class of "Cartan actions" has been accomplished in the recent work of Qian [25]. More generally, rigidity of hyperbolic actions of higher-rank abelian groups [10, 11] should be brought into the play.

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REFERENCES

1. D. V. Anosov: Geodesic flows on closed Riemannian manifolds with negative curvature, *Proc. Steklov Inst. Math.*, No. 90, (1967) (Russian); English transl., A.M.S., Providence, RI (1969).
2. D. Berend: Multi-invariant sets on tori, *Trans. AMS* **280** (1983), 509–532.
3. A. Borel and J.-P. Serre: Théorèmes de finitude en cohomologie galoisienne, *Comm. Math. Helv.* **39** (1964), 111–164.
4. S. G. Dani: On ergodic quasi-invariant measures of group automorhisms, *Israel J. Math.* **43** (1982), 62–74.
5. R. Feres: Connection-preserving actions of lattices in SL(n, $\mathbb{R}$), *Israel J. Math.* (to appear).
6. H. Furstenberg: A Poisson formula for semisimple Lie groups, *Ann. Math.* **77** (1963), 335–383.
7. S. Hurder: Rigidity for Anosov actions of higher rank lattices, *Ann. Math.* **135** (1992), 361–410.
8. A. Katok and J. Lewis: Local rigidity for certain groups of toral automorphisms, *Israel J. Math.* **75** (1991), 203–241.
9. A. Katok and J. Lewis: Global rigidity for lattice actions on tori and new examples of volume-preserving actions, *Israel J. Math.* (to appear).
10. A. Katok and R. Spatzier: First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity, *Publ. Math. I.H.E.S.* **79** (1994), 131–156.
11. A. Katok and R. Spatzier: Subelliptic estimates of polynomial differential operators and applications to rigidity of abelian actions, *Math. Res. Lett.* **1** (1994), 193–202.
12. J. Lewis: Infinitesimal rigidity for the action of SL(n, $\mathbb{Z}$) on $\mathbb{T}^n$, *Trans. AMS* **324** (1991), 421–445.
13. A. N. Livshitz: Homology properties of Y-systems, *Math. Zametki* **10** (1971), 555–564 (Russian); English transl., *Math. Notes* **10** (1971), 758–763.
14. A. N. Livshitz and Ja. G. Sinai: Invariant measures that are compatible with smoothness for transitive C-systems, *Soviet Math. Dokl.* **13** (1972), 1656–1659.
15. R. De La Llave, J. M. Marco and R. Moriyon: Canonical perturbation theory of Anosov systems and regularity results for the Livshitz cohomology equation, *Ann. Math.* **123** (1986), 537–611.
16. R. Mañé: *Ergodic theory and differentiable dynamics*, Springer, Berlin (1987).
17. A. Manning: There are no new Anosov diffeomorphisms on tori, *Amer. J. Math.* **96** (1974), 422–429.
18. G. A. Margulis: *Discrete subgroups of semisimple Lie groups*, Springer, Berlin (1991).
19. C. C. Moore: Amenable subgroups of semisimple groups and proximal flows, *Israel J. Math.* **34** (1979), 121–138.
20. V. Nitica and A. Török: Cohomology of dynamical systems and rigidity of partially hyperbolic actions of higher rank lattices, *Duke Math. J.* (to appear).
21. V. Nitica and A. Török: Regularity results for solutions of livsic cohomology equation with values in diffeomorphism groups, *Ergod. Th. & Dynam. Syst.* (to appear).
22. V. I. Oseledec: A multiplicative ergodic theorem, *Trudy Moskov. Mat. Obsc.* **19** (1968), 179–210 (Russian); English transl., *Trans. AMS* **19** (1968), 197–231.
23. J. Palis and J. C. Yoccoz: Centralizers of Anosov diffeomorphisms on tori, *Ann. Sci. Ec. Norm. Sup.* **22** (1989), 99–108.
24. N. Qian: Topological deformation rigidity of higher rank lattice actions, *Math. Res. Lett.* **1** (1994), 485–499.
25. N. Qian: Tangential flatness and global rigidity of higher rank lattice actions, preprint.
26. G. Seydoux: University of Chicago Doctoral Dissertation (1991).
27. D. Stowe: The stationary set of a group action, *Proc. AMS* **79** (1980), 139–146.
28. R. Zimmer: Actions of semisimple groups and discrete subgroups, *Proc. Int. Congr. of Mathematicians*, (1986: Berkeley, CA), A.M.S., Providence, RI (1987) pp. 1247–1258.

29. R. ZIMMER: Ergodic theory and the automorphism group of a G-structure, Group representations, in *Ergodic theory, operator algebras, and mathematical physics*, C. C. Moore, Ed., Springer, New York (1987) pp. 247–278.
30. R. ZIMMER: *Ergodic theory and semisimple groups*, Birkhäuser, Boston (1984).
31. R. ZIMMER: On the algebraic hull of an automorphism group of a principal bundle, *Comment. Math. Helv.* **65** (1990), 375–387.

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