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ORIGINAL PAPER

Rigidity of higher rank abelian cocycles with values in diffeomorphism groups

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Abstract We consider cocycles over certain hyperbolic $\mathbb{R}^k$ actions, $k \geq 2$, and show rigidity properties for cocycles with values in a Lie group or a diffeomorphism group, which are close to identity on a set of generators, and are sufficiently smooth. The actions we consider are Cartan actions of $SL(n,\mathbb{R})/\Gamma$ or $SL(n,\mathbb{C})/\Gamma$, for $n \geq 3$, and Γ torsion free cocompact lattice. The results in this paper rely on a technique developed recently by D. Damjanović and A. Katok.

Keywords Rigidity · Cocycle · Cohomological equation · Higher-rank abelian actions · Diffeomorphism groups · Partially hyperbolic diffeomorphism · Cartan actions

Mathematics Subject Classifications 37D40, 37C85

1 Introduction

This paper is part of a research program that aims to classify Hölder and smooth cocycles over higher rank abelian hyperbolic actions of $\mathbb{Z}^k$ or $\mathbb{R}^k$, $k \geq 2$, up to cohomology. The results obtained so far show that such cocycles exhibit strong rigidity properties, that is, they are cohomologous to constant cocycles, or the cohomology classes are finite in number and easy to describe. We present here a quick overview of related known results.

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1.1 The harmonic analysis method

The first rigidity result for cocycles over higher rank abelian actions was proved by Katok and Spatzier [13] where it was shown that real valued smooth/Hölder cocycles over a large class of natural hyperbolic $\mathbb{Z}^k$ or $\mathbb{R}^k$, $k \geq 2$, actions are smoothly/Hölder cohomologous to constants.[1] Those results cover in particular actions by hyperbolic automorphisms of a torus, and Weyl chamber flows, e.g. Cartan actions of $SL(n, \mathbb{R})/\Gamma$ and $SL(n, \mathbb{C})/\Gamma$, for $n \geq 3$, and Γ torsion free cocompact lattice. The proofs rely mostly on harmonic analysis techniques, such as Fourier transform and group representations for semisimple Lie groups. These techniques are difficult to apply to cocycles with non-abelian range.

1.2 Geometric method for TNS actions

A geometric method for proving cocycle rigidity was developed in [12]. The basic idea is to construct an invariant differentiable form using the invariant structures that exist on the stable/unstable foliations, and the commutativity of the action. The solution is first constructed as a differential form which is then shown to be closed and exact. The class of actions for which this method works is more restrictive than in [13]. Those are abelian higher rank actions on infranilmanifolds that exhibit a rich structure on the family of stable/unstable foliations, so called totally non-symplectic (TNS) actions. This method was applied successfully in [12] not only to real valued cocycles, but also to small (i.e. sufficiently close to identity on a set of generators) cocycles with values in general Lie groups. A related paper is [24] which contains rigidity results for cocycles with values in compact Lie groups. The results show that for (TNS) actions the cohomology classes are finite in number and easy to describe.

Note that there are difficulties in extending the results for cocycles with values in Lie groups from [12] for (TNS) actions to cocycles with values in diffeomorphism groups. Those difficulties are partially overcome in [23] where the cohomological equation is solved under additional assumptions, like the existence of a fixed center fiber for the extended higher rank abelian action. That paper also presents applications of the cohomology theory of cocycles over abelian actions with values in diffeomorphisms. These results are used to prove local rigidity of certain partially hyperbolic higher rank actions on compact manifolds.

1.3 Geometric methods for Weyl chamber flows

An extension of the method [12] to real valued cocycles over chamber Weyl flows can be found in the unpublished paper [Ferleger, S., Katok, A.: Unpublished] which contains in a rudimentary form an important new idea of using commutation relations between stable and unstable directions of various elements of the action to show exactness of the differential form solution.

More recently, Damjanović and Katok developed a different geometric method that can be applied not only to Weyl chamber flows but also to their restrictions to higher rank subgroups in general position (those actions are only partially hyperbolic) [5]. This work uses the approach from [11] for finding cohomology invariants for cocycles over partially hyperbolic actions that satisfy accessibility property. Accessibility

[1] For a correction of an error in the original proof for the Hölder case see errata posted at www.math.psu.edu/katok_a.

roughly means that one can connect any two points belonging to the manifold supporting the partially hyperbolic dynamical system by transverse piecewise smooth pathes included in stable/unstable leaves. This notion was introduced by Brin and Pesin [3] and it is playing a crucial role in the recent surge of activity in the field of partially hyperbolic diffeomorphisms. See [4,9] for two recent surveys. The cohomology invariants described in [11], called *periodic cycle functionals*, are heights of the cocycle over cycles constructed in the base out of pieces belonging to stable/unstable leaves. They provide a complete set of obstructions for solving the cohomology equation.

An essential new ingredient introduced in [5] is the use of algebraic K-theory [17]. For Cartan actions on $SL(n,\mathbb{R})/\Gamma$ or $SL(n,\mathbb{C})/\Gamma$ many cycles are generated by a finite number of types of elementary cycles which are given by commutator relations between elementary unipotent matrices and their conjugates. Each elementary cycle lies in the stable manifold for some partially hyperbolic element of the action, and consequently the functional over it vanishes. This fact is used to reduce the description of cohomology for cocycles to the classification of homomorphisms from the fundamental group $\pi_1(M)$ into the group that appear in the fiber. After that standard results in rigidity theory, like Margulis normal subgroup theorem, can be employed to find the cohomology.

1.4 Outline of the paper

One advantage of the method developed in [5] is that it can be extended to deal with cocycles with non-abelian range. In the present paper we extend the results from [5] to small cocycles with values in a Lie group or the diffeomorphism group of a compact manifold. There are two improvements compared to the latter paper, one technical and one more essential.

First, we construct the periodic cycle functionals and the solution of the cohomological equation when those obstructions vanish in the infinite-dimensional setup of the diffeomorphism groups which requires certain norm estimates. Those technical elements do not appear in the finite-dimensional case. Second, we show that in our situations the solution of the cohomological equation, which, for the non-abelian range, has been constructed in [5] only on the universal cover, descends to the factor. This is new even for the simple Lie group case.

For this second and more important improvement an interesting twist appears. Rather than having homomorphisms from abelian groups into diffeomorphisms groups, as in [23], we have homomorphisms from lattices in higher rank Lie groups into diffeomorphisms groups. These homomorphisms are close to identity on a compact set of generators. Consequently, in the proof of the main theorem we can use a recent result of Fisher–Margulis [7] about local rigidity of isometric actions of higher rank lattices on compact manifolds. Thus a local rigidity result about higher rank lattice actions is applied to prove a cohomological rigidity result about abelian actions.

The paper is organized as follows: in Sect. 2 we review basic notions about cocycles and metrics on diffeomorphisms groups and Lie groups, in Sect. 3 we construct an invariant structure along stable/unstable leaves of any partially hyperbolic element of the action, in Sect. 4 we review Katok–Kononenko periodic cycle functionals theory [11] in the setup of cocycles with non-abelian range, in Sect. 5 we review basic facts from the theory of partially hyperbolic difeomorphisms, in Sect. 6 we describe Cartan actions on $SL(n,\mathbb{R})/\Gamma$ and $SL(n,\mathbb{C})/\Gamma$, in Sect. 7 we review relevant notions from K-theory, in Sect. 8 we review Fisher–Margulis local rigidity result for isometric actions, and in Sect. 10 we present the main results and their proofs which also use two auxiliary statements from Sect. 9.

2 Cocycles and cohomology

Let M, N be smooth compact Riemannian manifolds endowed with metrics dist_M, dist_N. Let Homeo(N) be the set of homeomorphisms of N, endowed with the operation of composition. Let K be an integer or $K = \infty$ and $\mathrm{Diff}^K(N) \subset \mathrm{Homeo}(N)$ be the set of C^K-diffeomorphisms of N. We introduce the standard C^0 metric on Homeo(N):

$$\mathbf{d}_N(u, v) := \sup_{y \in N} \mathrm{dist}_N(u(y), v(y)), \quad u, v \in \mathrm{Homeo}(N),^{2} \tag{2.1}$$

For $0 < \theta \leq 1$ and $u \in \mathrm{Homeo}(N)$, define the θ-Hölder norm

$$\|u\|_\theta = \sup_{x,y \in N} \frac{\mathrm{dist}_N(u(x), u(y))}{\mathrm{dist}_N(x, y)^\theta}, \tag{2.2}$$

which may be finite or infinite. The 1-Hölder norm is called *Lipschitz norm*. Note that for any $u \in \mathrm{Diff}^K(N)$, the Lipschitz norm $\|u\|_1$ is finite and $\|u\|_1 \leq \|Du\|$, where Du is the derivative. For $u, v, w, z \in \mathrm{Homeo}(N)$ the metric $\mathbf{d}_N$ has the following properties:

$$\begin{aligned} \mathbf{d}_N(vu, wu) &= \mathbf{d}_N(v, w), \\ \mathbf{d}_N(uv, uw) &\leq \|u\|_1 \mathbf{d}_N(v, w). \end{aligned} \tag{2.3}$$

Using (2.3) we deduce:

$$\begin{aligned} \mathbf{d}_N(wz, uv) &\leq \mathbf{d}_N(wz, uz) + \mathbf{d}_N(uz, uv) \\ &\leq \mathbf{d}_N(w, u) + \|u\|_1 \mathbf{d}_N(z, v) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \mathbf{d}_N(u^{-1}, v^{-1}) &= \mathbf{d}_N(u^{-1}v, v^{-1}v) \\ &= \mathbf{d}_N(u^{-1}v, u^{-1}u) \leq \|u^{-1}\|_1 \mathbf{d}_N(u, v). \end{aligned} \tag{2.5}$$

If K is a finite integer, one can introduce on $\mathrm{Diff}^K(N)$ a structure of Banach manifold, and in particular a structure of complete metric space. We briefly review this standard construction. Let $n = \dim(N)$. If $\Omega \subset \mathbb{R}^n$ is a compact set, then a function $f\colon \Omega \to \mathbb{R}$ is C^K if it has continuous derivatives of order K. The space $C^K(\Omega)$ of all C^K functions on Ω, endowed with the norm defined as the supremum of the derivatives of order up to K, has a structure of Banach space. Using coordinate charts that cover the manifold, and taking maximum over the C^K norms of the coordinate expressions, one can define a norm on the space of C^K vector fields on N that makes it a Banach space. After that the exponential map in Riemannian geometry can be used to construct charts from $\mathrm{Diff}^K(N)$ into the set of vector fields. Note that it is standard to endow $\mathrm{Diff}^\infty(N)$ with a structure of Frechet manifold [7]. The C^∞ topology is defined as the inverse limit of the C^K topologies. We recall that two C^∞ diffeomorphisms are close if they are C^K close for some large K.

[2] Notice that Homeo(N) is not complete in that metric; see the proof of Proposition 3.3.

Let α: $\mathbb{R}^k \times M \to M$ be a smooth action of $\mathbb{R}^k$ on M. For simplicity, in what follows we denote $\alpha(a,x), a \in \mathbb{R}^k, x \in M$, by ax. Throughout the rest of this section H is a connected Lie group endowed with a right invariant metric $\mathbf{d}_H$.

Definition 2.1 A continuous map β: $\mathbb{R}^k \times M \to \mathrm{Diff}^K(N)$ (or β: $\mathbb{R}^k \times M \to H$) is called a *cocycle* over α if it satisfies:

$$\beta(a+b,x) = \beta(a,bx)\beta(b,x), \;\; a,b \in \mathbb{R}^k, x \in M. \tag{2.6}$$

Definition 2.2 (a) A cocycle β: $\mathbb{R}^k \times M \to \mathrm{Diff}^K(N)$ is said to be *cohomologous to a constant cocycle* if there exists a homomorphism π: $\mathbb{R}^k \to \mathrm{Diff}^K(N)$ and a continuous map h: $M \to \mathrm{Diff}^{K'}(N)$, $K' \geq 0$, $K' \leq K$, such that

$$\beta(a,x) = h(ax)\pi(a)h(x)^{-1}. \tag{2.7}$$

(b) A cocycle β: $\mathbb{R}^k \times M \to H$ is said to be *cohomologous to a constant cocycle* if there exists a homomorphism π: $\mathbb{R}^k \to H$ and a continuous map h: $M \to H$ such that (2.7) holds.

The map h is called *transfer map*.

Definition 2.3 Let α: $\mathbb{R}^k \times M \to M$ be an $\mathbb{R}^k$-action.

(a) Let $\beta: \mathbb{R}^k \times M \to \mathrm{Diff}^K(N)$ be a cocycle over α. The *extended action* $\widetilde{\alpha}$: $\mathbb{R}^k \times (M \times N) \to M \times N$ is defined by

$$\widetilde{\alpha}(a,x,y) = (\alpha(a,x), \beta(a,x)(y)),$$

(b) Let β: $\mathbb{R}^k \times M \to H$ be a cocycle over α. The *extended action* $\widetilde{\alpha}: \mathbb{R}^k \times (M \times H) \to M \times H$ is defined by

$$\widetilde{\alpha}(a,x,g) = (\alpha(a,x), \beta(a,x)g), \; a \in \mathbb{R}^k, x \in M, g \in H.$$

Any cocycle β: $\mathbb{R}^k \times M \to \mathrm{Diff}^K(N)$ can be viewed as a map β: $\mathbb{R}^k \times M \times N \to N$. In order to state our results, we need to assume certain regularity for cocycles. We assume throughout the paper that the regularity in $\mathbb{R}^k$ and N variables is C^K. For the Mth variable we need at least Hölder regularity. One way to obtain this is to require that the cocycle, viewed as a map β: $\mathbb{R}^k \times M \times N \to N$ (or β: $\mathbb{R}^k \times M \to H$), to be C^K, or C^∞. If this is the case, we will call the cocycle C^K*-cocycle*, respectively *smooth cocycle*. A broader class of cocycles which we will consider later is that of θ-Hölder cocycles.

Definition 2.4 Let $\theta \in (0,1]$. (a) The cocycle β: $\mathbb{R}^k \times M \to \mathrm{Diff}^K(N)$ is said to be θ*-Hölder* if there exists a $C_1 > 0$ constant and a compact set of generators $S \subset \mathbb{R}^k$ such that for any $a \in S$:

$$\mathbf{d}_N(\beta(a,x), \beta(a,y)) \leq C_1 \mathrm{dist}_M(x,y)^\theta, \quad x,y \in M. \tag{2.8}$$

(b) The cocycle β: $\mathbb{R}^k \times M \to H$ is said to be θ*-Hölder* if there exists $C_1 > 0$ a constant and a compact set of generators $S \subset \mathbb{R}^k$ such that for any $a \in S$:

$$\mathbf{d}_H(\beta(a,x), \beta(a,y)) \leq C_1 \mathrm{dist}_M(x,y)^\theta, \quad x,y \in M. \tag{2.9}$$

In both (a) and (b), the smallest value C_1 is denoted by $\|\beta\|_\theta$ and is called the θ*-norm of* β.

Any C^K-cocycle is θ-Hölder for any $0 < \theta \leq 1$, as immediately follows from the mean value theorem.

Definition 2.5 Let $0 < \lambda < 1$. (a) A θ-Hölder cocycle β: $\mathbb{R}^k \times M \to \mathrm{Diff}^K(N)$ is said to be λ-*center bunched* if there is a compact set of generators $S \subset \mathbb{R}^k$ such that:

$$\Lambda := \sup_{a \in S, x \in M} \|D_N \beta(a,x)^{\pm 1}\| \lambda^\theta < 1, \tag{2.10}$$

where D_N is the derivative in the N direction.

(b) A θ-Hölder cocycle β: $\mathbb{R}^k \times M \to H$ is said to be λ-*center bunched* if there is a compact set of generators $S \subset \mathbb{R}^k$ such that: $\Lambda := \sup_{a \in S, x \in M} \|Ad(\beta(a,x)^{\pm 1})\| \lambda^\theta < 1$, where $\mathfrak{h}$ is the Lie algebra of H, Ad: $H \to Aut(\mathfrak{h})$ the adjoint representation and $\|\cdot\|$ represents the operatorial norm on $Aut(\mathfrak{h})$ with respect to some fixed norm on $\mathfrak{h}$.

In both cases, a C^K-cocycle is said to be λ-*center bunched* if it satisfies (2.10) with $\theta = 1$.

If β is θ-Hölder and close to the identity, then (2.5), (2.8) and (2.10) imply that β^{-1} is θ-Hölder as well, and

$$\|\beta^{-1}\|_\theta \leq \lambda^{-\theta} \|\beta\|_\theta. \tag{2.11}$$

We assume in the future, without loss of generality, that the set S appearing in Definitions 2.4 and 2.5 is the unit cube in $\mathbb{R}^k$. We denote by $Q(\mathbb{R}^k)$ the unit cube in $\mathbb{R}^k$.

3 Invariant structure along a stable foliation

The main result in this section allows to construct an invariant structure under the extended action by a cocycle in the presence of a contracting invariant foliation for the action in the base. Several techniques used in this section appeared before in [12,21,22]. We present the proofs for cocycles with values in diffeomorphism groups, and refer to [21] for the proofs for cocycles with values in Lie groups.

Definition 3.1 Let M be a compact manifold, and α: $\mathbb{R}^k \times M \to M$ a smooth action on M. Let W be a continuous foliation of M with smooth leaves $W(x), x \in M$. The foliation W is called α-*invariant* if

$$\alpha(t, W(x)) \subset W(\alpha(t,x)), \ \ x \in M, \ \ t \in \mathbb{R}^k.$$

Definition 3.2 Let M be a compact manifold. If α: $\mathbb{R} \times M \to M$ is a smooth flow on M, an α-invariant foliation W is called *contracting* if there exist constants $C_2 > 0, 0 < \lambda < 1$, such that

$$\mathrm{dist}_{W(\alpha(t,x))}(\alpha(t,x), \alpha(t,y)) \leq C_2 \lambda^t \mathrm{dist}_{W(x)}(x,y), \ \ x,y \in M, t \geq 0. \tag{3.1}$$

An α-invariant foliation W is called *expanding* if there exist constants $C_2 > 0, 0 < \lambda < 1$, such that

$$\mathrm{dist}_{W(\alpha(-t,x))}(\alpha(-t,x), \alpha(-t,y)) \leq C_2 \lambda^t \mathrm{dist}_{W(x)}(x,y), \ \ x,y \in M, t \geq 0. \tag{3.2}$$

Remark The contracting/expanding foliations we actually use in this paper are stable, respectively unstable, foliations of a partially hyperbolic element of an $\mathbb{R}^k$ action and the flow is the restriction of that action to the one-parameter group generated by the element. Those foliations have the property that the distance between pairs of points in the same local leaf is equivalent to the distance between points on the manifold. This will allow to replace $\mathrm{dist}_{W(x)}(x,y)$ by $\mathrm{dist}_M(x,y)$ in some of the future arguments.

Proposition 3.3 *Let M, N be compact manifolds, $\alpha\colon \mathbb{R}\times M \to M$ a smooth action, and W an α-invariant contracting foliation of M with constants $C_2 > 0, 0 < \lambda < 1$. Let $\beta\colon \mathbb{R}\times M \to \mathit{Diff}^K(N)$ be a θ-Holder cocycle over α that is λ-center bunched. For $x \in M, t \geq 0$, define $\gamma_{x,t}\colon W(x) \to \mathit{Diff}^K(N) \subset \mathit{Homeo}(N)$ by*

$$\gamma_{x,t}(y) = \beta(t,y)^{-1}\beta(t,x). \tag{3.3}$$

Then the following statements hold:

(1) *The family of homeomorphisms $\{\gamma_{x,t}(y)\}_{t\geq 0}$ converges in $\mathit{Homeo}(N)$ as $t\to\infty$ to a homeomorphism:*

$$\gamma_x(y) := \lim_{t\to\infty} \gamma_{x,t}(y). \tag{3.4}$$

(2) *The map $\gamma_x\colon (W(x), \mathit{dist}_{W(x)}) \to (\mathit{Homeo}(N), \mathbf{d}_N)$ is uniformly θ-Holder.*
(3) $\gamma_x(x) = \mathit{Id}_N$.
(4) *The family of functions $\{\gamma_x\}_{x\in M}$ is invariant under extended action on $M\times N$, that is,*

$$\beta(t,y)\gamma_x(y) = \gamma_{tx}(ty)\beta(t,x), y \in W(x), t \geq 0. \tag{3.5}$$

(5) *If $y \in W(x)$ and $\mu > \lambda^\theta$ then*

$$\lim_{t\to\infty} \mu^{-t}\mathbf{d}_N(\beta(t,x), \beta(t,y)\gamma_x(y)) = 0. \tag{3.6}$$

(6) *The family of functions $\{\gamma_x\}_{x\in M}$ is uniquely determined by* (2), (3), *and* (4).

Proof We use a version of the well-known "telescopic argument" which is more familiar and straightforward when the range of the cocycle is abelian but also appears in non-abelian situations e.g. in [12,21,22]. Notice that λ-center bunching condition is crucial for this argument.

(1) A complete metric on Homeo(N) is given by

$$\max\{\mathbf{d}_N(u,v), \mathbf{d}_N(u^{-1},v^{-1})\}, u,v \in \mathrm{Homeo}(N).$$

Let $x\in M, y \in W(x)$. We show that the family $\{\gamma_{x,t}(y)\}_t$ is uniformly convergent in a complete metric on Homeo(N) as $t\to\infty$. Let $t' \geq t > 0$. To simplify the notation, during the proof we denote $\beta(1,x)$ by $\beta(x)$. We denote by $[\tau]$ the integer part of the real number τ, and by $\{\tau\}$ the fractional part of τ. We estimate only $d := \mathbf{d}_N(\gamma_{x,t'}(y), \gamma_{x,t}(y))$. The estimation for $\mathbf{d}_N(\gamma_{x,t'}^{-1}(y), \gamma_{x,t}^{-1}(y))$ is similar. Using (2.6) for any $z\in M$ we have:

$$\beta(t',z) = \beta(t'-t+t,z) = \beta(t'-t,tz)\beta(t,z). \tag{3.7}$$

By (3.7), applied for $z = x$, by (2.3), and again by (2.6), we have:

$$\begin{aligned} d &= \mathbf{d}_N(\beta(t',y)^{-1}\beta(t',x), \beta(t,y)^{-1}\beta(t,x)) \\ &= \mathbf{d}_N(\beta(t',y)^{-1}\beta(t'-t,tx)\beta(t,x), \\ &\qquad \beta(t,y)^{-1}\beta(t'-t,ty)^{-1}\beta(t'-t,ty)\beta(t,x)) \\ &= \mathbf{d}_N(\beta(t',y)^{-1}\beta(t'-t,tx), \beta(t',y)^{-1}\beta(t'-t,ty)). \end{aligned} \tag{3.8}$$

Using now repeatedly (2.6), (2.4) and triangle inequality for the metric $\mathbf{d}_N$, (3.8) becomes:

$$\begin{aligned}
d &= \mathbf{d}_N(\beta(t',y)^{-1}\beta((t'-1)x)\beta((t'-2)x)\cdots \\
&\qquad \beta((t'-[t'-t])x)\beta(\{t'-t\},tx), \\
&\qquad \beta(t',y)^{-1}\beta((t'-1)y)\beta((t'-2)y)\cdots\beta((t'-[t'-t])y)\beta(\{t'-t\},ty)) \\
&\le \mathbf{d}_N(\beta(t',y)^{-1}\beta((t'-1)x)\beta((t'-2)x)\cdots\beta((t'-[t'-t])x), \\
&\qquad \beta(t',y)^{-1}\beta((t'-1)y)\beta((t'-2)y)\cdots\beta((t'-[t'-t])y)) \\
&\qquad +\mathbf{d}_N(\beta(t',y)^{-1}\beta((t'-1)y)\beta((t'-2)y)\cdots\beta((t'-[t'-t])y)\beta(\{t'-t\},ty), \\
&\qquad \beta(t',y)^{-1}\beta((t'-1)y)\beta((t'-2)y)\cdots\beta((t'-[t'-t])y)\beta(\{t'-t\},tx)) \\
&\le \sum_{k=0}^{[t'-t]} \mathbf{d}_N(\beta(t',y)^{-1}\beta((t'-1)y)\cdots\beta((t'-k+1)y)\beta((t'-k)x), \\
&\qquad\quad \beta(t',y)^{-1}\beta((t'-1)y)\cdots\beta((t'-k+1)y)\beta((t'-k)y)) \\
&\qquad\quad +\mathbf{d}_N(\beta(t,y)^{-1}\beta(\{t'-t\},ty)^{-1}\beta(\{t'-t\},ty), \\
&\qquad\quad \beta(t,y)^{-1}\beta(\{t'-t\},ty)^{-1}\beta(\{t'-t\},tx)).
\end{aligned} \tag{3.9}$$

Note that by (2.6) we have

$$\beta(t',y)^{-1}\beta((t'-1)y)\cdots\beta((t'-k+1)y) = \beta(t'-k+1,y)^{-1}. \tag{3.10}$$

Using now first (3.10) and (2.3), and later chain rule, (2.8), (2.10), and (3.1), one can bound the right hand side in (3.9) by

$$\begin{aligned}
d &\le \sum_{k=0}^{[t'-t]} \|\beta(t'-k+1,y)^{-1}\|_1 \mathbf{d}_N(\beta((t'-k)x),\beta((t'-k)y)) \\
&\quad + \|\beta(t,y)^{-1}\beta(\{t'-t\},ty)^{-1}\|_1 \mathbf{d}_N(\beta(\{t'-t\},tx),\beta(\{t'-t\},ty)) \\
&\le C_1C_2C_3\left(\sum_{k=0}^{[t'-t]} \sup_{x\in M}\|D_N\beta(1,x)^{-1}\|^{[t'-k+1]}\lambda^{([t'-k])\theta}\mathrm{dist}_M(x,y)^\theta\right. \\
&\quad \left. + \sup_{x\in M}\|D_N\beta(1,x)^{-1}\|^{[t]+1}\lambda^{([t]+1)\theta}\mathrm{dist}_M(y,x)^\theta\right) \\
&\le C_4\Lambda^t\mathrm{dist}_M(x,y)^\theta,
\end{aligned} \tag{3.11}$$

where C_3, C_4 are constants independent of x, y, t, t' and $\Lambda < 1$ is defined in (2.10). Formula (3.11) implies (1).

(2) We show that the map $y \to \gamma_x(y)$ is uniformly θ-Holder. Let $y', y'' \in W(x)$ and $t > 0$. Then:

$$\begin{aligned}
\mathbf{d}_N(\gamma_{x,t}(y'),\gamma_{x,t}(y'')) &= \mathbf{d}_N(\beta(t,y')^{-1}\beta(t,x),\beta(t,y'')^{-1}\beta(t,x)) \\
&= \mathbf{d}_N(\beta(t,y')^{-1},\beta(t,y'')^{-1}) \\
&\le C_1C_2\sum_{k=0}^{[t]}\sup_{x\in M}\|D\beta(1,x)^{-1}\|^k\lambda^{k\theta}\mathrm{dist}_M(x,y)^\theta \\
&\le C_3\mathrm{dist}_M(x,y)^\theta,
\end{aligned}$$

where C_3 is independent of x, y.

(3) This is obvious.
(4) This follows from the identity $\beta(t,y)\gamma_{x,t'}(y) = \gamma_{tx,t'-t}(ty)\beta(t,x)$, which is a consequence of (3.3), by taking limit as $t' \to \infty$.
(5) By (2), (3) and (4) follows:

$$\begin{aligned}\mathbf{d}_N(\beta(t,x),\beta(t,y)\gamma_x(y)) &= \mathbf{d}_N(\beta(t,x),\gamma_{tx}(ty)\beta(t,x))\\ &= \mathbf{d}_N(Id_N,\gamma_{tx}(ty))\\ &= \lim_{t'\to\infty}\mathbf{d}_N(\beta(t',ty)^{-1}\beta(t',tx)\\ &\leq C\lambda^{t\theta}\mathrm{dist}_M(x,y)^\theta, \end{aligned} \tag{3.12}$$

where the last inequality follows as in the proof of (2). Then (3.6) follows by taking limit as $t \to \infty$.
(6) Let Γ_x: $W(x) \to \mathrm{Homeo}(N)$ be another family of functions that satisfies (2)–(4). From (4) follows that:

$$\Gamma_x(y) = \beta(t,y)^{-1}\Gamma_{tx}(ty)\beta(t,x). \tag{3.13}$$

Then (3.13), (2), (3), and (2.4) imply

$$\begin{aligned}\mathbf{d}_N(\Gamma_x(y),\gamma_{x,t}(y)) &= \mathbf{d}_N(\beta(t,y)^{-1}\Gamma_{tx}(ty)\beta(t,x),\beta(t,y)^{-1}\beta(t,x)))\\ &\leq \|D_N\beta(t,y)^{-1}\|\mathbf{d}_n(\Gamma_{tx}(ty),Id_N)\\ &\leq \|D_N\beta(t,y)^{-1}\|\mathbf{d}_n(\Gamma_{tx}(ty),\Gamma_{tx}(tx))\\ &\leq \|D_N\beta(t,y)^{-1}\|\lambda^{t\theta}\|\Gamma\|_\theta d_M(x,y)^\theta\\ &\leq C_4\Lambda^t\mathrm{dist}_M(x,y)^\theta, \end{aligned} \tag{3.14}$$

where C_4 is a constant independent of t and $\Lambda < 1$ is defined in (2.10).

From (3.14) and (3.4) follows that $\Gamma_x = \gamma_x$. □

Remark One can think of the quantity $\gamma_x(y)$ as the "height" of a point on the leaf of a lifted foliation in the space $M \times N$. The lifted foliation projects on the contracting foliation W.

Note that a similar invariant structure can be introduced over an expanding foliation. In this case $\gamma_x(y)$ is defined by

$$\gamma_x(y) = \lim_{t\to-\infty}\beta(t,y)^{-1}\beta(t,x). \tag{3.15}$$

The analog of Proposition 3.3 for Lie group cocycles is shown below.

Proposition 3.4 *Let H be a connected Lie group that has a cocompact lattice Δ and a right invariant metric $\mathbf{d}_H$. Let α: $\mathbb{R}\times M \to M$ be a smooth action, and W an α-invariant contracting foliation of M with contraction constant $0 < \lambda < 1$. Let β: $\mathbb{R}\times M \to H$ be a θ-Holder cocycle over α that is λ-center bunched. For $x \in M, t \geq 0$, define $\gamma_{x,t}$: $W(x) \to H$ by $\gamma_{x,t}(y) = \beta(t,y)^{-1}\beta(t,x)$. Then the following are true:*

(1) *The family $\{\gamma_{x,t}(y)\}_{t\geq 0}$ converges in H as $t \to \infty$.*
(2) *The map γ_x: $(W(x), dist_{W(x)}) \to (H, \mathbf{d}_H)$ given by*

$$\gamma_x := \lim_{t\to\infty}\gamma_{x,t}$$

is uniformly θ-Hölder.

(3) $\gamma_x(x) = Id_H$.
(4) *The family of functions* $\{\gamma_x\}_{x\in M}$ *is invariant under the extended action, that is,* $\beta(t,y)\gamma_x(y) = \gamma_{tx}(ty)\beta(t,x), y \in W(x), t \geq 0$.
(5) *If* $y \in W(x)$ *and* $\mu > \lambda^\theta$ *then*

$$\lim_{t\to\infty} \mu^{-t}\mathbf{d}_H(\beta(t,x), \beta(t,y)\gamma_x(y)) = 0.$$

(6) *The family of functions* $\{\gamma_x\}_{x\in M}$ *is uniquely determined by* (2), (3), *and* (4).

Proof Proposition 3.4 follows from Proposition 3.3 if we observe that H acts on the compact manifold $N := H/\Delta$ by left multiplications, which are diffeomorphisms, so β can be seen as taking values in $\mathrm{Diff}^\infty(N)$. Full details can be found in [21]. □

Remark Notice that the existence of a lattice is not necessary to construct the functions γ_x in Proposition 3.4, but rather a technical condition used to reduce the proof to the diffeomorphism case. For general finite dimensional Lie groups one can use a right invariant metric and proceed along the proof in [21].

Assume now that the foliation W is contracting/expanding under the action of several $\mathbb{R}$-flows that are parts of a higher rank abelian action. The following proposition shows that the "height" $\gamma_x(y)$ introduced in Proposition 3.3 does not depend on the particular $\mathbb{R}$-subflow used to build it. In order to mark the dependence of γ_x on α, we sometimes write γ_x^a if $a \in \mathbb{R}^k$ is a generator of the $\mathbb{R}$-subaction α used to construct the flow.

Proposition 3.5 *Let* M, N *be compact manifolds,* $\alpha\colon \mathbb{R}^k \times M \to M$ *a smooth action, and* W *an* α*-invariant foliation of* M. *Let* $0 < \lambda < 1$ *and* $\beta\colon \mathbb{R}^k \times M \to \mathit{Diff}^K(N)$ *be a* θ*-Holder cocycle over* α. *Let* $a, b \in \mathbb{R}^k$. *Assume that* W *is contracting* (*with contraction constants* λ_a *and* λ_b) *for the subactions induced by* $\mathbb{R}a$ *and* $\mathbb{R}b$, *and that* β *is* $\max(\lambda_a, \lambda_b)$*-center bunched. Then:*

(1) $\beta(b,y)\gamma_x^a(y) = \gamma_{bx}^a(by)\beta(b,x)$, *for* $y \in W(x)$;
(2) $\gamma_x^a = \gamma_x^b$, *for all* $x \in M$.

Proof We start proving (1). For $x \in M$, define the function $\Gamma_x\colon W(x) \to \mathrm{Diff}^K(N)$ by

$$\Gamma_x(y) = \beta(b,y)^{-1}\gamma_{bx}^a(by)\beta(b,x). \tag{3.16}$$

Clearly $\Gamma_x(x) = Id_N$, and since the family $\{\gamma_x\}_x$ is uniformly θ-Hölder, the family $\{\Gamma_x\}_x$ is also uniformly θ-Hölder. We will show that Γ_x satisfies condition (4) in Proposition 3.3, and then Proposition 3.3, (5), implies that $\Gamma_x = \gamma_x$.

Since $a+b = b+a$, the cocycle equation (2.6) gives $\beta(b,ax)\beta(a,x) = \beta(a,bx)\beta(b,x)$. Together with (3.5) this gives:

$$\begin{aligned}\beta(a,y)\Gamma_x(y) &= [\beta(a,y)\beta(b,y)^{-1}]\gamma_{bx}^a(by)\beta(b,x)\\ &= \beta(b,ay)^{-1}\gamma_{(a+b)x}^a((a+b)y)[\beta(a,bx)\beta(b,x)]\\ &= [\beta(b,ay)^{-1}\gamma_{(a+b)x}^a((a+b)y)\beta(b,ax)]\beta(a,x)\\ &= \Gamma_{ax}(ay)\beta(a,x).\end{aligned} \tag{3.17}$$

To prove (2), observe that γ_x^a satisfies Proposition 3.3, (4), as applies to γ_x^b: this is exactly (1) in this proposition. So, using Proposition 3.3, (5) again, it follows that $\gamma_x^a = \gamma_x^b$. □

The analog of Proposition 3.5 for Lie group valued cocycles has a similar statement and proof.

4 Partially hyperbolic diffeomorphisms

The results in the previous section only show uniform Hölder regularity for the functions γ_x. If the cocycle β is C^K or smooth, one can prove higher regularity results. We will rely on the theory of partially hyperbolic diffeomorphisms as presented in [3,10,25].

Assume K finite. For C^K cocycles with values in $\mathrm{Diff}^K(N)$ we show in Theorem 4.2 the following:

- The functions γ_x takes values in $\mathrm{Diff}^K(N)$.
- The function $y \to \gamma_x(y)$ is C^K as a function $W(x) \to \mathrm{Diff}^K(N)$.
- If γ_x, γ'_x correspond to β, β' respectively, then γ_x as a function $W(x) \to \mathrm{Diff}^K(N)$ converges C^K to γ'_x as β converges to β' in C^K-topology.

For C^K cocycles with values in a Lie group one can show, using the reduction to cocycles in the diffeomorphism group of $N := H/\Delta$, that:

- The function $y \to \gamma_x(y)$ is C^K as a function $W(x) \to H$.
- If γ_x, γ'_x correspond to β, β' respectively, then γ_x as a function $W(x) \to H$ converges C^K to γ'_x as β converges to β' in C^K-topology.

If K is infinite and β takes values a diffeomorphism group then the functions γ_x takes values in $\mathrm{Diff}^\infty(N)$ and the functions $y \to \gamma_x(y)$ are C^∞ as functions $W(x) \to \mathrm{Diff}^\infty(N)$. These facts are proved in [21] using the construction of the stable foliation from [10, Theorem 5.5]. A similar statement is true for Lie group valued cocycles. Note that we do not know if the functions γ_x depend smoothly on β, but this fact is not needed in the sequel.

Let L be a linear transformation between two normed linear spaces. The norm and conorm of L are defined as $\|L\| := \sup\{\|Lv\|; \|v\| = 1\}$ and $m(L) := \inf\{\|Lv\|; \|v\| = 1\}$.

Let X be a compact Riemannian manifold. A C^1-diffeomorphism $f\colon X \to X$ is called *partially hyperbolic* if the derivative $Df : TX \to TX$ leaves invariant a continuous splitting $TX = E^s \oplus E^c \oplus E^u$, $E^s \neq 0 \neq E^u$, such that Df contracts E^s by a constant $0 < \lambda_- < 1$, Df^{-1} contracts E^u by a constant $0 < \lambda_+ < 1$, and the inequalities

$$\|D^s_pf\| < m(D^c_pf) \quad \text{and} \quad \|D^c_pf\| < m(D^u_pf)$$

hold for all $p \in X$.

A C^K-*foliation* is a continuous foliation that is C^K along the leaves.

Assume that f is a partially hyperbolic C^r-diffeomorphism, $1 \le r < \infty$, that leaves invariant a C^1-foliation $\mathcal{L}$ tangent to the central direction E^s. We call f *r-normally hyperbolic at* $\mathcal{L}$ if:

$$m(D^u_pf) > \|D^c_pf\|^k \quad \text{and} \quad \|D^s_pf\| < m(D^c_pf)^k, \tag{4.1}$$

for all $0 \le k \le r$ and $p \in X$.

Theorem 4.1 [10] *Let X be a compact manifold, $f \in \mathrm{Diff}^r(X), r \ge 1$, a diffeomorphism that is r-normally hyperbolic at a C^r-foliation $\mathcal{L}_f$.*

(1) *The distributions E^s and E^u are integrable. The corresponding foliations are called stable, respectively unstable, and are denoted by W^s, respectively W^u. The foliations are Hölder, and their leaves $W^s(x)$ and $W^u(x)$ are C^r and depend continuously on $x \in X$ in C^r-topology. The leaves can be characterized as follows. Let $0 < \lambda'_- < \lambda_-, 0 < \lambda'_+ < \lambda_+$. Then*

$$\begin{aligned} y \in W^s(x) \text{ if and only if } \lim_{n\to\infty} (\lambda'_-)^{-n} dist_M(f^n(y), f^n(x)) = 0, \\ y \in W^u(x) \text{ if and only if } \lim_{n\to\infty} (\lambda'_+)^{-n} dist_M(f^{-n}(y), f^{-n}(x)) = 0. \end{aligned} \tag{4.2}$$

(2) *If $g \in Diff^r(X)$ is C^r close to f, then g is r-normally hyperbolic at a unique C^r-foliation $\mathcal{L}_g$ and the stable and unstable leaves of g converge in C^r to those of f as g converges to f in the C^r-topology.*

We are ready to discuss the regularity properties of the functions γ_x. We restrict the discussion to the setup relevant for the proof of the main result.

Theorem 4.2 *Let M, N be compact manifolds, $\alpha: \mathbb{R} \times M \to M$ a smooth action, for which the time one map $\alpha(1)$ is an r-normally hyperbolic to a C^K foliation $W^c_{\alpha(1)}$ of M. Let $W^s_{\alpha(1)}$ be an α-invariant contracting foliation of M with contraction constant $0 < \lambda < 1$, which also is the stable foliation of $\alpha(1)$. Let $\beta: \mathbb{R} \times M \to Diff^K(N)$ be a C^K-cocycle that is C^K-close to Id_N on the set of generators S. Then the map $y \to \gamma_x(y)$ takes values in $Diff^K(N)$ and is C^K as a function from $W^s_{\alpha(1)}(x) \to Diff^K(N)$. Moreover, if γ_x and γ'_x correspond to β and β' respectively, then γ_x converges C^K to γ'_x as β converges to β' in C^K-topology.*

Proof We start by observing that formula (3.4) gives the same limit γ_x if we take the limit using discrete time $n \in \mathbb{N}$. Note also that β C^K-close to Id_N on the set of generators S implies that β is λ-center bunched. The idea of the proof is to construct a partially hyperbolic diffeomorphism f for which the stable foliation is given by γ_x, and then read the regularity properties of γ_x from the regularity properties of the stable foliation.

Let $X = M \times N, f: X \to X, f(x,z) = (\alpha(1)(x), z)$ and $g: X \to X, g(x,z) = (\alpha(1)(x), \beta(1,x)(z))$. The map f is a partially hyperbolic diffeomorphism with center foliation with leaves $W^c_f(x,z) = W^c_{\alpha(1)}(x) \times N$. It is immediate that f is K-normally hyperbolic to the foliation W^c_f. Since β is C^K-close to Id_N on the set of generators S, the map g is a C^K-perturbation of f. Observe now that the graph of the function $W^s_{\alpha(1)}(x) \ni y \to (y, \gamma_x(y)(z)) \in X$ coincides with the stable leaf $W^s_g(x,z)$ of g. This follows from the characterization of the stable leaf given in formula (4.2) and from the contraction properties of the functions γ_x shown in Proposition 3.3, (5). □

5 Cycle functionals

In this section we review the theory developed by Katok and Kononenko in [11] which allows to study cocycles over partially hyperbolic actions that have accessibility property. Our set-up is that of cocycles over abelian actions with values in diffeomorphism groups. The statements here have analogs for cocycles with values in Lie groups and the proofs are similar.

In what follows M and N are compact manifolds, $\alpha\colon \mathbb{R}^k \times M \to M, k \geq 1$, is a smooth action on M, and $\beta\colon \mathbb{R}^k \times M \to \mathrm{Diff}^K(N)$ a θ-Hölder cocycle over α which is λ-center bunched. All the foliations that appear in this section are assumed to be α-invariant, continuous, and with smooth leaves, and contracting or expanding under the action of certain subflows $\mathbb{R}a, a \in Q(\mathbb{R})$. Therefore the construction of the functions γ_x introduced in Sect. 3 can be carried over. In order to emphasize the dependence of the function γ_x on a certain contracting/expanding foliation W, we introduce the notation γ_x^W.

Definition 5.1 Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be a family of foliations of M. An ordered set of points

$$(x_1, \ldots, x_l, x_{l+1}), x_i \in M, \quad 1 \leq i \leq l+1,$$

is called an $\mathcal{F}_{1,\ldots,r}$*-path* of length l if for every $i = 1, \ldots, l$ there exists $j(i) \in \{1, \ldots, r\}$ such that $x_{i+1} \in \mathcal{F}_{j(i)}(x_i)$. If $x_{l+1} = x_1$, the path is called $\mathcal{F}_{1,\ldots,r}$*-cycle.*

Definition 5.2 Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be a family of foliations of M, each $\mathcal{F}_i$ either contracting or expanding under the action of a subflow $\mathbb{R}a_i \subset \mathbb{R}^k, a_i \in Q(\mathbb{R}^k)$, and $\mathcal{P} = (x_1, \ldots, x_l, x_{l+1})$ an $\mathcal{F}_{1,\ldots,r}$*-path.* We define *the height of* β *over the path* $\mathcal{P}$ to be

$$H(\beta, \mathcal{P}) = \gamma_{x_l}^{\mathcal{F}_{j(l)}}(x_{l+1}) \ldots \gamma_{x_2}^{\mathcal{F}_{j(2)}}(x_3)\gamma_{x_1}^{\mathcal{F}_{j(1)}}(x_2). \tag{5.1}$$

Remark It follows from Proposition 3.5 that the height $H(\beta, \mathcal{P})$ does not depend on the particular subflows $\mathbb{R}a_i$. A different choice of the flows for which the foliations are still contracting/expanding gives the same height.

The following proposition shows a necessary condition for the triviality of a cocycle.

Proposition 5.3 *Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be a family of foliations of M, each $\mathcal{F}_i$ either contracting or expanding under the action of a subflow $\mathbb{R}a_i \subset \mathbb{R}^k, a_i \in Q(\mathbb{R}^k)$. Assume that the cocycle β is cohomologous to a constant cocycle. Then all the heights of β over $\mathcal{F}_{1,\ldots,r}$-cycles are trivial, that is, equal to Id_N.*

Proof Let $\pi\colon \mathbb{R}^k \to \mathrm{Diff}^K(N)$ be a homomorphism and $h\colon M \to \mathrm{Diff}^K(N)$ a transfer map such that (2.7) holds. Let $\mathcal{C} = (x_1, \ldots, x_l, x_{l+1})$, $x_{l+1} = x_1$, be a $\mathcal{F}_{1,\ldots,r}$-cycle. Assume that the foliation $\mathcal{F}_{j(i)}$ is contracting (the proof for expanding is similar) under the action of a subflow $\mathbb{R}a \subset \mathbb{R}^k$. Then

$$\begin{aligned}
\gamma_{x_i}^{\mathcal{F}_{j(i)}}(x_{i+1}) &= \lim_{t\to\infty} \beta(ta, x_{i+1})^{-1}\beta(ta, x_i) \\
&= \lim_{t\to\infty} h(x_{i+1})\pi(ta)^{-1}h(tax_{i+1})^{-1}h(tax_i)\pi(ta)h(x_i)^{-1} \\
&= h(x_{i+1})h(x_i)^{-1},
\end{aligned} \tag{5.2}$$

where for the last equality we use the continuity of h and that

$$\lim_{t\to\infty} \mathrm{dist}_M(tax_i, tax_{i+1}) = 0.$$

Thus

$$\begin{aligned}
H(\beta, \mathcal{C}) &= \gamma_{x_l}^{\mathcal{F}_{j(l)}}(x_1)\gamma_{x_{l-1}}^{\mathcal{F}_{j(l-1)}}(x_l) \ldots \gamma_{x_2}^{\mathcal{F}_{j(2)}}(x_3)\gamma_{x_1}^{\mathcal{F}_{j(1)}}(x_2) \\
&= h(x_1)h(x_l)^{-1}h(x_l)h(x_{l-1})^{-1} \ldots h(x_3)h(x_2)^{-1}h(x_2)h(x_1)^{-1} \\
&= Id_N.
\end{aligned} \tag{5.3}$$

□

Another instance when all the heights are trivial appears when we work with only one foliation.

Proposition 5.4 *Let $\mathcal{F}$ be a contracting or expanding foliation under the action of a subflow $\mathbb{R}a \subset \mathbb{R}^k$. Then the heights of β over all $\mathcal{F}$-cycles are trivial.*

Proof We assume that $\mathcal{F}$ is contracting. Let $\mathcal{C} = (x_1, \ldots, x_l, x_{l+1})$, $x_{l+1} = x_1$, be a $\mathcal{F}$-cycle. Then:

$$\begin{aligned} H(\beta, \mathcal{C}) &= \gamma^{\mathcal{F}}_{x_l}(x_1)\gamma^{\mathcal{F}}_{x_{l-1}}(x_l)\ldots\gamma^{\mathcal{F}}_{x_2}(x_3)\gamma^{\mathcal{F}}_{x_1}(x_2) \\ &= \lim_{t\to\infty} \beta(ta, x_1)^{-1}\beta(ta, x_l)\beta(ta, x_l)^{-1}\beta(ta, x_{l-1})\ldots \\ &\quad \beta(ta, x_3)^{-1}\beta(ta, x_2)\beta(ta, x_2)^{-1}\beta(ta, x_1) \\ &= Id_N. \end{aligned} \tag{5.4}$$

□

Under additional assumptions on the family of foliations, the necessary condition presented in Proposition 5.3 is also sufficient for the cocycle to be cohomologous to a constant.

Definition 5.5 Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be a family of foliations of M. The family is called *transitive* if for any $x, y \in M$ there exists $(x, x_2, \ldots, x_l, y)$ an $\mathcal{F}_{1,\ldots,r}$-path joining x and y. The family is called *locally transitive* if there exists an integer $N \geq 1$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in M$, $y \in B_M(x, \delta)$, there is a $\mathcal{F}_{1,\ldots,r}$-path $(x = x_1, \ldots, x_l = y), l \leq N$, such that $d_{\mathcal{F}_{j(i)}(x_i)}(x_{i+1}, x_i) < \varepsilon$ for $i = 1, \ldots, l$ and $j(i) \in \{1, \ldots, r\}$.

Proposition 5.6 *Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be a family of transitive locally transitive foliations, each foliation $\mathcal{F}_i$ either contracting or expanding under the action of a subflow $\mathbb{R}a_i \subset \mathbb{R}^k$, $a_i \in Q(\mathbb{R}^k)$. Assume that $H(\beta, \mathcal{C}) = Id_N$ for all cycles $\mathcal{C}$ determined by the family. Then β is cohomologous to a constant cocycle.*

Proof Let $x \in M$ fixed and $y \in M$ arbitrary. Since the family of foliations is transitive, there is a $\mathcal{F}_{1,\ldots,r}$-path $\mathcal{C}$ connecting x and y. Define the function h: $M \to \mathrm{Diff}^K(N)$ by

$$h(y) = H(\beta, \mathcal{C}). \tag{5.5}$$

From $H(\beta, \mathcal{C}) = Id_N$ for all cycles $\mathcal{C}$ it follows that the function h is well defined. Indeed, if $\mathcal{C}'$ is another path connecting x and y, then the concatenation of $\mathcal{C}$, listed from x to y, and $\mathcal{C}'$, listed from y to x, gives a cycle. Thus $H(\beta, \mathcal{C}')^{-1}H(\beta, \mathcal{C}) = Id_N$, and $H(\beta, \mathcal{C}) = H(\beta, \mathcal{C}')$.

Continuity of h is a consequence of the local transitivity of the family of continuous foliations with smooth leaves and the fact that the cocycle β is Hölder.

We verify now that h is a transfer map. Let $a \in \mathbb{R}^k$. Note that if $\mathcal{C} = (x = x_1, ..., x_l = y)$ is an $\mathcal{F}_{1,\ldots,r}$-path connecting x and y, then it follows from the α-invariance of the family of foliations that $a\mathcal{C} = (ax_1, ..., ax_l)$ is a $\mathcal{F}_{1,\ldots,r}$-path connecting ax and ay. Hence

$$\begin{aligned} h(ay) &= H(\beta, a\mathcal{C})h(ax) \\ &= \gamma^{\mathcal{F}_{j(l-1)}}_{ax_{l-1}}(ax_l)\cdots\gamma^{\mathcal{F}_{j(2)}}_{ax_2}(ax_3)\gamma^{\mathcal{F}_{j(1)}}_{ax_1}(ax_2)h(ax) \\ &= \lim_{t\to\infty} \beta(\varepsilon_{l-1}ta_{j(l-1)}, ax_l)^{-1}\beta(\varepsilon_{l-1}ta_{j(l-1)}, ax_{l-1})\cdots\beta(\varepsilon_2 ta_{j(2)}, ax_3)^{-1} \\ &\quad \beta(\varepsilon_2 ta_{j(2)}, ax_2)\beta(\varepsilon_1 ta_{j(1)}, ax_2)^{-1}\beta(\varepsilon_1 ta_{j(1)}, ax_1)h(ax_1), \end{aligned} \tag{5.6}$$

where $\varepsilon_i \in \{\pm 1\}$, depending on the foliation $\mathcal{F}_{j(i)}$ being contracting or expanding.
Observe now that:

$$\begin{aligned} &\beta(ta_{j(i)}, ax_{i+1})^{-1}\beta(ta_{j(i)}, ax_i) \\ &= \beta(a, x_{i+1})\beta(ta_{j(i)} + a, x_{i+1})^{-1}\beta(ta_{j(i)} + a, x_i)\beta(a, x_i)^{-1}, \end{aligned} \tag{5.7}$$

for $1 \leq i \leq l-1$. So (5.6) becomes:

$$\begin{aligned} h(ay) = \beta(a, x_l) \lim_{t\to\infty} \Big(&\beta(\varepsilon_{l-1}ta_{j(l-1)} + a, x_l)^{-1}\beta(\varepsilon_{l-1}ta_{j(l-1)} + a, x_{l-1}) \\ &\cdots \beta(\varepsilon_2 ta_{j(2)} + a, x_3)^{-1}\beta(\varepsilon_2 ta_{j(2)} + a, x_2)\beta(\varepsilon_1 ta_{j(1)} + a, x_2)^{-1} \\ &\beta(\varepsilon_1 ta_{j(1)} + a, x_1)\Big)\beta(a, x_1)^{-1}h(ax_1). \end{aligned} \tag{5.8}$$

Note that

$$\begin{aligned} &\lim_{t\to\infty} \beta(\varepsilon_{l-1}ta_{j(l-1)} + a, x_l)^{-1}\beta(\varepsilon_{l-1}ta_{j(l-1)} + a, x_{l-1}) \\ &= \lim_{t\to\infty} \beta(\varepsilon_{l-1}ta_{j(l-1)}, x_l)^{-1}\beta(a, \varepsilon_{l-1}ta_{j(l-1)}x_l)^{-1} \\ &\quad \beta(\varepsilon_{l-1}ta_{j(l-1)}, x_{l-1})\beta(\varepsilon_{l-1}ta_{j(l-1)}, x_{l-1}) \\ &= \lim_{t\to\infty} \beta(\varepsilon_{l-1}ta_{j(l-1)}, x_l)^{-1}\beta(\varepsilon_{l-1}ta_{j(l-1)}, x_{l-1}), \end{aligned}$$

because β θ-Hölder implies that

$$\lim_{t\to\infty} \beta(a, \varepsilon_{l-1}ta_{j(l-1)}x_l)^{-1}\beta(a, \varepsilon_{l-1}ta_{j(l-1)}x_{l-1}) = Id_N.$$

Similar identities hold for the other products on the right hand side of (5.8), so (5.8) becomes:

$$h(ay) = \beta(a, x_l)h(y)\beta(a, x_1)^{-1}h(ax_1) = \beta(a, y)h(y)\beta(a, x)^{-1}h(ax). \tag{5.9}$$

Define $\pi\colon \mathbb{R}^k \to \mathrm{Diff}^K(N)$ by

$$\pi(a) = h(ax)^{-1}\beta(a, x). \tag{5.10}$$

Note that π is well defined because x is fixed. We show that π is a representation, that is

$$\pi(a + b) = \pi(a)\pi(b). \tag{5.11}$$

Formula (5.11) is equivalent to

$$h((a + b)x)^{-1}\beta(a + b, x) = h(a)^{-1}\beta(a, x)h(b)^{-1}\beta(b, x), \tag{5.12}$$

which follows immediately from (5.9) if we replace y by bx and take into account that $\beta(a + b, x)\beta(b, x)^{-1} = \beta(a, b)$.
We finish the proof by observing that (5.9) is equivalent to

$$\beta(a, y) = h(ay)\pi(a)h(y)^{-1}, \tag{5.13}$$

that is, β is cohomologous to a constant cycle. □

Remark Under better accessibility properties for the foliations, [11] presents Hölder regularity results for h.

6 Cartan actions on $SL(n,\mathbb{R})/\Gamma$ and $SL(n,\mathbb{C})/\Gamma$

For the rest of the paper $\mathbb{K}$ is either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Let $G = SL(n,\mathbb{K})$. Let $D_n^+ \subset G$ be the group of diagonal matrices with positive elements. We parameterize D_n^+ as follows:

$$D_n^+ = \left\{ \mathrm{diag}(e^{t_1},\ldots,e^{t_n}) | \mathbf{t} = (t_1,\ldots,t_n), \sum_{i=1}^{n} t_i = 0 \right\}.$$

The $n-1$ dimensional subspace of $\mathbb{R}^n$ given by

$$\mathcal{D}_n = \left\{ (t_1,\ldots,t_n) | \sum_{i=1}^{n} t_i = 0 \right\}$$

can be viewed as the Lie algebra of D_n^+ via the inverse of the usual exponential map. So D_n^+ is isomorphic to $\mathbb{R}^{n-1}$.

Let $\Gamma \subset G$ be a torsion free co-compact lattice, that is a discrete group of co-finite volume without elements of finite order. The quotient space G/Γ has a structure of compact manifold. We consider the action of D_n^+ on the space G/Γ by left translations. This type of actions is called *Cartan action*.

Let $\alpha\colon D_n^+ \times G/\Gamma \to G/\Gamma$ be a Cartan action. Introduce a right invariant metric $d(\cdot,\cdot)$ on $SL(n,\mathbb{K})$, and denote in the same way the induced metric on G/Γ. Let $1 \le i,j \le n, i \ne j$, be two fixed indices, and let exp be the exponential map in $SL(n,\mathbb{K})$. Let $v_{i,j}$ be the elementary $n \times n$ matrix with only one non-zero entry, that in position (i,j). We denote $e_{ij}(s) = \exp(sv_{i,j})$ and define a foliation F_{ij} on G/Γ with leaves:

$$F_{ij}(x) = \{e_{ij}(s)x | s \in \mathbb{K}\}. \tag{6.1}$$

Note that it is immediate from the definition of the foliation F_{ij} that its leaves are invariant under left multiplication by $e_{ij}(s)$. Vice versa, since the leaves are one $\mathbb{K}$-dimensional, the motion along the leaves can be described in terms of multiplication by $e_{ij}(s)$.

The foliation F_{ij} is invariant under the action α. Indeed, $e_{ij}(s) = \mathrm{Id} + sv_{i,j}$, and a direct calculation shows that

$$\alpha(\mathbf{t})(\mathrm{Id} + sv_{i,j})x = (\mathrm{Id} + se^{t_i - t_j} v_{i,j})\alpha(\mathbf{t})x. \tag{6.2}$$

Formula (6.2) also shows that the foliation F_{ij} is contracting under the action of $\alpha(\mathbf{t})$ if $t_i < t_j$, and is expanding if $t_i > t_j$. Consequently, any element in D_n^+ that has the entries pairwise different acts as a partially hyperbolic diffeomorphism on G/Γ. The dimension of the center distribution is $n-1$ if $\mathbb{K} = \mathbb{R}$ and $2(n-1)$ if $\mathbb{K} = \mathbb{C}$.

7 Generating relations and Steinberg symbols

Proofs for the results in this section can be found in [17]. See also [16,26]. Throughout this section we assume $n \ge 3$.

The abstract Steinberg group $St_n(\mathbb{K})$ is defined by generators and relations. The generators are denoted by $x_{ij}(t), t \in \mathbb{K}, i,j \in \{1,2,\ldots,n\}, i \ne j$, and are subject to the relations:

$$x_{ij}(t)x_{ij}(s) = x_{ij}(t+s) \tag{7.1}$$

and

$$[x_{ij}(t), x_{kl}(s)] = \begin{cases} 1, & j \neq k, i \neq l \\ x_{il}(st) & j = k, i \neq l \\ x_{kj}(-st), & j \neq k, i = l, \end{cases} \tag{7.2}$$

Steinberg obtained the following presentation of the special linear group $SL(n, \mathbb{K})$.

Theorem 7.1 *The group $SL(n, \mathbb{K})$ is generated by $e_{ij}(t), i \neq j, t \in \mathbb{K}$, subject to the relations:*

$$[e_{ij}(t), e_{kl}(s)] = \begin{cases} 1, & j \neq k, i \neq l \\ e_{il}(st) & j = k, i \neq l \\ e_{kj}(-st), & j \neq k, i = l, \end{cases} \tag{7.3}$$

where $[\cdot, \cdot]$ denotes the commutator,

$$e_{ij}(t)e_{ij}(s) = e_{ij}(t+s), \tag{7.4}$$

and

$$h_{12}(t)h_{12}(s) = h_{12}(ts), \tag{7.5}$$

where

$$h_{12}(t) = e_{12}(t)e_{21}(-t^{-1})e_{12}(t)e_{12}(-1)e_{21}(1)e_{12}(-1)$$

for each $t \in \mathbb{K}^*$.

The natural map ϕ: $St_n(\mathbb{K}) \to SL(n, \mathbb{K})$ defined by $\phi(x_{ij}(t)) = e_{ij}(t)$ is a homomorphism. Its kernel is denoted by $K_2(\mathbb{K})$. The kernel coincides with the center of the Steinberg group. We use for it multiplicative notation, and denote the neutral element by 1.

Here is a way to construct elements in $K_2(\mathbb{K})$. Let $u, v \in \mathbb{K}^*$. Then the diagonal matrices

$$D_u = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}, D'_v = \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix},$$

commute and belong to $SL(3, \mathbb{K})$. Using an embedding of the $SL(3, \mathbb{K})$ in the upper left corner of $SL(n, \mathbb{K}), n \geq 3$, it follows that D_u, D'_v belong to any $SL(n, \mathbb{K}), n \geq 3$. Choose now representatives $U, V \in St_n(\mathbb{K})$, that is $\phi(U) = u, \phi(V) = v$, and define $\{u, v\} = UVU^{-1}V^{-1}$. Then $\{u, v\}$ is an element in $K_2(\mathbb{K})$.

Alternatively, for any unit in $u \in \mathbb{K}$ and $i \neq j$ one can define

$$w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$$

and

$$h_{ij}(u) = w_{ij}(u)w_{ij}(-1).$$

Then $\{u, v\} := [h_{ij}(u), h_{ik}(v)]$ is an element in $K_2(\mathbb{K})$. The map

$$\mathbb{K}^* \times \mathbb{K}^* \ni (u, v) \to \{u, v\} \in K_2(\mathbb{K})$$

is bi-multiplicative, that is $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$ and $\{u, v_1v_2\} = \{u, v_1\}\{u, v_2\}$, and skew-symmetric, that is $\{u, 1-u\} = 1$. Moreover, it is shown in [17] that

$$\{u, v\} = h_{ij}(uv)h_{ij}(u)^{-1}h_{ij}(v)^{-1}. \tag{7.6}$$

A presentation for the $K_2(\mathbb{K})$ in terms of relations and generators was found by Matsumoto [16].

Theorem 7.2 *Let $\mathbb{K}$ be a field. Then the kernel $K_2(\mathbb{K})$ of the natural map ϕ: $St_n(\mathbb{K}) \to SL(n, \mathbb{K})$ is generated by the elements $\{u, v\}, u, v \in \mathbb{K}^*$ subject to the relations:*

(1) $\{u, 1-u\} = 1$, *for* $u \neq 0, 1$,
(2) $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$,
(3) $\{u, v_1v_2\} = \{u, v_1\}\{u, v_2\}$.

Any bi-multiplicative map $c(\cdot, \cdot)$: $\mathbb{K}^* \times \mathbb{K}^* \to A$ into an abelian group A satisfying $c(u, 1-u) = 1_A$ is called *Steinberg symbol*. If A has a structure of Hausdorff space, and the Steinberg symbol is continuous as a function $\mathbb{K}^* \times \mathbb{K}^* \to A$, then the symbol is called *continuous*. The following results belong to Milnor:

Theorem 7.3 *(a) Every continuous Steinberg symbol on the field $\mathbb{C}$ of complex numbers is trivial.*

(b) If $c(x, y)$ is a continuous Steinberg symbol on the field $\mathbb{R}$ of real numbers, then $c(x, y) = 1$ if x or y is positive, and $c(x, y) = c(-1, -1)$ has order at most 2 if x and y are both negative.

8 Fisher–Margulis local rigidity result for isometric actions

Definition 8.1 Let Γ be a finitely generated discrete group, H a connected Lie group and N a compact manifold. On the set of representations from Γ into H, or on the set of group representations from Γ into $\mathrm{Diff}^K(N)$, one can introduce the compact-open topology.

A representation ρ_0: $\Gamma \to H$ is called *locally rigid* if for any representation ρ: $\Gamma \to H$, that is close to ρ_0, there exists an element in H that conjugates ρ and ρ_0.

Let $k_1, k_2 \geq 0$, integers. A smooth representation ρ_0: $\Gamma \to \mathrm{Diff}^\infty(N)$ is called C^{K,k_1,k_2}*-locally rigid* if for any representation ρ: $\Gamma \to \mathrm{Diff}^K(N)$, that is C^{k_1}-close to ρ_0, there exists a diffeomorphism $h \in C^{k_2}(N)$ that conjugates ρ and ρ_0.

The following theorem of Fisher–Margulis [6] will be used in the proof of the main result.

Theorem 8.2 *Let Γ be a discrete group with property (T). Let M be a compact smooth manifold, and let ρ_0 be a smooth action of Γ on M by Riemannian isometries. Then the action ρ_0 is $C^{K,K,K-\kappa}$ locally rigid for every $\kappa > 0$ for $K > 1$.*

Remark The previous theorem has a smooth version as well. For our application we need only the finite smoothness version.

9 Two auxiliary statements

The following regularity result can be found in [14,Theorem 2.1]. The proof of the finite regularity version follows immediately from the proof of the smooth version.

Theorem 9.1 *Let $X_1, X_2, \dots, X_k$ be C^∞ vector fields on a manifold M, of dimension N, such that their sum $\sum_{i=1}^{k} X_i$ is a totally non-integrable distribution and, for each $j \le r$, the dimension of the space spanned by the commutators of length at most j at each point is constant in a neighborhood. Let P be a distribution on M. Assume that for any positive integer $p \le n$ the p's partial derivatives $X_i^p(P)$ exist as continuous or local L^2 functions. Then P is a $C^{[n/r-N/2]}$ function on M. Moreover, if all partial derivative $X_i^p(P)$ exists as continuous or local L^2 function, then P is a C^∞ function on M.*

The following theorem of Newman [18] will be used in the proof of the main result. See [2], Sect. 9 for a proof.

Theorem 9.2 *Let N be a connected topological manifold endowed with a metric. Then there is $\varepsilon > 0$ such that any non-trivial action of a finite group on X has an orbit of diameter larger that ε.*

10 Main results

Theorem 10.1 *Let H be a semisimple real Lie group with first cohomology group $H^1(H, Lie(H)) = 0$, or $H = GL(n, \mathbb{R})$. Let $n \ge 3, G = SL(n, \mathbb{K})$, $\Gamma \subset G$ a co-compact torsion free lattice, and $M = G/\Gamma$. Let α: $D_n^+ \times M \to M$ be the Cartan action. Let β: $D_n^+ \times M \to H$ be a C^K-cocycle that is λ-center bunched, and close enough to the identity of H on the set of generators S. Then β is cohomologous to a constant cocycle via a $C^{[K/2-dim(M)/2]}$ transfer function h: $M \to H$. Moreover, if β is Hölder or smooth, then the transfer function is Hölder, respectively smooth.*

Theorem 10.2 *Let N be a compact manifold and $K \ge 2$ integer.. Let $n \ge 3, G = SL(n, \mathbb{K})$, $\Gamma \subset G$ a co-compact torsion free lattice, and $M = G/\Gamma$. Let α: $D_n^+ \times M \to M$ be the Cartan action. Let β: $D_n^+ \times M \to Diff^K(N)$ be a C^K-cocycle that is λ-center bunched, and close enough to the identity $Id(N)$ on the set of generators S. Then β is cohomologous to a constant cocycle via a $C^{[K/2-dim(M)/2]}$ transfer function h: $M \to Diff^K(N)$. Moreover, if β is smooth, then the transfer function is smooth.*

Remark (a) It was proved by Weil [27,28] that $H^1(H, Lie(H)) = 0$ for all semisimple Lie groups without compact or three dimensional factors.

(b) Let $S \subset \mathcal{D}$ be a subspace that contains a two dimensional subspace in general position, that is a subspace that intersects each hyperplane given by the equation $t_i = t_j, i \ne j$, along a different line. Using [5], one can show that Theorems 10.1 and 10.2 hold for abelian actions on M given by $\exp S \subset D_n^+$.

(c) We do not have a result for Hölder or C^1 cocycles with values in diffeomorphisms groups because a counterpart of the Fisher–Margulis result for representations in Hölder homeomorphism groups or C^1 diffeomorphism groups was not found yet.

(d) One should compare these results, in particular the regularity for the transfer map, with the Livsic type results from [19]. The loss or regularity there appears in N direction.

We prove Theorem 10.2 in full detail and then explain the changes needed for the proof of Theorem 10.1.

Proof of Theorem 10.2 Let $F_{i,j}, i \ne j$, be the α-invariant foliations introduced in Sect. 6. These foliations are smooth and their brackets generate the whole tangent space. As shown in [3], this facts imply that the system of foliations is locally transitive. Each F_{ij}-path built using these foliations can be described by a product of elements of type $e_{ij}(t)$. Indeed, each piece of an F_{ij}-leaf can be parameterized by $t \to e_{ij}(t)x, t \in I$, for some $x \in G$ and a compact interval I. The path is a cycle if and only if the product of these elements belongs to Γ.

It follows from Proposition 5.6 that if the heights $H(\beta, \mathcal{C})$ are equal to Id_N for all cycles $\mathcal{C}$ determined by a family of locally transitive foliations, then the cocycle β is cohomologous to a constant cocycle. Furthermore, it follows from Proposition 5.4 that if the cycle $\mathcal{C}$ is included in a stable or unstable leaf then the height $H(\beta, \mathcal{C})$ is equal to $Id(N)$.

The height over a cycle is, so far, dependent of the word in $e_{ij}(t)$'s describing the cycle. Changing the word, without changing the value of the product, can produce a different height. We show first that if for a cycle the product of $e_{ij}(t)$'s is equal to identity then the height over such a cycles is trivial. Using the presentation for $SL(n, \mathbb{R})$ from Sect. 7, each word in $e_{ij}(t)$'s representing the product can be written as a concatenation of conjugates of the basic relations (7.3), (7.4), and (7.5). Each of these relations defines an F_{ij}-cycle.

The relations of type (7.3) or (7.4) give cycles that are contained in stable leaves for elements of the action α. Indeed, in the case of (7.4), the motion along the cycle is described by multiplication by $e_{ij}(t)$, for various t's, so the cycle is included in the stable leaf of an element $\mathbf{t} \in D_n^+$ with $t_i < t_j$. In the case of (7.3), we split the proof into three cases.

(1) If $j \neq k, i \neq l$ then the cycle is contained in the stable leaf of an element $\mathbf{t} \in D_n^+$ with $t_i < t_j, t_k < t_l$.
(2) If $j = k, i \neq l$ then the cycle is contained in the stable leaf of an element $\mathbf{t} \in D_n^+$ with $t_i < t_j < t_k$.
(3) If $j \neq k, i = l$ then the cycle is contained in the stable leaf of an element $\mathbf{t} \in D_n^+$ with $t_k < t_l < t_j$.

Consider now relations (7.5) for two cases, $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ separately.

Assume $\mathbb{K} = \mathbb{R}$.

We fix a small neighborhood $\mathcal{U}$ of the trivial cocycle in which the cocycle β needs to be in order for the argument to work.

Let $\mathcal{V}$ be the neighborhood of Id_N in Homeo(N) that does not contain any map of period 2 (according to Theorem 9.2). Let $A = \{(1,1), (-1,1), (1,-1), (-1,-1)\}$. Let $\mathcal{C}(s,t)$ be the cycle given by the product $\{t,s\} := h_{12}(t)h_{12}(s)h_{12}(ts)^{-1}$ for some $s, t \in \mathbb{R}$. Since the functions γ_x depend continuously on the cocycle β, there is a neighborhood $\mathcal{U}$ of the trivial cocycle such that the heights $H(t,s)$ over $\mathcal{C}(t,s)$, $(t,s) \in A$, belongs to $\mathcal{V}$ if β belongs to $\mathcal{U}$.

Recall from Sect. 7 that $(t,s) \to \{t,s\}$ is a Steinberg symbol with values in $K_2(\mathbb{R})$. We show that when we vary (t,s) over $\mathbb{R}^* \times \mathbb{R}^*$ the height $H(t,s)$ over $\mathcal{C}(t,s)$ gives a continuous Steinberg symbol with values in an abelian subgroup of Diff$^K(N)$. For $t_1, t_2, s \in \mathbb{R}^*$ consider the product $\Pi := \{t_1t_2, s\}\{t_2, s\}^{-1}\{t_1, s\}^{-1}$. The Steinberg symbol $\{t,s\}$ is bi-multiplicative in $St_n(\mathbb{R})$, so Π is equal to identity of $St_n(\mathbb{R})$. Using the presentation for $St_n(\mathbb{R})$ from Sect. 7, any word representing Π can be written as a concatenation of conjugates of the basic relations (7.3) and (7.4) (which respectively coincide with (7.1) and (7.2)). So using the discussion above, the height over the cycle determined by Π is trivial. The height over a concatenation of two cycles is the product of the heights over the cycles. This implies $H(t_1t_2, s) = H(t_1, s)H(t_2, s)$. In a similar way one can show $H(s,t)$ is multiplicative in the second variable. To show that the height satisfies skew-symmetry, consider the relation $\{t,1-t\}$. This relation is equal to identity in the Steinberg group, because the Steinberg symbol $\{t,s\}$ is skew-symmetric, so any word representing $\{t,1-t\}$ is a product of conjugates of the standard relations (7.3) and (7.4). Using the discussion above it follows that the height $H(t,1-t)$ over the cycle determined by $\{t,1-t\}$ is trivial. We show that $H(s,t)$ takes values in an abelian group. This follows from the fact that any two symbols $\{t_1,s_1\}$ and $\{t_2,s_2\}$ belong to the abelian group $K_2(\mathbb{R})$, so the following equality $\{t_1,s_1\}\{t_2,s_2\} = \{t_2,s_2\}\{t_1,s_1\}$ holds in $St_n(\mathbb{R})$. As before, the identity in $St_n(\mathbb{R})$ implies $H(t_1,s_1)H(t_2,s_2) = H(t_2,s_2)H(t_1,s_1)$. The continuity of the symbol $H(t,s)$ follows from its definition and from the fact that the abelian group in which the symbol takes values has a Hausdorff topology induced from $\mathrm{Diff}^K(N)$.

So $(t,s) \to H(t,s)$ is a continuous Steinberg symbol. Due to Theorem 7.3,b), the only possible values for the height are $Id(N)$ or an element of order 2 in $\mathrm{Diff}^K(N)$.

We show now that if the cocycle β belongs to the neighborhood $\mathcal{U}$ described above, then the height is trivial. The height is continuous in $(s,t) \in \mathbb{R}^* \times \mathbb{R}^*$. We look at each connected component of $\mathbb{R}^* \times \mathbb{R}^*$. Let $(t,s) \in (0,\infty) \times (0,\infty)$. The other cases are similar. Since $(0,\infty) \times (0,\infty)$ is connected, the image of the height is connected. The image belongs to the union of the sets $\{h \in \mathrm{Homeo}(N) | h = Id_N\}$ and $\{h \in \mathrm{Homeo}(N) | h^2 = Id_N, h \neq Id_N\}$, which are both closed. It follows from Theorem 9.2 that the sets are disjoint. So the image is included in one of the sets. If the image is included in the first set, we are done. Otherwise, let $(1,1) \in (0,\infty) \times (0,\infty)$. Since $\beta \in \mathcal{U}$, the height over the cycle $\{1,1\}$ belongs to $\mathcal{V}$. But $\mathcal{V}$ does not contain any map of period 2, in contradiction to our assumption.

If $\mathbb{K} = \mathbb{C}$ the proof is similar, but simpler, because Theorem 7.3(a), implies that the continuous Steinberg symbol is trivial in this case.

After eliminating the contribution to the height that appears due to the relations, the product contains only the elements that conjugate the relations. Cancellations of type $e_{ij}(t)e_{ij}(-t) = Id_N$ do not change the height because the cycle determined by the product $e_{ij}(t)e_{ij}(-t)$ is contained in a stable leaf of an element of the action α, so the height over it has to be trivial.

We consider now the height over an arbitrary cycle, not necessarily with the product of e_{ij}'s trivial. Any cycle induces an element in the first fundamental group $\pi_1(G/\Gamma)$. One has an exact sequence

$$1 \to \pi_1(G) \to \pi_1(G/\Gamma) \to \pi_1(\Gamma) \to 1.$$

It is well known that for $n \geq 3$ one has $\pi_1(SL(n,\mathbb{R})) = \mathbb{Z}_2$ and $\pi_1(SL(n,\mathbb{C})$ is trivial. If $\mathbb{K} = \mathbb{R}$ then the cycles that induce the nontrivial element in $\pi_1(SL(n,\mathbb{R}))$ are homotopic to the cycle determined by the extra relation. See [17].

Two cycles that induce the same element in the fundamental group have the same height. Indeed, if their products are Π_1 and Π_2, then the concatenated product $\Pi_1\Pi_2^{-1}$ gives a word that is equal to identity, so the height over the cycle determined by $\Pi_1\Pi_2^{-1}$ is trivial, so the heights over Π_1 and Π_2 are equal. Since the height over a concatenation of two cycles is the product of the heights over the cycles, the height determines a homomorphism ψ from $\pi_1(G)$ into $\mathrm{Diff}^K(N)$. The above remarks about the fundamental group of G imply that ψ factors to a homomorphism from Γ into $\mathrm{Diff}^K(N)$.

If the cocycle β is C^K-small on a set of generators S, then ψ is C^K close to identity on a set of generators of Γ. Indeed, this follows from the fact that the functions γ_x used to construct the height are C^K continuous as functions of β and from the fact that the height depends only on the homotopy class of the cycle. Consider now the trivial representation π_0 of Γ into $\mathrm{Diff}^K(N)$ as an isometric action on the smooth manifold N. Then ψ is C^K close to π_0. Note that any cocompact lattice in $SL(n, \mathbb{K}), n \geq 3$, has property (T)[8]. Thus Fisher–Margulis local rigidity result for isometric actions (Theorem 8.2) can be applied, and ψ is $C^{K-\kappa}$ conjugate to π_0. But this implies that ψ coincides with π_0. So all the heights over the cycles are trivial, and the cocycle is cohomologous to a constant cocycle. Note that for this argument we only need the C^2-version of Fisher–Margulis result, that is we can assume $K = 2$. So this argument works for smooth cocycle even though we do not have a result on smooth dependence of the stable foliation on the cocycle β.

So far, the transfer map h: $M \to \mathrm{Diff}^K(N)$ is only continuous. To show higher regularity for h we employ standard results in rigidity. Look at h as a map $M \times N \to N$. It is standard to show that for any partially hyperbolic element in D_n^+, h is C^K along its stable and unstable directions. See for example [20]. This gives C^K regularity along a finite set of directions, that have the vectors tangent to their distributions, and their length 2 commutators, generating the whole tangent space TM. The commutators needed to consider are of type $[e_{ij}(t), e_{ji}(s)]$. Now Theorem 9.1, with $r = 2$, implies that h is $C^{[K/2-\dim(M)/2]}$ in the M direction. The statement about smooth cocycles follows from Theorem 9.1 as well. □

Proof of Theorem 10.1 The proof is similar to the proof of Theorem 10.2. Note that from a result of Borel [1] follows that any semisimple Lie group as well as $H = \mathrm{GL}(n, \mathbb{R})$ has a cocompact lattice. So Proposition 3.4 can be applied and the functions γ_x can be defined. To show that the height over the extra relation is not an element of order 2 in H we use the fact that the cocycle is small on a set of generators. As before, this implies that the height belongs to a small neighborhood of identity in H. But H is a Lie group and consequently does not have small subgroups. So the height is trivial.

When we study the height over general cycles, instead of a homomorphism from $\pi_1(M)$ into $\mathrm{Diff}^K(N)$ we have a homomorphism from Γ into the fiber H. Note that continuity of the functions γ_x is enough to guarantee the smallness of this representation. This is why we obtain here a Hölder result as well. Fisher–Margulis result is replaced either by the rigidity result of Weil [27,28]: a homomorphism π from a finitely generated group Γ to a semisimple Lie group H is locally rigid whenever the cohomology group $H^1(H, Lie(H)) = 0$; or by the result of Margulis [15] that $H^1(\Gamma, V) = 0$ for every homomorphism of Γ to $\mathrm{GL}(V)$, where V is finite dimensional and Γ is a lattice in a higher rank connected semisimple algebraic $\mathbb{R}$-group without compact factors. In our case Γ is finitely generated because it has property (T), and it is also a lattice in $\mathrm{SL}(n, \mathbb{R})$. Since the homomorphism is close to identity on a set of generators it has to be trivial.

The smooth and C^K regularity results for h follows as before. For the Hölder regularity result one can apply the Hölder regularity result from [11]. □

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