http://dx.doi.org/10.1090/pspum/069/1858534-a

Proceedings of Symposia in Pure Mathematics
Volume **69**, 2001

Cocycles, Cohomology and Combinatorial Constructions in Ergodic Theory

Anatole Katok in collaboration with E. A. Robinson, Jr.

Dedicated to the memory of Michel Herman (1942–2000)

Introduction

Cocycles and cohomological equations play a central role in ergodic theory as well as in its applications to other areas of dynamics. Among the questions which are reduced to cohomological considerations are existence of invariant measures and invariant geometric structures, time change for flows and, more generally, actions of continuous groups, orbit equivalence and its restricted versions, existence of eigenfunctions, classification of various kinds of product actions and many others. The subject is very diverse and includes many measure–theoretic, algebraic, analytic and geometric aspects. Very broadly, cohomological considerations produce two types of conclusions:

(i) if the set (it is very often not a group) of cohomology classes under consideration allows a reasonable structure then the corresponding objects allow a nice classification. In the extreme case when the set of cohomology classes is very small (e.g. if there is only one class) one speaks about *rigidity*;

(ii) if there is no good structure in the set of cohomology classes then individual classes are usually very "chaotic" and this often leads to construction of objects within a given class with various, often exotic properties. Conclusions of this type thus lead to genericity statements, as well as of counterexamples to certain naturally sounding hypotheses.

In the classical ergodic theory, which deals with measure preserving or nonsingular actions of $\mathbb{Z}$ and $\mathbb{R}$, conclusions of the first type never appear; the same is true in the more general context of ergodic theory for actions of amenable groups. In the topological context the situation changes only slightly. On the other hand, there is a variety of interesting situations in finer categories such as Hölder, C^r, $1 \leq r \leq \infty$, or real analytic, where a description of cohomology classes is possible, and produces crucial insights into various classification problems.

It is interesting to notice a difference between the classical cases of a $\mathbb{Z}$ or an $\mathbb{R}$ action, where such a classification only very rarely amounts to rigidity, and other cases, such as actions of higher rank abelian groups (i.e. $\mathbb{Z}^k$ or $\mathbb{R}^k$ for $k \leq 2$), where rigidity is quite widespread and is often crucial for understanding of structural questions.

Furthermore, for actions of groups which are both "large" and "rigid", rigidity appears and becomes, in many respects, prevalent already at the ergodic theory level, e.g. in the measurable category. In fact, one of the central issues in dynamics of actions of such groups is a translation of the available measurable rigidity results into topological and especially differentiable context. The prime examples of such groups are noncompact semisimple Lie groups of real rank greater than one and lattices (discrete subgroups of cofinite volume) in such groups. Kazhdan property (T) is often a key ingredient in producing various rigidity properties, although by itself it leads only to a limited array of results. Still some rigidity results are being extended to specific property (T) groups beyond semisimple Lie groups of higher rank and their lattices.

In the present work we almost completely concentrate on the classical cases. A more general setting appears only in basic definitions and in some comments meant to underline the contrast with those. The first two chapters deal with the measurable setting and hence all the results we prove or discuss are of type (ii). In the third chapter we introduce a general framework for the cohomological phenomena of type (i) (stability and effectiveness) with rigidity as a special case. We illustrate how these phenomena appear in several topological, symbolic and smooth settings. In the last two chapters we return to the type (ii) phenomena but in more specialized settings. Our aim there is to demonstrate how a "controlled chaos" can be produced in a variety of simple smooth or other special situations. The crucial concept here is very fast periodic approximation in the number–theoretic (Liouvillean numbers), differentiable and measurable contexts.

The purpose of this work is to introduce the main concepts and principal techniques and illustrate those by a variety of interesting example including both general statements and the treatment of particular classes of systems. We do not aspire to present a comprehensive survey of the subject. Accordingly, we keep references to a minimum.

The present work is an updated, revised and expanded version of the second of the four parts of our work "Constructions in Ergodic Theory" originally intended to appear as a book form, which was mostly written on 1982-83, appended during the eighties and which has been circulated in the manuscript form.

"Constructions in Ergodic Theory" is dedicated to a systematic although by no means exhaustive development of several principal classes of combinatorial constructions of measure-preserving transformations which allow to obtain some non-trivial properties and which are well adapted to the realization of abstract measure–preserving transformations as smooth or real-analytic systems on compact manifolds preserving an absolutely continuous or smooth measure, and as other special classes of dynamical systems. An updated version of the first part which contains a definitive account of the general concept of periodic approximation as well as its applications to establishing genericity of various ergodic properties in a variety of categories is about to appear as [**K1**]. The third and fourth parts were left unfinished and their fate at the time of writing remains uncertain.

The developments of the last decade, especially those dealing with actions of groups other than $\mathbb{Z}$ and $\mathbb{R}$, changed the appearance, and, to a certain extent, even the basic perception of the area which is the subject of the present work. Still the program outlined and illustrated in Part II of "Constructions in Ergodic

Theory" has proved to be fundamentally sound. In fact, this program influenced some of the later developments such as the systematic use of, and the search for, invariant distributions for various classes of dynamical systems. We would like to repeat though that, revisions and additions notwithstanding, the present text to a large extent reflects the perspective of the early eighties and hence primarily describes and refers to work published by that time. We apologize for probably not sufficiently emphasizing certain more recent results.

We would like to thank Alistair Windsor who carefully read the text and made some valuable suggestions.

Michel Herman made fundamental contributions to differentiable dynamics, smooth ergodic theory, and, specifically the area covered in the present work. Aside from numerous published papers his thinking made a huge influence on the way we view these subjects. His untimely death is a tremendous loss for the world dynamics community.

1. Definitions and principal constructions

We are going to discuss a group of constructions which appear frequently both in the general theory of measure-preserving transformations and in various concrete situations. The central concept for this circle of ideas is the notion of an (untwisted) cocycle over a measure-preserving transformation (and more generally over a group action), and the corresponding notions of coboundary and cohomology. In order to explain this concept in the most natural way, we will leave, for the moment, the confines of classical ergodic theory, which deals with measure-preserving transformations and flows, and consider the actions of more general groups. We will be able to touch only a few aspects of this subject. For a systematic review of basic notions in the area see [**HaK**]. A useful introductory discussion can be found [**KH**], section 2.9.

Cocycles play a particularly important role in the ergodic theory and dynamics of actions of groups other than $\mathbb{Z}$ and $\mathbb{R}$. Various aspects of this subject are treated in e.g. [**Sch1**], [**Z1**], [**HK1**], [**KSp1**].

Another aspect of the subject which we will not be able to discuss has been developed in the remarkable papers of Herman [**H4, H5**]. It involves the use of cocycles in smooth ergodic theory, in particular to obtain below estimates of the Lyapunov characteristic exponents in nonuniformly hyperbolic situations.

1.1 Cocycles, coboundaries and Mackey range. Let us consider a Lebesgue space (X, μ) and let Γ and G be two locally compact second countable groups. Let us suppose that $S = \{S_\gamma\}_{\gamma\in\Gamma}$ is a measurable right action of Γ on (X, μ) by measure-preserving transformations. The action S is *ergodic* if any S–invariant set has measure zero or full measure or, equivalently, if any S–invariant measurable function is almost everywhere constant.

DEFINITION 1.1. A G *cocycle* over the action S is a measurable function $\alpha : X \times \Gamma \to G$ such that

$$\alpha(x, \gamma_1\gamma_2) = \alpha(x, \gamma_1)\alpha(S_{\gamma_1}x, \gamma_2) \tag{1.1}$$

It follows from (1.1) that $\alpha(x, id_\gamma) = id_G$ and $\alpha(x, \gamma^{-1}) = (\alpha(S_{\gamma^{-y}}x, \gamma))^{-1}$. From a purely formal point of view, this is a special case of a concept familiar in algebra, topology and other branches of mathematics. The same is true for the next two definitions.

DEFINITION 1.2. Two G cocycles α and β over a Γ action S are called *cohomologous* if there exists a measurable map $\psi : X \to G$ such that

$$\beta(x,\gamma) = \psi^{-1}(x)\alpha(x,\gamma)\psi(S_\gamma x). \tag{1.2}$$

Let us note that for any cocycle α and for any ψ the function β defined by (1.2) is also a cocycle.

DEFINITION 1.3. A cocycle α is called a *coboundary* if there exists a measurable map $\psi : X \to G$ such that

$$\alpha(x,\gamma) = \psi^{-1}(x)\psi(S_\gamma x). \tag{1.3}$$

Curiously, in various cohomology theories there seems to be no established name for a cochain which provides the equivalence between two cocycles, i.e. in our situation for the function ψ in (1.2). Following a prevalent, but by no means universal, usage in ergodic theory we will call those functions *transfer functions.*

Notice that if the action S is ergodic then a transfer function ψ is uniquely defined up to left multiplication by a constant function.

DEFINITION 1.4. We will call a cocycle α an *almost coboundary* if it is cohomologous to a cocycle with constant coefficients, i.e. if there exists a homomorphism $\phi : \Gamma \to G$ such that for some measurable $\psi : X \to G$

$$\alpha(x,y) = \psi^{-1}(x)\phi(\gamma)\psi(S_\gamma x) \tag{1.4}$$

For a given cocycle, it is natural to ask whether it is a coboundary or at least an almost coboundary. The answer to this question depends on the solvability of equations (1.3) and (1.4) for ψ. We will refer to such equations as *cohomological equations.*

Since the function defined by (1.3) is always a cocycle, it is easy to construct a lot of cocycles which are coboundaries. As algebraic intuition would suggest, and as is confirmed by the discussion below, coboundaries should be regarded as trivial cocycles. As long as homomorphisms of Γ into G are known one finds many almost coboundaries too. The existence of cocycles other than almost coboundaries in many cases represents a formidable problem. This may not sound surprising to those who are familiar with cohomology theories in algebra and topology. However, the structure (or rather the absence of any reasonable structure) of the set of all cohomology classes of cocycles in many cases is strikingly different from what might be expected from familiar analogies.

If G is an abelian group then the product of two cocycles (coboundaries) is again a cocycle (corr. coboundary); thus the cocycles form a group and the coboundaries a subgroup. Hence one can define the corresponding (first) cohomology group.

If $\Gamma = \mathbb{Z}$, all cocycles can be described rather easily. Namely, the cocycles are in a one-to-one correspondence with measurable functions $h : X \to G$. The formula

$$\alpha(x,n) = \begin{cases} h(x)h(Sx)\dots h(S^{n-1}x) & n \geq 0 \\ h^{-1}(S^{-1}x)\dots h^{-1}(S^n x) & n < 0 \end{cases} \tag{1.5}$$

determines a cocycle and every cocycle α can be represented in this way by making $h(x) = \alpha(x, 1)$. A similar description is possible for the continuous time case ($\Gamma = \mathbb{R}$), subject to certain inessential restrictions (cf. Section 1.3).

The question of whether a cocycle over a $\mathbb{Z}$ action is a coboundary will be one of our central topics. The classification, up to cohomology, of cocycles over a $\mathbb{Z}$ action in a purely measurable setting does not make any sense as results of the next chapter will demonstrate. However, as will be seen in Chapter 3, in some finer categories it does become feasible for particular cases.

As we have noted, the construction of cocycles other than almost coboundaries for larger groups may be difficult. This is related to various rigidity phenomena which assert that within various classes any cocycle is an almost coboundary. For actions of some sufficiently large groups, such as semisimple Lie groups or lattices in such groups the situation is strikingly different from the classical cases and rigidity phenomena appear already in the measurable category, the best known example being Zimmer cocycle superrigidity theorem [**Z1**] [**FK**]. For actions of amenable groups the orbit equivalence theory [**CFW**] implies that the situation in measurable category is essentially as chaotic as for the classical cases. However for abelian groups of higher rank, e.g. $\Gamma = \mathbb{R}^h$ on $\mathbb{R}^h$ $h \geq 2$ nontrivial rigidity phenomena appear in Hölder and smooth categories [**KSp1**] [**KSch**] [**Sch1**].

The following construction, which is sometimes called the Mackey range [**Z1**], [**FK**], [**HaK**] allows us to associate with a G cocycle over a right Γ action, a left action of G. It generalizes the notion of induced action well known in the theory of group representations as well as constructions of the special flow (flow under a function), and the induced and special (integral) automorphisms, familiar in ergodic theory.

Any right Γ action $S = \{S_\gamma\}_{\gamma\in\Gamma}$ and G cocycle α over S determine a G extension $S^\alpha = \{S^\alpha_\gamma\}_{\gamma\in\Gamma}$ of S which acts on $X \times G$ by the following formula

$$S^\alpha_\gamma(x, g) = (S_\gamma x, g\alpha(x, \gamma)) \tag{1.6}$$

The cocycle equation (1.1) is equivalent to the group property for the extension

$$S^\alpha_{\gamma_1} S^\alpha_{\gamma_2} = S^\alpha_{\gamma_2\gamma_1}$$

since

$$S^\alpha_{\gamma_1} S^\alpha_{\gamma_2} = (S_{\gamma_1} S_{\gamma_2} x, g\alpha(x, \gamma_2)\alpha(S_{\gamma_2} x, \gamma_1)) = (S_{\gamma_2\gamma_1} x, g\alpha(x, \gamma_2\gamma_1)) = S^\alpha_{\gamma_2\gamma_1}(x, g).$$

There is a natural notion of isomorphism between two G extensions S^α and S^β of a Γ action S, namely an isomorphism which preserves every fiber $\{x\} \times G$, shifting it by an element of G

$$\psi(x, g) = (x, g\psi(x)). \tag{1.7}$$

Clearly

$$\psi \circ S^\beta_\gamma(x, g) = (S_\gamma x, g\beta(x, y)\psi(S_\gamma x))$$

and

$$S^\alpha_\gamma \circ \psi(x, g) = (S_\gamma x, g\psi(x)\alpha(x, \gamma)).$$

Thus two cocycles α and β are cohomologous if and only if S^α and S^β are isomorphic extensions and in particular α is a coboundary if and only if S^α is isomorphic to the trivial extension S^{id}:

$$S^{id}_\gamma(x, g) = (S_\gamma x, g) \tag{1.8}$$

http://dx.doi.org/10.1090/pspum/069/1858534-b

and α is an almost coboundary if and only if S^α is isomorphic to a product extension S^ϕ

$$S_\gamma^\phi(x,g) = (S_\gamma x, g\phi(\gamma)) \tag{1.9}$$

where $\phi : \Gamma \to G$ is a group homomorphism.

The group G acts on $X \times G$ in a natural way by the left shifts $L_{g_0}(x,g) = (x, g_0 g)$, and this action obviously commutes with any extension S^α of the form (1.6). In particular it preserves the decomposition of $X \times G$ into orbits of the action S^α and thus, at least formally, we can consider the factor action of G on the space of these orbits. We will denote this factor action by L^α. In general, the space of its orbits may not have a good measurable structure. Even if it does, the natural invariant measure may be infinite. For example, for the trivial extension (1.8) the factor is naturally isomorphic to the group G itself, and if G is not compact there is no finite translation invariant measure. In general one takes *the measurable hull* of the partition into the orbits of S^α, i.e. the measurable partition corresponding to the σ–algebra of measurable sets consisting of whole orbits of S^α. The factor action L^α restricted to this measurable hull is called the *Mackey range* of the cocycle α.

In the case of the constant coefficient cocycle ϕ (cf. (1.9)) the action L^ϕ is known as the action induced by the homomorphism ϕ. This action has a natural finite invariant measure if the subgroup $\phi(\Gamma) \subseteq G$ is closed, unimodular, and has cofinite volume in G. In particular, this is true if $\phi(\Gamma)$ is discrete and the factor $G/\phi(\Gamma)$ is compact. We proceed to discuss a natural generalization of this last condition to arbitrary cocycles.

1.2 Lipschitz cocycles, Pseudo-isometries and the Ambrose–Kakutani theorem. Let us consider two metrics d and d' on the same topological space X. we will call these metric *uniformly equivalent* if there are positive constants A, B and C such that for $x, x' \in X$,

$$Ad(x,x') - C < d'(x,x') < Bd(x,x') + C \tag{1.10}$$

A map f between metric spaces (X, d_X) and (Y, d_Y) is called a *pseudo isometry* if for some positive constants A, B and C, and for every $x, x' \in X$, $x \neq x'$,

$$Ad_X(x,x') - C < d_Y(f(x), f(x')) < Bd_X(x,x') + C.$$

It is clear that if we replace the metrics d_X and d_Y with uniformly equivalent metrics d'_X and d'_Y then the pseudo isometry f remains a pseudo isometry with respect to these new metrics.

Let us now return to group actions and cocycles. We will assume for the remainder of this section that Γ is a finitely generated discrete group and that G is a locally compact Lie group. Under these assumptions, both G and Γ possess natural classes of uniformly equivalent left-invariant metrics. Namely for Γ we take the word-length metric determined by any finite system of generators and for G, any left invariant Riemannian metric.

DEFINITION 1.5. A G cocycle α over a Γ action S is called a *Lipschitz cocycle* if for almost any $x \in X$, the map $\alpha_x : \alpha_x(\gamma) = \alpha(\gamma, x)$ is a pseudo isometry from Γ to G with constants A, B, C independent of x.

The special cases $\Gamma = \mathbb{Z}^n$ and $G = \mathbb{R}^n$ or $\mathbb{Z}^n$ are discussed in [**K2**]. In these cases the Lipschitz condition can be replaced by weaker "integrability" conditions.

Some important properties of Lipschitz cocycles in a more general setting can be derived from the results in the unpublished paper [**BKM**] which concern the behavior of nets in metric spaces under pseudo isometries. A *net* is a discrete subset of a metric space with the property that every point is a bounded distance away from it.

THEOREM 1.6. [**BKM**], *Let* Γ *be a discrete cocompact (and hence finitely generated) subgroup of a connected Lie group* G. *Then for every Lipschitz* G *cocycle* α *over a* Γ *action* S *on a Lebesgue space* (X, μ), *the extension* S^α *has a fundamental domain* $D = \bigcup_{x \in X} \{x\} \times D_x$ *where* D_x *is a bounded set in* G *whose boundary has Haar measure zero.*

By taking the restriction of $\mu \times \lambda_G$ to D , where λ_G is Haar measure in G, we obtain a natural finite invariant measure for the G action L^α in the factor of $X \times G$ into orbits of S^α. Obviously this conclusion holds for any cocycle β cohomologous to a Lipschitz cocycle α since the factor actions L^α and L^β for cohomologous cocycles α and β are in a natural correspondence.

Let us illustrate this situation by the classical case $\Gamma = \mathbb{Z}$, $G = \mathbb{R}$. According to (1.5) any $\mathbb{R}$ cocycle α over a $\mathbb{Z}$ action $S = \{S^n\}_{n \in \mathbb{Z}}$ is determined by a function $h : X \to \mathbb{R}$. If we assume that

$$0 < A < h(x) < B \tag{1.11}$$

then the cocycle determined by h is Lipschitz. In this case the extension of S to $X \times \mathbb{R}$ is generated by the automorphism

$$S^h(x,t) = (Sx, t + h(x)).$$

The set

$$D = \{(x,t) : 0 \leq t \leq h(S^{-1}x)\}$$

is a fundamental domain for S^h and the factor action $L^h = \{L^h_t\}_{t \in \mathbb{R}}$ acts on D as the vertical flow, where pairs of points of the form $(x, 0)$ and $(S^{-1}x, h(S^{-1}x))$ are identified. In this way, the factor action can be naturally identified with the special flow over S^{-1} built under the function $h(S^{-1}x)$ and the invariant measure is induced from $X \times \mathbb{R}$. As we will show in the next chapter, for every L^1 function g such that $\int_X g d\mu > 0$, one can find a function h satisfying (1.11) such that

$$g(x) = h(x) + \psi(Sx) - \psi(x) \tag{1.12}$$

for a measurable function ψ. In other words, the real-valued cocycle generated by any integrable function with non-zero average is cohomologous to a Lipschitz cocycle. For a generalization of this fact to other groups see [**HK1**].

The concept of a Lipschitz cocycle suggests a natural notion of equivalence for Γ actions.

DEFINITION 1.7. Let Γ be a finitely generated discrete group. Two ergodic right Γ actions $S = \{S_\gamma\}_{\gamma \in \Gamma}$ and $\mathcal{T} = \{T_\gamma\}_{\gamma \in \Gamma}$ are called *Kakutani equivalent* if there exists a Lipschitz Γ cocycle α over S such that the corresponding action L^α is isomorphic to the left action $T^{-1} = \{T_{\gamma^{-1}}\}_{\gamma \in \Gamma}$.

THEOREM 1.8. *(i) Kakutani equivalence is an equivalence relation.*
(ii) Let Γ be a cocompact discrete subgroup of a Lie group G. Then two ergodic right Γ actions S and $\mathcal{T}$ are Kakutani equivalent if and only if there exist Lipschitz G cocycles α and β over S and $\mathcal{T}$ correspondingly such that the corresponding left G actions L^{α} and L^{β} are isomorphic.

This theorem, which is a relatively easy corollary of the results in [**BKM**], provides a generalization of the classical result of Kakutani [**Ka**] (cf. also [**ORW**], [**K3**]) concerning the case $\Gamma = \mathbb{Z}$, $G = \mathbb{R}$.

Another interesting question is the extent to which the Ambrose–Kakutani theorem can be generalized. This theorem asserts in particular that every ergodic flow is isomorphic to a special flow over an ergodic automorphism. It is easy to see that in addition (1.11) can be satisfied, so that the Ambrose–Kakutani theorem essentially says that every ergodic flow ($\mathbb{R}$ action) is isomorphic to a flow obtained from a Lipschitz cocycle over an ergodic automorphism via the construction of Mackey range. The generalization of this theorem to the case $\Gamma = \mathbb{Z}^n$ and $G = \mathbb{R}^n$ is obtained in [**K2**]. For further discussion of that case see [**JR**].

On the other hand, for many sufficiently large groups the Ambrose–Kakutani theorem is not true and orbit equivalence classes tend to contain lots of information about the acting group (cf. [**Z1**], [**Z2**], [**Fu**]).

1.3 Cohomological equations for measure-preserving transformations and flows. From now on, we will consider only cocycles over $\mathbb{Z}$ and $\mathbb{R}$ actions, i.e. measure-preserving transformations and flows. Any G cocycle over a $\mathbb{Z}$ action is determined by a measurable function $h : X \to G$ via (1.5). For this reason, we will sometimes call the function h itself a cocycle.

1.3.1 Cocycles over flows. Let us give a similar characterization for a special class of cocycles over a flow. We will consider the case where G is a Lie group and we will assume the existence for a.e. x of the derivative

$$a(x) \overset{\text{def}}{=} \left.\frac{d\alpha(x,t)}{dt}\right|_{t=0}. \tag{1.13}$$

Here a is a measurable map from X to the Lie algebra $\mathfrak{g} = T_{id}G$ of G, and the cocycle α can be recovered from a by solving the differential equation

$$\frac{d\alpha(x,t)}{dt} = \mathcal{L}_{\alpha(x,t)}a(S_t x)$$

where $\mathcal{L}_g$ is the differential of the left shift by g, which carries the Lie algebra $\mathfrak{g}$ to T_gG.

This condition, differentiability along the orbits of an action, is not very restrictive. For example any real–valued cocycle α (and hence any vector–valued cocycle too) is cohomologous to a cocycle satisfying (1.13), namely the cocycle

$$\alpha_s(x,t) = \int_0^s \alpha(S_\tau x, t)d\tau.$$

for any positive s.

We are now going to review some of the main cases where cocycles and cohomology appear in ergodic theory. See [**HaK**] for an additional discussion.

1.3.2 Ergodicity and eigenvalues. Let us first observe that T is not ergodic if and only if the $\frac{\psi(Tx)}{\psi(x)} = 1$ has a non-constant solution. Furthermore, the eigenvalue problem for the ergodic transformation T is equivalent to the question of whether the constant S^1 cocycle λ, $\lambda(x,n)=\lambda^n$, is a coboundary. A similar characterization holds for flows.

1.3.3 Compact group extensions. Next consider the compact group extension S^h of a measure-preserving transformation S

$$S^h(x,g)=(Sx,gh(x)),$$

where $g \in G$, a compact group, and $h: X \to G$ is measurable. The extension S^h can be viewed as a measure-preserving transformation on $(X\times G, \mu\times\lambda_G)$, where λ_G is the normalized Haar measure on G. If the G cocycles defined by h_1 and h_2 are cohomologous then the corresponding extensions are isomorphic as measure-preserving transformations.

If G is abelian then

$$L^2(X\times G,\mu\times\lambda_G)=\bigoplus_{\chi\in G^*} H_\chi.$$

Here G^* is the group of characters of G and $H_\chi=\{f(x)\chi(g) : f\in L^2(X,\mu)\}$. This decomposition is orthogonal and is invariant under the unitary operator U_{S^h} corresponding to S^h. This means that any eigenfunction for S^h lies in one of the subspaces H_χ. This leads to the equation

$$\chi(h(x))f(Sx)=f(x) \tag{1.14}$$

for an invariant function $f(x)\chi(g)\in H_\chi$ and

$$\chi(h(x))f(Sx)=\lambda f(x) \tag{1.15}$$

for an eigenfunction $f(x)\chi(g)$. In other words, the questions reduces to determining whether the S^1 cocycle $\chi\circ h$ over S is correspondingly a coboundary or an almost coboundary.

1.3.4 Induced and special transformations. The induced map T_A is defined on a subset A of positive measure by

$$T_A x = T^{\min\{n>0:T^n x\in A\}}.$$

The special transformation $T_{n(\cdot)}$ (or integral transformation) is defined by a measure-preserving transformation T and a positive integrable integer valued "roof" function n on X. $T_{n(\cdot)}$ is a transformation of the set $X_{n(\cdot)}=\{(x,j): x\in X,\ 1\le j\le n(x)\}$ defined by

$$T_{n(\cdot)}(x,j)=\begin{cases}(x,j+1) & \text{if} \quad j<n(x)\\ (Tx,1) & \text{if} \quad j=n(x)\end{cases} \tag{1.16}$$

It preserves the measure induced on $X_{n(\cdot)}$ by the measure $\mu\times\lambda$ on $X\times\mathbb{Z}$, where λ is the uniform measure on $\mathbb{Z}$.

The induced and special transformations correspond to particular cases of the of the Mackey range construction, discussed in 1.1, for the case $\Gamma=G=\mathbb{Z}$. To obtain them in this way, we take for the induced map a $\mathbb{Z}$ cocycle over a transformation T determined by a function with values 0 and 1, (i.e. the characteristic function of a set), and for the special transformation, a $\mathbb{Z}$ cocycle with positive values. Let us note that a natural generalization of both constructions is provided by non-negative integer valued cocycles. Cocycles of this type play a central role in the theory of Kakutani equivalence (cf. [**K3**], [**ORW**]).

The isomorphism of induced maps T_A and T_B of the form $x \to T^{h(x)}x$ leads to the following cohomological equation for a $\mathbb{Z}$ valued function h:

$$\chi_A(x) = \chi_B(x) + h(Tx) - h(x). \tag{1.17}$$

Similarly, for special automorphisms with roof functions $n(x)$ and $m(x)$ this kind of isomorphism is equivalent to the existence of an integer valued solution ψ for the cohomological equation

$$n(x) = m(x) + \psi(Tx) - \psi(x). \tag{1.18}$$

The existence of an eigenfunction with eigenvalue λ for an induced or a special transformation also has a simple cohomological formulation. If the special transformation $T_{n(\cdot)}$ has an eigenvalue λ then

$$\lambda^{n(x)} = f(x)(f(Tx))^{-1} \tag{1.19}$$

where f is the restriction of an eigenfunction with eigenvalue λ to the base. In other words, the cocycle $\lambda^{n(x)}$ is a coboundary. Likewise for an induced map T_A, the existence of an eigenfunction with eigenvalue λ implies

$$\lambda^{\chi_A(x)} = f(x)(f(Tx))^{-1} \tag{1.20}$$

where f is the following extension to X of an eigenfunction g on $A : f(x) = \lambda g(T^{-i(x)}x)$, where $i(x)$ is the smallest positive number such that $T^{-i(x)}x \in A$.

1.3.5 Special flows. Two special flows built over T, $\{T_t^{h_1(\cdot)}\}_{t\in\mathbb{R}}$ and $\{T_t^{h_2(\cdot)}\}_{t\in\mathbb{R}}$ are isomorphic if h_1 and h_2, considered as $\mathbb{R}$ cocycles, are cohomologous, i.e. if

$$h_1(x) = h_2(x) + \psi(Tx) - \psi(x) \tag{1.21}$$

for a real valued measurable function ψ. In particular, if h is an almost coboundary, meaning the following cohomological equation has a measurable solution ψ

$$h(x) = h_0 + \psi(Tx) - \psi(x), \tag{1.22}$$

then the special flow $\{T_t^{h(\cdot)}\}_{t\in\mathbb{R}}$ is isomorphic to the suspension flow with a constant function. It is easy to see that $h_0 = \int_X h d\mu$. (see Proposition 2.1). The existence of an eigenfunction for the special flow implies that for some real r,

$$\exp\ irh(x) = f(x)f^{-1}(Tx) \tag{1.23}$$

for some measurable $f : X \to S^1$, so that the S^1 cocycle $\exp\ irh(x)$ is a coboundary. Let us note that by exponentiating (1.22) we obtain (1.23) for any $r = \frac{2\pi k}{h_0}$, $k \in \mathbb{Z}$; however, (1.23) may be true without (1.22).

Isomorphism conditions (1.17), (1.18), and (1.21) are only necessary but not sufficient. They correspond to special types of isomorphisms which, in a sense, preserve orbits and preserve order on these orbits (cf. [**K3**], §3). It follows from the theory of Kakutani equivalence that there are many other cases of isomorphism.

2. Structure of equivalence classes

Most of the material of this chapter is an amalgamation of results by Kočergin [**Kc**] and Ornstein and Smorodinsky [**OS**]. In [**HK1**] a completely different method is developed which yields most of these results. In the first section we arrive at the conclusion that in the measurable category each equivalence class of cocycles over an ergodic measure-preserving transformation is "uniformly distributed" within the subspace of all cocycles with a given average. Moreover, by modifying a cocycle on an arbitrarily small set, even subject to extra (sufficiently flexible) conditions, one can bring it within an arbitrary cohomology class. In the second section we push the same line of argument even further and show that every cocycle is cohomologous to a "regular" one, e.g. continuous or even continuously differentiable except for one point. Later in sections 3.3 and 3.4 we will show that the results of that kind can not be extended much further. Even a mild uniform condition stronger than continuity (e.g. a Hölder condition) in many cases changes the picture completely.

2.1 Majorization and density in L^1. We begin with two preliminary results.

PROPOSITION 2.1. *Let $h_1, h_2 : X \to \mathbb{R}^n$ be two measurable L^1 cocycles over an ergodic measure-preserving transformation T. If h_1 is cohomologous to h_2 then*

$$\int_X h_1 d\mu = \int_X h_2 d\mu.$$

PROOF. Let us assume that there is a measurable transfer function $\psi : X \to \mathbb{R}^n$

$$h_1(x) - h_2(x) = \psi(Tx) - \psi(x) \tag{2.1}$$

The statement of the proposition is obvious if ψ is integrable. In the general situation we apply the ergodic theorem for vector-valued functions to the function $h_1 - h_2$. Since by (2.1)

$$\frac{1}{n}\sum_{k=0}^{n-1} h_1(T^k x) - h_2(T^k x) = \frac{1}{n}(\psi(T^n x) - \psi(x))$$

so that by the ergodic theorem the left-hand side is close to $\int_X (h_1 - h_2) d\mu$ on a set of large measure. On the other hand, the right-hand side must be close to 0 except for a set of small measure. This implies that $\int_X h_1 d\mu = \int_X h_2 d\mu$. □

The next lemma is useful in handling cocycles with values in compact abelian groups.

Let $G = \mathbb{T}^k \oplus \bigoplus_{i=1}^{\ell} \mathbb{Z}/k_i$. There is a natural epimorphism $\exp : H = \mathbb{R}^k \oplus \mathbb{Z}^\ell \to G$ so that if the H cocycles h_1 and h_2 over a measure-preserving transformation T are cohomologous via the transfer function ψ then G cocycles $\exp \circ h_1$ and $\exp \circ h_2$ are cohomologous via $\exp \psi$.

LEMMA 2.2. *Given a G cocycle over T, $g : X \to G$ and $h_0 \in \mathbb{R}^{k+\ell}$ there exists an L^1 H cocycle h such that $\exp h = g$ and $\int_X h(x) d\mu = h_0$.*

REMARK. Although the cocycle h has discrete components we assume that the lattice $\mathbb{Z}^\ell$ is embedded into $\mathbb{R}^\ell$ so that integration makes sense.

PROOF. Since exp is a surjection on a compact subset of H we can find a bounded measurable branch for $\exp^{-1}$. By pulling back g along that branch we obtain a bounded and consequently integrable h such that $\exp h = g$. In order to correct the value of the integral we can add to h any integrable cocycle h' whose values lie in the kernel of exp. This kernel is a lattice in the continuous part of H and is a sublattice of finite rank in the discrete part. Obviously, one can find a lattice valued function with a given value of the integral so that the correction is possible. □

Now we proceed to the central result of this section. It is very close to Theorem 2 from [**Kc**]. However, the proof of that theorem indicated in [**Kc**] looks incomplete. We use instead a slight modification of the proof of a similar statement – Lemma 1 from [**OS**].

THEOREM 2.3. *Let f, g be two $\mathbb{R}$ or $\mathbb{Z}$ cocycles over an ergodic measure-preserving transformation T such that $\|f\|_{L^1} < \|g\|_{L^1}$. There exists a cocycle h cohomologous to f and such that $|h(x)| \le |g(x)|$ almost everywhere. If f is non-negative, h also can be chosen non-negative. Furthermore, if the function g is fixed then for every $\epsilon > 0$ one can find $\delta > 0$ such that if $\|f\|_{L^1} < \delta\|g\|_{L^1}$ then the transfer function ψ connecting f and h vanishes on a set of measure greater than $1-\epsilon$.*

PROOF. We will treat the cases of $\mathbb{R}$ and $\mathbb{Z}$ cocycles simultaneously.

Let $x \in X$. Consider the set of all pairs of integers (k, ℓ) satisfying the following properties

(i) $k \le 0 \le \ell$

(ii) For every $m = k,\ k+1, \dots, \ell-1$

$$\sum_{i=k}^{m} |f(T^i x)| \ge \sum_{i=k}^{m} |g(T^i x)|$$

(iii) $\sum_{i=k}^{\ell} |f(T^i x)| < \sum_{i=k}^{m} |g(T^i x)|$

It follows from the Birkhoff ergodic theorem and from the inequality $\|f\|_{L^1} < \|g\|_{L^1}$ that for almost every $x \in X$ at least one such pair exists and the values of k for all such pairs are bounded from below. Let $(k(x), \ell(x))$ be the pair satisfying (i)–(iii) with the minimal value of k. Such a pair is obviously unique. For $k = 0, -1, -2, \dots,$ $\ell = 0, 1, 2, \dots$ let

$$A_{k,\ell} = \{x \in X,\ k(x) = k,\ \ell(x) = \ell\}$$

The sets $A_{k,\ell}$ form a partition of X up to a set of measure 0 and $T^m A_{0,\ell} = A_{-m,\ell-m}$ for $m = 1, \dots, \ell$. We are going to construct a function h, such that $|h| \le |g|$ and

$$\sum_{i=k(x)}^{\ell(x)} h(T^i x) = \sum_{i=k(x)}^{\ell(x)} f(T^i x) \tag{2.2}$$

The last condition implies that h is cohomologous to f via the transfer function

$$\psi(x) = \sum_{i=k(x)}^{0} h(T^i x) - f(T^i x). \tag{2.3}$$

In particular, $\psi = 0$ on the set $\bigcup_{k=0}^{\infty} A_{k,0}$. Let us show how to construct h satisfying (2.2). The solutions are slightly different for the real-valued and integer-valued cocycles. In the real case the easiest way is to put

$$h(x) = |g(x)| \frac{\sum_{i=k(x)}^{\ell(x)} f(T^i x)}{\sum_{i=k(x)}^{\ell(x)} |g(T^i x)|}.$$

Condition (iii) implies that $|h| < |g|$. Condition (2.2) follows from this definition automatically.

In the case of integer valued cocycles let us consider the following measurable sets

$$B_{\ell, f_0,\dots,f_\ell, g_0,\dots,g_\ell} = \{x \in A_{0,\ell}, f(T^i x) = f_i,\ |g(T^i x)| = g_i,\ \ i = 0, \dots, \ell\}$$

Since by (iii) $\sum_{i=0}^{\ell} |f_i| < \sum_{i=0}^{\ell} g_i$ one can find integers $h_0, \dots, h_\ell$ of the same sign such that $|h_i| \le g_i$ $i = 0, \dots, \ell$ and $\sum_{i=0}^{\ell} h_i = \sum_{i=0}^{\ell} f_i$. We put for $x \in B_\ell, f_0, \dots, f_\ell, g_0, \dots, g_\ell$, $i = 0, \dots, \ell$, $h(T^i x) = h_i$. Condition (2.2) is obviously satisfied.

It remains to estimate the measure of the support of the transfer function ψ defined by (2.3). Since $\psi = 0$ on the set $\bigcup_{k=0}^{\infty} A_{k,0}$ it is enough to estimate the measure of that set. We have from (ii)

$$\begin{aligned} \|f\|_{L^1} &\ge \int_{X \setminus \cup A_{k,0}} |f(x)| d\mu = \sum_{\ell=0}^{\ell} |f(T^i x)| d\mu \ge \\ &\ge \sum_{\ell=0}^{\ell} \int_{A_{0,\ell}} \sum_{i=0}^{\ell-1} |g(T^i x)| d\mu = \int_{X \setminus \cup A_{k,0}} |g(x)| d\mu \end{aligned} \tag{2.4}$$

Since $|g|$ is an integrable function there is a function $\omega(\epsilon)$ decreasing to zero at $\epsilon \to 0$ such that if $\mu(B) < \epsilon$ $\int_B g d\mu < \omega(\epsilon)$. Thus (2.4) implies that $\|f\|_{L^1} > \omega(\mu(\text{supp } \psi))$. $\square$

Next two corollaries are also valid for both $\mathbb{R}$ and $\mathbb{Z}$ cocycles over an ergodic measure-preserving transformation T.

COROLLARY 2.4. *Every L^1 cocycle f is cohomologous to a bounded cocycle.*

COROLLARY 2.5. *Suppose f and g are L^1 cocycles such that $\int_X f d\mu = 0$ and $\|g\|_{L^1} > 0$. Then there exists an L^1 cocycle h cohomologous to f such $|h(x)| \le |g(x)|$ almost everywhere.*

There is one more corollary for $\mathbb{R}$-cocycles.

COROLLARY 2.6. *Given any two L^1 cocycles f and g with $\int_X f d\mu = \int_X g d\mu$ and $\epsilon > 0$, there exists an L^1 cocycle h cohomologous to f such that $|h(x) - g(x)| < \epsilon$.*

Furthermore, h can be required to coincide with g on a given set of measure less than 1.

A counterpart of that statement for $\mathbb{Z}$-cocycles is the following.

COROLLARY 2.7. *Given any two integrable $\mathbb{Z}$ cocycles f and g with $\int_X f d\mu = \int_X g d\mu$, there exists an integrable $\mathbb{Z}$ cocycle h cohomologous to f and such that*

$$|h - g| \leq 1.$$

Furthermore, h can be required to coincide with g on a given set A of measure less than 1. *If g is not constant outside of A the range of h may be required to be contained in the range of g.*

PROOF OF COROLLARIES 2.4–2.7. Corollary 2.4 follows directly from Theorem 2.3 if we take $g = (1+\epsilon)\int_X f d\mu$ for a positive ϵ.

To obtain Corollary 2.5 it is enough to show that f is cohomologous to a function with arbitrary small L^1-norm, because then Theorem 2.3 directly applies. To do that we represent $f = f_+ - f_-$ where $f_+ = \max(f, 0)$. Since $\int_X f d\mu = 0$, $\|f_+\|_{L^1} = \|f_-\|_{L^1}$. Let us fix an $\epsilon > 0$ and apply Theorem 2.3 to the pair f_-, $(1+\epsilon)f_+$. Thus, the function f_- is cohomologous to a nonnegative function h such that $h(x) \leq (1+\epsilon)f_+(x)$ and consequently $f = f_+ - f_-$ is cohomologous to $f_+ - h > -\epsilon f_+$. The last inequality implies that $\|f_+ - h\|_{L^1} = \int |f_+ - h| d\mu = 2\int \max(0, h - f_+) d\mu < 2\epsilon \|f_+\|_{L^1}$. Since ϵ can be chosen arbitrarily small this finishes the proof of Corollary 2.5.

Corollary 2.6 follows from the previous one applied to the pair of functions $f - g$ and s where s is nonnegative, less than ϵ everywhere and is equal to zero on the given set. Thus, $f = g + (f - g)$ is cohomologous to $g + h$ where $|h(x)| \leq s(x)$.

A very similar argument applies to Corollary 2.7. Here we take as s a characteristic function of a set $B \subset X \backslash A$ such that the values of g on B are not equal to $\sup_{X\backslash A} g$. □

Applying Theorem 2.3 and its four corollaries to each coordinate of a cocycle with values in $H = \mathbb{R}^k \oplus \mathbb{Z}^\ell$ we obtain similar results for cocycles with values in H. In particular, the following theorem follows immediately.

THEOREM 2.8. *Given a set $U \subseteq X$, $\mu(U) < 1$, a measurable function $f : U \to H = \mathbb{R}^k \oplus \mathbb{Z}^\ell$, a vector $h_0 \in \mathbb{R}^{k+\ell}$ and an L^1 H cocycle g over an ergodic measure-preserving transformation T, the set of all cocycles cohomologous to g is dense in the set*

$$A(f, h_0) = \{h \in L^1(X, H),\ h = f \ \text{ on } U,\ \int_X h d\mu = h_0\}.$$

In particular, for $U = \emptyset$ we obtain that every cohomology class is dense in the subspace of all cocycles with a fixed average.

This theorem together with Lemma 2.2 implies a similar but even more universal density result for cocycles with values in a compact abelian group of the form

$$G = \mathbb{T}^k \times \bigoplus_{i=1}^{\ell} \mathbb{Z}/k_i.$$

COROLLARY 2.9. *Given a set $U \subseteq X$, $\mu(U) < 1$, a measurable function $f : U \to G$ and a measurable G cocycle g over T, the set of all cocycle cohomologous to g is dense in the L^1 topology in the set*

$$A(f) = \{h : h = f \text{ on } U\}.$$

In particular for $U = \emptyset$ we obtain that every cohomology class of G cocycles is dense in the space of all measurable cocycles.

2.2 Continuous and almost differentiable representations. In this section we consider only real-valued cocycles.

THEOREM 2.10. *Let $\mathcal{L} \subset L^1\ (X, \mu)$ be a linear subspace of L^1 dense in the L^1 topology and closed in the L^∞ topology (uniform convergence almost everywhere). Then for every $f \in L^1(X, \mu)$ the set $\mathcal{L}_f = \{h \in \mathcal{L},\ h$ is cohomologous to $f\}$ is dense in the L^∞ topology in the set $\{h \in \mathcal{L},\ \int h d\mu = \int f d\mu\}$.*

PROOF. Let us fix a function $h \in \mathcal{L}$ such that $\int f d\mu = \int h d\mu$ and $\epsilon > 0$. By Corollary 2.6 one can find a cocycle f_1 cohomologous to f and such that $|f_1 - h| < \frac{\epsilon}{2}$. Then we began to apply Theorem 2.3 inductively. First we approximate f_1 by a function $h_1 \in \mathcal{L}$ such that

$$\|h_1 - f_1\|_{L^1} < \frac{\delta_1 \epsilon}{4}$$

where δ_1 is chosen sufficiently small to ensure that, by Theorem 2.3, $h_1 - f_1$ is cohomologous to a function r_1 such that $|f_1| < \frac{\epsilon}{4}$ via a transfer function ψ_1 whose support has measure less than $\frac{1}{2}$. The function $h_1 + r_1 \stackrel{\text{def}}{=} f_2$ is thus cohomologous to f_1 and consequently to f. Then we approximate f_2 by $h_2 \in \mathcal{L}$ such that $\|f_2 - h_2\|_{L^1} < \delta_2 \frac{\epsilon}{8}$ with an appropriately chosen δ_2 so that $f_2 - h_2$ is cohomologous to a function r_2, $|r_2| < \frac{\epsilon}{8}$ via a transfer function supported by a set of measure less than $\frac{1}{4}$, denote $h_2 + r_2 = f_3$, etc. In the limit we obtain

$$f' = \lim f_n = \lim h_n$$

and in both cases the convergence is uniform. Since $h_n \in \mathcal{L}$ and $\mathcal{L}$ is L_∞ closed, $f' \in \mathcal{L}$. On the other hand, since the transfer function connecting f_n and f_{n+1} has support of measure less 2^{-n}, by the Borel–Cantelli Lemma the sequence of transfer functions between f and f_n converges in probability to a transfer function between f and f'. □

COROLLARY 2.11. **[OS]** *Let X be a compact metric space, μ be a Borel probability nonatomic measure on $X, T : X \to X$ be a measure-preserving transformation (not necessarily continuous). Then every real-valued cocycle $f \in L^1(X, \mu)$ is cohomologous to a continuous cocycle. Moreover the set of continuous cocycles cohomologous to f is dense in uniform topology in the space of all continuous functions with the same integral as f.*

This statement follows immediately from Theorem 2.10 if we put $\mathcal{L} = C(X)$, the space of all continuous functions.

Corollary 2.11 can be strengthened by specifying the values of a continuous function cohomologous to f on any closed set F so that $\mu(X \backslash F) > 0$. The proof repeats that of Theorem 2.10 with an extra observation that f_1 may be already made to coincide with the given function on F and all the successive approximations may be chosen in order not to change that.

Pushing the method described above a bit further one obtains the following result which looks quite striking at first glance.

THEOREM 2.12. *Let M be a compact differentiable manifold, μ be a Borel probability measure on M, $T : M \to M$ be a measure-preserving transformation. Then every real-valued cocycle $f \in L^1(M, \mu)$ is cohomologous to a continuous cocycle $\bar{f}$ which is continuously differentiable except at a single point.*

SKETCH OF PROOF. First one finds a continuous cocycle f_1 cohomologous to f which is continuously differentiable outside a ball B_1 of radius, say, $1/2$ and can be extended to a continuously differentiable function. This is possible by a stronger version of Corollary 2.11 mentioned above. Then one approximates f_1 in uniform topology by a continuously differentiable cocycle g_1 which coincides with f outside B_1. If the L^1 norm of $f_1 - g_1$ is small enough one can find a cocycle f_2 cohomologous to f_1 (and hence to f) which coincides with f_1 outside a smaller ball $B_2 \subset B_1$ of radius $1/4$ and extends to a continuously differentiable function and such that the support of the transfer function ψ_1 has measure less than $1/2$. Continuing by induction one constructs on the nth step the cocycle f_n continuously differentiable outside of ball $B_n \subset B_{n-1}$ of radius 2^{n+1} which coincides with f_{n-1} outside of the ball B_{n-1} and extends to a continuously differentiable function and such that a transfer function ψ_n connecting f_n with f_{n-1} is supported on a set of measure less than 2^n. In the limit the function $\bar{f} = \lim_{n\to\infty} f_n$ is continuous everywhere and continuously differentiable outside of the single point $\bigcap_{n=1}^{\infty} B_n$. By the Borel–Cantelli lemma the series $\sum_{n=1}^{\infty} \psi_n$ converges and hence gives a transfer function between f_1 and $\bar{f}$. Since f_1 is cohomologous to f this finished the proof. □

REMARK. As many instances discussed in the next chapter will show, the single point of nondifferentiablity cannot be removed.

The transfer functions involved in the equivalence between continuous cocycles will very often be discontinuous, even for a homeomorphism.

For a homeomorphism f we will denote the set of f invariant Borel probability measures by $\mathcal{M}(f)$.

PROPOSITION 2.13. *Let f be a homeomorphism of a compact metric space X. The following three subspaces of $C(X)$ coincide:*
(i) $E_1 = \{\phi : \int \phi d\nu = 0 \text{ for all } \nu \in \mathcal{M}(f)\}$
(ii) $E_2 = \bar{\mathcal{C}}$, where $\mathcal{C} = \{\phi : \phi = h \circ f - h \text{ for a continuous function } h\}$
(iii) $E_3 = \bar{\mathcal{B}}$, where $\mathcal{B} = \{\phi : \phi = h \circ f - h \text{ for a bounded Borel function } h\}$.

PROOF. Obviously $E_3 \supseteq E_2$; since the space E_1 is closed we can apply Proposition 2.1 and conclude that $E_1 \supseteq E_3$. Thus it is enough to show that $E_2 \supseteq E_1$. This is a fairly straightforward argument using duality and the Hahn-Banach Theorem. We will present it here, because the same argument will appear several times in subsequent discussions. Consider the continuous linear operator $A : C(X) \to C(X)$ such that $(A\phi)(x) = \phi(f(x)) - \phi(x)$. E_2 is the closure of the image of A. The dual operator A^* acts on the dual space $C^*(X)$. Since every element of $C^*(X)$ is the difference of two measures, $\mathcal{M}(f)$ spans KerA^*. Thus $\chi \in$ Ker A^* if and only if for

http://dx.doi.org/10.1090/pspum/069/1858535

every $\phi \in C(X)$,

$$(A^*\chi)\phi = \chi(A\phi) = 0.$$

and $\phi \in E_1$ if and only if for every $\chi \in \mathrm{Ker}A^*$, $\chi(\phi) = 0$. In other words $\chi(\phi) = 0$ as soon as χ annihilates $\mathrm{Im}A$. By the Hahn-Banach Theorem, if $\phi \notin E_2$ one can construct χ such that $\chi(\phi) = 1$ and $\chi(\mathrm{Im}A) = 0$. □

REMARK. If the transformation f is minimal, i.e. all of its orbits are dense, then $\mathcal{C} = \mathcal{B}$. (Gottshalk, Hedlund, see e.g. [**KH**], Theorem 2.9.4.)

In view of Theorem 2.10 and Proposition 2.13, we can see that unless f is uniquely ergodic there are many continuous cocycles which are coboundaries via discontinuous and even unbounded transfer functions. Theorem 4.2 below shows that this is the case in the uniquely ergodic situation as well. At the end of section 3.3 we will give an example of a topologically transitive but not minimal transformation and a naturally defined continuous function which is a coboundary with a bounded but discontinuous transfer function.

3. Rigidity and stability

We proceed to explore various situations where equivalence classes of cocycles have a reasonable structure. As the discussion in the previous chapter shows, one can hope to find such situations only if the topology in the corresponding space of cocycles is considerably stronger than the topology of uniform convergence. The most natural situations to look upon are analytic, smooth, Lipschitz or Hölder cocycles over a transformation or a flow preserving the corresponding structure. Two basic concepts for the subsequent discussion are those of rigidity and stability. We will define these notions for cocycles over actions of fairly general groups but our discussion will be almost completely restricted to the case of a single transformation ($\mathbb{Z}$-action), with several glimpses of the case of flows.

3.1 Definitions. Let H be a topological space of G-cocycles over a measurable measure-preserving action S of a locally compact second countable group Γ. We will assume that the topology in H is stronger than the topology of convergence in probability uniform on compact subsets of Γ. In other words, if a sequence of cocycles $\alpha_n \in H$ converges in H to a cocycle α, then for any compact subset $K \subseteq \Gamma$ and any neighborhood U of the identity in G, and for any sufficiently large n $\alpha_n^{-1}(x,\gamma)\alpha(x,\gamma) \in U$ for all $\gamma \in K$ and for a set of x close to full measure.

DEFINITION 3.1. The space H is *rigid with respect to the action* S if every cocycle $\alpha \in H$ is an almost coboundary. If in addition, for every $\alpha \in H$, the transfer function ψ which establishes the equivalence between α and a constant cocycle, can be chosen from a given class Ψ of maps from X to G, we will call H *Ψ-rigid with respect to* S .

It is natural to allow a certain freedom of terminology and to speak of continuous, Hölder, smooth, (C^r or C^∞), or analytic rigidity when the transfer functions possess the corresponding properties.

For $\mathbb{R}^n$-cocycles over a measure-preserving transformation rigidity means that all of the functions from H with the fixed average are cohomologous to each other.

For actions of certain groups, e.g. semisimple Lie groups of rank greater than one, or lattices in such groups, rigidity occurs for large classes of measurable cocycles

[**Z1**], but for actions of $\mathbb{Z}$, and more generally, of amenable groups, it can appear only if H has a sufficiently fine topology. There is however a big difference between the classical case ($\mathbb{Z}$ and $\mathbb{R}$ actions) and the situation for actions of higher rank abelian groups.

In the latter case cocycle rigidity in Hölder, smooth and analytic categories is quite widespread, appears for many natural examples of actions, both smooth [**KSp1**] and symbolic [**Sch2**], and is related with hyperbolic behavior in the smooth situation and with expansiveness in the symbolic one. Cocycle rigidity is a key link in the whole panoply of rigidity results for such actions including local and global differentiable rigidity [**KSp2**].

For the classical cases Hölder cocycle rigidity very likely never happens and smooth cocycle rigidity appears for Diophantine translation of the torus and probably only there. We will discuss this issue in the next section.

A property which is much more common and definitely more useful in the classical cases is stability:

DEFINITION 3.2. Under the same assumptions as in Definition 3.1, we will call the space H *stable with respect to* S if every class of cohomologous cocycles in H is closed.

In this definition we assume that arbitrary measurable G-valued functions are allowed as transfer functions between cocycles from the space H. However, equivalence classes within H may shrink considerably if we put certain restrictions on the transfer functions. In particular, it is conceivable that unrestricted equivalence classes are not closed while the restricted ones are and vice versa. This possibility suggests certain ramifications of the concept of stability. As before, let Ψ be a certain class of maps from X to G.

DEFINITION 3.3. The space H is called Ψ-*stable with respect to the action* S if every class of cocycles from H, cohomologous via transfer functions from Ψ, is closed.

Let us note that in general, Ψ-stability may not imply stability.

DEFINITION 3.4. The space H is called Ψ-*effective with respect to* H if for any two cohomologous cocycles $\alpha, \beta \in H$ the transfer function ψ belongs to Ψ.

Obviously, if H is Ψ-effective then Ψ-stability implies stability.

Terms such as smooth, analytic or Hölder stability and effectiveness have the obvious meaning. We observe that if equivalent cocycles are assumed to be defined everywhere and the transfer function is only assumed to be measurable, then the corresponding cohomological equation needs only be satisfied almost everywhere.

If both spaces H and Ψ consist of continuous maps, then there are certain obvious invariants which are preserved under cohomology via a transfer function from Ψ. For example, if G is a subgroup of $\mathbb{R}^n$ and ν is any measure on X invariant with respect to the action S, then for any $\gamma \in \Gamma$ the average

$$\int_X \alpha(x,\gamma)d\nu$$

is a cohomology invariant (Proposition 2.13). This remains true even if ψ is a bounded Borel function. These remarks together with Corollary 2.11 show that for continuous cocycles over transformations which are not uniquely ergodic equivalence classes with continuous or bounded Borel transfer functions must be much smaller than the unrestricted equivalence classes.

For $\Gamma = \mathbb{Z}$ and arbitrary G, the conjugacy class in G of the product of the values of the cocycle along any periodic orbit is invariant. In sections 3.3 and 3.4 we will discuss situations when these invariants determine the equivalence classes, thus producing stability.

While no general theory exists for stability and effectiveness of smooth or other natural classes of cocycles, there is a good understanding of these phenomena for smooth dynamical systems with hyperbolic behavior and their counterparts in topological and symbolic dynamics. To a lesser extent stability and effectiveness are understood for partially hyperbolic systems and for parabolic systems with sufficiently regular features. For a general discussion of hyperbolic, partially hyperbolic, parabolic and elliptic behavior in dynamics, see [**HaK**]. A general outline of stability and effectiveness properties is also discussed there. Presently we will glance at some characteristic phenomena which appear for each type of behavior.

3.2 Translations of the torus and smooth rigidity.

3.2.1 Diophantine and Liouvillean numbers. The example which we are going to describe plays a basic role in the theory of perturbations of completely integrable Hamiltonian systems. The rigidity result was known to Kolmogorov in the early fifties [**Ko**].

Let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{T}^m$ so that each λ_i is a complex number of modulus 1. We will call the vector λ *Diophantine* if for some positive k and c and for arbitrary integers $n_1, \ldots, n_m$ such that $\sum_{j=1}^{m} |n_j| > 0$,

$$\left|\sum_{j=1}^{m} \lambda_j^{n_j} - 1\right| > c\left(\sum_{j=1}^{m} |n_j|\right)^{-k}. \tag{3.1}$$

If $\lambda_j = \exp 2\pi i \alpha_j$, $j = 1, \ldots, m$, then (3.1) is equivalent to the following condition for the vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ of phases: for arbitrary integers $n_1, \ldots, n_m$ such that $\sum_{j=1}^{m} |n_j| > 0$ and for every integer n,

$$\left|\sum_{j=1}^{m} \alpha_j n_j - n\right| > c\left(\sum_{j=1}^{m} |n_j|\right)^{-k}.$$

The vector of phases which is of course defined up to an integer vector is also called Diophantine. Often the vectors α (and corresponding λ's) which are not Diophantine are called *Liouvillean.*

Let $R_\lambda : \mathbb{T}^m \to \mathbb{T}^m$ be the rotation* of the torus by λ,

$$R_\lambda(z_1, \ldots, z_m) = (\lambda_1 z_1, \ldots, \lambda_m z_m).$$

This is the prototype example of elliptic behavior in dynamics (see [**HaK**], Chapter 7).

*This terminology suits multiplicative notations on the torus we used so far. When additive notations are used as later in this section the same map R_λ is called a *translation* and is denoted by T_α.

3.2.2 Cohomological equation over a rotation.

THEOREM 3.5. *If λ is Diophantine then for every C^∞ function (real or complex) h on $\mathbb{T}^n$ there exists a C^∞ function ψ (real or complex) such that*

$$h(z) - \int_{\mathbb{T}^m} h d\mu = \psi(R_\lambda z) - \psi(z) \tag{3.2}$$

where μ is Lebesgue measure on $\mathbb{T}^m$.

Furthermore, Diophantine vectors are the only ones for which this property holds.

PROOF. We will use Fourier expansion. For $n_1, \ldots, n_m \in \mathbb{Z}$, let us denote $\chi_{n_1,\ldots,n_m} = \prod_{i=1}^m z_i^{n_i}$. Then

$$h = \sum_{n_1,\ldots,n_m} h_{n_1,\ldots,n_m} \chi_{n_1,\ldots,n_m}$$

and h is a C^∞ function if and only if the Fourier coefficients $h_{n_1,\ldots,n_m}$ decrease faster than any power of $|n_1| + \cdots + |n_m|$.

Let us look for the Fourier expansion for the solution ψ of (3.2).

$$\psi = \sum_{\substack{n_1,\ldots,n_m \\ \Sigma|n_i|>0}} \psi_{n_1}, \ldots, n_m \, \chi_{n_1,\ldots,n_m}.$$

Its Fourier coefficients $\psi_{n_1,\ldots,n_m}$ are found from (3.2):

$$\psi_{n_1,\ldots,n_m} = \frac{h_{n_1,\ldots,n_m}}{\prod_{i=1}^m \lambda_i^{n_i} - 1}. \tag{3.3}$$

If h is C^∞ and λ is Diophantine, then it follows from (3.1) that the coefficients $\psi_{n_1,\ldots,n_m}$ decrease faster than any power of $|n_1| + \cdots + |n_m|$, so that ψ defined through (3.3) is a C^∞ function.

Conversely, if λ is Liouvillean, then there exists a sequence of vectors $n^k = (n_1^k, \ldots, n_m^k)$ such that for any natural number s as $k \to \infty$,

$$\left| \sum_{j=1}^m \lambda_j^{n_j^k} - 1 \right| = o \left(\sum_{j=1}^m |n_j^k| \right)^{-s}.$$

Define h by choosing its only nonzero Fourier coefficients as

$$h_{n_1^k,\ldots,n_m^k} = \left| \sum_{j=1}^m \lambda_j^{n_j^k} - 1 \right|^{1/2} \quad \text{for } k = 1, \ldots.$$

These Fourier coefficients decrease faster than any power of $\sum_{j=1}^m |n_j^k|$, hence the function h is C^∞. On the other hand, for a solution ψ of the cohomological equation (3.2) the Fourier coefficients $\psi_{n_1^k,\ldots,n_m^k}$ given by (3.3) go to infinity as $k \to \infty$ which contradicts existence of even an L^1 solution to (3.2). $\square$

REMARKS. 1. In the Liouvillean case the Fourier series for h constructed above is lacunary. In the case of one variable Herman [**He1**] showed that if the cohomological equation (3.2) with an L^2 function h given by a lacunary series has a measurable solution, then the solution is actually L^2. Thus, in our case with $m = 1$ for a Liouvillean α there is no measurable solution at all. This can also be shown using the methods developed in Section 5.1 below.

2. Herman also obtained sharp results for the equation (3.2) for Diophantine α with functions h of finite regularity. Namely if h is C^r, (3.1) is satisfied and $r - k$ is not an integer, then there is a C^{r-k} solution, and if $r - k$ is an integer, there is a solution which is $C^{r-k-\epsilon}$ for any $\epsilon > 0$ ([**He2**], Proposition A.8.1). However, typically (for a dense G_δ of h's in the corresponding spaces) there is no $C^{r-k+\epsilon}$ solution for any positive ϵ and no C^{r-k} solution if $r - k$ is an integer ([**He2**], Proposition XIII.4.5).

3. Theorem 5.6 below contains a generalization of our argument for the Liouvillean case.

3.2.3 Smooth rigidity and effectiveness. Using the terminology introduced in section 3.1 we can reformulate Theorem 3.5 by saying that for a Diophantine λ, the rotation R_λ is both C^∞ rigid and C^∞ effective. We do not know any other examples of measure-preserving diffeomorphisms which possess these properties.

CONJECTURE 3.6. *Any C^∞ diffeomorphism of a compact connected manifold which is C^∞ rigid and C^∞ effective is C^∞ conjugate to a Diophantine translation of a torus.*

There are similar questions concerning rigidity in C^r category for finite r.

PROPOSITION 3.7. *A C^∞ rigid and C^∞ effective diffeomorphism f is uniquely ergodic and its only invariant measure is a volume with C^∞ density.*

PROOF. Unique ergodicity immediately follows from Proposition 2.13 and the density of C^∞ functions in the space of continuous functions. For, given two measures μ and ν one can always find a C^∞ function such that its integrals with respect to these measures are different. But if a function is cohomologous to a constant even via a continuous transfer function this constant must be equal to the integral of the function with respect to any invariant measure, a contradiction.

Now consider a measure μ given by a positive C^∞ density. We have $f_*\mu = J\mu$ where the Radon–Nikodym derivative J (called the Jacobian) is C^∞ and positive. The C^∞ function $\log J$ is by assumption cohomologous to a constant:

$$\log J(x) = H(fx) - H(x) + c.$$

But then the measure $\nu = \exp H\mu$ satisfies $f_*\nu = \exp c\nu$. Since the total measure is preserved this implies that $c = 0$, hence ν is invariant. □

COROLLARY 3.8. *A C^∞ rigid and C^∞ effective diffeomorphism f is minimal.*

A part of the conjecture is that a C^∞ rigid and C^∞ effective diffeomorphism have to act on a torus. A possible approach would be to try to prove that as a measure preserving transformation such a map has to have pure point spectrum. It is not difficult to show that every ergodic diffeomorphism with pure point spectrum and C^∞ eigenfunctions is conjugate to a translation on a torus. An alternative possibility would be a smoothly rigid diffeomorphism with a continuous part in the spectrum or one with pure point spectrum but with non-smooth eigenfunctions. Even without rigidity the last possibility constitutes an interesting question. Such "non-standard" models do exist for R_λ with some λ which are very well approximated by rational vectors [**AK**], but their existence even for a single Diophantine λ is an open question.

Now consider diffeomorphisms of the torus $\mathbb{T}^m$. We use additive notation now. Every such diffeomorphism is homotopic to an automorphism F_A, whose lift to $\mathbb{R}^m$ is the linear map given by A, an $m \times m$ matrix with determinant ± 1.

There are two extreme cases in which our conjecture can be proven. If the matrix A has no roots of unity among the eigenvalues then any diffeomorphism homotopic to F_A has infinitely many periodic points ([**KH**], Theorem 8.7.1) and hence no cocycle rigidity is possible.

PROPOSITION 3.9. [**LSa**] *A C^∞ rigid and C^∞ effective diffeomorphism f of $\mathbb{T}^m$ homotopic to identity is C^∞ conjugate to a Diophantine translation.*

PROOF. Since f is homotopic to identity its lift F to the universal cover $\mathbb{R}^m$ has the form $\mathrm{Id} + H$ where H is a periodic function, hence it projects to function h on the torus. Using rigidity and effectiveness we obtain $h = \psi \circ f - \psi + c$, where ψ is a C^∞ function and c is a constant. Lift ψ to the function Ψ on $\mathbb{R}^m$. Obviously $H = \Psi \circ F - \Psi + c$. Consider the map $S = \mathrm{Id} - \Psi : \mathbb{R}^m \to \mathbb{R}^m$. We have

$$S \circ F = \mathrm{Id} + H - \Psi \circ F = \mathrm{Id} + \Psi \circ F - \Psi + c - \Psi \circ F = \mathrm{Id} - \Psi + c = (\mathrm{Id} + c) \circ S$$

Let s be the projection of the map S to the torus. Projecting the last equality on the torus we obtain

$$s \circ f = T_c \circ s \tag{3.4}$$

where $T_c = \mathrm{Id} + c$ is the translation by c.

It remains to show that s is a diffeomorphism. Taking the derivative of (3.4) we see that the set of singular points of s is f–invariant and hence dense by Corollary 3.8. By the Sard Theorem the set of regular points is open and dense hence the set of singular points is empty. Thus s is a covering map but since it is homotopic to identity it is a diffeomorphism. By Theorem 3.5 c must be a Diophantine vector. □

Using a similar method one can treat the case where the matrix A is unipotent, i.e. all of its eigenvalues are equal to one. In this case one can show that C^∞ cocycle rigidity and effectiveness implies that the diffeomorphism is C^∞ conjugate to an affine map. Using the method of Section 3.6 below one can show that such a map with a unipotent $A \neq \mathrm{Id}$ has infinitely many invariant distributions and hence cannot be rigid.

The remaining difficult case is when A is reducible with eigenvalues of absolute value greater than one coexisting with roots of unity.

3.3 Stability of Hölder cocycles for transformations with specification. In this section we will consider a rather robust situation of cocycle stability in the context of topological dynamics. Dynamical conditions which appear in this and the next section are abstract versions of hyperbolicity. Thus in these sections we essentially describe certain aspects of cocycle stability and effectiveness for dynamical systems with hyperbolic behavior.

Let f be a homeomorphism of a compact metric space X.

DEFINITION 3.10. We will say that f satisfies the *strong specification property* if for any $\epsilon > 0$ there exists $N(\epsilon)$ such that for any collection of orbit segments

$$\{f^i(x_j)\}_{j=1,\dots,L,i=0,\dots,m_j-1}$$

and for any $n_1, \dots, n_L > N(\epsilon)$, there exists a periodic orbit

$$\{f^i y\}_{i=0,\dots,P-1}$$

of period

$$P = \sum_{j=1}^{L} m_j + n_j$$

such that for $j = 1, \dots, L$,

$$\mathrm{dist}(f^i x_j, f^{\ell(i,j)} y) < \epsilon$$

where

$$\ell(i,j) = i + \sum_{k=1}^{j-1} m_k + n_k.$$

The two main examples with this property are:

(i) A topological Markov chain (subshift of finite type) Σ_A, where A is a zero-one matrix such that for some n, A^n has all positive entries; the latter condition is called transitivity (see [**KH**], Section 1.2.).

(ii) The restriction of a diffeomorphism of a compact manifold to a locally maximal (basic) hyperbolic set Λ satisfying one of the following equivalent conditions ([**KH**], Section 18.3):

(a) $F\big|_\Lambda$ is topologically mixing, i.e. for every two nonempty open sets U and V, there exists $N(U,V)$ such that $f^n U \cap V \neq \emptyset$ for all $n > N(U,V)$.

(b) Every power of f has a dense orbit on Λ.

(c) At least one stable or unstable manifold for f is dense in Λ.

Now assume that the homeomorphism f satisfies the strong specification property.

DEFINITION 3.11. We will say that a vector-function $h : X \to \mathbb{R}^m$ satisfies a *dynamical Hölder condition* if there exist $\epsilon > 0$ and $K > 0$ such that for any n, if $x, y \in X$,

$$d_n^f(x,y) \le \epsilon \ \text{ then } \sum_{i=0}^{n-1} \|h(f^i x) - h(f^i y)\| \le K \tag{3.5}$$

The reason for calling this property a Hölder condition is that in both examples (i) and (ii) above, this property is satisfied by Hölder functions with respect to a natural metric. In the symbolic case (i) the metric in the space

$$\Omega_A = \{\omega = (\dots\omega_{-1}\omega_0\omega_1,\dots) : \omega_n \in \{0,\dots,N-1\},\ A_{\omega_{n-1}\omega_n} = 1,\ n \in \mathbb{Z}\}$$

can be chosen as $\operatorname{dist}(\omega,\omega') = \sum_{n=-\infty}^{\infty} \frac{|\omega_n - \omega'_n|}{\lambda^{|n|}}$ for any $\lambda > 1$. In the smooth case (ii) any metric on Λ induced by a smooth Riemannian metric on the ambient manifold can be chosen.

All vector-valued functions on X satisfying (3.5) with a fixed ϵ form a Banach space H^f_ϵ with respect to the norm

$$\|h\|^f_\epsilon = \max_{x\in X}\|h(x)\| + \sup_n \sup_{x,y: d^f_n(x,y)\le\epsilon} \sum_{i=0}^{n-1}\|h(f^i x) - h(f^i y)\|.$$

The spaces H^f_ϵ will serve as H in our study of stability of cocycles.

PROPOSITION 3.12. *If f satisfies the strong specification property and $h : X \to \mathbb{R}^m$ satisfies the dynamical Hölder condition then the following two properties are equivalent*

(i) $\sum_{i=0}^{n-1} h(f^i x) = 0$ *for every periodic point $x \in X$ such that $f^n x = x$.*

(ii) *There exists a bounded Borel function $\psi : X \to \mathbb{R}^m$ such that*

$$h(x) = \psi(fx) - \psi(x).$$

This proposition immediately implies the following stability result.

COROLLARY 3.13. *Let f be a homeomorphism of a compact metric space X satisfying the strong specification property and Ψ be the set of all bounded Borel maps from X to $\mathbb{R}^m$. Then the spaces H^f_ϵ are Ψ-stable.*

PROOF OF COROLLARY 3.13. By Proposition 3.12 two $\mathbb{R}^m$ cocycles $h_1, h_2 \in H^f_\epsilon$ are equivalent with the transfer function from Ψ if and only if the sums of their values along each periodic orbit are equal. Those conditions define a closed affine subspace of H^f_ϵ.

PROOF OF PROPOSITION 3.12. Condition (ii) implies (i) because if $f^n x = x$ then $\sum_{i=0}^{n-1} h(f^i x) = \psi(f^n x) - \psi(x) = 0$. The converse follows easily from Lemma 3.14 and 3.15 which are formulated and proved below.

LEMMA 3.14. *Under the assumptions of Proposition 3.12, if (i) is satisfied then there exists a constant $L > 0$ such that for every $x \in X$ and every positive integer n*

$$\left\|\sum_{i=0}^{n-1} h(f^i x)\right\| < L.$$

PROOF. We apply the specification property to the orbit segment $\{f^i x\}$ $i = 0, \dots, n-1$, with $\epsilon > 0$ small enough to use the dynamical Hölder condition for h. This gives us a periodic point y of period, say, $n + N(\epsilon)$ such that $d^f_n(x,y) < \epsilon$. We have

$$\sum_{i=0}^{n-1} h(f^i x) = \sum_{i=0}^{n+N(\epsilon)-1} h(f^i y) +$$
$$+ \sum_{i=0}^{n-1}(h(f^i x) - h(f^i y)) - \sum_{n}^{n+N(\epsilon)-1} h(f^i y).$$

The first sum is equal to zero by (i); the second is uniformly bounded by the dynamical Hölder property; the third contains a bounded number of terms each of them bounded by the sup $\|h\|$. This proves the lemma. $\square$

LEMMA 3.15. *(D. Rudolph). If $\{a_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers such that for any integers $n, m, n < m$,*

$$\left|\sum_{i=n}^{m-1} a_i\right| < L$$

then

$$a_n = b_{n+1} - b_n$$

where

$$b_n = -\sup_{N\in\mathbb{Z}} S_n^N$$

and

$$S_n^N = \begin{cases} \sum_{i=0}^{N-1} a_{n+i}, & N > 0 \\ 0, & N = 0 \\ -\sum_{i=1}^{-N} a_{n-i}, & N < 0 \end{cases}$$

so that b_n is uniformly bounded.

PROOF. One can easily see that

$$S_{n+1}^{N-1} = S_n^N - a_n$$

so that

$$b_{n+1} - b_n = -\sup_{N\in\mathbb{Z}} S_{n+1}^{N-1} + \sup_{N\in\mathbb{Z}} S_n^N = a_n. \square$$

In order to finish the proof of Proposition 3.12, we first use Lemma 3.14 and then apply Lemma 3.15 to every coordinate of h along every orbit of f. This gives us the solution $\psi = (\psi_1, \ldots, \psi_m)$ of the cohomological equation where

$$\psi_i(x) = -\sup_{N\in\mathbb{Z}} \begin{cases} \sum_{n=0}^{N-1} h_i(f^n x), & N > 0 \\ 0, & N = 0 \\ -\sum_{n=-1}^{N} h_i(f^n x), & N < 0. \end{cases} \tag{3.6}$$

These functions are obviously Borel, and by Lemma 3.14 they are bounded. $\square$

The proof above demonstrates in the simplest possible form a general and fruitful method of establishing triviality of real- or vector-valued cocycles over group actions. If we denote the cocycle $h(x, n)$ defined by a given function h then formula (3.6) gives the value of the ith coordinate of the transfer function simply as $-\sup_n h_i(x, n)$. Of course, such a definition is only possible if the values of the cocycle at every point are bounded. They do not have to be uniformly bounded though; the resulting transfer function would then be unbounded.

Moreover, there are situations when the set of values of the cocycle at a particular point is unbounded but the cocycle at the that point as a function of n (i.e. a function on $\mathbb{Z}$, or, for a more general action, function of the acting group), has some equivariant characteristics. Here are examples of such characteristics which are actually useful in ergodic theory and topological dynamics. We restrict ourselves to $\mathbb{R}$ valued cocycles over $\mathbb{Z}$ actions.

(i) Suppose the set of values of the cocycle at x has a Cesaro average or, more generally, can be averaged using some algorithmic procedure. If $A(x)$ is the value of such an average, then $-A(x)$ can be used as the solution of the coboundary equation. This construction can also be used for actions of amenable groups. Aside from an obvious generalization to vector valued cocycles it extends to situations where a natural notion of center of gravity can be introduced in the range of the cocycle.

(ii) Suppose there is a set of of values on n of positive lower density, say $d(x)$, for which the values of the cocycle are bounded from above and below. In this case one can define the distribution function $F_x(t)$ of the values of cocycle at x, namely, the lower density of a set for which the values of the cocycle are $\leq t$. This function is monotone non-decreasing and has asymptotic values $0 \leq F_x(-\infty) < F_x(\infty) \leq 1$. Now let

$$t(x) \stackrel{\text{def}}{=} \sup\{t : \ F_x(t) < \frac{F_x(-\infty) + F_x(\infty)}{2}\}.$$

By assumption $t(x)$ is a finite number and $t(f(x)) = t(x) + h(x)$ thus producing a solution of the coboundary equation.

In the next section we will show that in certain cases, including topological Markov chains (subshifts of finite type) and hyperbolic sets for diffeomorphisms, one can prove that the transfer function ψ is continuous and even Hölder. In general, however, this is not true. We will now describe a counterexample due to B. Marcus.

EXAMPLE. The space X will be a shift-invariant closed subset of the set of doubly infinite $(+1, -1)$ sequences, containing all sequences $\omega = \{\omega_n\}$, $n \in \mathbb{Z}$, such that for all $m, n \in \mathbb{Z}$, $m < n$,

$$\left|\sum_{i=n}^{n-1} \omega_i\right| \leq K \tag{3.7}$$

for a fixed positive integer $K \geq 2$. Naturally, the transformation will be the shift on two symbols σ_2. We will consider the zero coordinate ω_0 as a function on X, so that $\omega_0(\sigma_2^n \omega) = \omega_n$.

PROPOSITION 3.16. *The shift σ_2 restricted to the space X satisfies the strong specification property, the function ω_0 satisfies dynamical Hölder condition, and ω_0 is not a coboundary with a continuous transfer function.*

PROOF. If $\omega \in X$ is periodic, i.e. if $\omega_{n+k} = \omega_n$ for some k and for all n then $\sum_{i=0}^{k-1} \omega_i = 0$, since otherwise (3.7) cannot be true. The function ω_0 satisfies the dynamical Hölder condition because, as we mentioned before, this is true for the full shift space. The specification property can be established directly, but it is easier to deduce it from a general argument.

The map we are considering is a *sofic system*, i.e. a symbolic system which is continuous factor of a subshift of finite type. To see this, we take the shift on $K+1$ symbols $\{0, 1, \ldots, K\}$, and define the transition matrix $A = (a_{ij})$ where

$$a_{ij} = 1 \quad \text{if} \quad |i-j| = 1.$$

and is zero otherwise. Denote by σ_A the restriction of the shift to Ω_A. Every element $\alpha \in \Omega_A$ determines a sequence $\omega(\alpha)$ of $+1$ and -1, namely

$$\omega_i(\alpha) = \alpha_{i+1} - \alpha_i. \tag{3.8}$$

Obviously any $\omega(\alpha)$ satisfies (3.7) and conversely, for any sequence ω satisfying (3.7) we can find $\alpha \in \Omega_A$ such that $\omega = \omega(\alpha)$. Finally, it follows from the definition that any factor of a system with specification satisfies the specification property. Thus our system has specification.

We will now show that the function ψ defined by (3.6) for $h(\omega) = \omega_0$ is not continuous. For, if ω_0 is cohomologous to zero with a continuous transfer function ψ, then the cohomological equation can be lifted to a cohomological equation on Ω_A, so that the lift $\widetilde{\psi}$ of ψ is constant on preimages of points and is still continuous. But from (3.8), we see that $\omega_0(\alpha) = \alpha_1 - \alpha_0 = \alpha_0(\sigma_A\alpha) - \alpha_0(\alpha)$. Since we assume that $\omega_0(\alpha) = \widetilde{\psi}(\sigma_A\alpha) - \widetilde{\psi}(\alpha)$, then by the uniqueness, up to a constant c of a continuous (but not a bounded Borel!) transfer function, we have $\widetilde{\psi}(\alpha) = \alpha_0 + c$. This implies that α_0 is the same for any α projected to a given ω.

If $\sup\limits_{m<n} \left| \sum\limits_{i=m}^{n} \omega_i \right| < K$ then there exist more than one such α with different 0 coordinates. This shows that $\widetilde{\psi}$ cannot be continuous. □

3.4 Livshitz* theory.

3.4.1 Closing lemma and continuous solutions. Let us now show how a condition slightly different from the specification property, enables us to insure the continuity of the transfer function, and also to generalize the results of the previous section to cocycles with values in more general groups. Proposition 3.18 below is an abstract and somewhat diluted version of an extract from Livshitz's work [**L1**].

DEFINITION 3.17. A homeomorphism f of a compact metric space X satisfies the *closing lemma* if there exists an $\epsilon_0 > 0$ such that for any ϵ, $0 < \epsilon \leq \epsilon_0$, there is a $\delta > 0$ such that for any orbit segment $\{f^i(x)\}_{i=0,\ldots,m}$ with $\text{dist}(f^m x, x) < \delta$, there exists a periodic orbit $\{f^i(y)\}_{i=0,\ldots,m-1}$, $f^m(y) = y$, such that $d^f_m(x,y) < \epsilon$.

The closing lemma is satisfied for subshifts of finite type and for locally maximal hyperbolic sets.

We will consider a G-cocycle h, where G is an arbitrary locally compact abelian group, and we will need a slightly stronger assumption on h than the equivalent of the dynamical Hölder condition. Namely, we will assume that for any neighborhood U of the identity in G there exists $\epsilon > 0$ such that if $d^f_n(x,y) \leq \epsilon$ then

$$h(x) \ldots h(f^{n-1}x) h(f^{n-1}y)^{-1} \ldots h(y)^{-1} \in U. \tag{3.9}$$

*We use a natural English spelling of the name as in [**KH**] instead of the phonetic spelling Livšic which appears in English translations of [**L1**] and [**L2**]

PROPOSITION 3.18. *Suppose that f is topologically transitive* * *and satisfies the closing lemma. A G-cocycle h satisfying (3.9) can be represented as $\psi(fx)\psi^{-1}(x)$, where ψ is continuous, if and only if the product of the values of h along every periodic orbit is equal to the identity.*

PROOF. Again as in Proposition 3.12, the statement is obvious in one direction, i.e. the cohomology condition implies the statement about periodic orbits.

Let $x \in X$ be a point with a dense orbit. We define ψ along this orbit by

$$\psi(f^n x) = \begin{cases} id, & n = 0 \\ h(f^{n-1}x)\dots h(x), & n > 0 \\ h(f^n x)^{-1} \cdot h(f^{n+1}x)^{-1} \cdot \dots \cdot h(f^{-1}x)^{-1}, & n < 0. \end{cases}$$

we will show that the ψ defined in this way is uniformly continuous on the orbit. Then it can be extended to be a continuous function on X. Since $(\psi \circ f) \cdot \psi^{-1}$ coincides with h along the orbit, the continuity of h implies the desired result.

In order to show uniform continuity, we assume that the points $f^n x$ and $f^m x$, $n < m$ are sufficiently close. Then by the closing lemma, we can approximate the segment $\{f^{n+k}x\}_{k=0,\dots,m-n}$ by a periodic orbit $\{f^k y\}_{k=0,\dots,m-n-1}$.

We have

$$\begin{aligned}
&\psi(f^m x)\psi(f^n x)^{-1} \\
&= h(f^n x)h(f^{n+1}x)\dots(h(f^{m-1}x) \\
&= (h(f^n x)\dots h(f^{m-1}x))(h(y)\dots h(f^{m+n-1}y))^{-1}(h(y))\dots h(f^{m+n-1}y)) \\
&= (h(f^n x)\dots h(f^{m-1}x)) \cdot (h(f^{m+n-1}y)^{-1}\dots h(y)^{-1}) \cdot (h(y))\dots h(f^{m+n-1}y))
\end{aligned}$$

The third product is the identity and if $d^f_{m-n}(f^n x, y)$ is small enough, then by (3.9) the product of the first and second products belong to a given small neighborhood of the identity. But by the closing lemma, we can guarantee that $d^f_{m-n}(f^n x, y)$ is small by making $\mathrm{dist}(f^n x, f^m x)$ small. □

3.4.2 Hölder regularity. The core results of the Livshitz theory establishing both stability and effectiveness of cocycles in the primary hyperbolic situations can be summarized as follows.

THEOREM 3.19. [**L1**] [**L2**] [**LS**] [**KH, Section 19.2**] *Suppose that the group G has equivalent left and right invariant metrics and let f be any topological Markov chain or a restriction of a C^2 diffeomorphism to a locally maximal hyperbolic set. In both cases we assume that all points are nonwandering (see* [**KH, Section 18.3**]. *Let furthermore h be an α-Hölder cocycle over f, $0 < \alpha \le 1$. Then the following conditions are equivalent*

(i) *If $f^n x = x$ then $h(x,n) = Id_G$, i.e. the product of the values of h along every periodic orbit is equal to the identity.*
(ii) *$h(x) = \psi(fx)\psi^{-1}(x)$, where ψ is an α-Hölder function*
(iii) *$h(x) = \psi(fx)\psi^{-1}(x)$, where ψ is a measurable function with respect to the Gibbs measure (or the equilibrium state, see* [**KH, Chapter 20**]*) μ_ϕ for a Hölder function ϕ.*

---

*Topological transitivity follows from the specification property.

REMARK. Notice that by Corollary 2.11 the continuity of h is not sufficient to make any conclusion about existence of transfer functions. However for cocycles satisfying the Dini condition which is weaker than Hölder the assumptions of Proposition 3.18 can be verified.

Obviously (ii) implies both (i) and (iii). To deduce (ii) from (i) one uses the same argument as in the proof of Proposition 3.18 but employs a stronger version of the closing lemma which gives the necessary estimates ([**KH, Section 6.4c**]). The deduction (ii) from (iii) uses the key *a priori* regularity argument which is also the basis of all higher regularity results. In order to avoid cumbersome notations we will only discuss cocycles with values in abelian groups.

Consider the stable and unstable manifolds $W^s(x)$ and $W^u(x)$ of a point x (see [**KH, Sections 6.2, 6.4**]). In the symbolic case these are simply the sets of points with the same nonnegative (corr. nonpositive) coordinates as x.

If ϕ is a solution of cohomological equation then for any $y \in W^s(x)$ one has for any $n > 0$,

$$\psi(y) = \psi(x) \cdot \psi^{-1}(f^n y) \cdot \psi(f^n x) \cdot \prod_{i=0}^{n-1} h(f^i x) \cdot h^{-1}(f^i y).$$

Since the distance between $f^n x$ and $f^n y$ exponentially decreases the product converges exponentially as $n \to \infty$. If ψ is uniformly continuous then $\psi^{-1}(f^n y) \cdot \psi(f^n x)$ converges to Id_G so that

$$\psi(y) = \psi(x) \cdot \lim_{n\to\infty} \prod_{i=0}^{n-1} h(f^i x) \cdot h^{-1}(f^i y). \tag{3.10}$$

If ψ is only measurable a justification for taking is limit is needed.

One can write a similar expression for the values of the transfer function along the unstable manifold $W^u(x)$ using the negative iterates of f, namely for $y \in W^u(x)$,

$$\psi(y) = \psi(x) \cdot \lim_{n\to\infty} \prod_{i=1}^{n} h(f^{-i} x) \cdot h^{-1}(f^{-i} y). \tag{3.11}$$

Legitimacy of these procedures in the case of a measurable transfer function ψ requires an argument which is based on the product structure for the measures μ_ϕ.

Notice also that these expressions allow to obtain Holder regularity of ψ as long as existence of a solution is known.

Theorem 3.19 has a natural extension for flows

3.4.3 Differentiable stability. In order to discuss further regularity let us assume that f is an Anosov diffeomorphism of a compact manifold. If the cocycle h is differentiable one can differentiate expressions (3.10) and (3.11) with respect to y. Since individual stable and unstable manifolds have the same degree of differentiability as the map, one can show using the chain rule that for a C^∞ diffeomorphism the solution ψ has as many derivatives along the stable and unstable directions as the map f, and that those derivatives are continuous This is sufficient to deduce that if h is C^1 then ψ is also C^1.

Higher regularity follows from the remarkable general result by Journé [**Jo**]:

JOURNÉ'S THEOREM. *Assume F and E are two transverse Hölder foliations with C^∞ leaves such all derivatives of all orders along the leaves are uniformly continuous. Let h be a function which has all derivatives of order up to r along the leaves of both foliations and theses derivatives are α Hölder. Then h is C^r and rth derivatives are β Hölder for some $\beta > 0$.*

This, in particular, implies C^r stability for C^{r+1} cocycles and hence C^∞ stability for C^∞ cocycles. The latter together with other regularity results was originally proved in [**dLMM**] using elliptic regularity theory. Another proof based on the Fourier transform is in [**HK2**]. Analytic stability was proved in [**dL**] based on some ideas from [**HK2**].

Thus in the hyperbolic case for cocycles with values in abelian groups the understanding of stability issues is virtually complete.

Without going into details we would like alert the reader to the fact that the picture becomes more complicated and less clear-cut for cocycles with values in nonabelian groups. The main issue here is that the exponential convergence of some orbits within the system, on which stability results are based, has to overcome possible exponential divergence within the group. Thus if the group G itself has a subexponential growth (e.g. if it is a nilpotent Lie group) or if the cocycle is sufficiently small (i.e. its values are close enough to the identity) this divergence can be overcome and stability results hold. The basic source here is the seminal paper by Livshitz [**L2**], where in addition to the correct results along the above lines a wrong claim for the general case is made. The paper by V. Niţică and A. Török [**NT**] reflects the present state of the subject including both optimal regularity results and examples showing that in general even the transfer functions between analytic cocycles may have only limited regularity.

3.5 Invariant distributions and stability of partially hyperbolic systems.

3.5.1 Invariant distributions and periodic orbits. In all cases of stability which we have discussed, from toral rotations to Anosov systems, the cohomology classes of cocycles were determined by the values of integrals with respect to the invariant measures. Invariant measures can be viewed as linear functionals on the space of continuous functions, but as we have seen, stability can only occur in smaller spaces. It is natural to wonder whether there are other invariant functionals defined on these spaces which cannot be extended to the space of all continuous functions, in particular whether there are any invariant distributions of positive order, i.e. invariant linear functionals defined on the spaces of smooth functions.

In fact, invariant distributions represent the most general possible source of stability of spaces of smooth vector-valued cocycles. For, let f be a C^r or C^∞ diffeomorphism of a compact manifold X and H be the space of C^r on C^∞ functions on X. Stability in H means that the operator $U_f - \mathrm{Id} : H \to H$ has closed image, say H_0. But then by the duality H_0 corresponds to the kernel of the dual operator in the space of all continuous linear functionals on H, i.e. H_0 is the common zero level for all f-invariant distributions (cf. Proposition 2.13)). For the flow case we consider instead of $U_f - \mathrm{Id}$ the differential operator generated by the flow.

Since in all cases mentioned in Sections 3.3 and 3.4 the space H_0 was determined by invariant measures the stability implies that in those cases invariant measures are dense in the weak topology in the corresponding spaces of invariant distributions. On the other hand, specification property implies that invariant measures carried by individual periodic orbits are dense in weak topology in the space of invariant measures. These remarks together with the results discussed in subsection 3.4.3 prove the following fact.

COROLLARY 3.20. *For any transitive Anosov diffeomorphism invariant measures carried by individual periodic orbits are dense in the weak topology in the space of all invariant distributions of all orders.*

REMARK. In [**V3**] W. Veech proved a similar statement for an arbitrary (not necessarily hyperbolic) automorphism F_A of the torus $\mathbb{T}^m$ which is ergodic with respect to Lebesgue measure. Equivalently, the matrix A has no roots of unity among its eigenvalues. Such an automorphism is necessarily partially hyperbolic which in this case means that there are eigenvalues both inside and outside of the unit circle. Each orbit $\mathcal{O}$ of the dual automorphism A^* of the lattice $\mathbb{Z}^m$ produces an invariant distribution $\delta_{\mathcal{O}}$ of correlation type similarly to the special case discussed in subsection 3.6.1 below. The system $\delta_{\mathcal{O}}$ generates the space of invariant distributions. Although periodic orbits for an ergodic toral automorphism are dense, specification property does not hold. This is the reason why in order to show that those distributions can be weakly approximated by linear combinations of invariant measures on periodic orbits, Veech has to use arithmetic techniques which are much more subtle and specialized than the arguments in the hyperbolic case.

In the next two subsections we will describe two different types of general constructions of invariant distributions which are not measures. These constructions produce interesting examples even in the hyperbolic case.

3.5.2 Periodic cycle distributions for partially hyperbolic systems. We discuss some of the constructions and results from [**KK**].

A diffeomorphism g of a manifold M with a Riemannian norm $||\cdot||$ is called *partially hyperbolic* if there exist real numbers $\lambda_1 > \mu_1 > 0$, $i = 1, 2$, $K, K' > 0$ and a continuous splitting of the tangent bundle

$$TM = E^+ \oplus E^0 \oplus E^-$$

such that for all $x \in M$, for all $v \in E^+(x)$ ($v \in E^+(x)$ respectively) and $n > 0$ ($n < 0$ respectively) we have for the differential $g_* : TM \to TM$

$$||g_*(v)|| \leq Ke^{-\lambda_1 n}||v||,$$

and, respectively

$$||g_*(v)|| \leq Ke^{-\lambda_2|n|}||v||,$$

and for all $n \in \mathbb{Z}$ and $v \in E^0(x)$ we have

$$||g_*(v)|| \geq K'e^{-\mu_1 n}||v||,\ n > 0\,\text{and}\,||g_*(v)|| \geq K'e^{-\mu_2|n|}||v||,\ n < 0.$$

This property does not depend on the choice of Riemannian metric if the manifold M is compact.

We will call E^+ and E^- *the stable and unstable distributions* respectively. They are uniquely integrable to foliations with smooth leaves which we will denote W^s and W^u.

For any x, y such that $y \in W^s(x)$ the distance $d_M(f^i(x), f^i(y))$ decreases exponentially, thus $\phi(f^i(x)) - \phi(f^i(y))$ also decreases exponentially. Therefore, for any Hölder function ϕ the series

$$P^+(x,y)(\phi) = \sum_{i=0}^{\infty}(\phi(f^i(x)) - \phi(f^i(y)))$$

converges absolutely. (cf. (3.10)). Similarly for any x, y such that $y \in W^u(x)$ the series

$$P^-(x,y)(\phi) = -\sum_{i=-1}^{-\infty}(\phi(f^i(x)) - \phi(f^i(y))$$

also converges absolutely (cf. (3.11)).

We will call a set $S(x,y)$ of points $x_1 = x, x_2, \dots, x_k = y \in M$ a *broken path* from x to y, if $x_{i+1} \in W^a(x_i)$, $i = 1, \dots, k-1$, where $a = s$ or u. A closed broken path, i.e a set C of points $x_1, x_2, \dots, x_{2n}, x_{2n+1} = x_1 \in M$ if $x_{2k} \in W^s(x_{2k-1})$ and $x_{2k+1} \in W^u(x_{2k})$, for $k = 1, \dots, n$. will be called a *periodic cycle.*

For an Anosov system there are many periodic cycles of length four; however in the general partially hyperbolic case periodic cycles must be longer.

DEFINITION 3.21. For a periodic cycle C, we will denote by $F(C)$ the following continuous functional on the space L of Hölder functions:

$$F(C)(\phi) = P^+(x_1, x_2)(\phi) + P^-(x_2, x_3)(\phi) + \cdots$$

$$+P^+(x_{2n-1}, x_{2n})(\phi) + P^-(x_{2n}, x_1)(\phi).$$

We will call this functional a *periodic cycle functional.*

The periodic cycle functionals are invariant distributions. Thus any such functional must vanish on the space of coboundaries.

If ψ is a solution of the cohomological equation

$$\phi(x) = \psi(fx) - \psi(x),$$

then the values of ψ along every broken path are uniquely determined by its value at the beginning and are obtained by adding the value of P^+ or P^- depending on whether consecutive points on the path belong to the same stable or unstable manifold. In particular, the periodic cycle functionals represent obstructions to this process which must vanish to ensure uniqueness of the result.

One can reverse the argument and try to construct the solution for a given ϕ for which the periodic cycle functionals vanish. The best chances of success appear if any two points can be connected by a broken path. This is not always true; counterexamples are time t maps for suspensions of Anosov diffeomorphisms, and, more interestingly, ergodic but not hyperbolic automorphisms of a torus. However those cases are rather special. There are many partially hyperbolic systems for which not only connection is possible but the length of a minimal broken path connecting two points can be estimated from above by a positive power α of the distance. Such systems are called *locally (α) Hölder transitive* and in this case for any function for which the periodic cycle functionals vanish the construction of a solution along broken paths succeeds and produces a Hölder function albeit with a smaller Hölder exponent. To summarize

THEOREM 3.22. [**KK**] *If f is a partially hyperbolic diffeomorphism which is locally α-Hölder transitive, then, for any $\beta \in (0,1]$, the space of β-Hölder cocycles is $\alpha\beta$-Hölder stable and C^0-effective. The space of Hölder coboundaries is the common kernel of periodic cycle functionals.*

Coming back to the hyperbolic case one can use the periodic cycle functionals coming from the *quadrangles*, i.e. the cycles of period four. Moreover, due to the local product structure this construction works not only for Anosov diffeomorphisms but for the locally maximal hyperbolic sets as well. It this case the pair of foliations is locally Lipschitz transitive (i.e. locally 1-Hölder transitive) so that no loss in the Hölder exponent appears in agreement with Theorem 3.19(ii). It would be interesting to find a reasonably explicit procedure to recover the sums of a function along periodic orbits through the periodic cycle functionals.

3.5.3 Correlation functions. Let $f : (X,\mu) \to (X,\mu)$ be a measure preserving transformation, ϕ and ψ be L^2 functions. The inner products

$$\int_X \phi \cdot \overline{\psi \circ f^{-n}} d\mu = \int_X \phi \circ f^n \cdot \overline{\psi} d\mu$$

are called the *correlation coefficients* of ϕ and ψ. The map f is mixing if the correlation coefficients of any two functions with zero average go to zero as $n \to \infty$. It may happen that for a given ϕ and dense set of functions ψ with zero average the series

$$\sum_{n=-\infty}^{\infty} \int_X \phi \circ f^n \cdot \overline{\psi} d\mu \tag{3.12}$$

converge. Obviously exponential decay of correlation is sufficient for convergence. If this happens for all zero average C^∞ functions, (3.12) defines an invariant distribution.

THEOREM 3.23. *Let f be a transitive topological Markov chain or a restriction of a C^2 diffeomorphism to a topologically mixing locally maximal hyperbolic set and let μ_θ be the Gibbs measure for a Hölder function θ. For any two Hölder functions the correlation coefficients*

$$\int_X \phi \cdot \overline{\psi \circ f^{-n}} d\mu$$

decay exponentially.

Correlation decay provides yet another way to characterize cohomological behavior of Hölder functions for hyperbolic dynamical systems.

THEOREM 3.24. *Let f be a transitive topological Markov chain or a restriction of a C^2 diffeomorphism to a topologically mixing locally maximal hyperbolic set and let μ_θ be the Gibbs measure for a Hölder function θ. For an α-Hölder function ϕ the following conditions are equivalent*

(i) *$\phi(x) = h(fx) - h(x)$, where h is an α-Hölder function,*
(ii) *For any Hölder function ψ,*

$$\sum_{n=-\infty}^{\infty} \int_X \phi \circ f^n \cdot \overline{\psi} d\mu_\theta = 0$$

Sketch of proof. Assume (i). Then

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\int_X \phi\circ f^n\cdot\overline{\psi}d\mu_\theta &= \sum_{n=-\infty}^{\infty}\int_X \phi\cdot\overline{\psi\circ f^{-n}}d\mu_\theta\\ &= \sum_{n=-\infty}^{\infty}\int_X (h\circ f-h)\cdot\overline{\psi\circ f^{-n}}d\mu_\theta\\ &= \lim_{n\to\infty}\sum_{k=-n}^{n-1}\int_X (h\circ f-h)\cdot\overline{\psi\circ f^{-k}}d\mu_\theta\\ &= \lim_{n\to\infty}\int_X (h\circ f^n-h\circ f^{-n})\cdot\overline{\psi}d\mu_\theta=0.\end{aligned}$$

Now assume (ii). By Theorem 3.19 it is sufficient to proof that the sum of values of ϕ along any periodic orbit is equal to zero. Assume the opposite. Then there exists a periodic orbit with nonzero, say, positive sum. One can assume then that ϕ is actually positive at all points of the orbit. Using certain care one can find a function ψ with zero average and with very large positive values at the orbit for which sufficiently many correlation coefficients will be positive and fairly large which prevents vanishing of (3.12)

3.6 Stability determined by invariant distributions in parabolic systems.

3.6.1 A simple example: an affine map of the torus. Now we are going to discuss in detail an example of a different sort, namely a uniquely ergodic transformation where invariant distributions provide a complete system of invariants for the cohomology, but the single invariant measure does not give enough information for this purpose. Invariant distributions which we will encounter will be of the correlation type (3.12).

Consider the affine A map of $\mathbb{T}^2$ given by

$$A(z,w)=(\lambda z, zw).$$

We will consider only real or complex valued cocycles.

If we assume that λ is Diophantine (3.1), i.e for some positive k and c, and for every positive integer q,

$$|\lambda^q-1|>c/q^k. \tag{3.13}$$

then by Theorem 3.5 functions depending only on z are C^∞ rigid and in the case of finite regularity there is a fixed loss of regularity (see remarks in subsection 3.2.2). Since the decomposition into functions depending only on z and their orthogonal complement is invariant it is sufficient to consider functions in that complement.

Let us define a family of first order A-invariant distributions $\delta_{m,n}$, for $0\le m<|n|$, $n\ne 0$, by their Fourier expansions:

$$\delta_{m,n}=\sum_{k=-\infty}^{\infty}\lambda^{km+\frac{k(k-1)n}{2}}z^{m+kn}w^n.$$

Equivalently, for $f \in C^1(\mathbb{T}^2)$, let $z = e^{2\pi i\phi}$ and $w = e^{2\pi i\theta}$ and let as before $\chi_{m,n} = z^m w^n$. Then

$$\begin{aligned}
\delta_{m,n}(f) &= \lim_{K\to\infty} \int_{\mathbb{T}^2} f(z,w) \sum_{k=-K}^{K} \lambda^{-km-\frac{k(k-1)n}{2}} \overline{z}^{m+kn}\overline{w}^n \, d\phi d\,\theta \\
&= \lim_{K\to\infty} \int_{\mathbb{T}^2} f(z,w) \sum_{k=-K}^{K} \overline{\chi_{m,n} \circ A^k} d\phi d\theta \\
&= \lim_{K\to\infty} \sum_{k=-K}^{K} f(A^k(z,w))\overline{z}^m\overline{w}^n \, d\phi d\,\theta.
\end{aligned}$$

Obviously every distribution $\delta_{m,n}$ is a derivative of an L^2-function.

Let H^r, $r \geq 0$, be the space of functions ψ on $\mathbb{T}^2$ such that ψ is r times differentiable and all of its r^{th} derivatives have absolutely convergent Fourier series.

THEOREM 3.25. *Let $f, g \in C^{r+\epsilon}$, $r \geq 3$ and $\epsilon > 0$. Assume that both functions are orthogonal to the functions depending only on z. Then there exists $\psi \in H^{r-3}$ such that*

$$f = g + \psi \circ A - \psi \tag{3.14}$$

if and only if

$$\delta_{m,n}(f) = \delta_{m,n}(g)$$

for all $n \in \mathbb{Z}$, $0 \leq m < |n|$. In particular, if f and g are C^∞, then ψ is C^∞. In addition, if f and g are real analytic then so is ψ. Furthermore, for $f, g \in C^1$ if equation (3.14) has an L^1 solution then $\delta_{m,n}(f) = \delta_{m,n}(g)$ for all $0 \leq m < |n|$, so that the solution actually belongs to H^{r-3}.

Our estimate of the loss of regularity for the solution of the cohomological equation is not sharp but it is sufficient to establish C^∞ stability.

PROOF. It is enough to consider only the case $g = 0$. We look for the Fourier coefficients $\psi_{m,n}$ for the transfer function ψ. By our assumption $f_{m,0} = 0$ for all $m \in \mathbb{Z}$ hence we need only consider the case $n \neq 0$. Since $U_A\chi_{m,n} = \lambda^m \chi_{m+n,n}$ for $n \neq 0$ equation (3.14) implies the following infinite system of algebraic equations.

$$f_{m,n} = \lambda^{m-n}\psi_{m-n,n} - \psi_{m,n} \tag{3.15}$$

so that

$$\psi_{m,n} = -f_{m,n} + \lambda^{m-n}\psi_{m-n,n}.$$

Proceeding by induction,

$$\begin{aligned}
\psi_{m,n} &= -f_{m,n} - \lambda^{m-n} f_{m-n,n} + \lambda^{2m-3n}\psi_{m-2n,n} \\
&= -\sum_{k=0}^{N-1} \lambda^{e_k(m,n)} f_{m-kn,n} + \lambda^{e_N(m,n)}\psi_{m-Nn,n}
\end{aligned} \tag{3.16}$$

for all $N \geq 0$, where

$$e_k(m,n) = e_{k-1}(m,n) + m - kn$$

and

$$e_0(m,n) = 0$$

so that

$$e_k(m,n) = km - \frac{1}{2}k(k+1)n.$$

Similarly, using (3.15) again, and repeating the steps above

$$(3.17)\qquad \begin{aligned}\psi_{m,n} &= \lambda^{-m}(f_{m+n,n} + \psi_{m+n,n})\\ &= \sum_{k=1}^{N}\lambda^{-e'_k(m,n)}f_{m+kn,n} + \lambda^{-e'_N(m,n)}\psi_{m+Nn,n}\end{aligned}$$

for all $N \geq 1$, where

$$e'_k(m,n) = km + \frac{1}{2}k(k-1)n.$$

Assuming that $\psi_{m+Nn,n} \to 0$ as $N \to \pm\infty$, which is true even if $\psi \in L^1$, we obtain from (3.16) and (3.17) two expressions for $\psi_{m,n}$:

$$(3.18)\qquad \psi_{m,n} = -\sum_{k=0}^{\infty}\lambda^{km-\frac{1}{2}k(k+1)n}f_{m-kn,n}$$

and

$$(3.19)\qquad \psi_{m,n} = \sum_{k=1}^{\infty}\lambda^{-km-\frac{1}{2}k(k-1)n}f_{m+kn,n}$$

If the solution exists, these two expressions must coincide, so that

$$(3.20)\qquad \sum_{k=-\infty}^{\infty}\lambda^{-km-\frac{1}{2}k(k-1)n}f_{m+kn,n} = 0.$$

For $0 \leq m < |n|$, the left hand side of (3.20) is exactly what we call $\delta_{m,n}(f)$. It is easy to see that if equation (3.20) is satisfied for some (m,n), it is also satisfied for $(m+kn,n)$ for every k. This gives the last statement in the Theorem.

Assuming (3.20), we can estimate $\psi_{m,n}$. For $mn < 0$ we will use (3.18) and for $mn \geq 0$ we will use (3.19). We have respectively

$$|\psi_{m,n}| \leq \sum_{k=0}^{\infty}|f_{m-kn,n}|$$

$$|\psi_{m,n}| \leq \sum_{k=1}^{\infty}|f_{m+kn,n}|.$$

If $f \in C^{r+\epsilon}$, then

$$|f_{m,n}| \leq \frac{c}{(|m|+|n|)^{r+\delta}}$$

for some $\delta > 0$, and so

$$|\psi_{m,n}| \leq \frac{c'}{(|m|+|n|)^{r-1+\delta/2}}.$$

This implies that $\psi_{m,n} \in H^{r-3}$.

If f is real analytic, then

$$|f_{m,n}| < c\,\exp(-\alpha(|m|+|n|))$$

for $\alpha, c > 0$. It follows that

$$|\psi_{m,n}| \leq \sum_{k=0}^{\infty} c \exp(-\alpha(|m| + k|n| + |n|) \leq c' \exp -\alpha(|m| + |n|) \,.$$

$\square$

Thus, we have shown that various spaces of smooth cocycles over A are stable and "almost" effective with a certain loss of regularity which is natural to expect. The C^∞ cocycles are both C^∞ stable and C^∞ effective. We do not know whether the existence of a measurable solution to (3.14), for sufficiently smooth f and g implies that the solution is L^1, which is then smooth by the theorem. The difficulty here is that non-integrable measurable functions are not distributions so that Fourier analysis can not be applied in this case. Probably, in order to solve this problem one needs to develop a more geometric approach to the solution of (3.14). This comment applies also to the cocycles with values in groups which are not linear spaces.

It is interesting to notice that if functions f and g have absolutely convergent Fourier series and equal averages then equation (3.14) has solutions ψ which are distributions of order 1. Either of the two formulas (3.18) and (3.19) gives Fourier coefficients for such a solution.

The method of Theorem 3.25 applies to arbitrary ergodic automorphisms and affine maps on a torus of any dimension. The only restriction is the Diophantine condition for the rotational part of the affine map. In all those cases there is a natural one-to-one correspondence between invariant distributions which determine stability and infinite orbit of the lattice automorphism dual to the automorphism part of the considered map.

3.6.2 Horocycle flows. Another and even more interesting example of C^∞ stability and C^∞ effectiveness determined by invariant distributions is given by the horocycle flow on a compact surface of constant negative curvature. This flow can be equivalently described as the action of the one-parameter subgroup

$$H_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

by right multiplications on the left factor space $B = PSL(2,\mathbb{R})/\Gamma$, where Γ is a discrete cocompact subgroup of $PSL(2,\mathbb{R})$.

The horocycle flow is a part of the right action of $PSL(2,\mathbb{R})$ on B. This action generates a unitary representation of $PSL(2,\mathbb{R})$ in $L^2(B,\chi)$ where χ is Haar measure. This representation decomposes into a direct sum of countably many irreducible ones. Obviously every representation space is invariant with respect to the operators corresponding to the horocycle flow. The space of each irreducible representation carries certain invariant distributions for the horocycle flow. Namely,

(i) for a representation from the principal series there are two invariant distributions of order less than one similar to $\delta_{m,n}$ for the affine map on $\mathbb{T}^2$;
(ii) for a representation from the discrete series there is one invariant distribution of order less than one;
(iii) for a representation form the complementary series there are infinitely many representations of growing order.

In every representation space there is an intrinsic notion of a C^∞ function which agrees with the geometric notion coming from B. More precisely, let $f \in C^\infty(B)$; and for any irreducible representation θ of $PSL(2,\mathbb{R})$, let P_θ be the orthogonal projection of $L^2(B)$ to the representation space. Then $P_\theta f$ is a C^∞ vector in the sense of the representation and the norms $\|P_\theta f\|$ decrease faster than any power of a parameter naturally ascribed to the irreducible representations. Furthermore, this condition is necessary and sufficient for f to be a C^∞ function. These remarks show that the problem can be treated in a way very similar to the way affine maps of $\mathbb{T}^2$ were treated; using certain (not always elementary) facts about irreducible unitary representations of $PSL(2,\mathbb{R})$ instead of the elementary Fourier analysis of the previous proof. See [**FlFo**] for detailed rigorous arguments in this case.

3.6.3 Flows on surfaces. Without trying to explain the substance of the situation we have to mention for the sake of completeness another remarkable instance of stability determined by invariant distributions.

Consider a compact orientable surface S of genus $g > 1$ with a smooth positive area form Ω. Since the Euler characteristic $\chi = 2 - 2g$ of S is negative any vector field on S has to have zeroes. The case which is the most interesting from the dynamical point of view is those of area preserving topologically transitive flows with finitely many fixed points. These fixed points have to be of saddle type, either simple, or multiple saddles. A topologically transitive area preserving flow with finitely many saddle points exhibit parabolic behavior since its derivative has triangular form outside of the fixed points. However this behavior is less uniform than that of affine maps or horocycle flows. Aside from point masses at each saddle there are no more than g ergodic probability measures [**KH, Theorem 14.7.6**].The transverse behavior of area preserving flows on surfaces is defined by a finite set of parameters [**KH, Theorem 14.7.4**] and is closely related to the interval exchange transformations discussed in Section 4.4. For the set of parameters of full measure there is only one nontrivial probability ergodic invariant measure. There are invariant distributions defined by the jets at the fixed points; they are invariants of the normal form of the vector field near these points. Forni [**Fo**] discovered additional invariant distributions which are supported outside of fixed points and give a complete description of these distributions. The picture is quite different from the cases we discussed before. There is only a finite dimensional space of invariant distributions for any given order r but the dimension grows to infinity with r. For certain conditions which can be interpreted as counterparts of Diophantine conditions Forni showed C^∞ stability of C^∞ cocycles.

4. Wild cochains with tame coboundaries

In this and the next chapter we will discuss the situations where correspondingly effectiveness and stability break down.

Thus, we will presently consider "regular" (e.g. continuous, smooth or piecewise constant) cocycles which are coboundaries in measure-theoretic sense but such that the transfer functions behave very wildly from the topological point of view. This phenomenon is often connected with abnormally good periodic approximation. It was known to Kolmogorov [**Ko**] who used it to construct an example of a time change for a linear flow of the two–torus with pure point spectrum but discontinuous eigenfunctions. Furstenberg [**F**] used it for his construction of minimal but not uniquely ergodic diffeomorphisms of $\mathbb{T}^2$. Anosov [**A**] studied the case of circular rotation thoroughly and systematically and in particular found examples of analytic coboundaries with wild transfer functions.

Cocycles of this sort are used to construct various examples of dynamical systems whose topological and ergodic properties differ wildly. They can also be used for more positive tasks, e.g. for producing realizations of measure-preserving transformations by volume preserving homeomorphisms of various manifolds.

We will illustrate this phenomenon with four examples. The first one deals with continuous cocycles over a topologically transitive homeomorphism of a compact metric space preserving a Borel measure positive on open sets. As we noticed in Section 2.2 if the homeomorphism is not uniquely ergodic then the existence of cohomologous continuous cocycles equivalent via discontinuous transfer functions follows from Corollary 2.11 and Proposition 2.13. The construction of Theorem 4.2 shows a specific mechanism for the occurrence of wild transfer functions between continuous cocycles. This mechanism is universal and in particular is independent of the existence of more than one ergodic invariant measure.

The results of Chapter 3 show that in general such mechanism cannot be extended to smooth cocycles. However, an extension of that sort is possible if the diffeomorphism allows a very good periodic approximation in C^∞ sense (Theorem 4.5). We will also indicate how a similar construction works for real analytic cocycles over a rotation of the circle with exceptionally well approximable rotation number (Theorem 4.7).

Our last example deals with even smaller, actually a finite-dimensional space of cocycles, namely some piecewise constant cocycles over interval exchange transformations (Theorem 4.9).

The method of proof in all four cases is rather uniform. The desired cocycle together with the transfer function will be constructed by a converging iterative (or inductive) procedure. On every step of the construction we will have a cocycle which is a coboundary within the given category. In other words, the transfer function will be correspondingly continuous, smooth, real-analytic or piecewise constant. The sequence of cocycles will converge within the category while the sequence of transfer functions will converge only in probability or in L^1 and will diverge in a certain prescribed fashion in the uniform topology. The control over the convergence will by provided by making the modifications on any step insignificant enough in measure-theoretic sense in comparison with the previous steps. On the other hand, the divergence of the sequence of transfer functions is controlled by making subsequent additions sufficiently uniformly distributed.

4.1 Continuous cocycles over measure-preserving homeomorphisms. Let X, Y be two topological spaces and ν be a Borel measure on X positive on open sets.

DEFINITION 4.1. We will say that a Borel map $\psi : X \to Y$ is *metrically dense with respect to* ν if for any pair of non-empty open sets $V \subset X$ and $W \subset Y$,

$$\nu(V \cap \psi^{-1}(W)) > 0.$$

Equivalently, this property means that the lift of the measure ν to graph $\psi \subset X \times Y$ is a measure positive on open sets.

THEOREM 4.2. *Let X be a compact metric space and U be an open subset of X. Suppose $f : X \to X$ is a topologically transitive homeomorphism preserving a Borel probability measure μ which is positive on open sets. Let G be a path connected separable locally compact abelian group. Then there exists a continuous map $\phi : X \to G$ such that*

(i) *Outside of U, $\phi = e$, the identity in G.*
(ii) $\phi(x) = \psi(f(x))\psi^{-1}(x) = \lim_{n\to\infty} \psi_n(f(x))\psi_n^{-1}(x)$ *where the functions ψ_n are continuous and ψ is measurable.*
(iii) *ψ is metrically dense with respect to μ.*

This theorem can be used to prove that every abstract measure-preserving transformation with finite entropy can be realized as a volume-preserving homeomorphism of any compact Riemannian manifold of dimension greater than one.

PROOF. *Step 1.* The desired function ϕ will be constructed as a uniform $\lim_{n\to\infty} \phi_n$ where inductively

$$\phi_{n+1}(x) = \phi_n(x)\eta_n(x) \tag{4.1}$$

such that outside of U, $\eta_n = e$ and

$$\eta_n(x) = \theta_n(f(x))\theta_n^{-1}(x). \tag{4.2}$$

In other words

$$\phi_n(x) = \psi_n(f(x))\psi_n^{-1}(x)$$

where

$$\psi_n(x) = \prod_{j=1}^{n} \theta_j(x). \tag{4.3}$$

We will ensure that the sequence ψ_n converges in probability so that (ii) will be satisfied. On the other hand, we will make the sequence ψ_n wildly divergent in the uniform topology so that on the n'th step, the function ψ_n will satisfy the property (iii$_n$), stated below, which can be considered as an approximate version of (iii). In order to formulate this property let us fix a translation invariant metric d_G on G and denote by $D(r, g)$ the ball of radius r around $g \in G$. Let $g_n,\ n = 1, 2, \dots$ be a dense sequence of elements of G. We will assume that there exists a sequence of positive numbers δ_n, $\delta_n < \delta_{n-1}/2$, such that for every $k - 1, \dots, n$ and for every disc D of radius 2^{-k} in X,

(iii$_n$) $\mu(D \cap \psi_n^{-1} D(2^{-k}, g_k)) > \delta_k \left(\frac{1}{2} + \frac{1}{2^{n-k+1}}\right)$.

In order to ensure that subsequent steps do not destroy this condition we will assume

(a_n) $\mu\{x \in X; \theta_n(x) \neq e\} < \delta_{n-1}/10$.

It is clear that the sequence ψ_n defined by (4.3), and satisfying (iii$_n$) and (a_n) converges in probability and that the limit function ψ satisfies (iii).

Step 2. We will show how to construct θ_n and δ_n inductively. The base of the induction is trivial. Now let us assume that θ_{n-1} and δ_{n-1} have been constructed. This allowed us to construct ψ_{n-1}, and by the uniform continuity of this function, we can find a positive number δ'_n such that if $d_X(x_1, x_2) < \delta'_n$ where d_X is the distance in X then

$$d_G(\psi_{n-1}(x_1), \psi_{n-1}(x_2)) < 2^{-n-10}. \tag{4.4}$$

Let

$$H = \{g \in G : gg_n^{-1} \in (\psi_{n-1}(X))^{-1}\}.$$

This set is compact, so that we can cover it by a finite set of balls of radius 2^{-n-5} and pick an element g in each of the balls. S will denote the set of all such g. For each $g \in S$ we are going to construct a function θ_n^g. The supports of θ_n^g will be disjoint for different g and θ_n^g will be supported by a very small neighborhood of a segment of a single orbit in such a way as to make θ_n^g reach the value g inside every ball of radius δ_n' in X. Furthermore the function

$$\eta_n^g(x) = \theta_n^g(f(x))(\theta_n^g(x))^{-1} \tag{4.5}$$

will be equal to the identity outside the set U and will be uniformly within $1/2^n$ of the identity everywhere. Then we will put

$$\theta_n = \prod_{g\in S} \theta_n^g$$

so that the function $\psi_n = \psi_{n-1}\theta_n$ will reach a value within 2^{-n-4} of g_n inside every δ_n' ball in X. This follows from the choice of δ_n'. On the other hand, $\mu(\mathrm{supp}(\theta_N))$ will be chosen so small that the values near g_k, $k = 1, \dots, n-1$ achieved on the previous stages of the construction will persist.

Step 3. We will now show how to construct the functions θ_n^g and insure (a_n) and (iii_n). Let us connect every $g \in S$ by a path γ_g to the identity $e \in G$. This path may be divided into intervals such that each interval lies within a ball of radius 2^{-n}. Let K_n be the maximal number of such intervals for any $g \in S$.

For every $g \in S$ let us take a sufficiently long segment of a dense orbit, say $\Gamma_g = \{x_g, f(x_g) \dots f^N(x_g)\}$ such that all Γ_g are disjoint for different g and in addition each segment consists of three consecutive parts so that the first and the last parts contain at least K_n points from U, and the middle part intersects every ball in X of radius $\delta_n'/10$. Take the segment Γ_g and mark the first K_n iterates where $f^j x_g \in U$. Call these times $t_1(g), \dots, t_{K_n}(g)$. Do the same for the last K_n visits calling them $s_{K_n}(g) \dots s_1(g)$, where $s_1(g)$ is the last visit to U. The function θ_n^g will be different from the identity only in very small neighborhoods of the points $f^{t_i(g)}(x_g)$ and $f^{s_i(g)}(x_g)$, $i = 1, \dots, K_n$. The range of this function will be a neighborhood of the path γ_g. Pick points $e = h_1, \dots, h_{K_n} = g$ so that each interval $[h_j, h_{j+1}] \subseteq \gamma_g$ lies within a ball of radius 2^{-n}, and construct the function θ_n^g by induction in $j = 1, \dots, K_n$. Suppose that it has been constructed through time t_j. We multiply it by a continuous function η_n^g whose support $U(g,j) \subset U$ is a small neighborhood of $f^{t_{j+1}(g)^n}(x_g)$, whose range is $h_j^{-1}([h_j, h_{j+1}])$, and such that

$$\eta_n^g(f^{t_{j+1}(g)}(x_g)) = h_j^{-1}h_{j+1}$$

so that

$$\theta_n^g(f^{t_{j+1}(g)}(x_g)) = h_{j+1}.$$

We continue this procedure until $t_{K_n}(g)$ has been reached so that $\theta_n^g(f^{t_{K_n}(g)}(x_g)) = g$, stop for the middle section of Γ_g, and then "undo" the function θ_n^g along the last section of Γ_g by putting θ_n^g in a neighborhood of $f^{s_j(g)}(x_g)$ equal to

$$\theta_n^g(x) = (\theta_n^g(f^{t_j(g)-s_j(g)}(x)))^{-1}.$$

Thus, if the neighborhoods $U(g,j)$, $j = 1 \ldots K_n$ are chosen sufficiently small then for any integer t, $t_{K_n}(g) \le t \le s_{K_n}(g)$

$$\theta_n^g(f^t(x_g)) = g.$$

By the choice of the interval Γ_g the function θ_n^g reaches value g inside any disc of radius $\delta'_n/10$ in X. By the choice of $U(g,j)$ one can also guarantee, that the total measure of the support of each θ_n^g (and consequently of θ_n) can be made arbitrary small and the supports of θ_n^g, $g \in S$, disjoint. In particular, the condition (a_n) will be satisfied. That ensures that the condition (iii_n) is satisfied for $k = 1, \ldots, n-1$. Let us show that if δ_n is chosen sufficiently small then (iii_n) is also satisfied for $k = n$.

Let $x \in X$ and $g \in S$ be such that

$$d_G(g\psi_{n-1}(x), g_n) < 2^{-n-5} \tag{4.6}$$

By the choice of the orbit segments Γ_g the ball of radius $\frac{\delta'_n}{10}$ about x contains a point from the middle segment of Γ_g, say y, so that $\theta_n(y) = \theta_n^g(y) = g$. So that by (4.4) and (4.6)

$$d_G(\psi_n(y), g_n) < 2^{-n-4}.$$

Since $\delta_n < 2^{-n}$ this implies that for the ball $D(x)$ about x of radius 2^{-n}

$$\mu(x) \overset{\text{def}}{=} \mu(D(x) \cap \psi_n^{-1}(D_G(2^{-n}, g_n)) > 0$$

and by the compactness of X the measure $\mu(x)$ is bounded away from 0 for all x. Any positive lower bound for $\mu(x)$ may serve as δ_n. Obviously this choice of δ_n insures (iii$_n$) for $k = n$. □

Let us consider the following continuous G extension f^ϕ of f acting in the space $X \times G$:

$$f^\phi(x,g) = (f(x), g\phi(x))$$

and let $\nu = (id \times \psi)_* \mu$.

COROLLARY 4.3.

(i) *The measure ν is positive on open sets in $X \times G$.*

(ii) *The extension f^ϕ preserves ν.*

(iii) *(f^ϕ, ν) is metrically isomorphic to (f, μ).*

PROOF. By Theorem 4.2, (ii), the map $R = id \times \psi : X \to X \times G$ satisfies $R \circ f^\phi \circ R^{-1} = f$. This proves (ii) and (iii). By Theorem 4.2, (iii), the measure ν is positive on open sets, since for $V \subseteq X$, $W \subseteq G$, $\nu(V \times W) = \mu(V \cap \psi^{-1}(W))$. □

4.2 Fast approximation and C^∞ cocycles. A construction similar to that of Theorem 4.2 works in C^∞ category if the transformation in consideration possesses a C^∞ equivalent of periodic homogeneous approximation with a speed which is faster than any negative power of the characteristic parameter. While we do not need the above notions in our presentation a proper discussion can be found in [**K1**]. To simplify notations we will consider only real-valued cocycles. Let M be a compact connected m-dimensional C^∞ manifold, $C^r(M)$ be the Banach space of r times continuously differentiable real-valued functions on M provided with a norm $\|\cdot\|_r$. We will use the same notation for the associated operator norm.

DEFINITION 4.4. A C^∞ diffeomorphism $f : M \to M$ *admits fast* C^∞ *periodic approximation* if there exists a sequence $q_n \to \infty$ such that for every pair of positive integers k, r there exists a constant $c(k,r)$ such that

$$\|U_{f^{q_n}} - Id\|_r < \frac{c(k,r)}{q_n^k} \tag{4.7}$$

Equivalently, for every $h \in C^\infty(M)$, $h \neq 0$,

$$\|h \circ f^{-q_n} - h\|_r < \frac{\|h\|_r \cdot c(k,r)}{q_n^k}. \tag{4.8}$$

EXAMPLE. For the rotation of the circle $R_\lambda : z \to \lambda z$ fast C^∞ periodic approximation is equivalent to λ being Liouvillean (cf. Section 3.2).

THEOREM 4.5. *Suppose that a* C^∞ *diffeomorphism* $f : M \to M$ *preserves a measure* μ *given by a density bounded between two positive constants, is topologically transitive and admits fast* C^∞ *periodic approximation. Let* U *be any open non-empty subset of* M*. Then there exists a* C^∞ *cocycle* h *equal to* 0 *outside* U *and such that*

$$h = \psi \circ f - \psi = \lim_{n\to\infty} \psi_n \circ f - \psi_n$$

where the functions ψ_n *are* C^∞ *and the function* ψ *is measurable, discontinuous at every point and for every* $\lambda \in \mathbb{R}\backslash\{0\}$ *the function* $\exp i\lambda\psi : M \to S^1$ *is metrically dense with respect to* μ *(cf. Definition 4.1). In addition,* h *can be chosen in such a way that* ψ *either belongs to* L^1 *or not.*

PROOF. Let $x_0 \in U$ be a point whose semi-orbit $\{f^n x_0\}$ $n = 0, 1, 2, \dots$ is dense in M. Let $t_1, \dots, t_m$ be C^∞ local coordinates in a neighborhood $V \subset U$ of the point x_0 such that $t_i(x_0) = 0$, $i = 1, \dots, m$. Obviously, every C^∞ function of the local coordinates equal to zero near ∂V can be extended to a C^∞ function on M by making it equal to zero outside V. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a standard C^∞ bump function such that $\theta(0) = 1$, $\theta(t) = 0$ for $|t| \geq 1$ and θ is non-increasing for positive values of t and non-decreasing for the negative ones. Then for any sufficiently large positive q the function h_q defined by

$$h_q(x) = \begin{cases} \theta\left(q^6\left(\sum\limits_{i=1}^m t_i^2(x)\right)\right) & \text{if} \quad x \in V \\ 0 \quad \text{otherwise} \end{cases} \tag{4.9}$$

is a C^∞ function on M.

For two versions of the statement (ψ is integrable or not) the function h is defined by

$$h = \sum_{k=1}^\infty q_{n_k}^2 \left(h_{q_{n_k}^2} \circ f^{-q_{n_k}} - h_{q_{n_k}^2}\right) \tag{4.10}$$

and

$$h = \sum_{k=1}^\infty q_{n_k}^{3m} \left(h_{q_{n_k}^2} \circ f^{-q_{n_k}} - h_{q_{n_k}^2}\right) \tag{4.11}$$

correspondingly, where q_n is a sequence which appears in (4.8) and the sequence $n_k \to \infty$ will be chosen later. To verify all the assertions of the theorem except for the metric density of the function $\exp\, i\lambda\psi$ it is enough to observe that $q_{n_k} > k$.

First let us show that the series in the right hand part of (4.10) and (4.11) converge in the C^∞ topology.

By applying the chain rule in (4.9) we obtain

$$\|h_q\|_r < c_1(r)q^{6r} \tag{4.12}$$

where $c_1(r)$ depends on θ but not on q. Let us put in (4.8) $k = 6r + 3m + 10$. We obtain from (4.8) and (4.12)

$$\|h\|_r \le c_1(r) \cdot c(k,r) \sum_{n=1}^{\infty} q_n^{-10}.$$

Let us show that the cocycle h is a measurable coboundary. Define

$$\psi_q \stackrel{\text{def}}{=} -\sum_{\ell=1}^{q} h_{q^2} \cdot f^{-\ell}.$$

It follows immediately from the definition that

$$h_{q^2} \circ f^{-q} - h_{q^2} = \psi_q \cdot f - \psi_q.$$

Since $\mu(\text{supp } h_q) < c_2 \cdot q^{-3m}$ we have

$$\mu(\text{supp } \psi_q) \le q\mu(\text{supp } h_q) < c_2 q^{1-3m} \le c_2 q^{-2} \tag{4.13}$$

so that $\sum_q \mu(\text{supp}\psi_q) < \infty$.

Consequently for every sequence of constants a_q the series $\sum_q a_q \psi_q$ converges in probability. This implies that $h = \psi \circ f - \psi$ where for h defined by (4.10)

$$\psi = \sum_{k=1}^{\infty} q_{n-k}^2 \psi_{q_{n_k}} \tag{4.14}$$

and for h defined by (4.11)

$$\psi = \sum q_{n-k}^{3m} \psi_{q_{n_k}}. \tag{4.15}$$

We have from (4.9) $c_r q^{-3m} < \|h_q\|_{L^1} < c_3 q^{-3m}$ and consequently

$$\|\psi_q\|_{L^1} \le q\|h_q\|_{L^1} < c_3 q^{1-3m}$$

so that it follows that ψ defined by (4.14) is an L^1 function. On the other hand, since h_q is non-negative and

$$\|\psi_q\|_{L^1} > c_r q^{1-3m}$$

the function defined by (4.15) is not integrable. The discontinuity of ψ at every point follows from the fact that for every positive m $\psi_{q_{n_k}}(f^m x_0) \to -\infty$ so that since all $\psi_{q_{n_k}}$ are non-positive functions

$$\lim_{x \to f^m x_0} \psi(x) = -\infty$$

Since the semi-orbit of x_0 is dense this proves discontinuity.

It remains to show how an appropriate choice of the sequence q_{n_k} provides the metric density of the function $\exp i\lambda\psi$. This can be achieved by an inductive argument very similar to the one used in the corresponding part of the proof of Theorem 4.2. Since ψ_q is a non-positive continuous function, $\mu(\text{supp } \psi_q) \to 0$ as $q \to \infty$ and $\psi_q(f^\ell x_0) < -1$ for $\ell = 1, \ldots, q$, it follows that for every $\epsilon > 0$ there exists $Q(\epsilon)$ such that for every $q > Q(\epsilon)$ the functions $\exp i\lambda^2\psi_q$ and $\exp i\lambda q^{3m}\psi_q$ map every disc of radius ϵ onto the circle. Moreover, this remains true for every function χ of the form $\exp i\lambda(q^2\psi_2 + \phi)$ or $\exp i\lambda(q^{3m}\psi_q + \phi)$ where ϕ is continuous and the oscillation of ϕ on any disc of radius ϵ does not exceed $1/\lambda$. If ϕ is fixed this implies that for any fixed $\delta > 0$, any ϵ-disc D in X and any δ-interval $\Delta \subset S^1$,

$$\mu(\chi^{-1}(\Delta) \cap D) > \alpha(\epsilon, \delta, q) > 0.$$

We can think now of ϕ as $\sum_{\ell=1}^{k-1} q_{n_\ell}^{3m}\psi_{q_\ell}$. Fixing ϵ_n such that the oscillation of ϕ in every ϵ-disc is less than $1/\lambda$, we can find k such that $q_{n-k} > Q(\epsilon_n)$ and apply the construction described above. In order to keep the approximate density achieved on the previous steps intact, we assume in addition that q_{n_k} grow so fast that

$$\sum_{\ell=k+1}^{\infty} \psi(\text{supp } \mu_{q_{n_\ell}}) < \frac{\alpha(\epsilon_k, w^{-k}, q_{n_K})}{2}.$$

This implies that for any interval Δ of length 2^{-k} and any ϵ_k-disc $D \subset X$

$$\mu(\psi^{-1}(\Delta) \cap D) > \frac{1}{2}\alpha(\epsilon_k, 2^{-k}, q_{n_k}). \ \Box$$

COROLLARY 4.6. *If a diffeomorphism f satisfies all the assumptions of Theorem 4.5 and is ergodic with respect to the measure μ then there exists a C^∞ S^1 extension of f which is topologically transitive on $M \times S^1$ but not ergodic with respect to the product measure.*

PROOF. We define the extension via the cocycle $\exp ih$ where h is given by Theorem 4.5. Since this cocycle is a coboundary the extension is measure-theoretically isomorphic to the trivial extension and thus is not ergodic. On the other hand, the measure $(id \times \psi)_*\mu$ is invariant and ergodic with respect to the extension and since ψ is metrically dense this measure is positive on open subsets of $M \times S^1$. This implies topological transitivity. $\Box$

4.3 Minimal nonergodic diffeomorphisms of $\mathbb{T}^2$. In certain cases, the construction of a wild coboundary can be made more global and uniform. Here is an example of how such a construction works for rotations of the circle allowing good periodic approximation. Since the result is not new (cf. [**F**], [**A**]), we give only a very brief sketch of the argument. See [**KH**], Sections 2.9 and 12.6 for a more detailed argument.

THEOREM 4.7. *There exists a λ and an analytic function ϕ such that*

$$\phi(x) = \psi(\lambda z) - \psi(z) \tag{4.16}$$

where ψ is measurable, metrically dense with respect to Lebesgue measure (cf. Definition 4.1), and ψ can be made either integrable or not integrable.

COROLLARY 4.8. *For every $r > 0$, the S^1 extension*

$$R_\lambda^{\exp ir\phi}(z,w) = (\lambda z, w \exp ir\phi(z))$$

is minimal but not strictly ergodic.

One can deduce the corollary directly from the statement of the theorem, however, it is easier to apply the general result of Gottshalk and Hedlund, (cf. [**Pa**], where Furstenberg's construction can also be found, or [**KH**], Section 2.9), which asserts that if the extension $R^{\exp ir\phi}$ is not minimal then for some positive integer n, the cocycle $\exp(irn\phi)$ is a coboundary with a continuous transfer function.

Let us outline one of the constructions which leads to a proof of Theorem 4.7. This construction is very close in spirit to the general constructions in sections 4.1 and 4.2. We will use Dirichlet kernels

$$D_{q,n}(z) = \sum_{\substack{j=-n+1 \\ j\neq 0}}^{n-1} z^{jq} + z^{-jq}$$

to produce an analytic counterpart of the bump functions used in those arguments. The transfer function is build as

$$\sum_{n=1}^{\infty} W_n D_{q_n,m_n}(z) \tag{4.17}$$

where by making $q_n, m_n \to \infty$ fast enough, we guarantee convergence in probability, since most of the mass of the Dirichlet kernel $D_{q,n}$ is concentrated around the qth roots of unity, and by controlling W_n, we can make the function (4.17) either integrable or not. The number $\lambda = \exp 2\pi i\alpha$ is built inductively as the limit of very good approximations

$$\alpha \stackrel{\text{def}}{=} \lim_{n\to\infty} p_n/q_n.$$

The inductive step works as follows. Having constructed p_n, q_n, m_n and W_n, we consider the function

$$\psi_n(z) = \sum_{j=1}^{n} W_j D_{q_j,m_j}(z)$$

as given, and assume that we have a good estimate in some complex neighborhood of S^1 for

$$\psi_n(\lambda_n z) = \psi_n(z),$$

where $\lambda_n = \exp 2\pi i p_n/q_n$. That estimate will persist if we replace λ_n by any number sufficiently close to it, in particular by $\exp 2\pi i(p_n/q_n + 1/Lq_n)$ for any sufficiently large L. We can choose $L = L_n$ and m_{n+1} in such a way that the Dirichlet kernel

$$D_{L_n q_n, m_{n+1}}$$

is on the one hand, sufficiently close to zero in probability, and on the other hand sufficiently close to being metrically dense. The first property guarantees convergence in probability, and also guarantees the persistence of the degree of metric density achieved in the previous step. The last property guarantees a better degree of metric density for

$$\psi_{n+1} = \psi_n + W_{n+1} D_{L_n q_n, m_{n+1}},$$

so that we put $q_{n+1} = L_n q_n$. Since the Dirichlet kernel has period $2\pi/q_{n+1}$,

$$\psi_{n+1}(\lambda_{n+1}z) - \psi_{n+1}(z) = \psi_n(\lambda_{n+1}z) - \psi_n(z)$$

and as we have seen the expression on the right hand side can be controlled in a complex neighborhood of S^1.

The arguments in section 5.4 below follow a very similar pattern with cosines instead of Dirichlet kernels. However, those arguments are generic, whereas the property we are discussing now can be carried out only by a construction where the next step depends thoroughly on the previous ones. Similar phenomena appear in the conjugation–approximation construction [**AK**], [**HaK**] which is the most powerful general method of constructing diffeomorphisms with prescribed, often exotic properties based on fast periodic approximation.

REMARK. Herman [**He1**] pointed out that for rotations of the circle, cohomology with nonintegrable transfer functions can not be effected by a lacunary Fourier series. This conforms well with the highly non-lacunary nature of the Dirichlet kernels and also underlines the difference between the above construction and that of section 5.4.

4.4 Minimal nonergodic interval exchange transformations. An *interval exchange transformation* (i.e.t.) is a map T of an interval Δ onto itself which preserves Lebesgue measure λ and has only a finite number of discontinuities. It owes its name to an equivalent description as a rearrangement in some fixed new order of subintervals $\Delta_1, \dots, \Delta_m$, which form a partition of Δ, with a possible change of orientation on some of the subintervals. If the orientation does not change we will call T on *oriented interval exchange transformation.* There is a certain ambiguity in the definition of i.e.t. at the points of discontinuity but we fix this ambiguity for an oriented i.e.t. by assuming that the interval Δ is right half-open and that T is continuous from the right. Interval exchange transformations appear as natural section maps for area preserving flows on the surfaces of higher genus.

If an i.e.t. T has a dense orbit then every other orbit, except possibly the orbits of the points of discontinuity, is also dense [**Ke**], see also [**KH**], Section 14.5. We will call such an i.e.t. *quasi-minimal.* If T is an oriented i.e.t. and all its semi-orbits are dense we will call it a *minimal* i.e.t. If we fix m and the order in which the subintervals $\Delta_1 \dots \Delta_m$ are permuted then an oriented i.e.t. is determined by a probability vector

$$\lambda(\Delta_1), \dots, \lambda(\Delta_m)$$

i.e. by a point of an $(m-1)$-dimensional simplex σ_{m-1}. Under some natural combinatorial assumptions all elements of the simplexes except for those belonging to a countably many linear submanifolds determine minimal i.e.t.

The total number of ergodic probability invariant measures for a quasi-minimal i.e.t. is always finite ([**Ve1**], [**KH**], section 14.5). If there is only one such measure we will call T strictly ergodic. Since Lebesgue measure is always invariant this unique measure must be Lebesgue. The central result in the theory of interval exchange transformations is that under the same natural combinatorial assumptions which yield minimality, almost every point of the simplex σ_{m-1} with respect to Lebesgue measure determines a strictly ergodic i.e.t. This was proved independently and simultaneously by W. Veech [**V2**] and H. Masur [**M**].

We are going to show that the minimal but not strictly ergodic i.e.t., although nongeneric, are fairly common.*

We will deal with two-point extension of a minimal oriented i.e.t. Let us call a function $n : \Delta \to \mathbb{Z}/2 = \{0,1\}$ a *k-step function* if it has exactly k points of discontinuity. We associate to any $n : \Delta \to \mathbb{Z}/2$ a $\mathbb{Z}/2$ extension $T^{n(\cdot)}$ of T.

$$T^{n(\cdot)}(x,j) = (Tx, j + n(x)). \tag{4.18}$$

If T is an exchange transformation of m intervals and n is a k-step function then $T^{n(\cdot)}$ can be easily interpreted as an i.e.t. exchanging not more than $2m+2k$ intervals.

THEOREM 4.9. *Let T be a minimal oriented interval exchange transformation. There exists a 3-step function $n : \Delta \to \mathbb{Z}/2$ such that*

(i) $n(x) = h(Tx) - h(x)$
where $h : \Delta \to \mathbb{Z}/2$ is a measurable function and

(ii) *For each subinterval $I \subset \Delta$,*

$$\begin{aligned} &\lambda(h^{-1}(\{0\}) \cap I) > 0 \quad \textit{and} \\ &\lambda(h^{-1}(\{1\}) \cap I) > 0. \end{aligned}$$

Before proving this theorem we will show how it allows us to construct minimal but not strictly ergodic i.e.t.'s.

COROLLARY 4.10. *The two-point extension $T^{n(\cdot)}$ (cf. (4.18)) is minimal but not uniquely ergodic.*

The proof of the corollary is essentially the same as that of Corollary 4.3. By Theorem 4.9 (i) $n(x)$ is a coboundary so that $T^{n(\cdot)}$ is metrically isomorphic to $T \times Id$ via $R(x,j) = (x, j + h(x))$. Therefore $T^{n(\cdot)}$ preserves two sets of positive product measure: graph (h) and graph $(1-h)$. By Theorem 4.9 (ii) and by the minimality of T every orbit of $T^{n(\cdot)}$ visits every open subset of $\Delta \times \mathbb{Z}/2$.

PROOF OF THEOREM 4.9. Since T^{-1} is also an i.e.t. with the same number of points of discontinuity we can replace (i) by

(i′) $n(x) = h(T^{-1}x) - h(x)$,

then apply the result to T^{-1} and obtain the statement of the theorem.

In all the subsequent arguments we omit the minus sign because we are working in $\mathbb{Z}/2$.

Let us call an interval $I \subseteq \Delta$ *k-clear* if T^i is continuous on the interior of I for $|i| =\leq k$. Let a be the left endpoint of Δ and $a_n \overset{\text{def}}{=} T^n a$. By the minimality of T the sequence $\{a_n\}, n = 0, 1, \ldots$ is dense in Δ. If $a_n < a_m$ we will denote by $\chi_{m,n} : \Delta \to \mathbb{Z}/2$ the characteristic function of the interval $[a_n, a_m)$.

As in the proof of Theorem 4.2 the function $n(x)$ is constructed inductively with very similar methods of control over convergence and divergence. However, the whole construction is much more explicit. Let $k_0 = 0$ and $k_1 > 0$ be chosen in such a way that the interval $[a, a_{k_1}]$ is disjoint from its image $[a_1, a_{k_1+1}]$. The inductive construction is determined by an increasing sequence of natural numbers

*The rest of this section appeared with minor modifications in [**KH**] as subsection 14.5e

k_m, $m = 0, 1, \ldots$. In order to formulate the conditions on k_m we need another sequence ℓ_m defined by

$$\ell_0 = 1, \ell_1 = k_1 + 1 \qquad (4.19)$$
$$\ell_{m+1} = k_{m+1} - k_m + \ell_{m-1}$$

so that

$$\ell_m = 1 + \sum_{i=0}^{m-1} k_{m-i}(-1)^i.$$

We will assume that

(I_1) $a < a_{k_1} < a_{k_2} < \cdots < a_{k_m} < a_1 < a_{\ell_2} < \cdots < q_{\ell_{2\left[\frac{m}{2}\right]}}$

$$< a_{\ell_1} < a_{\ell_3} < \cdots < a_{\ell_{2\left[\frac{m+1}{2}\right]-1}}$$

(I_2) The interval $[a_{k_m}, a_{k_{m+1}}]$ is $2k_m-$ clear

(I_3) $a_{k_{m+1}} - a_{k_m} < \frac{a_{k_m} - a_{k_{m-1}}}{3k_m}$.

Let us now show how conditions (I_1), (I_2) and (I_3) can be satisfied inductively. Let us assume that they are already satisfied up to m. Then by (I_1) we can find e,

$$a_{k_m} < e < c_m \stackrel{\text{def}}{=} \begin{cases} a_{\ell_m}, & m \text{ even} \\ a_{\ell_{m-1}}, & m \text{ odd} \end{cases}$$

such that $[a_{k_m}, e)$ is a k_m-clear interval. By the minimality of T, we can find $k_{m+1} > k_m$ such that $a_{k_{m+1}}$ belongs to the left half of the interval $[a_{k_m}, e)$. This implies (I_2) for $m+1$. From the definition of ℓ_m, c_{m+1} cannot be more to the left of c_m than $a_{k_{m+1}}$ is to the right of a_{k_m}. This implies (I_1). Finally, (I_3) can be achieved by simply choosing $a_{k_{m+1}}$ each time very close to a_{k_m}.

Now we are prepared to construct inductively the cocycle $n(x)$ and the transfer function $h(x)$. We begin with a 3-step function

$$\begin{aligned} n_1(x) &= \chi_{0,k_1}(x) + \chi_{1,k_1+1}(x) \\ &= \chi_{0,k_1}(x) - \chi_{0,k_1}(T^{-1}x) \end{aligned}$$

and proceed by induction

$$\begin{aligned} n_{m+1}(x) &= n_m(x) + \chi_{k_m,k_{m+1}}(x) + \chi_{\ell_{m-1},\ell_{m+1}}(x) \\ &= n_m(x) + \chi_{k_m,k_{m+1}}(x) - \chi_{k_m,k_{m+1}}(T^{\ell_{m+1}-k_{m+1}}x) \\ &= n_m(x) + g_{m+1}(x) - g_{m+1}(T^{-1}x) \end{aligned} \qquad (4.20)$$

where

$$g_{m+1}(x) = \sum_{i=0}^{k_m - \ell_{m-1} - 1} \chi_{k_m+j,k_{m+1}+j}(x).$$

We will represent $n_m(x)$ on one hand as a 3-step function and on the other hand as a coboundary.

To show that it is a 3-step function we have from (4.20) and (I_1),

$$\begin{aligned} n_m(x) &= \sum_{i=1}^{m} \chi_{k_{i-1},k_i}(x) = \chi_{\ell_0,\ell_1}(x) + \sum_{i=1}^{m-1} \chi_{\ell_{i-1},\ell_{i+1}}(x) \\ &= \chi_{0,k_m}(x) + \chi_{\ell_{2\left[\frac{m}{2}\right]},\ell_{2\left[\frac{m+1}{2}\right]-1}}(x). \end{aligned}$$

Thus $n_m(x)$ converges pointwise to

$$n(x) = \chi_{[a,b)}(x) + \chi_{[c,d)}(x)$$

where

$$b = \lim_{m\to\infty} a_{k_m},\ c = \lim_{m\to\infty} a_{\ell_{2m}},\ d = \lim_{m\to\infty} a_{\ell_{2m+1}}.$$

To represent n_m as a coboundary, we use (4.20) to obtain

$$n_m(x) = h_m(x) - h_m(T^{-1}x)$$

where

$$\begin{aligned} h_m(x) &= \chi_{0,k_1}(x) + \sum_{i=2}^{m} g_i(x) \\ &= \chi_{0,k_1}(x) + \sum_{i=2}^{m} \sum_{j=0}^{k_{i-1}-\ell_{i-2}-1} \chi_{k_{i-1}+j,k_i+j}(x). \end{aligned} \tag{4.21}$$

Since $h_{m+1}(x) = h_m(x) + g_{m+1}(x)$ we have from (4.20) and (I_3),

$$\begin{aligned} &\lambda(\{x \in \Delta : g_{m+1}(x) = 1\} \\ &\quad \le (k_m - \ell_{m-1})(a_{k_{m+1}} - a_{k_m}) < \frac{a_{k_m} - a_{k_{m_1}}}{3} \end{aligned}$$

so that the sequence h_m converges in L^1 to a function h which obviously satisfies (i'). Thus, it remains to prove (ii). Again the argument resembles very much those from section 4.1, being only simpler and more explicit.

Let us call an interval $[a_{k_{m-1}+j}, a_{k_m+j})j = 0,1,\ldots,k_{m-1}-\ell_{m-2}-1$ *of rank* m. It follows from (I_3) that all intervals of a given rank are pairwise disjoint. Moreover for $\ell < m$ any interval of rank m is either disjoint with an interval of rank ℓ or is contained inside such an interval. To prove that let us assume that the opposite is true so that for some

$$j \in \{0,1,\ldots,k_{m-1} - \ell_{m-2} - 1\}$$

and

$$s \in \{0,1,\ldots,k_{\ell-1} - \ell_{\ell-2} - 1\}$$

$$a_{k_{m-1}+j} < a_{k_{\ell-1}+s} < a_{k_m+j}.$$

Since $k_{\ell-1} + s < 2k_{\ell-1} < 2k_{m-1}$ we can apply $T^{-(k_{\ell-1}+s)}$ and obtain from (I.2) that the point a_0 lies between

$$a_{k_{m-1}} + (j - k_{\ell-1} - s)$$

and

$$a_{k_m+(j-k_{\ell-1}-s)}.$$

But this is impossible because a is the left end of Δ.

It follows from the last statement and from (4.21) that the function h_m is constant on every interval of rank m.

Let us fix an interval $I \subset \Delta$. By the minimality of T every orbit segment of length N for sufficiently large N intersects I, in particular I contains an interval I' of rank m for a sufficiently large m . h_m is constant on I' and from (I_3) for $n > m$

$$\begin{aligned}
&\lambda(\{x \in I' \; ; \; h_n(x) = h_m(x)\}) \geq \\
\text{(4.22)} \qquad &\lambda(I') - \sum_{k=1}^{n-m} \lambda(g_{n+k}^{-1}(1)) \geq \\
&\lambda(I')\left(1 - \frac{1}{3} - \frac{1}{9} - \cdots - \frac{1}{3^{m-n}}\right)
\end{aligned}$$

so that

$$\lambda\{x \in I' : h(x) = h_m(x)\} \geq \frac{\lambda(I')}{3}.$$

Let us now take the smallest $m' > m$ such that I' contains an interval I'' of rank m'. Then the value of $h_{m'}$ on I'' is different from that of h_m on I' so that applying the same argument to I'' we obtain

$$\lambda\{x \in I'' : h(x) = h_{m'}(x)\} > \frac{\lambda(I'')}{3}.$$

Inequalities (4.22) and (4.23) imply (ii). □

REMARK. A slight modification of the argument allows us to replace minimality by quasi-minimality in the assumption with the outcome being a 4-step function instead of a 3-step function.

5. Non-trivial cocycles

In this chapter we discuss methods of proving that a certain cocycle is not a coboundary or an almost coboundary, and various applications of those methods. In contrast with situations considered in Chapter 3 where the non-triviality of cocyles was derived from non-vanishing of certain invariant distributions we will concentrate now on the non-stable case, i.e. we will look for non-trivial cocycles which are the limits of coboundaries in various topologies. The subtlety of this problem is that there are very few means to disprove the existence of a transfer function which is assumed to be only measurable with no assumptions about continuity or integrability.

5.1 Two general criteria. The following propositions summarize the two main approaches to the problem. In both cases G is an arbitrary second countable topological group.

PROPOSITION 5.1. *Let ϕ be a G-valued cocycle over a measure-preserving transformation $T : (X, \mu) \to (x, \mu)$.*

(i) *If for a sequence of integers q_n, $T^{q_n} \to Id$ in the weak topology* and ϕ is a coboundary then the product $\phi(x)\phi(Tx) \ldots \phi(T^{q_n-1}x) \to Id_G$ in probability.*
(ii) *If for any $q_n \to \infty$, any $\alpha > 0$, and for every measurable set A*

$$\mu(T^{q_n} A \cap A) > \alpha\mu(A),$$

*A transformations T for which such a sequence exists is called rigid [**HaK**], [**K1**].

then for every coboundary ϕ and every neighborhood U of $e \in G$,

$$\varlimsup_{n\to\infty} \mu\{x : \phi(x)\phi(Tx)\dots\phi(T^{q_n-1}x) \in U\} > \alpha.$$

PROOF. (i) If $\phi(x) = \psi(x)^{-1}\psi(Tx)$ then

$$\phi(x)\phi(Tx)\dots\phi(T^{q_n-1}x) = \psi^{-1}(x)\psi(T^{q_n}x)$$

and since $\psi(T^{q_n}x)$ converges in probability to $\psi(x)$ so that the product converges to identity.

ii) Let us take a countable partition η of G such that for every element $c \in \eta$ and $g_1, g_2 \in c$, $g_1g_2^{-1}$ and $g_1^{-1}g_2$, belong to U. Let $A = \psi^{-1}(c)$, for all $c \in \eta$ and apply a diagonal argument. □

PROPOSITION 5.2. *Let G be provided with a metric compatible with the uniform structure. If a G-valued cocycle over T is a coboundary then for every $\epsilon > 0$ there exists R such that for all n*

$$\mu\{x \in X; dist(\phi(x)\phi(Tx)\dots\phi(T^{n-1}x), e) \le R\} > 1-\epsilon$$

PROOF. Again if ϕ is a coboundary

$$\phi(x)\phi(Tx)\dots\phi(T^{n-1}x) = (\psi(x))^{-1}\psi(T^nx).$$

Since ψ and ψ^{-1} have the property in question, T^n preserves the measure so that $\psi \cdot T^n$ also has that property. Finally, since the product is continuous for every R there is $R_1 = R_1(R)$ such that if $\mathrm{dist}(x_i, e) < R$ $i = 1, 2$ then $\mathrm{dist}(x_1x_2, e) < R_1$. □

Since we will be interested in finding cocycles which are not almost coboundaries we will formulate explicitly the contrapositives of Proposition 5.1(i) and 5.2 for that case.

COROLLARY 5.3. *Under the same conditions as Proposition 5.1, if there exits a neighborhood U of the identity $e \in G$, a sequence $q_n \to \infty$, sequences of closed subsets V_n and W_n of G, and $\epsilon > 0$ such that*

(i) $T^{q_n} \to Id$ *in the weak topology*
(ii) $V_nW_n^{-1} \cap U = \phi$
(iii) $\mu\{x : \phi(x)\phi(Tx)\dots\phi(T^{q_n-1}x) \in V_n\} > \epsilon$
(iv) $\mu\{x : \phi(x)\phi(Tx)\dots\phi(T^{q_n-1}x) \in W_n\} > \epsilon.$

Then ϕ is not cohomologous to any constant.

COROLLARY 5.4. *Under the same conditions as Proposition 5.2, let U_k be an increasing sequence of neighborhoods of the identity in G such that $\bigcup_{k=1}^{\infty} U_k = G$. If there exists a sequence $n_k \to \infty$, sequences of closed subsets V_k and W_k of G, and $\epsilon > 0$ such that*

(i) $V_kW_k^{-1} \cap U_k = \phi$
(ii) $\mu(\{x : \phi(x)\phi(Tx)\dots\phi(T^{n_k-1}x) \in V_k\}) > \epsilon$
(iii) $\mu(\{x : \phi(x)\phi(Tx)\dots\phi(T^{n_k-1}x) \in W_k\}) > \epsilon$

Then ϕ is not cohomologous to any constant.

The method based on Corollary 5.3 applies to cocycles with values in arbitrary groups including compact ones, but it requires the transformation to be rigid. The second method is free of that restriction but it works only for cocycles with values in some non-compact (e.g. compactly generated) groups.

In sections 5.2–5.5 we will explore various situations where a cocycle $\phi = \lim \phi_n$ fails to be an almost coboundary, although ϕ_n's are coboundaries and L^1 convergence of ϕ_n to ϕ is very fast. Our strategy will be as follows. Although ϕ_n is a coboundary, for large values of n the corresponding transfer function ψ_n will be huge so that ϕ_n will satisfy the condition (ii)–(iv) of Corollaries 5.3 or 5.4 for a finite but growing with n number of iterates.

Due to fast convergence all the alterations on the subsequent steps will be small enough to keep the properties for the limit cocycle for the given iterates and, naturally, they will produce similar properties for higher iterates.

In addition to Corollary 5.4 the following observation is useful in this situation. Let $T : (X, \mu) \to (X, \mu)$ be an ergodic measure-preserving transformation and let U and V be two disjoint closed subsets of G. Let $\phi(x) = (\psi(x))^{-1}\psi(Tx)$ and let $u = \mu(\psi^{-1}(U))$, $v = \mu(\psi^{-1}(V))$.

LEMMA 5.5. *Suppose that*

$$\epsilon_0 = u(1-u) - 1 + u + v > 0. \tag{5.1}$$

Then there exists an arbitrarily large k such that for every $\delta > 0$

$$\mu(\{x : \phi(x)\phi(Tx)\dots\phi(T^{k-1}x) \in V^{-1}U\}) \geq \epsilon_0 - \delta$$

and

$$\mu(\{x : \phi(x)\phi(Tx)\dots\phi(T^{k-1}x) \in U^{-1}V\}) \geq \epsilon_0 - \delta.$$

REMARK. Condition (5.1) is satisfied if $u + v$ is sufficiently close to one and neither u or v is very small.

PROOF. We write as usual

$$\phi(x)\phi(Tx)\dots\phi(T^{k-1}x) = \psi(x)^{-1}\psi(T^k x)$$

and take sets $A = \psi^{-1}(U)$ and $B = \psi^{-1}(V)$. From the ergodic theorem,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mu(A \cap T^{-k}A) = \mu(A)^2 = u^2. \tag{5.2}$$

In particular we can find an arbitrarily large k such that $\mu(A \cap T^{-k}A) \leq u^2 + \delta$ so that $\mu((X\backslash A) \cap T^{-k}A) \geq 1 - u(1-u) - \delta$ and by (5.1), $\mu(B \cap T^{-k}A) \geq \mu((X\backslash A) \cap T^k A) - \mu(X\backslash(A \cup B)) \geq u(1-u) - \delta - 1 + u + v = \epsilon_0 - \delta$. If $x \in B$ and $T^k x \in A$ then

$$\phi(x)\phi(Tx)\dots\phi(T^{k-1}x) \in V^{-1}U$$

Similarly, we show that $\mu(A \cap T^{-k}B) \geq$

$$\begin{aligned}
&\mu(A \cap T^{-k}(X\backslash A)) - \mu(X\backslash(A \cup B)) \\
&= \mu(T^k A \cap (X\backslash A)) - 1 + u + v \\
&= \mu(T^k A) - \mu(T^k A \cap A) - 1 + u + v \\
&= \mu(A) - \mu(A \cap T^{-k} A) - 1 + u + v \\
&\geq u(1-u) - 1 + u + v - \delta = \epsilon_0 - \delta.
\end{aligned}$$

□

This lemma can be used to verify the conditions of Corollary 5.4 if, e.g. G is an abelian group provided with a translation invariant metric and U, V are such that

$$\operatorname{dist}(U^{-1}V,\ V^{-1}U)$$

is sufficiently large. (Cf. the proof of Theorem 5.6 below).

5.2 The case of fast C^∞ approximation. We now return to the situation discussed in section 4.2.

THEOREM 5.6. *Suppose that f is a C^∞ diffeomorphism of a compact connected m-dimensional manifold M which preserves a measure μ given by a density bounded between two positive constants, and is ergodic with respect to μ. If f admits a fast C^∞ periodic approximation then there exists a real valued C^∞ cocycle $h = \lim_{n\to\infty}(\psi_n \circ f - \psi_n)$, where ψ_n is C^∞, which is not cohomologous to any constant.*

PROOF. We begin with a "cubic triangulation" of M; in other words we form a mod 0 partition of M into diffeomorphic images of the standard m-dimensional cube $I^m = \{(t_1, \ldots t_m) : 0 \leq t_i \leq 1,\ i = 1 \ldots m\}$. Then for a given number n we can subdivide the standard cube by n^m equal cubes by dividing the value of each coordinate into n equal intervals. The images of this subdivision form a partition of M which we will denote by ξ_n. We associate with every element $c \in \xi_n$ a function α_c which is somewhat similar to the function defined by (4.9).

The element c is an image under a fixed (independent on n) diffeomorphism of a cube in $\mathbb{R}^m$ with a center at a point $(t_1^0 \ldots t_m^0)$ and with sides parallel to the coordinate axes. We define α_c as the image under that diffeomorphism of the function $n\prod_{i=1}^{m}\theta_n(t_i - t_i^0)$ extended by zero outside of the image of I^m. Here $\theta_n(t)$ is a C^∞ bump function equal to 1 for $|t| < \frac{1}{2n}$, to 0 for $|t| > \frac{1}{2n} + \frac{1}{2n^{10}}$ and such that its r-th derivative does not exceed $C(r)n^{100}$ for $r = 1, 2, \ldots$.

Due to the fast periodic approximation for $c \in \xi_{q_n}$ the function $\alpha_c \circ f^{q_n} - \alpha_c$ allows the C^r estimate

$$\|\alpha_c \circ f^{q_n} - \alpha_c\|_r < c_1(k,r)q_n^{-k} \tag{5.3}$$

(cf. (4.8)). Details of this estimate are the same as in the proof of Theorem 4.5.

We have

$$\alpha_c \circ f^{q_n} - \alpha_c = \psi_c \circ f - \psi_c$$

where

$$\psi_c = \sum_{k=0}^{q_n-1} \alpha_c \circ f^k.$$

We are going to show that one can choose a collection of elements of ξ_{q_n}, $c_1, \dots, c_\ell$ so that the function $\psi = \sum_{i=1}^{\ell} \psi_{c_i}$ has the following property

$$\mu(\psi^{-1}([q_n,\infty))) > \frac{2}{5} \qquad \mu(\psi^{-1}(\{0\})) > \frac{2}{5}. \tag{5.4}$$

Then we can apply Lemma 5.5 and obtain a number $k = k_n$ such that for the function

$$\phi = \sum_{i=1}^{\ell} \alpha_{c_i} \circ f - \alpha_{c_i} = \psi \circ f - \psi, \tag{5.5}$$

$$\mu(\{x : \sum_{j=0}^{k_n-1} \phi(T^j x) \le -q_n\}) > \frac{1}{50} \quad \text{and} \quad \mu(\{x : \sum_{j=0}^{k_n-1} \phi(T^j x) \ge q_n\}) > \frac{1}{50}. \tag{5.6}$$

To verify (5.4) we order the elements of ξ_{q_n} and consider successively the functions $\psi_{c_1}, \psi_{c_1} + \psi_{c_2}, \psi_{c_1} + \psi_{c_2} + \psi_{c_3}, \dots$.

We denote $\psi^{(k)} = \sum_{i=1}^{k} \psi_{c_i}$.

Let us look at the distribution of the values of two successive functions $\psi^{(k)}$ and $\psi^{(k+1)}$. Both functions are non-negative and $\psi^{(k+1)} - \psi^{(k)} = \psi_{c_k}$. the support of ψ_{c_k} is the union of q_n successive iterates of c_k with its neighborhood. We will disregard all the neighborhoods and their images since the total measure of their union does not exceed a constant multiple of q_n^{-9}.

By neglecting this small set we can reduce the question to considering distributions of iterates for characteristic functions of the elements $c_k \in \xi_{q_n}$ multiplied by the constant q_n. These iterates cover increasingly large fractions of the total measure. The distribution of a sum is the sum of the distributions. Adding the sum of the iterates of $q_n \chi_{c_k}$, we change the value by at least q_n on a set of measure between $\mu(c_k)$ and $q_n \mu(c_k)$.

The last number is small since by our assumptions $\mu(c_n) < q_n^{-m} \cdot d$ for a constant d. Eventually the value on a set of measure close to 1 becomes greater than q_n. Since we start from zero, that means that for some k the measure of support is within dq_n^{-m+1} of $1/2$. By making n large enough we obtain (5.4).

Let us point out that

$$\|\phi\|_r \le d_1 q_n^{m-1} \max_{c \in \xi_{q_n}} \|\alpha_c \circ f - \alpha_c\|_r$$

for another constant d_1 and thus $\|\phi\|_r$ decreases faster than any power of q_n as $n \to \infty$ by (4.8).

Now we proceed to build the cocycle h inductively as a sum of $\phi^{(k)}$ where each $\phi^{(k)}$ is defined by (5.5) for a certain recurrence time q_{n_k}. By choosing the numbers q_{n_k} sufficiently quickly increasing we guarantee the convergence of the series $h = \sum_{k=1}^{\infty} \phi^{(k)}$. On the other hand we want to insure that for every k

$$\mu(\{x : \sum_{j=0}^{q_{n_k}-1} h(T^j x) \leq -\frac{q_{k_n}}{2}\}) > \frac{1}{50}$$

(5.7) and

$$\mu(\{x : \sum_{j=0}^{q_{n_k}-1} h(T^j x) \geq \frac{q_{k_n}}{2}\}) > \frac{1}{50}.$$

By Corollary 5.4 these inequalities imply that h is not cohomologous to a constant. We have

$$h = \sum_{i=0}^{k-1} \phi^{(i)} + \phi^{(k)} + \sum_{i=k+1}^{\infty} \phi^{(i)}.$$

Since $\phi^{(i)} = \psi^{(i)} \circ f - \psi^{(i)}$ (cf. (5.5)) for every ℓ

$$\left| \sum_{j=0}^{\ell-1} \sum_{i=0}^{k-1} \phi^{(i)}(T^j x) \right| < 2 \sum_{i=1}^{k-1} \max |\psi^{(i)}| \tag{5.8}$$

and the expression in the right hand part of the last inequality is known by the time we make the choice of q_{n_k} so that we can choose

$$q_{n_k} > 20 \sum_{i=1}^{k-1} \max |\psi^{(i)}|.$$

On the other hand by choosing the subsequent return times far away we can insure that for $i > k$

$$|\psi^{(i)}| < 2^{-1} \cdot k_{n-K}^{-1}$$

so that

$$\left| \sum_{j=0}^{k_{n_k}-1} \sum_{i=k+1}^{\infty} \phi^{(i)}(T^j x) \right| < 1. \tag{5.9}$$

Since

$$\left| \sum_{j=0}^{k_{n_k}-1} h(t^j x) \right| > \left| \sum_{j=0}^{k_{n_k}-1} \phi^{(k)}(T^j(x)) \right| -$$
$$\left| \sum_{j=0}^{k_{n_k}-1} \sum_{i=0}^{k-1} \phi^{(i)}(T^j x) \right| - \left| \sum_{j=0}^{k_{n_k}-1} \sum_{i=k+1}^{\infty} \phi^{(i)}(T^j(x)) \right|,$$

inequalities (5.6), (5.8) and (12.9) imply (5.7). □

5.3 Weakly mixing flows on $\mathbb{T}^2$. Kolmogorov's Theorem 3.5 implies that for any Diophantine (in other words, not exceptionally well rationally approximable) number α every C^∞ special flow build over the rotation R_λ, $\lambda = \exp 2\pi i\alpha$ has discrete spectrum (cf. (1.22)). Now we are going to show that for very well approximable rotation numbers there exist not only C^∞ but even real-analytic special flows which have continuous spectrum. To that end we will find a criterion which guarantees that for a real-valued function h on the circle the function exp irh for every real r is not a coboundary. We will use the approach suggested by Corollary 5.3. Let us consider the special flow over R_λ build under a function $h_0 + h(z)$ where h_0 is a constant and $h(z)$ is a function with zero average

$$h(z) = \sum_{n\neq 0} h_n z^n.$$

We consider a sequence of iterates, say q_n, $n = 1, 2, \ldots$ of the rotation R_λ corresponding to a very good rational approximation p_n/q_n, $0 \le p_n < q_n$ of α. To make the notation lighter we omit the index n in the subsequent computation.

We have

$$\begin{aligned}\sum_{k=0}^{q-1} h(\lambda^k z) = \sum_{m=-\infty}^{\infty} h_m \left(\frac{1-\lambda^{qm}}{1-\lambda^m}\right) z^m = \sum_{q\neq lm} h_m \left(\frac{1-\lambda^{qm}}{1-\lambda^m}\right) z^m + \\ \sum_{\ell\neq 0}^{\infty} h_{\ell q} \left(\frac{1-\lambda^{q^2\ell}}{1-\lambda^q}\right) z^{\ell q}.\end{aligned} \tag{5.10}$$

We approximate λ by $\lambda_0 = \exp 2\pi i\frac{p}{q}$ and write the expression for $\sum\limits_{k=0}^{q-1} h(\lambda_0^h z)$ corresponding (5.10). The sum over m not divisible by q vanishes and the second sum becomes

$$\sum_{\ell=-\infty}^{\infty} q h_{\ell q} z^{\ell q}.$$

We denote by a_q the L^2 norm of this power series, by b_q the expression $q \sum\limits_{k=-\infty}^{\infty} |h_{\ell q}|$ and by $e_q = |\alpha - \frac{p}{q}|$. Obviously $a_q \le b_1$. We will assume a certain regularity condition on the decrease of Fourier coefficients $h_{\ell q}$, namely

$$\frac{|h_q|}{\sum\limits_{\ell=1}^{\infty} |h_{\ell q}|} > c > 0 \tag{5.11}$$

where the constant c is independent of q. Since $|h_q| < a_q$ (5.11) implies that

$$b_1 \le \sqrt{2} c^{-1} a_q \tag{5.12}$$

Obviously for every integer m

$$|\lambda^m - \lambda_0^m| \le |m| \cdot |\lambda - \lambda_0| \le |m| \cdot e_q$$

so that if h' is absolutely summable

$$\begin{aligned}&\left|\sum_{k=0}^{q-1} h(\lambda^k z) - h(\lambda_0^k z)\right| \\ (5.13) \qquad &\leq \left|\sum_{m=-\infty}^{\infty}\right| h_m \left|\sum_{k=0}^{q-1}\right| \lambda^{km} - \lambda_0^{km} \mid \mid \\ &\leq e_q \frac{q(q-1)}{2} \sum_m m|h_m| < c' e_q\, q^2\end{aligned}$$

where c' is a constant which depends on h but not on q.

Now we make an assumption on the relationship between the speed of approximation for α and the decrease of Fourier coefficients for h. Namely we assume

$$\lim_{n\to\infty} \frac{e_{q_n} q_n^2}{b_{q_n}} = 0. \tag{5.14}$$

By (5.14) and (5.12) we see that for the function

$$h^{(q)}(z) = \sum_{k=0}^{q-1} h(\lambda^k z)$$

which has zero average both the maximum and the L^2-norm are of order b_q. Thus it has to have both positive and negative values of that order on sets of measure bounded away from zero by a constant independent of q. Up to an error of order $e_q q^2$ the function $h^{(q)}$ coincides with the function

$$\widetilde{h}^{(q)} = q \sum_{\ell=-\infty}^{\infty} h_{\ell q} z^{\ell q}.$$

Let us now take $L = \epsilon_0 \left[\frac{1}{a_q}\right]$ where ϵ_0 is a sufficiently small constant and consider

$$\sum_{k=0}^{Lq-1} h(\lambda^k z) = \sum_{j=0}^{L-1} \sum_{k=0}^{q-1} h(\lambda^k(\lambda^{jq} z)) = \sum_{j=0}^{L-1} h^{(q)}(\lambda^{jq} z).$$

Replacing in the last expression $h^{(q)}$ by $\widetilde{h}^{(q)}$ we allow an error of order $e_q \cdot q^2 \cdot L = o(1)$.

On the other hand, we have

$$\sum_{j=0}^{L-1} \widetilde{h}^{(q)}(\lambda^{jq} z) = q \sum_{\ell=-\infty}^{\infty} h_{\ell q} \sum_{j=0}^{L-1} \lambda^{j\ell q^2} z^{\ell q}. \tag{5.15}$$

We want again to compare uniform and L^2 norms for a function, this time for the one given by (5.15).

The uniform norm does not exceed Lb_q. The L^2 norm is greater than

$$q|h_q| \sum_{j=0}^{L-1} \lambda^{jq^2}.$$

We have by (5.12) and (5.14) for $0 \le j \le L-1$

$$|\lambda^{jq^2} - 1| \le jq^2|\lambda - \lambda_0| \le Lq^2 e_q = o(1).$$

Thus if q is sufficiently large,

$$\left|\sum_{j=0}^{L-1} \lambda^{jq^2}\right| > \frac{L}{2}$$

so that by (5.11) and the definition of L the L^2 norm of our function is greater than a certain constant which depends on h and ϵ_0 and can be arbitrary small by the choice of ϵ_0. What is important is that the ratio of the above estimate of the uniform norm and the below estimate for the L^2 norm is a constant independent of q ϵ_0. Since the average of the function (5.15) is zero it reaches both positive and negative values of order ϵ_0 on sets of the measure separated from zero. Taking into account the remark about the error we conclude that the same is true for the function

$$\sum_{k=0}^{Lq-1} h(\lambda^k)$$

But by our choice of L and by (5.14) $\lambda^{Lq} \to 1$ as $q \to \infty$ so that the function

$$\sum_{k=0}^{Lq-1} h_0 + h(\lambda^k z)$$

is not close in probability to any constant. However the variation of that function is estimated from above by a multiple of ϵ_0. When we pass to the function $\exp\, ir(h_0 + h(z))$ we see that the spread of values for

$$\prod_{k=0}^{Lq-1} \exp\, ir(h_0 + h(\lambda^k z))$$

persists if ϵ_0 is chosen small enough. Applying Corollary 5.3 we see that the special flow over R_λ build under $h_0 + h$ is weakly mixing. Let us summarize the discussion.

THEOREM 5.7. *Let* $h(z) = \sum_{n \ne 0} h_n z^n$ *be a* C^2 *real valued function on* S^1 *with zero average. Let* R_λ *be a rotation on* S^1 $\lambda = \exp 2\pi i\alpha$. *Suppose for a certain sequence of rational numbers* p_n/q_n,

$$\frac{q_n|\alpha - p_n/q_n|}{\sum_{k=1}^{\infty} |h_{kq_n}|} \to 0 \tag{5.16}$$

and

$$\frac{|h_{q_n}|}{\sum_{k=1}^{\infty} |h_{kq_n}|} > c > 0. \tag{5.17}$$

Then for any h_0 *and* r *the cocycle* $\exp\, ir(h_0 + h(z))$ *is not a coboundary and consequently the special flow over* R_λ *build under the function* $h_0 + h(z)$ *is weakly mixing.*

REMARK. Since conditions (5.16) and (5.17) remain true if we multiply h by a constant, those conditions also guarantee that S^1 extension of R_λ determined by the function exp ih has no eigenfunctions except for those lifted from the base (cf. (1.15)).

It is interesting to notice that for real analytic functions with a regular decrease of Fourier coefficients the sufficient condition for weak mixing given by (5.16) is very close in terms of the speed of approximation for λ to the negation of the sufficient condition for the discrete spectrum namely

$$\sum_n \frac{|h_n|^2}{|\lambda^n - 1|^2} < \infty \tag{5.18}$$

(cf. Theorem 3.5).

Let us consider for example the function

$$h(z) = \sum_{n\neq 0} 2^{-|n|} z^n = \frac{2\cos\phi - 2}{5 - 2\cos\phi}$$

where $z = \exp 2\pi i\phi$. This function obviously satisfies (5.17) and (5.16) becomes for it

$$2^{q_n} q_n \left| \alpha - \frac{p_n}{q_n} \right| \to 0$$

which is sufficient for weak mixing for the special flow. On the other hand (5.18) converges if for some $c > 0$ and for all $n > 0$

$$\frac{2^{-n}}{|\lambda^n - 1|} < \frac{c}{n} \quad \text{or, equivalently,} \quad 2^q \left| \alpha - \frac{p}{q} \right| > c$$

for all p and q.

Let us conclude with several comments.

Weak mixing is typical in the category sense for the special flows over circle rotations and hence for time changes in linear flows on the torus [**Fa1**].

Mixed spectrum can appear even for an analytic roof function (A. Katok; unpublished) while it is not known whether for an analytic of smooth roof function an exotic pure point spectrum can appear, i.e. whether it is possible that the special flow has a pure point spectrum, but the roof function is not additively cohomologous to a constant.

It is possible that for the roof functions with a sufficiently regular decay of Fourier coefficients there is a dichotomy between the solvability of the additive cohomological equation and weak mixing.

Finally mixing is possible for a smooth time change in a linear flow on a torus of dimension greater than two [**Fa2**].

5.4 Ergodicity of analytic cylindrical cascades. A problem closely connected with that discussed in section 5.3 concerns the $\mathbb{R}$ extension of the rotation R_λ

$$R_\lambda^h(z,t) = (\lambda z, t + h(z)). \tag{5.19}$$

This transformation, which is sometimes called a cylindrical cascade, preserves infinite Lebesgue measure. We are going to discuss the possibility of T being ergodic. It is obviously necessary that h have zero average. On the other hand, if the cocycle h is a coboundary, then R_λ^h is isomorphic to the direct product $R_\lambda \times Id$, and thus not ergodic. Furthermore, if

$$\exp\ irh(z) = \psi(z)^{-1}\psi(\lambda z)$$

then the function

$$\psi(z)^{-1} \exp\ irt$$

is R_λ^h invariant. Thus the ergodicity on R_λ^h implies that the special flow over R_λ build under the function $h(z) + 2\pi\alpha k + 2\pi\ell$ is weakly mixing for any integers k and ℓ, where $\lambda = \exp\ 2\pi i\alpha$. It looks as though the ergodicity of R_λ^h is a stronger property than the weak mixing of the special flows; however, no counterexamples are known.

Krygin [**Kr**] proved the existence of an analytic cocycle over an irrational rotation such that the extension (5.19) is ergodic. This result was generalized by Herman [**He3**] to cocycles with values in more general groups. Herman's proof is essentially categorical. Another interesting feature of that proof is that it interprets the approximation of the cocycle by coboundaries with diverging transfer functions as the approximation of a diffeomorphism by elements of the actions of a compact group via diverging conjugations as in [**AK**].

We will now formulate and sketch a proof of a categorical version of Krygin's theorem. Let us fix a complex annular neighborhood U of the circle S^1 and consider the space of all functions analytic in U and continuous on the boundary, with the topology of uniform convergence. Let $\mathcal{A}$ be the product of this space with circle, with the product topology.

THEOREM 5.8. *The set of pairs $(h, \lambda) \in \mathcal{A}$ such that the extension (5.19) is ergodic is a residual set in $\mathcal{A}$.*

We follow the same course as in the categorical theorems in [**K1**]. First, the ergodicity of R_λ^h on the cylinder follows from the ergodicity of the induced map $(R_\lambda^h)_{B_M}$ on every band $B_M = [-M, M] \times S^1$. The categorical argument rests on the following observation which is an almost immediate corollary of the ergodic theorem.

PROPOSITION 5.9. *Let $T : (X, \mu) \to (X, \mu)$ be a measure-preserving transformation of a Lebesgue probability space and let $\{\phi_1, \phi_2 \dots\}$ be a countable dense subset of $L^1(X, \mu)$. Suppose for any positive integers K and N and any $\epsilon > 0$, there exists a set $A_{K,N,\epsilon} \subset X$ of measure greater than $1 - \epsilon$ such that for every $x \in A_{K,N,\epsilon}$ there exists $n(x) > N$ such that for $k = 1, \dots, K$*

$$\left| \frac{1}{n(x)} \sum_{j=0}^{n(x)-1} \phi_k(T^j x) - \int \phi_k d\mu \right| < \epsilon. \tag{5.20}$$

Then T is ergodic.

To set up the categorical argument, we fix a band B_M, a natural number N, an $\epsilon > 0$, a continuous function ϕ on B_M, and a neighborhood A in $\mathcal{A}$. We will construct below a pair $(h, \lambda) = (h^{(A)}, \lambda^{(A)}) \in A$ such that if for $T = (R_\lambda^h)_{B_M}$ and for the given function (5.20) holds, then (5.20) remains true for $T(R_{\lambda'}^{h'})_{B_M}$ for any other (h', λ') from a sufficiently small neighborhood of (h, λ). Let us denote this neighborhood by

$$P(h, \lambda, \phi, N, \epsilon).$$

Let $A_1, A_2, A_3, \dots$ be a base of neighborhoods in $\mathcal{A}$ and let $\phi_1, \phi_2, \dots$ be a dense set in $C(B_M)$.

The set

$$\mathcal{B} = \bigcap_{J=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} P(h^{(A_\ell)}, \lambda^{(A_\ell)}, \phi_k, N, 2^{-J}).$$

is a dense G_δ set in $\mathcal{A}$.

According to Proposition 5.9, for every $(h, \lambda) \in \mathcal{B}$ $(R^h_\lambda)_{B_M}$ is ergodic. Taking the intersection over all integers M, we obtain the result.

It remains to prove the approximation step.

We can think of A as a neighborhood of a pair (h, λ) where h is a coboundary and λ is rational. This is because for every irrational λ, the trigonometric polynomials are all coboundaries with real analytic transfer functions, and thus coboundaries are dense for the given λ. Keeping the transfer function fixed, we approximate λ by a rational number. Thus we can think of A as a neighborhood of $(h, \exp 2\pi i\, p_0/q_0)$ where $h(z) = \psi(z \exp 2\pi i\, p_0/q_0) - \psi(z)$.

Let us note that if g is a coboundary, $g(z) = \phi(\lambda z) - \phi(z)$, then the corresponding cylindrical cascade R^g_λ satisfies

$$R^g_\lambda = \Phi \circ (R_\lambda \times Id) \circ \Phi^{-1}$$

where

$$\Phi(z, t) = (z, t + \phi(z)).$$

The map R^g_λ is embedded into the periodic flow

$$S_s = \Phi \circ (R_x \times Id) \circ \Phi^{-1},$$

$s \in \mathbb{R}$, whose orbits are graphs of the functions $\phi + const$. In particular, for any sufficiently large q and p such that $(p, q) = 1$, the orbits of $S_{p/q}$ fill those graphs with high uniform density. Both of these remarks remain true when we pass from the flow S_s to the flow induced by S_s on any band B_M.

For any $\epsilon > 0$, any M and any given function ψ, one can find Q and W such that for any $q \geq Q$ and $w \geq W$ the function

$$\psi_{w,q}(z) = \psi(z) + w(z^q + z^{-q})$$

has the property that all but a set of measure of less than ϵ of the points in B_M belong to graphs of the functions $\psi_{w,q} + const.$ whose intersections with B_M are ϵ uniformly distributed. The last property means that for any intervals $\Delta \subseteq S^1$ and $\Sigma \subseteq [-M, M]$,

$$\left| \lambda(\{z \in \Delta : \psi_{w,q}(z) + c \in \Sigma\}) - \frac{\lambda(\Delta)\lambda(\Sigma)}{2M} \right| < \epsilon.$$

We first perturb p_0/q_0, replacing it with p_1/q_1 for q_1 sufficiently large and then replace h by

$$\begin{aligned} h_1(z) &= \psi(z \exp 2\pi i\, p_1/q_1) - \psi(z) \\ &= \psi(z \exp 2\pi i\, p_1/q_1) + w((z \exp 2\pi i\, p_1/q_1)^{q_1} \\ &\quad + (z \exp 2\pi i\, p_1/q_1)^{-q_1}) \\ &\quad - \psi(z) - w(z^{q_1} + z^{-q_1}) \\ &= \widetilde{\psi}(z \exp 2\pi i\, p_1/q_1) - \widetilde{\psi}(z) \end{aligned}$$

where

$$\widetilde{\psi}(z) = \psi(z) + w(z^{q_1} + z^{-q_1})$$

and we choose q_1 and w according to the previous remark. Then we choose a very large (actually arbitrarily large) L, and replace h_1 by

$$\widetilde{h}(z) = \widetilde{\psi}\left(z \exp 2\pi i\left(\frac{p_1}{q_1} + \frac{1}{Lq_1}\right)\right) - \widetilde{\psi}(z).$$

Now the pair $(h^{(A)}, \lambda^{(A)})$, needed for the category argument, can be taken as $(\widetilde{h}, \widetilde{\lambda})$ where $\widetilde{\lambda} = \exp\ 2\pi i\left(\frac{p_1}{q_1} + \frac{1}{Lq_1}\right)$. Due to the remark about graphs, the map induced by $R_{\widetilde{\lambda}}^{\widetilde{h}}$ on B_M satisfies the assumptions of Proposition 5.9 with good precision. □

Let us note that in a constructive version of this argument, the cocycle may be built as a lacunary Fourier series.

5.5 Weak mixing of special flows over interval exchange transformations. Let $T : \Delta \to \Delta$ be an interval exchange transformation (cf. 4.4). We will consider a special flow built over T under a function h which we assume to be piecewise continuously differentiable; so that at every point of nondifferentiability, $h'(x)$ has limits from the left and right. let us identify the ends of the interval Δ and fix an orientation on the circle obtained this way. We denote

$$\mathcal{J}(h) = \sum f(x+0) - f(x-0)$$

where the summation is taken over the finite set of all points of discontinuity.

Let us remark that the first examples of measure-preserving transformations and flows with continuous spectrum were constructed by J. Von Neumann in his celebrated paper [**N**] which established the foundation of modern ergodic theory. He proved that the special flow built over an irrational rotation under a piecewise C^1 function h such that $\mathcal{J}(h) \neq 0$ is weakly mixing. In his proof, von Neumann used the rigidity of the base transformation (i.e. the rotation). Although not every interval exchange transformation is rigid*, they always possess a recurrence property which is strong enough to carry through an argument similar to von Neumann's. Namely, we will use the following property [**K4**].

PROPOSITION 5.10. *There exists a positive α and a sequence of positive integers $n_k \to \infty$ such that for every measurable set $A \subseteq S^1$*

$$\mu(A \cap T^{n_k} A) > \alpha\mu(A)$$

We can now formulate the generalization of von Neumann's result.

THEOREM 5.11. *If $T : \Delta \to \Delta$ is an interval exchange transformation ergodic with respect to Lebesgue measure and h is a piecewise C^1 function such that $\mathcal{J}(h) \neq 0$, then the special flow over T build under h is weakly mixing but not mixing.*

*cf. Del Junco [**J**], who proved that some exchanges of 3 intervals have minimal self-joinings and consequently their centralizers consist only of powers. On the other hand, a rigid transformation always has an uncountable centralizer.

PROOF. Since h is a function of bounded variation, the absence of mixing follows from [**K4**].

To prove weak mixing, we show as usual that for every $r \neq 0$ the function $\exp\ irh(x)$ is not cohomologous to any constant. We will use Proposition 5.1, ii), whose assumption in this case, is satisfied due to Proposition 5.10. Let $\theta_n(x) = \exp\ ir\sum_{k=0}^{n-1} h(T^k x)$. We will show that any weak limit point of the sequence of measures

$$(\theta_n)_*\lambda$$

is a nonatomic measure, where λ denotes Lebesgue measure on Δ.

Let ζ be the partition of Δ into subintervals formed by all points of discontinuity of T and all points where the derivative $h'(x)$ is discontinuous and let $\zeta_n = \zeta \vee T^{-1}\zeta \vee \dots T^{-n+1}\zeta$. Let $\omega(\epsilon)$ be a common modulus of continuity for h' on all its intervals of continuity. If $c \in \zeta_n$ then all functions $h \circ T^i$; $i = 0, \dots, n-1$ are differentiable on c and since $T^i\big|_c$ has derivative ± 1 we have for $x, y \in c$

$$\left|h'(T^i x) - h'(T^i y)\right| \leq \omega(|x-y|)$$

and consequently

$$\left|\sum_{i=0}^{n-1} h'(T^i x) = \sum_{i=0}^{n-1} h'(T^i y)\right| < n\omega(|x-y|). \tag{5.21}$$

We have

$$\int_\Delta h'(x)dx = -\mathcal{J}(h). \tag{5.22}$$

Let us denote for an integer n and $\epsilon > 0$

$$A_{n,\epsilon} = \left\{ x \in \Delta : \left| \frac{1}{n}\sum_{i=0}^{n-1} h(T^i x) + \mathcal{J}(h) \right| < \epsilon \right\}. \tag{5.23}$$

By the ergodicity of T we have for any $\epsilon > 0$

$$\lambda(A_{n,\epsilon}) \to 1 \quad \text{or} \quad n \to \infty.$$

Let us now fix $\beta > 0$ and consider all elements of the partition ζ_n of length $> \frac{\beta}{n}$. Let $R = \text{card}\ \zeta$. Since the total number of elements in ζ does not exceed Rn (every iterate of T adds as many new elements as the number of points of discontinuity of T), the total measure of these elements is greater than $1 - R\beta$. Let us divide now every element into intervals of length between $\frac{\beta}{n}$ and $\frac{w\beta}{n}$ and call those of the intervals, which intersect the set $\Delta_{n,\epsilon}$, (n, ϵ, β)-admissible ones. The total measure of (n, ϵ, β)-admissible intervals is clearly greater than $\lambda(\Delta_{n,\epsilon}) - \beta R$. For every (n, ϵ, β)-admissible interval σ we have by (5.21) and (5.22)

$$\begin{aligned} \left(-\mathcal{J}(h) - \epsilon - \omega\left(\frac{2\beta}{n}\right)\right) n < \min_\sigma \sum_{i=0}^{n-1} h'(T^i x) \leq \max_\sigma \sum_{i=0}^{n-1} f'(T^i x) < \\ \left(-\mathcal{J}(h) + \epsilon + \omega\left(\frac{2\beta}{n}\right)\right) n \end{aligned} \tag{5.24}$$

If ϵ is chosen small enough and n is big enough then in particular (5.24) implies that

$$\max_{\sigma}\sum_{i=0}^{n-1} h'(T^i x) < 2|\mathcal{J}(h)|n \tag{5.25}$$

so that by choosing $\beta < \frac{1}{4r|\mathcal{J}(h)|}$ (recall that the length of σ is less than $\frac{2\beta}{n}$) we can guarantee that the function θ_n is injective on every (n,ϵ,β)-admissible interval.

On the other hand, the length of the image $\theta_n(\sigma)$ is bigger than $\frac{r}{2}\beta\mathcal{J}(h)|$ if we guarantee by our choice of n, ϵ that

$$\min_{\sigma}\sum_{i=0}^{n-1} h'(T^i x) > \frac{1}{2}|\mathcal{J}(h)|n \tag{5.26}$$

It follows from (5.25) and (5.26) that the density of the measure $(\theta_n)_*(\lambda|_\sigma) = \mu_{n,\sigma}$ can oscillate on the interval $\theta_n(\sigma)$ ratio of at most 4, so that for every interval $\delta \in \Delta$ of length ℓ

$$\mu_{n,\sigma}(\delta) < \frac{4\ell}{\theta_n(\sigma)} < \frac{4\ell}{r\beta|\mathcal{J}(h)|}.$$

This means in particular that

$$\begin{aligned}
&\lambda(\{x \in \Delta : \theta_n(x) \in \delta\}) \leq \\
&\lambda(\{x \in \Delta : \theta_n(x) \in \delta\ ,\ x \text{ belongs to an } (n,\epsilon,\beta)\text{-admissible} \\
&\text{(interval)}\}) + \mu(\{x \in \Delta\ ,\ x \text{ does not belong to an} \\
&(n,\epsilon,\beta)\text{-admissible interval}\}) < \frac{4\ell}{r\beta|\mathcal{J}(h)|} + R\beta + 1 - \lambda(A_{n,\epsilon}).
\end{aligned} \tag{5.27}$$

Fixing first a sufficiently small ϵ and a sufficiently big n, and then sufficiently small β we can always find ℓ such that the right hand part of (5.27) is less than any given positive number. This shows that any weak limit power of the sequence of measures $(\theta_n)_*\lambda$ cannot contain an atom. □

References

[A] D. V. Anosov, *Functional homology equation connected with an ergodic rotation of the circle*, Math. USSR Izvestija **7** (1973), 1257–1271.

[AK] D. V. Anosov and A. B. Katok, *New examples in smooth ergodic theory. Ergodic diffeomorphisms*, Transactions of the Moscow Mathematical Society **123** (1970), 1–35.

[BKM] D. Bernstein, A. Katok, and G. D. Mostow, *Nets and Quasiisometries*, unpublished.

[CFW] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Erg. Th. & Dynam. Syst. **1** (1981), 431–450.

[Fa1] B. Fayad, *Weak mixing in reparametrized linear flows on the torus*, to appear, Erg. Th. & Dynam. Syst..

[Fa2] B. Fayad, *Analytic mixing reparametrizations of the irrational flows on the torus $\mathbb{T}^n$*, $n \geq 3$, to appear, Erg. Th. & Dynam. Syst..

[FK] R. Feres and A. Katok, *Ergodic theory and dynamics of G-spaces*, Handbook in Dynamical Systems, vol. 1, Elsevier, 2001.

[FlFo] G. Forni, *Invariant distributions for horocycles flows*, in preparation.

[Fo] G. Forni, *Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus* **146** (1997), 295–344.

[Fu] A. Furman, *Orbit equivalence rigidity*, Ann. of Math **150** (1999), 1083–1108..

[F] H. Furstenberg, *Strict ergodicity and transformations of the torus*, Amer. J. Math. **83** (1961), 573–601.

[HaK] B. Hasselblatt and A. Katok, *Principal structures*, Handbook in Dynamical Systems, vol. 1, Elsevier, 2001.

[He1] M. R. Herman, *L^2 regularity of measurable solutions of a finite difference equation of the circle*, preprint, University of Warwick (1976).

[He2] M. R. Herman, *Sur la conjugaison diffrentiable des diffomorphismes du cercle des rotations*, Publ. Math. IHES **49** (1979), 5–233.

[He3] M. R. Herman, *Constructions de diffeomorphismes ergodiques*, Unpublished manuscript.

[He4] M. R. Herman, *Construction d'un diffeomorphisme minimal d'entropie topologique non nulle*, Erg. Th. & Dynam. Syst. **1** (1981), 65–76.

[He5] M. R. Herman, *Une mthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractre local d'un thorme d'Arnol'd et de Moser sur le tore de dimension* 2, Comm. Math. Helv. **58** (1983), 453–502..

[HK1] S. Hurder and A. Katok, *Ergodic theory and Weil measures for foliations*, Ann. Math. **126** (1987), 221–275.

[HK2] S. Hurder and A. Katok, *Differentiability, rigidity and Godbillon-Vey classes for Anosov flows*, Publ. Math. IHES **72** (1990)), 5–61.

[Jo] J.-L. Journé, *On a regularity problem occurring in connection with Anosov diffeomorphisms*, Comm. Math. Phys. **106** (1986), 345–351.

[J] A. del Junco, *A family of counterexamples in ergodic theory*, Israel J. Math **44** (1983), 160–188.

[JR] A. Del Junco and D. Rudolph, *Kakutani equivalence of ergodic $\mathbb{Z}^n$ actions*, Erg. Th. & Dynam. Syst. **4** (1984), 89–104.

[Ka] S. Kakutani, *Induced measure preserving transformations*, Proc. Japan Acad. **19** (1943), 635–641.

[K1] A. Katok, *Combinatorial constructions in ergodic theory. I. Approximation and genericity*, Proc. Steklov Inst. Math (2001).

[K2] A. B. Katok, *The special representation theorem for multi-dimensional group actions*, Asterisque **49** (1977), 117-140.

[K3] A. B. Katok, *Monotone equivalence in ergodic theory*, Math. USSR, Izvestija **11** (1977), 99–146.

[K4] A. Katok, *Interval exchange transformations and some special flows are not mixing,*, Isr. J. Math **35** (1980), 301-310.

[KH] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge university press, 1995.

[KK] A.Katok and A. Kononenko, *Cocycles' stability for partially hyperbolic systems.*, Math. Res. Lett **3** (1996), 191–210.

[KSch] A. Katok and K. Schmidt, *The cohomology of expansive $\mathbb{Z}^d$-actions by automorphisms of compact abelian groups*, Pacific J. Math. **170** (1995), 105-142.

[KSp1] A.Katok and R. Spatzier, *First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity*, Publ. Math. IHES **79** (1994), 131-156.

[KSp2] A.Katok and R. Spatzier, *Differential rigidity of Anosov actions of higher rank abelian groups and algebraic lattice actions*, Proc. Steklov ath. Inst. **216** (1997), 287-314.

[Ke] M. Keane, *Interval exchange transformations*, Math. Zeitsch. **141** (1975), 25–31.

[Kc] A. V. Kočergin, *The homology of functions over dynamical systems (in Russian)*, Dokl. Akad. Nauk SSSR **231** (1976), 795–798.

[Ko] A. N. Kolmogorov, *On dynamical systems with integral invariant on the torus*, in Russian, Doklady Akad. Nauk SSSR **93** (1953), 763–766.

[Kr] A. Krygin, *Examples of ergodic cylindrical cascades*, Math. Notes USSR Acad. Sci **16** (1974), 1180–1186.

[dl] R. de la Llave, *Analytic regularity of solutions of Livsic's cohomology equation and some applications to analytic conjugacy of hyperbolic dynamical systems*, Erg. Th. & Dynam. Syst **17** (1997), 649–662.

[dLMM] R. de la Llave, J. M. Marco, and R.Moriyn, *Canonical perturbation theory of Anosov systems and regularity results for the Livšic cohomology equation*, Ann. of Math. **123** (1986), 537–611.

[L1] A. Livšic, *Some homology properties of U–systems*, Math. Notes USSR Acad. Sci **10** (1971), 758–763.

[L2] A. Livšic, *Cohomology of dynamical systems*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 1296–1320.

[LS] A. Livšic and Ja. G. Sinaĭ, *On invariant measures compatible with smooth structure for transitive U-systems. Soviet Math. Doklady* **13** (1972), 1656–1659.

[LSa] R. U. Luz, and N. M. dos Santos,, *Cohomology-free diffeomorphisms of low-dimension tori*, Erg. Th. & Dynam. Syst. **18** (1998), 985–1006.

[M] H. Masur, *Interval exchange transformations and measured foliations*, Ann. of Math **115** (1982), 169–200.

[N] J. von Neumann, *Zur Operatorenmethode in der klassischen Mechanik*, Ann. Math. **33** (1932), 587–642.

[NT] V. Niţică and A. Török, *Regularity of the transfer map for cohomologous cocycles*, Erg. Th. & Dynam. Syst. **18** (1998), 1187–1209.

[ORW] D. Ornstein, D. Rudolph, and B. Weiss, *Equivalence of measure preserving transformations*, Memoirs of the AMS **37** (1982), no. 262.

[OS] D. Ornstein and M. Smorodinsky, *Continuous speed changes for flows*, Israel J. Math. **31** (1978), 161–168.

[Pa] W. Parry, *Topics in ergodic theory*, Cambridge University Press, 1881.

[Sch1] K. Schmidt, *Dynamical systems of algebraic origin*, Birkhäuser, 1995.

[Sch2] K. Schmidt, *The cohomology of higher-dimensional shifts of finite type*, Pacific J. Math **170** (1995), 237–269.

[V1] W. Veech, *Interval exchange transformations*, J. Analyse Math. **33** (1978), 222–272.

[V2] W. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. **115** (1982), 201–242.

[V3] W. Veech, *Periodic points and invariant pseudomeasures for toral endomorphisms*, Erg. Th. & Dynam. Syst. **6** (1986), 449–473.

[Z1] R. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, 1984.

[Z2] R. Zimmer, *Actions of semisimple groups and discrete subgroups*, Proceedings of the International Congress of Mathematicians, Amer. Math. Soc., Providence, RI, 1987, pp. 1247–1258.

Department of Mathematics, Pennsylvania State University, University Park, PA 16802

Department of Mathematics, George Washington University, Washington, D. C. 20052