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ARITHMETICITY AND TOPOLOGY OF SMOOTH ACTIONS OF HIGHER RANK ABELIAN GROUPS

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ABSTRACT. We prove that any smooth action of $\mathbb{Z}^{m-1}$, $m \geq 3$, on an m-dimensional manifold that preserves a measure such that all non-identity elements of the suspension have positive entropy is essentially algebraic, i.e., isomorphic up to a finite permutation to an affine action on the torus or on its factor by $\pm$Id. Furthermore this isomorphism has nice geometric properties; in particular, it is smooth in the sense of Whitney on a set whose complement has arbitrarily small measure. We further derive restrictions on topology of manifolds that may admit such actions, for example, excluding spheres and obtaining lower estimate on the first Betti number in the odd-dimensional case.

INTRODUCTION

Let α be a smooth action of $\mathbb{Z}^{m-1}$, $m \geq 3$, on an m-dimensional manifold M, not necessarily compact. We assume that α is uniformly $C^{1+\theta}$, $\theta > 0$, with respect to a certain smooth Riemanninan metric on M, i.e., the generators of the action and their inverses have uniformly bounded derivatives satisfying Hölder condition with exponent θ and a fixed Hölder constant. Naturally, if M is compact this condition does not depend on the choice of the Riemannian metric. This regularity assumption allows us to apply standard results of smooth ergodic theory to any invariant measure of the action.

Following [14] we assume that α has an invariant probability ergodic measure μ such that

(1) Lyapunov characteristic exponents are non-zero and are in general position, i.e., the dimension of the intersection of any l of their kernels is the minimal possible, i.e., is equal to $\max\{k-l, 0\}$,
(2) at least one element in $\mathbb{Z}^{m-1}$ has positive entropy with respect to μ.

We will call such a pair (α, μ) a *maximal rank positive entropy action.*

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Let us notice that there is an equivalent definition of a maximal rank positive entropy action that only uses notions invariant under measure-theoretic isomorphism.

Proposition A. *A smooth action α of $\mathbb{Z}^{m-1}$, $m \geq 3$, on an m-dimensional manifold M is a maximal rank positive entropy action if and only if every non-identity element of the suspension of the action has positive entropy with respect to μ.*

Proof. In one direction this is proven in [14, Section 3.1]. Namely, it is shown there that is α is a maximal rank positive entropy action then any non-identity element of the suspension has positive entropy.

In the opposite direction the argument is given in [16, Proposition 3.1]. Since the statement of that proposition looks different we repeat the argument here.

We argue by contradiction. Assume that Lyapunov exponents are not in general position. Let k be the maximal number of Lyapunov hyperplanes whose intersection has dimension greater than minimal, i.e., greater than $m-1-k$. Consider the restriction of the action to the intersection of those k Lyapunov hyperplanes that we denote by L. Intersections of remaining $m-k$ hyperplanes with L are in general position; hence they divide L into 2^{m-k} domains where the exponents have all possible combinations of signs. In particular there is a domain D where all $m-k$ exponents are negative. Since the remaining exponents vanish on L all suspension action elements from D have zero entropy, contradicting the assumption that all non-identity elements of the suspension have positive entropy. □

The main result of [14] is absolute continuity of the measure μ for a maximal rank positive entropy action. In [14, Sections 8.1 and 8.2] a program of further study of such actions has been formulated.

In the present paper we mostly complete this program. Firstly we extend the description of maximal rank actions on the torus with Cartan homotopy data from [20], where a positive entropy hyperbolic measure always exists, to maximal rank positive entropy actions on arbitrary manifolds. Secondly we obtain substantial information on topology of manifolds that may admit such actions, in particularly excluding spheres and many other standard manifolds.

Let us call an *infratorus* a factor of $\mathbb{R}^m$ by a group E of affine transformations that contains a lattice L of translations as a finite index subgroup. Thus an infratorus is the factor of the torus $\mathbb{R}^m/L$ by a finite group G of affine transformations. In this definition infratorus is a varifold and not necessarily a smooth manifold since the group G may not act freely; in particular it may have fixed points. In fact, the only examples of infratori that admit maximal rank abelian actions by affine transformations and that are not tori are of that kind: such an infratorus is obtained by factorizing $\mathbb{T}^m$ by the involution $Ix = -x$ that has 2^m fixed points. Let us denote such an infratorus by $\mathbb{T}^m_{\pm}$.

REMARK 1. By blowing up the singular points and glueing in copies of the projective space of codimension one, one constructs a smooth action on a manifold that is diffeomorphic to the affine action on the infratorus outside of the singular points, see [18]. This can be considered as the "standard smooth model" of the infratorus action. Examples of infratori that are smooth manifolds can be found in [19, Section 2.1.4].

We formulate and present our results in two parts. The reason is that the first part is likely to hold with proper modifications (in particular, allowing more general infratori) in greater generality, namely, in the setting similar to that of [21] where no connection is assumed between the rank $k \geq 2$ of the action and the dimension of the ambient manifold. Most steps in the proof work in that generality and remaining difficulties, while substantial, are of technical nature. The second part heavily relies on existence of codimension-one stable manifolds and hence is specific for the maximal rank setting.

The first part (Theorem 1) states in particular that modulo a finite permutation any such action is "arithmetic", i.e., there is a measurable isomorphism between the restriction of the action to each ergodic component of a certain finite index subgroup $\Gamma \subset \mathbb{Z}^{m-1}$ and a Cartan action by affine automorphisms of the torus $\mathbb{T}^m$ or its factor $\mathbb{T}^m_\pm$. This isomorphism has nice topological and geometric properties that are described in detail below. This provides solutions of most conjectures and problems from [14, Section 8.1].

The second part asserts that the restriction of the above mentioned isomorphism to each ergodic component of the group Γ extends to a continuous map between an open set in M and the complement to a finite set on $\mathbb{T}^m$ or $\mathbb{T}^m_\pm$ that is a topological semi-conjugacy (a factor-map) between α and α_0 (Theorem 2). Furthermore, this map can be modified on a set of arbitrarily small measure and then extended to a homeomorphism between an open set in M and the complement to a finite set on $\mathbb{T}^m$ or $\mathbb{T}^m_\pm$. This has implications for the topology of M, in particular disproving Conjecture 4 from [14].

Technically the present paper builds upon the results of [14, 21]. We use background information from those papers without special references.

1. FORMULATION OF RESULTS

1.1. **The arithmeticity theorem.**

THEOREM 1. *For $r = 1+\theta$, $0 < \theta < 1$, or $r \geq 2$ an integer, let α be a C^r maximal rank positive entropy action on a smooth manifold M of dimension $m \geq 3$.*

Then there exist

- *disjoint measurable sets of equal measure $R_1, \dots, R_n \subset M$ such that $R = \bigcup_{i=1}^n R_i$ has full measure and the action α cyclically interchanges those sets. Let $\Gamma \subset \mathbb{Z}^{m-1}$ be the stationary subgroup of any of the sets R_i (Γ is of course isomorphic to $\mathbb{Z}^{m-1}$);*
- *a Cartan action α_0 of Γ by affine transformations of either the torus $\mathbb{T}^m$ or the infratorus $\mathbb{T}^m_\pm$ that we will call* the algebraic model*;*
- *measurable maps $h_i : R_i \to \mathbb{T}^m$ or $h_i : R_i \to \mathbb{T}^m_\pm$, $i = 1, \dots, n$;*

such that

(1) *h_i is bijective almost everywhere and $(h_i)_*\mu = \lambda$, the Lebesgue (Haar) measure on $\mathbb{T}^m$ (correspondingly $\mathbb{T}^m_{\pm}$);*
(2) *$\alpha_0 \circ h_i = h_i \circ \alpha \restriction_\Gamma$;*
(3) *for almost every $x \in M$ and every $\mathbf{n} \in \mathbb{Z}^{\mathbf{m-1}}$ the restriction of h_i to the stable manifold W^s_x of x with respect to $\alpha(\mathbf{n})$ is a C^r diffeomorphism;*
(4) *h_i is $C^{r-\epsilon}$ in the sense of Whitney on a set whose complement to R_i has arbitrarily small measure; those sets will be described in the course of proof; in particular, they are saturated by local stable manifolds.*

Sometimes, when this cannot cause confusion, we will call the actions satisfying assumptions of Theorem 1 simply *maximal rank actions*.

REMARK 2. Statement (4) implies that the measure μ is absolutely continuous. However, as we mentioned before, this fact is the principal result of [14] and it forms a basis of the proof of Theorem 1.

REMARK 3. Statements (1) and (2) imply that h_i is a measurable isomorphism between (α,μ), restricted to the set R_i and subgroup Γ, and the algebraic model (α_0,λ).

REMARK 4. It follows from (1) and (2) that for $n = 1$ the action α is weakly mixing (and, in fact, mixing); for $n > 1$ or equivalently, if $\Gamma \neq \mathbb{Z}^{m-1}$, Γ action is not ergodic but its restriction to any of its n ergodic components is weakly mixing (and mixing).

REMARK 5. Statement (3) immediately implies that Jacobians along Lyapunov foliations are measurably cohomologous to constants (exponents for the algebraic model), thus solving Conjecture 1 from [14]. Furthermore, the transfer function is smooth along the Lyapunov foliations.

1.2. **Corollaries from arithmeticity.**

1.2.1. *Entropy and Lyapunov exponents.* Theorem 1 immediately implies the solution of Problem 1 and Conjecture 2 from [14]. In fact, description of Cartan (maximal rank) actions on the torus via units in the algebraic number fields given in [17] provides more precise information. We consider the weakly mixing case first.

COROLLARY 1. *Let α be a $C^{1+\theta}, \theta > 0$ weakly mixing maximal rank positive entropy $\mathbb{Z}^{m-1}$ action. There exists a totally real algebraic number field K of degree m, that is a simple extension of $\mathbb{Q}$ uniquely determined by α, and, for any system of generators of α, an $(m-1)$-tuple of multiplicatively independent units $\lambda_1,\dots,\lambda_{m-1}$ in K such that the Lyapunov characteristics exponents for those generators of α are*

$$\log|\phi_1(\lambda_i)|,\dots,|\log\phi_m(\lambda_i)|,\quad i = 1,\dots,m-1,$$

where $\phi_1,\dots,\phi_m$ are different embeddings of K into $\mathbb{R}$.

In the general case one applies Corollary 1 to the restriction of the action to the stationary subgroup Γ for each of the sets R_i. Those restrictions for $i = 1,\dots,n$ are isomorphic and hence have the same entropy that is also equal to the entropy of $\alpha(\gamma)$ for any $\gamma \in \Gamma$ with respect to the non-ergodic measure μ. Let k be the index of Γ. Since the k-th power of any element of $\mathbb{Z}^{m-1}$ lies in Γ and every element is a power of a generator one immediately obtains a description of exponents in the general case.

COROLLARY 2. *Lyapunov exponents of any element of a maximal rank action α have the form*

$$\frac{|\log\phi_i(\lambda)|}{k},\dots,\frac{|\log\phi_m(\lambda)|}{k},$$

where λ is a unit in a totally real algebraic number field of degree m and k is a positive integer that depends only on α but not on the element. Here as before $\phi_1,\dots,\phi_m$ are different embeddings of the field into $\mathbb{R}$.

Since entropy of an action element is equal to the Mahler measure of the corresponding unit we can use exponential lower estimate for the Mahler measure for totally real fields [33, 8] to obtain a lower bound on entropy.

COROLLARY 3. *The entropy of any element of a weakly mixing maximal entropy action on an m-dimensional manifold is bounded from below by cm, where c is a universal constant.*

1.2.2. *Entropy and isomorphism rigidity.* Eigenvalues of an integer matrix A, when simple and real, determine its conjugacy class over $\mathbb{R}$ and hence over $\mathbb{Q}$. Assume that $\det A = \pm 1$. That in turn determines a conjugacy class of the automorphism of the torus F_A up to a common finite factor or finite extension. By [17, Theorem 5.2] for a broad class of $\mathbb{Z}^k$, $k \geq 2$, actions by automorphisms of a torus, that includes all Cartan actions, measure theoretic isomorphism (with respect to Lebesgue measure) implies algebraic isomorphism.

Notice that passing to a finite factor or finite extension does not change entropy. Likewise the entropy for affine actions with the same linear parts are the same. By symmetry the entropy does not change if all generators of an action are replaced by their inverses.

Theorem 1 allows us to show that the entropy function determines a maximal rank action action on a finite index subgroup up to a measurable isomorphism with above mentioned trivial modifications. We call the next statement a corollary, despite the length of the argument needed to deduce it from Theorem 1. The point is that the argument is purely algebraic and deals only with linear actions, modulo choosing appropriate finite index subgroups.

COROLLARY 4. *Let α and α' be two maximal rank actions. Assume that they are both weakly mixing and their entropy functions coincide. Then restrictions of α and α' to a certain subgroup $\Gamma \subset \mathbb{Z}^{m-1}$ of finite index are finite factors of measurably isomorphic actions, possibly with replacing all generators of one action by their inverses.*

Proof. Let α_0 and α'_0 be the algebraic models for α and α'. Weak mixing implies that for both of them $n = 1$. Now take finite covers $\tilde{\alpha}_0$ and $\tilde{\alpha}'_0$ (if necessary) that are actions by affine transformations of $\mathbb{T}^m$. Take finite index subgroups of $\mathbb{Z}^{m-1}$ for which $\tilde{\alpha}_0$ and $\tilde{\alpha}'_0$ act by automorphisms. Taking the intersection of those subgroups obtain a finite index subgroup Γ_1 for which both $\tilde{\alpha}_0$ and $\tilde{\alpha}'_0$ act by automorphisms. Those restrictions still have identical entropy functions. Now there is a subgroup Γ_2 of finite index such that eigenvalues for all generators of both actions are positive. Let $\Gamma = \Gamma_1 \cap \Gamma_2$. Since eigenvalues are simple they are thus determined for the Γ action by Lyapunov exponents. But by [17, Proposition 3.8] irreducible (in particular, Cartan) actions by automorphisms with the same eigenvalues of their generators are algebraically conjugate to finite factors of the same action.

Let us show that Lyapunov exponents are in turn determined by the entropy function, possibly with replacing all generators by their inverses. To see that, notice first that entropy function is not differentiable exactly at the union of the kernels of the Lyapunov exponents, the Lyapunov hyperplanes. Thus it determines every Lyapunov exponent up to a scalar multiple.

In the Cartan case for each Lyapunov exponent χ there is exactly one Weyl chamber $\mathscr{C}_\chi$ where this exponent is positive and all other negative. Inside this Weyl chamber entropy is equal to χ. This Weyl chamber and its opposite $-\mathscr{C}_\chi$ are determined from the configuration of Lyapunov hyperplanes as the only two whose boundaries intersect all Lyapunov hyperplanes except for $\ker\chi$. Thus for every Lyapunov exponents χ of α there is en exponent χ' of α' such that is equal to either α or $-\alpha$. Let us show that for all exponents the sign is the same. Suppose that for two exponents χ_1 and χ_2 of α there are exponents χ_1 and $-\chi_2$ of α'. Then in the Weyl chamber $\mathscr{C}_{\chi_1}$ χ_2 is negative, hence in this Weyl chamber the entropy of α' is at least $\chi_1 - \chi_2$, i.e., greater than the entropy of α. Thus all exponents of α and α' are either equal or have opposite signs. In the latter case we can change generators of α' to their inverses and obtain actions with equal exponents. □

In the case of maximal rank actions that are not weakly mixing one restricts the action to the stationary subgroup Γ for each of the sets R_i and applies Corollary 4 to each of those sets. An obvious additional invariant is the index of Γ. It can be determined, for example, from the discrete spectrum of the action. This spectrum determines how different elements of the action interchange the sets R_i. A conjugacy between restrictions of the action of γ to different sets R_i can be effected by using the action of elements from corresponding cosets of Γ. Hence Corollary 4 can be simultaneously applied to those sets.

COROLLARY 5. *Let α and α' be two maximal rank actions. Assume that they have the same discrete spectrum and their entropy functions coincide. Then restrictions of α and α' to a certain subgroup $\Gamma \subset \mathbb{Z}^{m-1}$ of finite index are finite factors of measurably isomorphic actions, possibly with replacing all generators of one action by their inverses.*

1.2.3. *Cocycle rigidity.* Proper classes of cocycles over actions that we consider are Lyapunov Hölder and Lyapunov smooth, i.e., measurable cocycles that are Hölder or smooth correspondingly along Lyapunov foliations almost everywhere; see [21, Definition 8.1] for precise definition. Any such cocycle over a C^∞ maximal rank action can be transferred to a cocycle over a finite extension of the linear model the same way as is described in the proof of [21, Theorem 2.8]. This proof works verbatim in our case and produces cocycle rigidity.

COROLLARY 6. *Any Lyapunov Hölder (corr. Lyapunov smooth) real valued cocycle over a C^r, $1+\theta \le r \le \infty$, maximal rank action is cohomologous to a constant cocycle via a Lyapunov Holder (corr. Lyapunov smooth) transfer function (with the obvious proviso that the Lyapunov regularity of the transfer function is less that regularity of the action).*

REMARK 6. Notice that in the proof of [14, Theorem 4.1] we use a special time change for the suspension action and for this time change the expansion coefficient of the original suspension action in the Lyapunov direction is indeed cohomologous to a constant. This however does not imply that expansion coefficient for the original action or its suspension is cohomologous to a constant. For example this is not the case for hyperbolic flows where W. Parry constructed synchronization time change that inspired our construction [28].

1.3. **The topology theorem.** Let L denote either a torus $\mathbb{T}^m$ or infratorus $\mathbb{T}^m_\pm$.

THEOREM 2. *Let α be a C^r, $r>1$, maximal rank positive entropy action, then*

(1) the sets $R_1,\dots,R_n$ in Theorem 1 can be chosen inside open Γ-invariant subsets $O_1,\dots,O_n$ that are also interchanged by α;
(2) each map h_i extends to a continuous map $\tilde{h}_i : O_i \to L\smallsetminus F$, where F is a finite α_0-invariant set;
(3) if L is a torus or if $x\in L$ is a regular point in the infratorus then there exists an arbitrarily small parallelepiped P_x (in some linear coordinates) containing x such that on the boundary of P_x the map h_i is invertible and the inverse is a diffeomorphism on every face of P_x;
(4) if $x\in F$ is a singular point in the infratorus $\mathbb{T}^m_\pm$ then there exists an arbitrarily small projective parallelepiped P_x (the factor of a centrally symmetric parallelepiped in some linear coordinates by the involution $t\to -t$) containing x such that on the boundary of P_x the map h_i is invertible and the inverse is a diffeomorphism on every face of P_x;
(5) if $\mathscr{R} = L\smallsetminus\bigcup_{x\in F}\mathrm{Int}P_x$ then $\tilde{h}_i^{-1}\mathscr{R}$ is homeomorphic to $\mathscr{R}$ via a homeomorphism H that coincides with h_i on ∂P_x.

REMARK 7. Notice that any singular point in L must be in the exceptional set F because the topology of a small neighborhood of a singular point is different from that of points in a manifold.

1.4. **Topological corollaries.** Theorem 2 allows us to make conclusions about topology of manifolds that admit maximal rank positive entropy actions. Right now we list only some of those properties that can be derived quickly. More detailed discussion of the consequences of Theorem 2 will appear in a separate paper.

COROLLARY 7. *Let M be a connected manifold of odd dimension $m \geq 3$ that admits a maximal rank positive entropy action of $\mathbb{Z}^{m-1}$. Then the following hold.*

If M is orientable, it is homeomorphic to the connected sum of the torus $\mathbb{T}^m$ with another manifold.

If M is non-orientable, its orientable double cover is homeomorphic to the connected sum of the torus $\mathbb{T}^m$ with another manifold.

In particular, in both cases the fundamental group $\pi_1(M)$ contains a subgroup isomorphic to $\mathbb{Z}^m$.

Proof. Consider the orientable case first. In odd dimension the infratorus $\mathbb{T}^m_\pm$ is not orientable and the same is true of its complement to a finite set or, equivalently, of the complement to the union of finitely many small balls. Since an open subset of an orientable manifold is orientable the open subset $S = \operatorname{Int} \tilde{h}_i^{-1}\mathscr{R} \subset M$ is orientable and hence L is the torus. Now take a disc $D \subset \mathbb{T}^m$ that contains the set F and consider closed set $H^{-1}(\mathbb{T}^m \setminus D)$, where the map H is defined in Theorem 2 (5). Its boundary is a sphere and thus M is the connected sum of $\mathbb{T}^m$ and a manifold that is obtained by glueing a disc to the boundary of $M \setminus H^{-1}(\mathbb{T}^m \setminus D)$.

Now assume that M is non-orientable and take the orientable double cover $\tilde{M}$ of M. Let $I : \tilde{M} \to \tilde{M}$ be the deck transformation. Consider lifts of the elements of the maximal rank action α to $\tilde{M}$. Each element f has two lifts f_1 and $f_2 = f_1 I$. The involution I commutes with all lifts. Either the group Γ consisting of all lifts is abelian or its commutator is the group of two elements generated by I.

Let us show that Γ has a finite index abelian subgroup isomorphic to $\mathbb{Z}^{m-1}$. If Γ is already abelian then it is isomorphic to the direct product $\mathbb{Z}^{m-1} \times \mathbb{Z}/2\mathbb{Z}$. Otherwise consider generators of the action α and let $f_1, \dots, f_{m-1}$ be their lifts to $\tilde{M}$. The centralizer $Z(f_i)$ of each of those elements in Γ is either the whole of Γ or an index two subgroup. This follows from the fact $I^2 = \operatorname{Id}$ and that I is in the center of Γ since this implies that the product of any two elements not in $Z(f_i)$ belongs to $Z(f_i)$. Thus $\mathscr{Z} = \bigcap_{i=1}^{m-1} Z(f_i)$ is a finite index abelian subgroup of Γ that belongs to its center. Notice that the index of $\mathscr{Z}$ in Γ is at most 2^{m-1}. Since the only finite order element of Γ is I, $\mathscr{Z}$ is isomorphic to $\mathbb{Z}^{m-1} \times \mathbb{Z}/2\mathbb{Z}$.

Thus $\mathbb{Z}^{m-1}$ acts on $\tilde{M}$ by lifts of elements of α. This is obviously a maximal rank positive entropy action so that from the argument for the orientable case $\tilde{M}$ is the connected sum of torus with another manifold. Since $\pi_1(\tilde{M})$ embeds into $\pi_1(M)$ the former is a subgroup isomorphic to $\mathbb{Z}^m$. □

A very similar argument allows us to partially extend Corollary 3 to actions that are not weakly mixing.

COROLLARY 8. *The entropy of any element of a maximal rank action on an m-dimensional manifold M is bounded from below by $\frac{cm^2}{\beta_1(M)}$ for orientable M and* *by* $\frac{cm^2}{\beta_1(M)+\beta_{m-1}(M)}$ *for non-orientable* M*, where* c *is a universal constant and* β_1 *is the first Betti number.*

Proof. As before, let us consider the orientable case first. The obvious lower estimate is the entropy for the elements of the action of Γ on each ergodic component, divided by the number n of ergodic components that is equal to the index of Γ. Hence this number needs to be estimated from above. Repeating the argument about connected sums from the proof of Corollary 7 we deduce that M is homeomorphic to the connected sum of n copies of the torus $\mathbb{T}^m$ and another manifold. Hence by Mayer-Vietoris theorem $\beta_1(M) \geq mn$ or $n \leq \frac{\beta_1(M)}{m}$. Now Corollary 3 implies the needed estimate.

In the non-orientable case we consider the orientable double cover, lift the action as in the proof of Corollary 7, notice that entropy does not change, and use the estimate in the orientable case. Since the first Betti number of the orientable double cover of the non-orientable manifold is $\beta_1(M)+\beta_{m-1}(M)$, the inequality follows. □

REMARK 8. Notice that there is no estimate from below that depends on dimension only as in the weak mixing case. To produce examples in a fixed dimension with an arbitrarily small entropy one modifies appropriately the suspension construction over a weakly mixing action on the torus by making holes around fixed points similarly to [18] and connecting them by cylinders, similarly to the filling of holes descried in [20], produces examples with arbitrarily low entropy.

The even-dimensional case is more complicated. While the case $L = \mathbb{T}^m$ of course works the same way, if $L = \mathbb{T}^m_{\pm}$ the manifold M may not be a connected sum with the infratorus as one of the components. Since in this case the infratorus is orientable the double cover trick does not work. Indeed, there are some simply-connected manifolds (for example, some $K3$ surfaces) that admit maximal rank actions. Still some conclusions can be drawn. Here is a simple example.

COROLLARY 9. *Maximal rank positive entropy actions to not exist on any sphere.*

Proof. Only the case $L = \mathbb{T}^m_{\pm}$ needs to be considered. In this case by Theorem 2 (4) there exists a smooth embedding of $\mathbb{R}P(m-1)$ to M. If $M = S^m$ this would imply existence of an embedding into $\mathbb{R}^m$, which is impossible [9]. □

1.5. **Structure and general remarks on the proof.** We conclude the introduction with a roadmap to the proof of Theorem 1.

1. We begin with [21, Theorem 2.11] (based on the main technical Theorem 4.1 from [14])[1], which states that in our setting conditional measures of the leaves on each Lyapunov foliation are absolutely continuous. This implies that the measure μ itself is absolutely continuous, see [21, Theorems 5.2 and 2.4].

[1] The proof of the latter theorem is technically the most difficult part in the whole construction and this is the basis on which everything else rests.

2. The next step is reduction to the leading weakly mixing case which is based on the fact that for a hyperbolic absolutely continuous measure ergodic components have positive measure [29].
3. Furthermore, there is a unique measurable system of smooth affine parameters on the leaves of a Lyapunov foliation [21, Proposition 3.3]. In fact, these affine parameters and conditional measures are closely related: the affine parameter is obtained by integrating the conditional measure. Naturally, both conditional measures and affine parameters are defined up to a scalar multiple. Fixing a measurable normalization smooth along the leaves of the Lyapunov foliation in question produces a cocycle; different normalizations produce cohomologous cocycles.
4. Affine structure on the (one-dimensional) leaves of the Lyapunov foliations can be uniquely extended to affine structures on leaves of unstable foliations for any element of the action. Those structures are invariant with respect to the whole action. Moreover, those affine structures have additional "diagonal property:" Lyapunov foliations correspond to coordinate lines. Conditional measure of those unstable foliations are equivalent to the Lebesgue measure defined by the affine structure.[2]
5. Next comes one of the key new ingredients in the proof: the holonomy between stable manifolds along the unstable manifolds is an affine diagonal map. This is proved by induction on the dimension of the stable manifold, with the case of Lyapunov manifolds (that are stable manifolds for some Weyl chambers in the maximal rank case) being the base of the induction.

   On a typical stable manifold corresponding unstable manifolds exist at Lebesgue almost every point. A priori, however, one can guarantee only that on a set of positive measure those manifolds have a certain size and change regularly. Thus even for nearby stable manifolds the unstable holonomy is only defined on set of positive measure while remaining unstable manifolds may stray away and never reach the target. We show that the holonomy is defined almost everywhere (and in fact extends to an affine map defined everywhere. This implies in particular that all unstable manifolds are large. The same of course applies to stable manifolds.
6. Fix a regular point $x \in M$ and a Weyl chamber. Let s be the dimension of the stable manifolds for that Weyl chamber. By the previous step, using x as the origin for the affine structures on its stable and unstable manifolds we can define a "development map"

$$h_x : \mathbb{R}^m \to \mathbb{R}^s \times \mathbb{R}^{m-s} \to \mathscr{W}^s(x) \times \mathscr{W}^u(x) \to M.$$

   This map is absolutely continuous, maps the standard affine structure in $\mathbb{R}^m$ onto the product of affine structures on $\mathscr{W}^s(x)$ and $\mathscr{W}^u(x)$, and moreover maps almost every coordinate line onto a leaf of the corresponding Lyapunov foliation. The map is however not injective and we explore this

[2] All of the above naturally takes place almost everywhere. In particular, not every coordinate line inside an unstable manifold a priori corresponds to an actual leaf of a Lyapunov foliation.

at the next step. At this stage we show that locally the map is injective and that locally properly normalized Lebesgue measure in $\mathbb{R}^m$ maps to our α-invariant measure μ.

7. Development maps with different base points differ by an invertible affine map of $\mathbb{R}^m$ with diagonal linear part. Since images of regular points under the action elements are regular, our action, restricted to a set of full measure, appears as factor of an affine action in $\mathbb{R}^m$ with diagonal linear parts.
8. The next crucial step is to study the the kernel of the development map h_x, the *homoclinic group* $\Gamma_x \subset \mathit{Aff}(\mathbb{R}^m)$. It follows from the local product structure of the Pesin set that the translation subgroup Tr_x is discrete. We show that the rank of Tr_x is maximal, which then implies that it is a finite index subgroup of the homoclinic group Γ_x. Thus $\mathbb{R}^m/Tr_x$ is a torus and $\mathbb{R}^m/\Gamma_x$ is an infratorus. Maximality of the rank of α implies that the infratorus is either a torus or $\mathbb{T}^m_{\pm}$. This immediately gives the linear model and the first two statements of Theorem 1.
9. Statement (3) follows from the smoothness of stable and unstable manifolds and from smoothness and invariance of affine structures. Finally statement (4) follows from some purely analytic results by Journé.

2. WEAK-MIXING REDUCTION

As we mentioned in Section 1.5 statement (1), measure μ is absolutely continuous w.r.t Lebesgue. Let $\alpha : \mathbb{Z}^{m-1} \to \mathit{Diff}(M^m)$ be a $C^{1+\theta}$ action as in Theorem 1. Take $\mathbf{n} \in \mathbb{Z}^{m-1}$ such that μ is a hyperbolic measure for $\alpha(\mathbf{n})$. This means that $\mathbf{n}$ does not lie on a Lyapunov hyperplane. By Pesin ergodic decomposition theorem [29] there is $k > 0$ and a $\alpha(k\mathbf{n})$-invariant set $R_1 \subset M$ of positive μ-measure such that $\alpha(k\mathbf{n})\restriction_{R_1}$ is a Bernoulli automorphism; in particular, it is weakly mixing. Set $\mathbf{n}_1 := k\mathbf{n}$. By ergodicity of $\alpha(\mathbf{n}_1)|R_1$ we have that for any $\mathbf{m} \in \mathbb{Z}^{m-1}$ either $\mu(\alpha(\mathbf{m})(R_1) \cap R_1) = 0$ or $\alpha(\mathbf{m})(R_1) = R_1 (\mathrm{mod}\, 0)$.

Since μ is an ergodic invariant measure for the whole action there are $\mathbf{n}_2, \ldots, \mathbf{n}_n \in \mathbb{Z}^{m-1}$ such that $\mu(\alpha(\mathbf{n}_l)(R_1) \cap \alpha(\mathbf{n}_k)(R_1)) = 0$ for $k, l = 1, \ldots n$, $k \neq l$ and $\mu(M \smallsetminus \alpha(\mathbf{n}_1)(R_1) \cup \cdots \cup \alpha(\mathbf{n}_n)(R_1) = 0$. Set $R_i = \alpha(\mathbf{n}_i)(R_1)$. Let $\Gamma \subset \mathbb{Z}^{m-1}$ be the finite index subgroup of $\mathbf{m} \in \mathbb{Z}^{m-1}$ such that $\alpha(\mathbf{m})(R_1)$=R_1 (mod 0). This is the decomposition and the finite index subgroup claimed in Theorem 1.

Thus it is enough to prove Theorem 1 for a weakly mixing action α. We will assume that without restating it explicitly until the final step of the proof of Theorem 1 at the end of Section 4.

LEMMA 2.1. *For any hyperbolic element* $\mathbf{n} \in \mathbb{Z}^{m-1}$,

(i) $\alpha(\mathbf{n})$ *is Bernoulli and*

(ii) there is a set of full measure R *such that for any Weyl chamber* $\mathscr{C}$ *if* $x \in R$

$$\bigcup_{z \in \mathscr{W}^u_{\mathscr{C}}(x) \cap R} \mathscr{W}^s_{\mathscr{C}}(z)$$

is a set of full measure.

Proof. Let us show first that $\alpha(\mathbf{n})$ is weakly mixing (and hence Bernoulli) for any hyperbolic element $\mathbf{n} \in \mathbb{Z}^{m-1}$. Using Pesin ergodic decomposition theorem we have a $k > 0$ and a set $\hat{R} \subset M$ of positive μ-measure invariant by $\alpha(k\mathbf{n})$ such that $\alpha(k\mathbf{n})|\hat{R}$ is weakly mixing (and Bernoulli). This set is an ergodic component of $\alpha(k\mathbf{n})$.

Now remember that there is an element of the action that we previously denoted by $\mathbf{n}_1$ such that $\alpha(\mathbf{n}_1)$ is weakly mixing. Since it commutes with $\alpha(k\mathbf{n})$, it interchanges ergodic components of the latter map. If there is more than one ergodic component for $\alpha(k\mathbf{n})$, $\alpha(\mathbf{n}_1)$ has a non-constant eigenfunction. Thus weak mixing of $\alpha(\mathbf{n}_1)$ implies that $\mu(M \setminus \hat{R}) = 0$ and $k = 1$, so that $\alpha(\mathbf{n})$ is weakly mixing and hence Bernoulli. This proves (i).

Let now $\mathscr{C}$ be a Weyl chamber and let $\mathbf{n} \in \mathscr{C}$. Observe that if P is a Pesin set for a hyperbolic measure then for every point $x \in P$ there is an open neighborhood $\mathscr{P}_x$ of a fixed size (a Pesin box) such that for every $y \in P \cap \mathscr{P}_x$ the local stable manifold of x intersects the local unstable manifold of y transversally (at a single point). Since $\alpha(\mathbf{n})$ is hyperbolic and weak mixing, by the previous observation, for a.e. x, y there is a non-negative integer $k \geq 0$ such that $\mathscr{W}^u_{\mathscr{C}}(\alpha(k\mathbf{n})(y))$ intersects transversally $\mathscr{W}^s_{\mathscr{C}}(x)$. To see that, take $k > 0$ such that $\alpha(k\mathbf{n})(y) \in P \cap \mathscr{P}_x$. Now let $k(x, y)$ be the minimum of such integers k. It is clear that $k(\alpha(\mathbf{n})(x), \alpha(\mathbf{n})(y)) = k(x, y)$ for $\mu \times \mu$-a.e. (x, y) and by the remark about Pesin boxes $k(x, y) = 0$ on a set of positive $\mu \times \mu$ measure. Now, since $\alpha(\mathbf{n})$ is weakly mixing, $\alpha(\mathbf{n}) \times \alpha(\mathbf{n})$ is ergodic and hence $k(x, y) = 0$ a.e. This statement is equivalent to statement (ii) of the lemma. □

3. AFFINE STRUCTURES AND HOLONOMIES

3.1. **Affine structures.**

3.1.1. *Affine structures for Lyapunov foliations.* Let χ be a Lyapunov exponent of α and $\mathscr{W} = \mathscr{W}^\chi$ be the corresponding Lyapunov foliation defined μ almost everywhere. Recall that for a maximal rank action all Lyapunov exponents are simple and hence leaves of all Lyapunov foliations are one-dimensional.

There is a unique α-invariant family of smooth affine parameters defined on almost every leaf of $\mathscr{W}$. Those affine structures change continuously within any Pesin set, see [14, Proposition 7.2], and hence they can be defined not only almost everywhere (at "typical" leaves with respect to recurrence or ergodic behavior of α) but at other specific important places such as leaves passing through periodic points that belong in a Pesin set. Those affine parameters are obtained by integrating telescoping products, see the proof of [13, Lemma 3.2]. But at the same time affine parameters define conditional measures of μ with respect to $\mathscr{W}$.

By [21, Proposition 4.2] those affine structures are invariant with respect to the holonomy along leaves of the stable foliation of any generic singular element $\alpha(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^{\mathbf{k}}$, for which $\chi(\mathbf{t}) = \mathbf{0}$.

3.1.2. *Affine structures for stable foliations.* We will now construct the affine structures along stable manifolds of any Weyl chamber $\mathscr{C}$ and prove their coherence. Originally such a structure is anchored at a base point and the structure constructed with another regular point at the stable foliation as the base point may a priori be different. Coherence means that all those structures coincide so that the structure is really associated with a stable manifold and is independent of the choice of the based point.

Since Lyapunov hyperplanes are in general position any combination of signs of Lyapunov exponents, except for all positive or all negative, is possible for elements of the action α, and hence there are no resonances. In particular for every Lyapunov exponent χ there is a Weyl chamber $\mathscr{C}_\chi$ such that inside $\mathscr{C}_\chi$, χ is the only positive Lyapunov exponent. Hence the stable manifolds $\mathscr{W}^s_{\mathscr{C}_\chi}(x)$ have codimension one. For any Weyl chamber $\mathscr{C}$ the manifolds $\mathscr{W}^s_{\mathscr{C}}(x)$ are the intersections of $\mathscr{W}^s_{\mathscr{C}_\chi}(x)$ over those χ that are negative in $\mathscr{C}$. In particular, leaves of any Lyapunov foliation are intersections of $m-1$ codimension one stable manifolds.

Moreover, we can take elements of the action such that ratios of negative Lyapunov exponents are arbitrarily close to one.

Let $\mathscr{C}$ be a Weyl chamber and $s = s(\mathscr{C})$ be the dimension of $\mathscr{W}^s_{\mathscr{C}}$. Given $s \geq 1$, let $\mathscr{D}_s$ be the group of invertible diagonal matrices on $\mathbb{R}^s$ and let $Emb^{1+\epsilon}(\mathbb{R}^s, M)$ be the space of $C^{1+\epsilon}$ embeddings of $\mathbb{R}^s$ into M with the topology of $C^{1+\epsilon}$ convergence on compact subsets, observe that in this way $Emb^{1+\epsilon}(\mathbb{R}^s, M)$ is a Polish space. Existence and coherence of affine structures has been known for a while. Both Proposition 3.1 and Proposition 3.2 are contained in [20, Proposition 2.7] that in turn refers to [12, Section 6.2]. These sources together provide what amounts to a sketch of a proof. These statements form a particular case of the more general theory of non-stationary normal forms and invariant geometric structures. We choose to formulate existence and coherence of affine structure in our case that is simpler than the general non-resonance case.

PROPOSITION 3.1. *Let α be a $C^{1+\theta}$, $0 < \theta < 1$ (respectively C^k, $k \geq 2$ integer) action as in Theorem 1. Then there is a set of full measure $R \subset M$ and a measurable map $H^{\mathscr{C}} : R \to Emb^{1+\theta}(\mathbb{R}^s, M)$ (respectively $H^{\mathscr{C}} : R \to Emb^k(\mathbb{R}^s, M)$) such that denoting $H^{\mathscr{C}}(x) = H_x$,*

1. *$H_x : \mathbb{R}^s \to \mathscr{W}^s_{\mathscr{C}}(x)$, i.e., $H_x(\mathbb{R}^s) = \mathscr{W}^s_{\mathscr{C}}(x)$;*
2. *$H_x(0) = x$;*
3. *$D_0 H_x : \mathbb{R}^s \to E^s_{\mathscr{C}}(x)$ sends the standard basis into the frame of invariant spaces E_{χ_i} where $E^s_{\mathscr{C}}(x) = E_{\chi_1} \oplus \cdots \oplus E_{\chi_s}$ for some ordering of the Lyapunov exponents;*
4. *there is a cocycle of diagonal maps of $\mathbb{R}^s$, $A : \mathbb{Z}^{m-1} \times R \to \mathscr{D}_s$ such that $H_{\alpha(\mathbf{n})(x)} \circ A(\mathbf{n}, x) = \alpha(\mathbf{n}) \circ H_x$ for every $\mathbf{n} \in \mathbb{Z}^{m-1}$ and $x \in R$.*

Such a family is unique modulo composition with a diagonal map $D : R \to \mathscr{D}_s$, i.e., if $\hat{H}$ is another affine structure then for a.e. x, $H_x^{-1} \circ \hat{H}_x \in \mathscr{D}_s$.

Proof of Proposition 3.1. The existence and uniqueness follows from Theorem 4 and its Addendum 1 in the Appendix once we find an element of the action with the right Lyapunov exponents. Let us find an element $\mathbf{n}\in\mathscr{C}$ such that $(1+\theta)\chi_+(\mathbf{n})<\chi_-(\mathbf{n})$ where for $\mathbf{n}\in\mathscr{C}$, $\chi_-(\mathbf{n})=\min_\chi\chi(\mathbf{n})$, $\chi_+(\mathbf{n})=\max_{\chi(\mathbf{n})<0}\chi(\mathbf{n})$. By the general position condition on Lyapunov exponents, we can define an invertible linear map $L:\mathbb{R}^{m-1}\to E_0$ where $E_0=\{(x_1,\dots,x_m)\in\mathbb{R}^m:\sum_i x_i=0\}$ so that $\chi_i(\mathbf{n})=L(\mathbf{n})_i$. Hence $L(\mathscr{C})$ corresponds to a choice of $+'s$ and $-'s$. Without loss of generality we can assume that $L(\mathscr{C})$ corresponds to $x_1,\dots,x_s<0$ and $x_{s+1},\dots,x_m>0$. Hence, it is enough to find $(x_1,\dots,x_m)$ such that

$$(\star)\qquad \begin{array}{l}\sum_i x_i=0,\\ x_1,\dots,x_s<0,\\ x_{s+1},\dots,x_m>0,\\ (1+\theta)x_i<x_j \text{ for } i,j\le s.\end{array}$$

Take $x_1=x_2=\dots=x_s<0$ and $x_{s+1}=x_{s+1}=\dots=x_m>0$ and such that $sx_1+(m-s)x_m=0$. This clearly works but may fail to correspond to $L(\mathbf{n})$ for some $\mathbf{n}\in\mathbb{Z}^{m-1}$. To solve this problem observe that $L(\mathbb{Q}^{m-1})$ is dense in E_0 and hence we can pick $\mathbf{n}\in\mathbb{Z}^{m-1}$ and $q\in\mathbb{N}$ such that $L(\frac{\mathbf{n}}{q})$ satisfies condition $(\star)$. By homogeneity of condition $(\star)$ we get that $L(\mathbf{n})$ also satisfies condition $(\star)$ and we are done. □

Now we prove coherence of the affine structures from Proposition 3.1 along stable manifolds.

PROPOSITION 3.2. *There is a set of full measure $R\subset M$ such that if $x,y\in R$ and $y\in\mathscr{W}^s_{\mathscr{C}}(x)$ then $H_y^{-1}\circ H_x$ is an affine map with diagonal linear part.*

Proof. Take some $\mathbf{n}$ in the Weyl chamber $\mathscr{C}$ such that $\mathscr{W}^s_{\mathscr{C}}(x)$ is the stable manifold for $\alpha(\mathbf{n})$. Let us number the Lyapunov exponents so that $\chi_1(\mathbf{n})<0,\dots,$ $\chi_s(\mathbf{n})<0$ for $\mathbf{n}\in\mathscr{C}$. Take L a Luzin set of continuity for $z\to H_z$ of μ-measure close to 1. Then there is a set of full measure $R_{\mathbf{n}}\subset M$ such that whenever $x,y\in R_{\mathbf{n}}$ then there are iterates $l_i\to+\infty$ such that $\alpha(l_i\mathbf{n})(x),\alpha(l_i\mathbf{n})(y)\in L$ for every i. Such a sequence of iterates can be found using Birkhoff ergodic theorem as long as L has μ-measure larger that $1/2$. The set of full measure R we claim in the proposition is the intersection of $R_{\mathbf{n}}$'s for finitely many choices of $\mathbf{n}\in\mathscr{C}$ according to some pinching of the Lyapunov spectrum to be determined later.

Take $x,y\in R_{\mathbf{n}}$ with $y\in\mathscr{W}^s_{\mathscr{C}}(x)$. Denote $A^{(l)}(x)=A(l\mathbf{n},x)$ and similarly with y. We have

$$H^{-1}_{\alpha(l\mathbf{n})(y)}\circ H_{\alpha(l\mathbf{n})(x)}=A^{(l)}(y)H_y^{-1}\circ H_x\circ(A^{(l)}(x))^{-1}.$$

By continuity on Luzin sets and convergence $d(\alpha(l_i\mathbf{n})(x),\alpha(l_i\mathbf{n})(y))\to 0$ that follows from the fact that $y\in\mathscr{W}^s_{\mathscr{C}}(x)$ we obtain

$$\lim_{l_i\to+\infty}\|H^{-1}_{\alpha(l_i\mathbf{n})(y)}\circ H_{\alpha(l_i\mathbf{n})(x)}-id\|_{C^1(B(1))}=0, \tag{3.1}$$

where $\|\cdot\|_{C^1(B(1))}$ stands for the sup C^1 norm on the unit ball.

Let $A^{(l)}(x) =: \mathrm{diag}(\lambda_k^{(l)}(x))$. Then

$$\lim_{l\to\pm\infty} |\lambda_k^{(l)}(x)|^{1/l} = \lim_{l\to\pm\infty} |\lambda_k^{(l)}(y)|^{1/l} = \exp(\chi_k(\mathbf{n})) =: \lambda_k < 1.$$

Set $P = H_y^{-1} \circ H_x$. Then it follows from (3.1) that

$$\lim_{l_i\to+\infty} \|A^{(l_i)}(y)P \circ (A^{(l_i)}(x))^{-1} - id\|_{C^1(B(1))} = 0. \tag{3.2}$$

Let $P = (P_1,\ldots,P_s)$, take P_k and let us show that for $j \neq k$ the partial derivative $\partial_j P_k$ vanishes. We have for any given l and $j \neq k$,

$$\partial_j \left(A^{(l)}(y)P \circ (A^{(l)}(x))^{-1} - id\right)_k = \frac{\lambda_k^{(l)}(y)}{\lambda_j^{(l)}(x)} (\partial_j P_k) \circ (A^{(l)}(x))^{-1}.$$

Applying (3.2) and that $\bigcup_{l_i} A^{(l_i)}(x)(B(1)) = \mathbb{R}^s$, we see that if $\lambda_k/\lambda_i > 1$ then $\partial_j P_k \equiv 0$ on $\mathbb{R}^s$. Hence if we take $\mathbf{n}_{k,j} \in \mathscr{C}$ such that $\chi_k(\mathbf{n}_{k,j}) > \chi_i(\mathbf{n}_{k,j})$, we get that $\partial_j P_k \equiv 0$ on $\mathbb{R}^s$ for $j \neq k$. Existence of such an element $\mathbf{n}_{k,j} \in \mathscr{C}$ follows from the maximal rank condition that implies that all Lyapunov exponents are in general position.

So, we have that $P_k(v) = P_k(v_k)$ only depends on the k-th variable. Denote with P_k' the derivative of P_k. Then again using formula (3.2) and arguing as before we get that for any $R > 0$,

$$\lim_{l_i\to+\infty} \sup_{t\in B(R)} \left| \frac{\lambda_k^{(l_i)}(y)}{\lambda_k^{(l_i)}(x)} P_k'(t) - 1 \right| = 0.$$

In particular this implies that

$$\frac{\lambda_k^{(l_i)}(y)}{\lambda_k^{(l_i)}(x)} \to 1/P_k'(t)$$

for any $t \in \mathbb{R}$. This gives the claim since the left hand side does not depend on t and then P_k' is constant and hence P_k is affine. □

The following lemma uses uniqueness of affine structures and absolute continuity of the invariant measure.

LEMMA 3.3. *(i) Given any two Weyl chambers $\mathscr{C}_1$ and $\mathscr{C}_2$ such that the invariant foliations $\mathscr{W}^s_{\mathscr{C}_1} \subset \mathscr{W}^s_{\mathscr{C}_2}$, affine structures on the leaves of $\mathscr{W}^s_{\mathscr{C}_1}$ are restrictions of affine structures on $\mathscr{W}^s_{\mathscr{C}_2}$.*
(ii) If $E^s_{\mathscr{C}} = E^s_{\mathscr{C}_1} \oplus\cdots\oplus E^s_{\mathscr{C}_l}$ we have that affine structures on $\mathscr{W}^s_{\mathscr{C}}$ is a product (direct sum) of affine structures on $\mathscr{W}^s_{\mathscr{C}_i}$, $i = 1,\ldots,l$.

Proof. The main point in the lemma is to prove that for a typical x, $\mathscr{W}^s_{\mathscr{C}_1}(x)$ is a coordinate plane in the affine structure of $\mathscr{W}^s_{\mathscr{C}_2}(x)$. Once we settle this the lemma follows from uniqueness of affine structures.

For a.e. point x we may assume that (Lebesgue) almost every point y in $\mathscr{W}^s_{\mathscr{C}_2}(x)$ is a regular point; moreover, we may assume also that for Lebesgue a.e. point $y \in \mathscr{W}^s_{\mathscr{C}_2}(x)$, (Lebesgue) a.e. point $z \in \mathscr{W}^s_{\mathscr{C}_1}(y)$ is a regular point. This follows from absolute continuity of invariant foliations and the fact that conditional measures are equivalent to Lebesgue.

Let us denote with $H_z : \mathbb{R}^s \to \mathcal{W}^s_{\mathscr{C}_2}(z)$ the affine structures on $\mathcal{W}^s_{\mathscr{C}_2}(z) = \mathcal{W}^s_{\mathscr{C}_2}(x)$ based at z.

Let x be a point as in the previous paragraph and y a regular point in $\mathcal{W}^s_{\mathscr{C}_2}(x)$. By Proposition 3.1(3) we have that there is a coordinate plane V such that $D_0H_x(V) = T_x\mathcal{W}^s_{\mathscr{C}_1}(x)$

Let us consider the manifold $W = H_x^{-1}(\mathcal{W}^s_{\mathscr{C}_1}(y)) \subset \mathbb{R}^s$ and let us show that this manifold is a plane parallel to V. Let us assume that (Lebesgue) a.e. $z \in \mathcal{W}^s_{\mathscr{C}_1}(y)$ is regular point. We shall show that for Lebesgue a.e. point $a \in W$, $T_aW = V$. By Proposition 3.1(3) $D_0H_z(V) = T_z\mathcal{W}^s_{\mathscr{C}_1}(z)$, and since $\mathcal{W}^s_{\mathscr{C}_1}(z) = \mathcal{W}^s_{\mathscr{C}_1}(y)$ we get that

$$T_z\mathcal{W}^s_{\mathscr{C}_1}(y) = D_0H_z(V).$$

On the other hand, by Proposition 3.2 $H_x^{-1} \circ H_z$ is an affine map with diagonal linear part, and hence the derivative at 0 of $H_x^{-1} \circ H_z$ is diagonal. So if $a \in W$ and $H_x(a) = z$ is a regular point then

$$\begin{aligned} T_aW = T_aH_x^{-1}(\mathcal{W}^s_{\mathscr{C}_1}(y)) &= D_zH_x^{-1}(T_z\mathcal{W}^s_{\mathscr{C}_1}(y)) = D_zH_x^{-1}(D_0H_z(V)) \\ &= D_0(H_x^{-1} \circ H_z)(V). \end{aligned}$$

Since $D_0(H_x^{-1} \circ H_z)$ is diagonal and V is a coordinate plane we get that

$$D_0(H_x^{-1} \circ H_z)(V) = V$$

and hence $T_aW = V$ for Lebesgue a.e. $a \in W$. Since W is a C^1 manifolds then we have that $T_aW = V$ for every $a \in W$ and hence W is a plane parallel to V as wanted. □

REMARK 9. Since not every point on a leaf of $\mathcal{W}^s_{\mathscr{C}}$ is regular not all coordinate lines correspond to actual leaves of Lyapunov foliations. But for a typical leaf this is true for almost every coordinate line. On the other hand, in the setting of Lemma 3.3, if $\mathcal{W}^s_{\mathscr{C}_1} \subset \mathcal{W}^s_{\mathscr{C}_2}$, then $\mathcal{W}^s_{\mathscr{C}_1}$ uniquely extends to a smooth foliation, indeed an affine foliation in the affine coordinates of $\mathcal{W}^s_{\mathscr{C}_2}(x)$

COROLLARY 3.4. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two Weyl chambers and let $\mathscr{C}_3$ be the Weyl chamber such that $E^s_{\mathscr{C}_1} \cap E^s_{\mathscr{C}_2} = E^s_{\mathscr{C}_3}$. Then there is a set of full measure $R \subset M$ such that if $x, y, z \in R$ with $z \in \mathcal{W}^s_{\mathscr{C}_1}(x) \cap \mathcal{W}^s_{\mathscr{C}_2}(y)$ then $\mathcal{W}^s_{\mathscr{C}_3}(z) \subset \mathcal{W}^s_{\mathscr{C}_1}(x) \cap \mathcal{W}^s_{\mathscr{C}_2}(y)$ is a linear subspace in the affine structures along $\mathcal{W}^s_{\mathscr{C}_1}(x)$ and $\mathcal{W}^s_{\mathscr{C}_2}(y)$ tangent to the space corresponding to $E^s_{\mathscr{C}_3}$.*

3.2. **Uniformity of the holonomies.** Now we will show that holonomy maps along unstable manifolds between two stable manifolds are almost everywhere defined w.r.t. Lebesgue measure on stable manifolds and are affine with respect to the affine structures defined in the previous section. The following proposition is a central ingredient in our construction of the global arithmetic structure on a set of full measure. Unlike the results of the previous section than can be extended to a general non-resonance case, its assertion depends on the maximal rank or a similar assumption.

PROPOSITION 3.5. *Let $\mathscr{C}$ be a Weyl chamber. There is a set of full measure $R := R_{\mathscr{C}} \subset M$ such that if $x, y \in R$ and $y \in \mathscr{W}^u_{\mathscr{C}}(x)$ then the holonomy along unstables $\mathrm{Hol}^{u,\mathscr{C}}_{x,y} : \mathscr{W}^s_{\mathscr{C}}(x) \to \mathscr{W}^s_{\mathscr{C}}(y)$ is defined for Lebesgue a.e. $z \in \mathscr{W}^s_{\mathscr{C}}(x)$ and is affine, i.e., there is a diagonal linear map B preserving the frame such that for a.e. $z \in \mathscr{W}^s_{\mathscr{C}}(x)$, there is a point $\mathrm{Hol}^{u,\mathscr{C}}_{x,y}(z) \in \mathscr{W}^u_{\mathscr{C}}(z) \cap \mathscr{W}^s_{\mathscr{C}}(y)$ and moreover*

$$\mathrm{Hol}^{u,\mathscr{C}}_{x,y} \circ H^{\mathscr{C}}_x = H^{\mathscr{C}}_y \circ B.$$

Lebesgue a.e.

Proof. The idea of the proof is that by Lemma 3.3 is that a holonomy is globally defined and affine if it takes place inside a single unstable manifold of a regular point. We will use induction on dimension $s(\mathscr{C})$ of the stable manifold $\mathscr{W}^s_{\mathscr{C}}$.

Let as assume first that $\mathscr{W}^s_{\mathscr{C}}$ is 1-dimensional. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be Weyl chambers such that $E^u_{\mathscr{C}_1} \oplus E^u_{\mathscr{C}_2} = E^u_{\mathscr{C}}$. By Lemma 3.3 $\mathscr{W}^u_{\mathscr{C}_1}$ and $\mathscr{W}^u_{\mathscr{C}_2}$ are a pair of transverse linear sub-foliaitons of $\mathscr{W}^u_{\mathscr{C}}$. Hence there is a set of full measure R_0 such if $x \in R_0$ and $y \in R_0$ then there are regular points $a, b \in \mathscr{W}^u_{\mathscr{C}}(x)$ such that $a \in \mathscr{W}^u_{\mathscr{C}_1}(x)$, $b \in \mathscr{W}^u_{\mathscr{C}_1}(y)$ and $a \in \mathscr{W}^u_{\mathscr{C}_2}(b)$. We have that

$$\mathrm{Hol}^{u,\mathscr{C}}_{x,y}(z) = \mathrm{Hol}^{u,\mathscr{C}}_{b,y} \circ \mathrm{Hol}^{u,\mathscr{C}}_{a,b} \circ \mathrm{Hol}^{u,\mathscr{C}}_{x,a}(z),$$

see Figure 1. Since $E^s_{\mathscr{C}} \oplus E^u_{\mathscr{C}_1} = E^u_{\mathscr{C}_3}$ for some Weyl chamber $\mathscr{C}_3$ and $E^s_{\mathscr{C}} \oplus E^u_{\mathscr{C}_2} = E^u_{\mathscr{C}_4}$ for some Weyl chamber $\mathscr{C}_4$ we have by Lemma 3.3 that $\mathscr{W}^u_{\mathscr{C}}$ and $\mathscr{W}^s_{\mathscr{C}_1}$ are transverse linear subfoliations of $\mathscr{W}^u_{\mathscr{C}_3}(x)$ and hence $\mathrm{Hol}^{u,\mathscr{C}}_{x,a} : \mathscr{W}^s_{\mathscr{C}}(x) \to \mathscr{W}^s_{\mathscr{C}}(a)$ is affine holonomy inside $\mathscr{W}^u_{\mathscr{C}_3}(x)$ which is everywhere defined. Similar argument applies to $\mathrm{Hol}^{u,\mathscr{C}}_{b,y}$ and $\mathrm{Hol}^{u,\mathscr{C}}_{a,b}$. Thus we proved the statement for the case when $\mathscr{W}^s_{\mathscr{C}}$ is 1-dimensional.

Now let the dimension of $\mathscr{W}^s_{\mathscr{C}}$, be $s(\mathscr{C})$, and assume by induction that we have proven the proposition for any Weyl chamber $\mathscr{C}'$ with $s(\mathscr{C}') < s(\mathscr{C})$. There are Weyl chambers $\mathscr{C}_1$ and $\mathscr{C}_2$ such that $E^u_{\mathscr{C}} = E^u_{\mathscr{C}_1} \cap E^u_{\mathscr{C}_2}$ and $E^s_{\mathscr{C}} = E^s_{\mathscr{C}_1} \oplus E^s_{\mathscr{C}_2}$. Obviously dimensions $s(\mathscr{C}_i)$, $i = 1, 2$ of $\mathscr{W}^s_{\mathscr{C}_i}$ are strictly smaller than $s(\mathscr{C})$.

By the induction hypothesis, we have that for a.e. points x and $y \in \mathscr{W}^u_{\mathscr{C}}(x) \subset \mathscr{W}^u_{\mathscr{C}_1}(x)$, $\mathrm{Hol}^{u,\mathscr{C}_1}_{x,y} : \mathscr{W}^s_{\mathscr{C}_1}(x) \to \mathscr{W}^s_{\mathscr{C}_1}(y)$ is everywhere defined and is affine.

Taking typical points x, y, we have that Lebesgue a.e. point $a \in \mathscr{W}^s_{\mathscr{C}_1}(x)$ is regular, and $\mathrm{Hol}^{u,\mathscr{C}_1}_{x,y}(a) \in \mathscr{W}^u_{\mathscr{C}_1}(a) \cap \mathscr{W}^s_{\mathscr{C}_1}(y)$ is also regular, see Figure 2. Moreover, $\mathrm{Hol}^{u,\mathscr{C}_1}_{x,y}(a) = \mathrm{Hol}^{u,\mathscr{C}}_{x,y}(a)$. Indeed, $E^u_{\mathscr{C}} \oplus E^s_{\mathscr{C}_1} = E^u_{\mathscr{C}_3}$ for some Weyl chamber $\mathscr{C}_3$ which gives by Lemma 3.3 that $\mathscr{W}^u_{\mathscr{C}}$ and $\mathscr{W}^s_{\mathscr{C}_1}$ are a pair of transverse affine foliations in $\mathscr{W}^u_{\mathscr{C}_3}(x) = \mathscr{W}^u_{\mathscr{C}_3}(y)$ and hence $\mathrm{Hol}^{u,\mathscr{C}_1}_{x,y}(a) = \mathrm{Hol}^{u,\mathscr{C}}_{x,y}(a) \in \mathscr{W}^u_{\mathscr{C}}(a) \cap \mathscr{W}^s_{\mathscr{C}_1}(y) \subset \mathscr{W}^u_{\mathscr{C}_1}(a) \cap \mathscr{W}^s_{\mathscr{C}_1}(y)$, see Figure 2.

Now, for Lebesgue a.e. $z \in \mathscr{W}^s_{\mathscr{C}}(x)$, $\mathscr{W}^s_{\mathscr{C}_2}(z) \cap \mathscr{W}^s_{\mathscr{C}_1}(x)$ intersects in a point $a \in \mathscr{W}^s_{\mathscr{C}_1}(x)$ in the conditions of the previous paragraph. Hence $\mathrm{Hol}^{u,\mathscr{C}}_{x,y}(a)$ is well defined and again by induction we have that

$$\mathrm{Hol}^{u,\mathscr{C}_2}_{a,\mathrm{Hol}^{u,\mathscr{C}}_{x,y}(a)} : \mathscr{W}^s_{\mathscr{C}_2}(a) \to \mathscr{W}^s_{\mathscr{C}_1}(\mathrm{Hol}^{u,\mathscr{C}}_{x,y}(a))$$

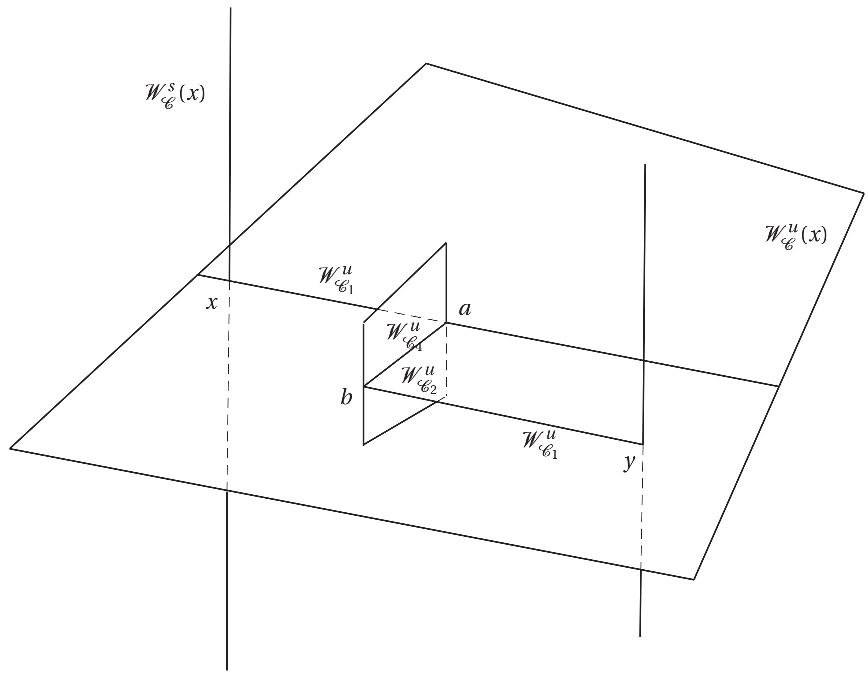

FIGURE 1. Proposition 3.5.i

is Lebesgue a.e. defined and is affine and

$$\mathrm{Hol}^{u,\mathscr{C}_2}_{a,\mathrm{Hol}^{u,\mathscr{C}}_{x,y}(a)}(z) \in \mathcal{W}^u_{\mathscr{C}_2}(z) \cap \mathcal{W}^s_{\mathscr{C}_2}(\mathrm{Hol}^{u,\mathscr{C}}_{x,y}(a)).$$

Moreover, since $E^u_{\mathscr{C}}$ and $E^s_{\mathscr{C}_2}$ are jointly integrable, arguing as before, we have that

$$\mathrm{Hol}^{u,\mathscr{C}_2}_{a,\mathrm{Hol}^{u,\mathscr{C}}_{x,y}(a)}(z) = \mathrm{Hol}^{u,\mathscr{C}}_{x,y}(z).$$

Therefore the holonomy $\mathrm{Hol}^{u,\mathscr{C}}_{x,y}$ is Lebesgue a.e. defined and the proposition follows. □

If a Weyl chamber $\mathscr{C}$ is fixed we simplify notations and write $\mathrm{Hol}^u_{x,y}$ for $\mathrm{Hol}^{u,\mathscr{C}}_{x,y}$.

PROPOSITION 3.6. *For $x, a, b \in R_{\mathscr{C}}$, $a \in \mathcal{W}^u_{\mathscr{C}}(x)$ and $b \in \mathcal{W}^s_{\mathscr{C}}(x)$ we have that*

$$\mathrm{Hol}^s_{x,b}(a) = \mathrm{Hol}^u_{x,a}(b).$$

Proof. The proof of this proposition is very similar to that of the previous one. Assume without loss of generality that dimension of $\mathcal{W}^s_{\mathscr{C}}$ is larger than 1 (otherwise consider $\mathcal{W}^u_{\mathscr{C}}$ instead). Even though we will use several different Weyl chambers in the proof of the proposition, all holonomies here will be w.r.t. invariant manifolds of the Weyl chamber $\mathscr{C}$.

Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be Weyl chambers such that $E^s_{\mathscr{C}_1} \oplus E^s_{\mathscr{C}_2} = E^s_{\mathscr{C}}$. Then there are regular points $z_1, z_2 \in \mathcal{W}^s_{\mathscr{C}}(x)$ such that $\bar{z}_1 := \mathrm{Hol}^u_{x,a}(z_1)$, $\bar{z}_2 := \mathrm{Hol}^u_{x,a}(z_2)$ are well

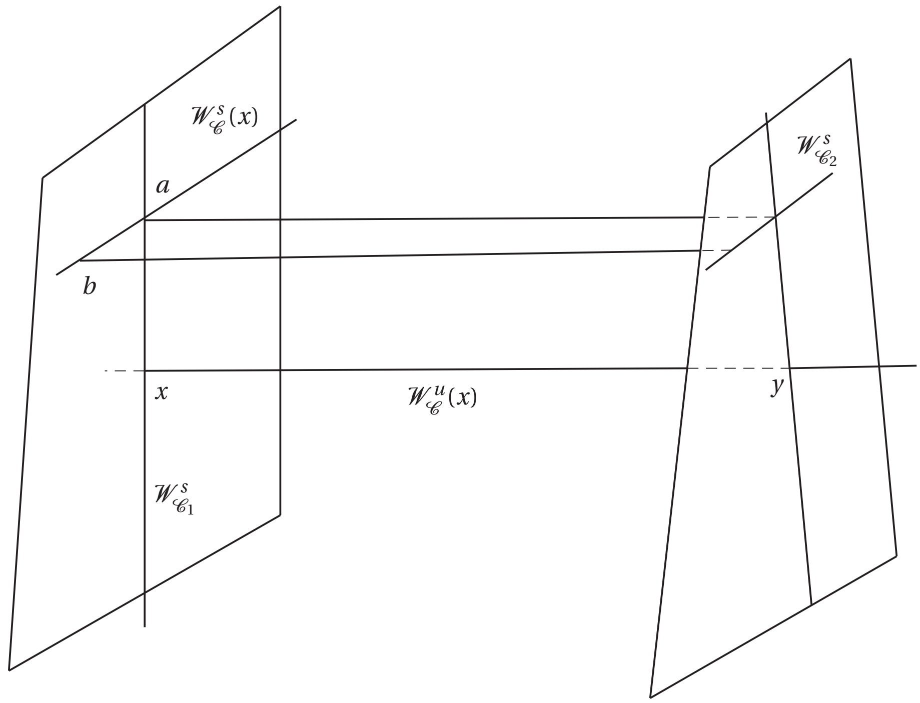

FIGURE 2. Proposition 3.5.ii

defined and regular points, $z_1 \in \mathscr{W}^s_{\mathscr{C}_1}(x)$, $z_2 \in \mathscr{W}^s_{\mathscr{C}_2}(z_1)$, and $b \in \mathscr{W}^s_{\mathscr{C}_1}(z_2)$, see Figure 3. We know on the one hand that $\mathrm{Hol}^s_{x,b} = \mathrm{Hol}^s_{z_2,b} \circ \mathrm{Hol}^s_{z_1,z_2} \circ \mathrm{Hol}^s_{x,z_1}$. We have also that $\mathrm{Hol}^u_{x,a} = \mathrm{Hol}^u_{z_1,\bar{z}_1} = \mathrm{Hol}^u_{z_2,\bar{z}_2}$.

Let $\mathscr{C}_3$ and $\mathscr{C}_4$ be Weyl chambers such that $E^u_{\mathscr{C}} \oplus E^s_{\mathscr{C}_1} = E^u_{\mathscr{C}_3}$ and $E^u_{\mathscr{C}} \oplus E^s_{\mathscr{C}_2} = E^u_{\mathscr{C}_4}$. Then we have, using that $x, a, z_1, \bar{z}_1 \in \mathscr{W}^u_{\mathscr{C}_3}(x)$ that

$$\bar{z}_1 = \mathrm{Hol}^u_{x,a}(z_1) = \mathrm{Hol}^s_{x,z_1}(a)$$

since all these holonomies take place inside $\mathscr{W}^u_{\mathscr{C}_3}(x)$. Following this argument we get also that $z_1, z_2, \bar{z}_1, \bar{z}_2 \in \mathscr{W}^u_{\mathscr{C}_4}(z_1)$ and hence

$$\bar{z}_2 = \mathrm{Hol}^u_{x,a}(z_2) = \mathrm{Hol}^u_{z_1,\bar{z}_1}(z_2) = \mathrm{Hol}^s_{z_1,z_2}(\bar{z}_1).$$

Finally, since $z_2, \bar{z}_2, b, \mathrm{Hol}^u_{z_2,\bar{z}_2}(b) \in \mathscr{W}^u_{\mathscr{C}_3}(z_2)$, we have

$$\mathrm{Hol}^u_{x,a}(b) = \mathrm{Hol}^u_{z_2,\bar{z}_2}(b) = \mathrm{Hol}^s_{z_2,b}(\bar{z}_2).$$

Putting all this together we get the proposition. □

4. THE ARITHMETIC STRUCTURE

4.1. **The development map.** Let us fix a Weyl chamber $\mathscr{C}$. Similarly we have affine parameters along the unstable foliation for points in $R_{-\mathscr{C}}$, where $-\mathscr{C}$ is the opposite Weyl chamber. Let $R = R_{\mathscr{C}} \cap R_{-\mathscr{C}}$, be the set of regular point for

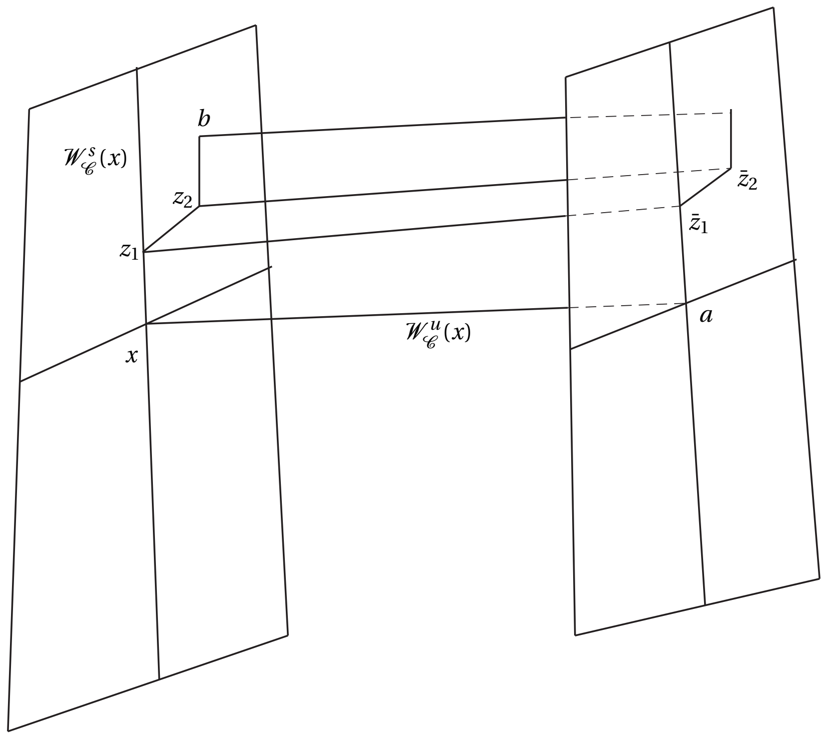

FIGURE 3. Proposition 3.6

both Weyl chambers. Notice that R intersected with a.e. Lyapunov manifold has full Lebesgue measure. Also, we may need to reduce R several times to sets still of full measure satisfying some additional properties. We omit any reference to $\mathcal{C}$ in our notations and denote by H^s_x the affine parameters along stable manifolds $\mathcal{W}^s$ and by H^u_x the affine parameters along unstable manifolds $\mathcal{W}^u$.

Observe that in the affine coordinates the conditional measure is standard Haar measure with some normalization.

We shall define a kind of covering or development map for M, defined almost everywhere. Roughly, the idea is as follows: for a given x we define $\hat{h}_x : \mathcal{W}^s(x) \times \mathcal{W}^u(x) \to M$, $\hat{h}_x(a,b) = \mathrm{Hol}^u_{x,b}(a) = \mathrm{Hol}^s_{x,a}(b)$ which is a point in $\mathcal{W}^u(a) \cap \mathcal{W}^s(b)$. Then we use affine parameters on $\mathcal{W}^s(x)$ and $\mathcal{W}^u(x)$ to define $h_x : \mathbb{R}^s \times \mathbb{R}^u \to M$. Since holonomies are only defined a.e. we need to take certain care, and that is what we do in the following paragraphs.

Let us fix $x \in R$ and assume that $\mathcal{W}^s(x) \cap R$ and $\mathcal{W}^u(x) \cap R$ have full Lebesgue measure. Let

$$R^s = (H^s_x)^{-1}(\mathcal{W}^s(x) \cap R) \subset \mathbb{R}^s$$

and

$$R^u = (H^u_x)^{-1}(\mathcal{W}^u(x) \cap R) \subset \mathbb{R}^u.$$

Given $z^u \in R^u$, using Proposition 3.5 we can take $D^x_{z^u} \in \mathscr{D}_s$ such that

$$\mathrm{Hol}^u_{x,H^u_x(z^u)} \circ H^s_x = H^s_{H^u_x(z^u)} \circ D^x_{z^u}$$

(recall that $\mathrm{Hol}^u_{x,H^u_x(z^u)} : \mathscr{W}^s(x) \to \mathscr{W}^s(H^u_x(z^u))$).

Let us define $h^u_x : \mathbb{R}^s \times R^u \to M$ by

$$h^u_x(z^s, z^u) = H^s_{H^u_x(z^u)}(D^x_{z^u}(z^s)).$$

Observe that by Proposition 3.5 for a.e. (z^s, z^u),

$$h^u_x(z^s, z^u) = \mathrm{Hol}^u_{x,H^u_x(z^u)}\left(H^s_x(z^s)\right). \tag{4.1}$$

Similarly, given $z^s \in R^s$, using Proposition 3.5 we can take $D^x_{z^s} \in \mathscr{D}_u$ such that

$$\mathrm{Hol}^s_{x,H^s_x(z^s)} \circ H^u_x = H^u_{H^s_x(z^s)} \circ D^x_{z^s}. \tag{4.2}$$

Let us define $h^s_x : R^s \times \mathbb{R}^u \to M$ by

$$h^s_x(z^s, z^u) = H^u_{H^s_x(z^s)}(D^x_{z^s}(z^u)).$$

We also have here that for a.e. (z^s, z^u),

$$h^s_x(z^s, z^u) = \mathrm{Hol}^s_{x,H^s_x(z^s)}\left(H^u_x(z^u)\right). \tag{4.3}$$

LEMMA 4.1. *For Lebesgue a.e. (z^s, z^u), $h^s_x(z^s, z^u) = h^u_x(z^s, z^u)$.*

Proof. This is an immediate consequence of Proposition 3.6 and formulas (4.1) and (4.3). □

Let us denote $h_x = h^s_x = h^u_x$.

Given $l \geq 1$, let $\mathscr{D}_l$ be the group of invertible diagonal matrices on $\mathbb{R}^l$ and let $\mathscr{A}_l$ be the group of affine maps on $\mathbb{R}^l$ whose linear parts are in $\mathscr{D}_l$. In the sequel, when we say almost everywhere (a.e.) we mean w.r.t. Lebesgue measure unless another measure is clearly specified.

LEMMA 4.2. *For μ a.e. x and for a.e. $(w^s, w^u) \in R^s \times R^u$ there is $L \in \mathscr{A}_m$ such that if we set $y = h_x(w^s, w^u)$ then*

(1) $L(0,0) = (w^s, w^u)$ and

(2) $h_x \circ L = h_y$ a.e.

Proof. Let $a = h_x(w^s, 0)$. Let us prove first the proposition for x and a. By Proposition 3.2 there is $B \in \mathscr{A}_s$ such that $H^s_a = H^s_x \circ B$, $B(0) = w^s$. Hence from the definition of h^s we get that

$$\begin{aligned} h^s_a(z^s, z^u) &= H^u_{H^s_a(z^s)}(D^a_{z^s}(z^u)) = H^u_{H^s_x(Bz^s)}(D^a_{z^s}(z^u)) \\ &= H^u_{H^s_x(Bz^s)}(D^x_{Bz^s}(Dz^u)) = h^s_x(Bz^s, Dz^u), \end{aligned}$$

where $D = (D^x_{Bz^s})^{-1} D^a_{z^s}$. We need to see that the map D does not depend on z^s.

From the definition of $D^x_{Bz^s}$, Proposition 3.5 and formula (4.2) we know that

$$\mathrm{Hol}^s_{x,H^s_x(Bz^s)} \circ H^u_x = H^u_{H^s_x(Bz^s)} \circ D^x_{Bz^s}.$$

Hence

$$(D^x_{Bz^s})^{-1} = (H^u_x)^{-1} \circ (\mathrm{Hol}^s_{x,H^s_x(Bz^s)})^{-1} \circ H^u_{H^s_x(Bz^s)}$$

and similarly

$$D^a_{z^s} = (H^u_{H^s_a(z^s)})^{-1} \circ \mathrm{Hol}^s_{a,H^s_a(z^s)} \circ H^u_a.$$

Since $H^s_a(z^s) = H^s_x(Bz^s)$ and for any b, $\mathrm{Hol}^s_{b,x} \circ \mathrm{Hol}^s_{a,b} = \mathrm{Hol}^s_{a,x}$ we get that

$$\begin{aligned} D = (D^x_{Bz^s})^{-1} D^a_{z^s} &= (H^u_x)^{-1} \circ (\mathrm{Hol}^s_{x,H^s_x(Bz^s)})^{-1} \circ \mathrm{Hol}^s_{a,H^s_a(z^s)} \circ H^u_a \\ &= (H^u_x)^{-1} \circ \mathrm{Hol}^s_{H^s_a(z^s),x} \circ \mathrm{Hol}^s_{a,H^s_a(z^s)} \circ H^u_a \\ &= (H^u_x)^{-1} \circ \mathrm{Hol}^s_{a,x} \circ H^u_a = D_{a,x}. \end{aligned}$$

So, take $L = (B, D) \in \mathscr{A}_m$ in this case.

Now the lemma follows from the observation $y = h_x(w^s, w^u) = h_a(0, D^{-1}w^u)$ and the previous argument with u and s interchanged. □

We have an immediate corollary of the previous lemma:

COROLLARY 4.3. *There is a set of full measure $R \subset \mathbb{R}^m$ such that if (z^s, z^u) and (w^s, w^u) are in R and $h_x(z^s, z^u) = h_x(w^s, w^u)$ then there is $L \in \mathscr{A}_m$ such that $h_x \circ L = h_x$ a.e.*

4.2. **The quotient.**

DEFINITION 1. Let Γ_x be the group of $L \in \mathscr{A}_m$ such that

$$h_x(L(z^s, z^u)) = h_x(z^s, z^u)$$

for Lebesgue a.e. $(z^s, z^u) \in \mathbb{R}^s \times \mathbb{R}^u = \mathbb{R}^m$. Γ_x should be thought as the group of deck transformations of the "covering" h_x. We call Γ_x the *homoclinic group* since there is a correspondence between the points in $\mathscr{W}^u(x) \cap \mathscr{W}^s(x)$ and Γ_x.[3]

We consider $\mathbb{R}^m$ with its natural additive group structure and let λ be Haar (equal to Lebesgue) measure on $\mathbb{R}^s \times \mathbb{R}^u = \mathbb{R}^m$. It is the product of Haar measure on $\mathbb{R}^s$ and Haar on $\mathbb{R}^u$.

LEMMA 4.4. *For Lebesgue a.e. $\bar{z} = (z^s, z^u)$ there is $c_x(\bar{z}) > 0$ and for any $\epsilon > 0$ there is $\delta > 0$ and a set $K_\epsilon(\bar{z}) \subset B_\delta(\bar{z})$, such that*

(1) $\dfrac{\lambda(K_\epsilon(\bar{z}) \cap B_\delta(\bar{z}))}{\lambda(B_\delta(\bar{z}))} \geq 1 - \epsilon$*;*
(2) $h_x \restriction_{K_\epsilon(\bar{z})}$ *is one-to-one;*
(3) $\mu(h_x(A)) = c_x(\bar{z})\lambda(A)$ *for any measurable set $A \subset K_\epsilon(\bar{z})$.*

Proof. By Lemma 4.2 and Corollary 4.3, it is enough to prove the lemma when $\bar{z} = (0,0)$. Given $\epsilon > 0$, since x is a regular point it belongs to some Pesin set, and we may assume it is a density point of a Pesin set. Hence we can take δ small so that for $K_\epsilon = h_x^{-1}(P \cup B_\delta(0,0))$ we have (1) and (2).

To prove (3) notice that the conditional measure of μ along stables and unstables is Haar measure (with some normalization). Hence, by holonomy invariance of the conditional measures, we have that, locally on Pesin sets, the measure μ is the product of Haar on stable and Haar on unstable which gives Haar in affine coordinates h_x. □

[3] It is a nice exercise for the reader to carry out this construction explicitly in the case of a hyperbolic automorphism of $\mathbb{T}^2_\pm$.

The next lemma is a direct corollary of Proposition 3.1(4) for both for s and u, the definition of h_x, Oseledets Multiplicative Ergodic Theorem, the definition of Γ_x and Corollary 4.3.

LEMMA 4.5. *For μ-a..e. point x the following holds:*

(1) There is a cocycle $\hat{\alpha}_x : \mathbb{Z}^{m-1} \times R \to \mathscr{D}_m$ such that $h_{\alpha(\mathbf{n})(x)} \circ \hat{\alpha}_x(\mathbf{n}) = \alpha(\mathbf{n}) \circ h_x$, where $R \subset \mathbb{R}^m$ is a set of full Lebesgue measure.
(2) $|\hat{\alpha}_x(k\mathbf{n})|^{1/k} \to D(\mathbf{n}) = \mathrm{diag}(\exp\chi_1(\mathbf{n}), \dots, \exp\chi_m(\mathbf{n}))$, as $k \to \pm\infty$.[4]
(3) For any $\mathbf{n} \in \mathbb{Z}^{m-1}$, $\hat{\alpha}_x(\mathbf{n})\Gamma_x = \Gamma_{\alpha(\mathbf{n})(x)}\hat{\alpha}_x(\mathbf{n})$.

PROPOSITION 4.6. *For μ-a.e. point $x \in M$, Γ_x contains a normal subgroup of finite index isomorphic to $\mathbb{Z}^m$ acting by translations on $\mathbb{R}^m$.*

Proof. Given x let $Tr_x \subset \Gamma_x$ be the normal subgroup of translations in Γ_x. We always regard $\mathbb{R}^m$ with the standard inner product. Let $E(x) \subset \mathbb{R}^m$ be the vector space generated by the translations in Tr_x. Lemma 4.4 implies that Tr_x is discrete and hence the quotient $E(x)/Tr_x$ is a torus. For, if Tr_x contains a sufficiently short vector γ this would imply by the statement (1) of the lemma that the sets K_ϵ and $K_\epsilon + \gamma$ overlap contradicting the statement (2).

Let $v(x)$ be the volume of $E(x)/Tr_x$. Notice that $y \to v(y)$ is a measurable map. Indeed, by Lemma 4.2 for a.e. y there is an $L_{x,y}$ such that $h_x \circ L_{x,y} = h_y$, and by the construction one can choose $L_{x,y}$ in such a way that $x \to L_{x,y}$ is measurable. Let $D_{x,y}$ be the linear part of $L_{x,y}$. A direct calculation shows that $D_{x,y}Tr_y = Tr_x$ which gives the measurability of $y \to v(y)$.

On the other hand, by Lemma 4.5 we have for any $\mathbf{n} \in \mathbb{Z}^{m-1}$ and a.e. $x \in M$,

$$\hat{\alpha}_x(\mathbf{n})Tr_x = Tr_{\alpha(\mathbf{n})(x)}. \tag{4.4}$$

Let $D(\mathbf{n}) = \mathrm{diag}(\exp\chi_1(\mathbf{n}), \dots, \exp\chi_m(\mathbf{n}))$.

Let y be a typical point. Let $d = \dim E(y)$. It follows from the (4.4) and ergodicity of α that d does not depend on y. Let us assume by contradiction that $0 < d < m$ then, since the Lyapunov exponents of α are in general position we can chose an element $\mathbf{n} \in \mathbb{Z}^{m-1}$ such that for $l \to +\infty$ the action of the iterates $D(l\mathbf{n})$ in the dth exterior product $\Lambda^d(\mathbb{R}^m)$ expands the volume element of $E(y)$ exponentially. Since the cocycle $\hat{\alpha}_y(l\mathbf{n})$ is asymptotically $D(l\mathbf{n})$ we have that $\hat{\alpha}(l\mathbf{n})_y$ also expands the volume element of $E(y)$ exponentially. Hence for $l \to +\infty$, $v(\alpha(l\mathbf{n})(y)) \to \infty$, a contradiction since by recurrence $\alpha(l\mathbf{n})(y)$ has to return to a region where $v(y)$ is finite.

Hence either $d = 0$ or $d = m$.

If $d = 0$ then Γ_x has no translation part and by considering the homomorphism $D_0 : \Gamma_x \to \mathscr{D}_m$, where D_0L is the derivative of L at 0, we deduce that Γ_x is abelian. So, Γ_x is conjugate by a translation to the action of a diagonal subgroup on $\mathbb{R}^m$.

Let $F(x) = \mathit{Fix}(\Gamma_x)$ be the set of points fixed by all the elements in Γ_x. Observe that $F(x)$ is an affine subspace parallel to some coordinate plane. We shall

[4] For a diagonal matrix D, $|D|$ is the matrix with entries its absolute values and $|D|^{1/k}$ is its real positive k-th root.

show that $0 \in F(x)$ for μ-a.e. x (i.e., $F(x)$ is a linear subspace) and then reach a contradiction.

From Lemma 4.5 we get that $F(\alpha(\mathbf{n})(x)) = \hat{\alpha}_x(\mathbf{n})F(x)$. By ergodicity of $\alpha(\mathbf{n})$ we get that there is a coordinate plane F, independent of x and a unique vector $p(x)$ perpendicular to F such that $F(x) = p(x) + F$ for μ-a.e. x. Moreover $p(x)$ is measurable, $\hat{\alpha}_x(\mathbf{n})F = F$ and $\hat{\alpha}_x(\mathbf{n})p(x) = p(\alpha(\mathbf{n})(x))$ for every $\mathbf{n} \in \mathbb{Z}^{m-1}$. By Lemma 4.5 the cocycle $\hat{\alpha}_x(\mathbf{n})$ is diagonal and $D(\mathbf{n}) = \operatorname{diag}(\exp\chi_1(\mathbf{n}), \ldots, \exp\chi_m(\mathbf{n}))$ asymptotically, so we deduce, considering each coordinate of $p(x)$ at a time, that $p(x) = 0$.

Thus $F(x) = F$ is a linear subspace independent of x and $0 \in F = F(x)$. In particular $\Gamma_x \subset \mathcal{D}_m$. Since $h_x \restriction_{\mathbb{R}^s \times \{0\}}$ coincides with the affine parameter along the stable manifold of x and h_x is injective on a local stable manifold of x we deduce that $\mathbb{R}^s \times \{0\} \subset F$. Similarly we get that $\{0\} \times \mathbb{R}^u \subset F$ and hence $F = \mathbb{R}^m$, i.e., $\Gamma_x = \{id\}$ is trivial.

In particular, by Corollary 4.3 we get that $h_x : \mathbb{R}^m \to M$ is one-to-one Lebesgue a.e. Let $\nu = (h_x)_*\mu$. By Lemma 4.4, ν is a a probability measure equivalent to Lebesgue measure and invariant by the action $\alpha_0(\mathbf{n}) := h_x^{-1} \circ \alpha(\mathbf{n}) \circ h_x$. By Lemmas 4.2 and 4.5 we have that $\alpha_0(\mathbf{n})$ is affine for every $\mathbf{n}$. But this is a contradiction since affine maps on $\mathbb{R}^m$ do not admit positive entropy invariant probability measures but $(\alpha_0(\nu\mathbf{n}), \nu)$ is measurably isomorphic through h_x to $(\alpha(\mathbf{n}), \mu)$.

So $d = m$ and hence $E(x) = \mathbb{R}^m$. Recall that from Lemma 4.4 we know that Tr_x is discrete. Let us take a linear map and conjugate Tr_x to $\mathbb{Z}^m$ and Γ_x to Γ. Since Tr_x is normal in Γ_x then we have that $\mathbb{Z}^m$ is normal in Γ. Hence we have that $\hat{\Gamma} = \Gamma/\mathbb{Z}^m$ is identified with a subgroup of affine maps on the torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$, and $\mathbb{R}^m/\Gamma_x \sim \mathbb{R}^m/\Gamma = \mathbb{T}^m/\hat{\Gamma}$. Again, using Lemma 4.4 we have that $\hat{\Gamma}$ cannot have any recurrence and hence it has to be finite, finishing the proof. □

Thus $\mathbb{R}^m/\Gamma_x$ is a well defined orbifold. Since Γ_x acts by volume preserving transformations $h_x : \mathbb{R}^m/\Gamma_x \to M$ is an isomorphism of measure spaces.

Let $\alpha_0 : \mathbb{Z}^{m-1} \to \mathit{Aff}(\mathbb{R}^m/\Gamma_x)$ be the abelian action defined by conjugating $\alpha(\mathbf{n})$ with h_x.

$$h_x \circ \alpha_0(\mathbf{n}) = \alpha(\mathbf{n}) \circ h_x \tag{4.5}$$

for any $\mathbf{n} \in \mathbb{Z}^{m-1}$. Let $\nu = (h_x)_*\mu$ be the pullback measure.

Corollary 4.7. *Γ_x is isomorphic either to $\mathbb{Z}^m$ or to $\mathbb{Z}^m \ltimes \{\pm id\}$, $\nu = \lambda$ is Haar measure (or projected Haar measure) on $L := \mathbb{R}^m/\Gamma_x$ and $h := h_x$ is a measurable conjugacy between (α_0, λ) and (α, μ).*

Proof. The only thing that needs a proof in this corollary is the property on the group and on ν. We know already that $\mathbb{Z}^m$ is a finite index normal subgroup of Γ_x. Let $\tilde{\alpha}_0$ be the lifting of the action α_0 to the finite covering $\mathbb{T}^m$ and let us lift also the measure ν to $\mathbb{T}^m$. By the generic position of the Lyapunov exponents for α we deduce that $\tilde{\alpha}_0$ is a restriction of a maximal Cartan action to a finite index subgroup and hence we get that the lifted measure is absolutely continuous w.r.t.

Lebesgue and invariant and hence is Haar measure. Hence we get the claim on the measure. Again using that $\tilde{\alpha}_0$ is a maximal Cartan action on $\mathbb{T}^m$ we get that the only possibility for $\Gamma_x/\mathbb{Z}^m$ is to be $\{\pm id\}$ and we get the corollary. □

4.3. **Conclusion of proof of Theorem 1.** By the weak-mixing reduction from Section 2 we obtain a set R_1 with $\mu(R_1) > 0$ and a finite index subgroup stabilizing R_1. By restricting the action to this finite index subgroup and normalizing the measure we may assume that the measure is weak-mixing and hence by Lemma 2.1 we get that there is a set of full measure R_2 such that for any $x \in R_2$,

$$R_3 := \bigcup_{z \in \mathscr{W}^u_{\mathscr{C}}(x) \cap R} \mathscr{W}^s_{\mathscr{C}}(z)$$

is a set of full measure.

As a consequence of the construction of h_x we see that the image of h_x contains R_3 and hence has full measure, and hence $h_x : (\mathbb{R}^m/\Gamma_x, \nu) \to (M, \mu)$ is an isomorphism conjugating α with α_0. Corollary 4.7 implies that $\mathbb{R}^m/\Gamma_x$ is either a torus or the infratorus $\mathbb{T}^m_{\pm}$.

Take some x and define $h = h_x^{-1}$. This gives the first part and items (1) and (2) of Theorem 1. Item (3) follows from construction, i.e., h^{-1} restricted to the affine spaces parallel to the axes is affine parameters of corresponding stable manifold.

Finally item (4) is a consequence the Journé Theorem, see [5, Theorem 5.7 and Proposition 5.13] and item (3). More precisely, consider a Pesin set Λ of large measure and take the set $W^u_{loc}(\Lambda) \cup W^s_{loc}(\Lambda)$, where $W^u_{loc}(\Lambda) = \bigcup_{y \in \Lambda} W^u_{loc}(y)$. Restrict h to Σ. Since affine structures and holonomies vary continuously on Λ we have that h is continuous on Σ. Moreover, since stable foliations are Hölder continuous along $W^s(\Lambda)$ we have that derivatives of h along the stable direction are Hölder. Similarly for $W^u(\Lambda)$ and along the unstable foliation. Finally we get by Journé's Theorem that h is smooth in the Whitney sense on $W^s(\Lambda) \cap W^u(\Lambda) \supset \Lambda$.

Observe that we can use h_x and its restriction to planes parallel to the axes as new affine parameters. These are still smooth parameters, and with these new affine parameters holonomies are isometries.

For future use, let us summarize some properties of the measurable conjugacy.

LEMMA 4.8. *There is an α_0-invariant set of full Lebesgue measure $R \subset \mathbb{R}^m/\Gamma_x$ in the infratorus such that for every $v \in R$ the measurable conjugacy h_x restricted to any invariant linear subspace $v + E \subset \mathbb{R}^m/\Gamma_x$ through v coincides with the affine structure on $\mathscr{W}_E(h_x(v)) \subset M$, where $\mathscr{W}_E(y) \subset M$ is the invariant manifold associated to E through y. In particular, for a.e. y and every Weyl chamber $\mathscr{C}$, $h_x^{-1} \restriction_{\mathscr{W}^u_{\mathscr{C}}(y)}$ is a diffeomorphism onto $h_x^{-1}(y) + E^u_{\mathscr{C}}$ (the corresponding unstable plane) and holonomies are isometries in these affine parameters.*

5. ANOSOV ACTIONS

An action $\alpha : \mathbb{Z}^k \to Diff(M)$ is an Anosov action if there is $\mathbf{n_0} \in \mathbb{Z}^k$ such that $\alpha(\mathbf{n_0})$ is an Anosov diffeomorphisms.

THEOREM 3. *Let (α, μ) be an action as in Theorem 1, i.e., a maximal rank action, and assume furthermore that α is an Anosov action. Then α is smoothly conjugate to α_0 and hence M is indeed diffeomorphic to a (standard) torus.*

We shall prove that the measurable conjugacy in Theorem 1 is indeed a homeomorphism.

Proof. Let $x \in M$ be a regular point and consider $h_x : \mathbb{R}^s \times \mathbb{R}^u \to M$ which is defined almost everywhere with respect to Lebesgue measure on $\mathbb{R}^s \times \mathbb{R}^u$. We consider here the Anosov element and take the Weyl chamber containing this Anosov element for the definition of h_x. First of all observe that by definition we get that there are $\epsilon > 0$ and $\delta > 0$ small such that if (z^s, z^u) is δ close to $(0,0)$ then

$$h_x(z^s, z^u) = W^s_\epsilon(h_x(z^s, 0)) \cap W^u_\epsilon(h_x(0, z^u)).$$

This implies that $h_x \restriction_{B_\delta(0,0)}$ is continuous. Now, using Proposition 4.2 we get that for Lebesgue a.e. (w^s, w^u) there is an isometry L such that if $y = h_x(w^s, w^u)$ then $L(0,0) = (w^s, w^u)$ and $h_x \circ L = h_y$ a.e. In particular h_x restricted to the δ neighborhood of (w^s, w^u), $B_\delta(w^s, w^u)$, is also continuous since h_y is continuous when restricted to the δ neighborhood $B_\delta(0,0)$ of $(0,0)$ and $h_x = h_y \circ L^{-1}$ and L is an isometry. Since δ is fixed we get that the union of the δ balls around Lebesgue a.e. point is $\mathbb{R}^s \times \mathbb{R}^u$ and hence h_x is continuous everywhere.

Following the same reasoning as in the proof of Theorem 1 we get that h_x is indeed a covering map, and taking the quotient by the group of deck transformations we get that h_x is a homeomorphisms and a conjugacy between the affine action α_0 on an infratorus and the action α.

Observe that here the infratorus is a manifold. Hence, applying the results in [31] or [32] on global rigidity of maximal Anosov rank actions, we obtain smooth conjugacy. □

6. PROOF OF THEOREM 2

6.1. **Boxes and their iterates.** From Theorem 1 we have a decomposition into weak mixing components, a corresponding finite index subgroup of $\mathbb{Z}^{m-1}$ and a measurable conjugacy $h : (M, \nu) \to (L, \lambda)$ between α and an affine action α_0 when restricted to this finite index subgroup. Here we shall show how h coincides with a continuous onto map from an α-invariant open set O and $L \setminus F$ for some finite α_0-invariant set F satisfying the conclusion of Theorem 2.

The first step is to identify the open set O and the finite set F. Given a Weyl chamber $\mathscr{C}$ and a regular point x we denote by $\mathscr{W}^\sigma_\mathscr{C}(x)$, $\sigma = s$ or u, the global stable and unstable manifolds through x corresponding to this Weyl chamber. If a regular Pesin set is fixed we will denote the local invariant manifold by $W^\sigma_{\mathscr{C},loc}(x)$.

Let $\mathscr{C}_1,\ldots,\mathscr{C}_m$ denote the Weyl chambers with only one positive exponent and hence stable mnanifolds have codimension one.

DEFINITION 2. We say that a closed set $B\subset M$ is a *box* or a *cube* if it is homeomorphic to the unit cube in $\mathbb{R}^m$ and its boundary ∂B is in the union of stable and unstable manifolds for different Weyl chambers, i.e., there are regular points $x^{i,\pm}$, $i=1,\ldots,m$, such that

$$\partial B\subset\bigcup_{i=1}^{m}\left(\mathscr{W}^s_{\mathscr{C}_i}(x^{i,-})\cup\mathscr{W}^s_{\mathscr{C}_i}(x^{i,+})\right).$$

We shall call each piece

$$\partial^{\pm}_{\mathscr{C}_i}B_l:=\partial B_l\cap\mathscr{W}^s_{\mathscr{C}_i}(x^{i,\pm})$$

a *face* of the cube B (or of its boundary ∂B). We assume that x_i^+ and x_i^- do not belong to the same stable manifold; if not, take connected components.

DEFINITION 3. Given a Pesin set P, if we can take $x_l^{i,\pm}\in P$ close enough to each other so that

$$\partial B\subset\bigcup_{i=1}^{m}\left(W^s_{\mathscr{C}_i,loc}(x^{i,-})\cup W^s_{\mathscr{C}_i,loc}(x^{i,+})\right),$$

then we say that B is a *good box* and we get as a consequence that

$$\partial^{\pm}_{\mathscr{C}_i}B=\partial B\cap W^s_{\mathscr{C}_i,loc}(x^{i,\pm}).$$

LEMMA 6.1. *For any given Pesin set P and for ν a.e. point $x\in P$ there is a sequence of good boxes B_l, $l\geq 1$, such that:*

(1) $x\in B_l\subset \operatorname{int}B_{l-1}$ and $\bigcap_{l\geq 1}B_l=\{x\}$,
(2) B_l is diffeomorphic to the closed unit cube,
(3) Each connected component of $W^u_{\mathscr{C}_i,loc}(x)\smallsetminus\{x\}$ intersects a corresponding face of $\partial^{\pm}_{\mathscr{C}_i}B_l\neq\emptyset$,
(4) h is defined a.e. w.r.t. Lebesgue measure on ∂B_l and coincides with a diffeomorphism with C^r norm bounded by a constant depending only on P and $h(\partial B_l)$ is the boundary of a linear cube $\hat{B}_l$,
(5) For $i=1,\ldots m$, $W^s_{\mathscr{C}_i,loc}(x)$ disconnects B_l into two connected components named $B^{\pm}_{i,l}$ which are also boxes and $h(\partial B^{\pm}_{i,l})$ is the boundary of a corresponding linear cube $\hat{B}^{\pm}_{i,l}$.

Moreover, the points $x_l^{i,\pm}\in P$ can be further required to belong to a given full measure set (e.g., has a dense orbit in the support of ν).

Proof. Let x be a density point on the Pesin set P intersected with the set of full measure in Lemma 4.8. Since $W^s_{\mathscr{C}_i,loc}(x)$ locally separates a neighborhood of x into two connected components, we can take the points $x_l^{i,\pm}$ from the same set as x and from both sides of $W^s_{\mathscr{C}_i,loc}(x)$, approaching x.

Parts (1), (2) and (3) follow from uniformity of foliations on Pesin sets. Parts (4) and (5) are a consequence Lemma 4.8 and uniformity on Pesin sets (Luzin set). □

Given a good box B, for $\sigma = s$ or u, let

$$W^{\sigma}_{\mathscr{C}_i,B}(x) = B \cap W^{\sigma}_{\mathscr{C}_i,loc}(x).$$

Let $\mathscr{W}^{(u,\pm)}_{\mathscr{C}_i}(x)$ be the connected component of $\mathscr{W}^{u}_{\mathscr{C}_i}(x) \smallsetminus \{x\}$ that intersects $\partial^{\pm}_{\mathscr{C}_i} B$ and for a regular point $y \in \mathscr{W}^{s}_{\mathscr{C}_i}(x)$ we define $\mathscr{W}^{u,\pm}_{\mathscr{C}_i}(y)$ accordingly. Finally, for $r > 0$ let $W^{u,\pm}_{\mathscr{C}_i,r}(y)$ be the segment inside $\mathscr{W}^{u,\pm}_{\mathscr{C}_i}(y)$ of length r with respect to the affine parameters given by h (see Lemma 4.8) with one endpoint y.

Let us fix a point x as in Lemma 6.1, and $l \geq 1$. We shall omit the subscript l in B_l in the sequel. Define

$$O = \bigcup_{\mathbf{n}\in\Gamma} \alpha(\mathbf{n})(int B).$$

The corresponding set

$$\hat{O} = \bigcup_{\mathbf{n}\in\Gamma} \alpha_0(\mathbf{n})(int \hat{B}),$$

where the linear cube $\hat{B}$ is defined in Lemma 6.1(4), is an open nonempty α_0-invariant set. By Berend's Theorem [2] we get that the set $\hat{O}$ is the complement to a finite α_0-invariant set F. Observe that singular points of the infratorus are contained in F since points in $L \smallsetminus F$ have a cube neighborhood. We may also assume that

$$O = \bigcup_{\mathbf{n}\in\Gamma} \alpha(\mathbf{n})(B) \quad \text{and} \quad \hat{O} = L \smallsetminus F = \bigcup_{\mathbf{n}\in\Gamma} \alpha_0(\mathbf{n})(\hat{B})$$

because the faces of the boundary of B (respectively of $\hat{B}$) are formed by stable manifolds of different elements of the action passing trough points which can be taken to have dense orbit on the support of the measure and hence each face of the boundary is mapped eventually completely inside $int B$ (respectively $int \hat{B}$).

For a point x as in Lemma 6.1, let $r^{i,\pm}$ be the length of the connected component of $W^{u}_{\mathscr{C}_i,B}(x) \cap B^{\pm} = W^{u}_{\mathscr{C}_i,B}(x) \cap \mathscr{W}^{(u,\pm)}_{\mathscr{C}_i}(x)$ measured with respect to the affine parameter in $\mathscr{W}^{u}_{\mathscr{C}_i}(x)$ (i.e., $W^{u}_{\mathscr{C}_i,B}(x) \cap B^{\pm} = W^{u,\pm}_{\mathscr{C}_i,r^{i,\pm}}(x)$).

The following lemma is crucial since it does not use uniformity on Pesin sets and hence allows us to go from a property valid on a positive measure set to one valid almost everywhere.

LEMMA 6.2. *For $1 \leq i \leq m$ and for ν a.e. x and for any full Lebesgue measure subset $R \subset W^{s}_{\mathscr{C}_i,B}(x)$,*

$$\bigcup_{z\in R} W^{u,\pm}_{\mathscr{C}_i,r^{i,\pm}}(z) = B^{\pm} \pmod 0$$

w.r.t. measure μ. In particular, for μ a.e. point in $y \in B$,

$$W^{u}_{\mathscr{C}_i,B}(y) \pitchfork W^{s}_{\mathscr{C}_i,B}(x) \neq \emptyset.$$

Proof. The assertion on the transverse intersection is an immediate consequence of the first assertion. The first assertion is an immediate consequence of Lemma 4.8, that h is a measurable conjugacy between ν and λ, and that the same assertion for the linear case is trivial. □

Let $E^{\sigma}_{\mathscr{C}_j}$, $\sigma = s$ or u, $j = 1, \ldots, m$, be the corresponding stable and unstable invariant spaces for the linear action. Let us use the same notation for their projection on the infratorus L. Observe that as long as $z + E^{\sigma}_{\mathscr{C}_j} \subset \mathbb{R}^m$ does not contain a point corresponding to a singular point of the infratorus, the natural projection p from $z + E^{\sigma}_{\mathscr{C}_j}$ into L is one-to-one and onto the corresponding affine space $E^{\sigma}_{\mathscr{C}_j}(p(z))$.

Given a box $\hat{B}$ as in Lemma 6.1 and $\hat{y} \in \hat{B}$, let $E^{\sigma}_{\mathscr{C}_j, \hat{B}}(\hat{y})$ be the connected component of $E^{\sigma}_{\mathscr{C}_j}(\hat{y}) \cap \hat{B}$ containing $\hat{y}$. Given a regular point $y \in B$ recall that $W^{\sigma}_{\mathscr{C}^i, B}(y)$ is the connected component of $B \cap \mathscr{W}^{\sigma}_{\mathscr{C}_i}(y)$ containing y.

LEMMA 6.3. *For μ a.e. point $y \in B$, $W^s_{\mathscr{C}_i, B}(y)$ is a k-dimensional box and*

$$h(W^s_{\mathscr{C}_i, B}(y)) = E^s_{\mathscr{C}_i, \hat{B}}(h(y)).$$

Moreover, for ν a.e. $y, z \in B$ with $z \in W^u_{\mathscr{C}_i, B}(y)$, $\mathrm{Hol}^s_{y,z} : \mathscr{W}^s_{\mathscr{C}_i}(y) \to \mathscr{W}^s_{\mathscr{C}_i}(z)$ is such that

$$\mathrm{Hol}^s_{y,z}(W^s_{\mathscr{C}_i, B}(y)) = W^s_{\mathscr{C}_i, B}(z).$$

Finally, $W^s_{\mathscr{C}_i, B}(y)$ separates B in two connected components, homeomorphic to boxes.

Proof. The first assertion follows from Lemma 4.8 and the constructions of the boxes in Lemma 6.2. The second is a direct consequence of the first and the same property for the linear case.

Finally, let us prove the third assertion. From Lemma 6.1 and the first part we get that $h(\partial B_l \cup W^s_{\mathscr{C}_i, B}(y)) = \partial \hat{B} \cup E^s_{\mathscr{C}_i, \hat{B}}(h(y))$ and on this domain h is a diffeomorphism by Lemma 4.8.

Taking B small enough so that it is in a neighborhood chart and using Schönflies Theorem [1, 4, 26, 27] we deduce that the pair $(B, W^s_{\mathscr{C}_i, B}(y))$ is homeomorphic to the pair $(I^m, I^{m-1} \times \{1/2\})$.[5] □

LEMMA 6.4. *Given a set R of full measure, for every $y \in B$ there is a sequence of boxes $y \in B_{n+1}(y) \subset \operatorname{int} B_n(y)$, $n = 1, 2, \ldots$, such that*

1. *$\partial B_n(y)$ is contained in the union of stable manifolds for different Weyl chambers through points from R;*
2. *for each $n \geq 1$, $h(\partial B_n(y))$ is the boundary of a parallelepiped; moreover, $\mathrm{diam}(h(\partial B_n(y))) \to 0$ as $n \to \infty$.*

Proof. This is an immediate consequence of Lemma 6.3 Since we can use standard binary subdivision of boxes. □

[5] In our earlier paper [20] we constructed similar boxes in the case of actions on the torus. Those boxes are also homeomorphic but may be not diffeomorphic to the cube.

6.2. **Conclusion of proof of Theorem 2.** From Lemma 6.4 it follows that h extends uniquely to a continuous map from B onto $\hat{B}$. Indeed for $y \in B$ take the nested sequence from Lemma 6.4 and define $h(y)$ to be the limit point of $h(\partial B_n(y))$. Continuity follows since the preimage of the box bounded by $h(\partial B_n(y))$ is a neighborhood of y for every $n \geq 0$.

From the definition of O and $L \smallsetminus F$ we get that h extends uniquely to a continuous surjective map $h : O \to L \smallsetminus F$ such that $\alpha_0 \circ h = h \circ \alpha$. Moreover, from Lemma 6.4 it also follows that for any $z \in L \smallsetminus F$, $h^{-1}(z)$ is the nested intersection of boxes and for λ a.e. $z \in L$ this nested intersection is a point by Lemma 6.1.

To complete the the proof of Theorem 2 we shall show that the restriction of h to a suitable $(m-1)$-dimensional skeleton is a diffeomorphism and that this restriction extends to a homeomorphism of O onto $L \smallsetminus F$.

We have the following topological lemma for the infratorus.

LEMMA 6.5. *Given $\epsilon > 0$ and a box $\hat{B} \subset L \smallsetminus F$ as in Lemma 6.1 there is a bounded subset $K \subset \mathbb{Z}^{m-1}$, $R > 0$ and a partition by rectangles C_i, $i = 1, \ldots, r$, of the complement of some neighborhood of the singularities $L_\epsilon := \bigcup_{1 \leq i \leq r} C_i$, such that*

(1) $\operatorname{diam} C_i < \epsilon$,
(2) $C_i \cap C_j \subset \partial C_i \cap \partial C_j$ *for* $i \neq j$,
(3) $C_i \subset \alpha_0(\mathbf{n})(\hat{B})$ *for some* $\mathbf{n} \in K$,
(4) $\partial C_i \subset \bigcup_{\mathbf{n} \in K, a \in \pm, 1 \leq j \leq m} \alpha_0(\mathbf{n})((E^s_{\mathscr{C}_j, R} + \partial^a_{\mathscr{C}_j}(\hat{B}))),$
(5) $L \smallsetminus L_\epsilon \subset \bigcup_{z \in F} B_\epsilon(z),$
(6) *int* L_ϵ *is homeomorphic to* $L \smallsetminus F$.

Proof. Consider $L \smallsetminus \bigcup_{z \in F} B_\epsilon(z)$ and the covering of this compact set by the iterates of $\hat{B}$, $\alpha_0(\mathbf{n})(\hat{B})$, $\mathbf{n} \in \mathbb{Z}^{m-1}$. Take a finite subcover, i.e., a finite subset $K \subset \mathbb{Z}^{m-1}$ so that $\alpha_0(\mathbf{n})(\hat{B})$, $\mathbf{n} \in K$ also covers. Now, $L_\epsilon = \bigcup_{\mathbf{n} \in K} \alpha_0(\mathbf{n})(\hat{B})$ admits a partition by rectangles R_i as desired. □

Let $\widehat{Sk} = \bigcup_i \partial C_i$ be the $(m-1)$-dimensional skeleton defined by the partition from Lemma 6.5.

LEMMA 6.6. *h^{-1} restricted to $\widehat{Sk}$ is a diffeomorphism onto a $(m-1)$-dimensional skeleton Sk. Moreover the diffeomorphism $h^{-1} : Sk \to \widehat{Sk}$ extends to a homeomorphism $g_\epsilon : L_\epsilon \to U_\epsilon$ from L_ϵ onto an open subset $U_\epsilon \subset O$, that is a diffeomorphism if $m - 1 = 2, 4, 5, 11, 60$, i.e., $m = 3, 5, 6, 12, 61$, see* [25].

Observe that for other dimensions $m-1$ the possible existence of exotic spheres and hence of nonstandard smooth embeddings of S^{m-1} into $\mathbb{R}^m$ [24, 23] may preclude the possibility of extending h^{-1} diffeomorphically to some cell of the partition.

Proof. It is a consequence of Lemma 4.8 that h^{-1} restricted to $\widehat{Sk}$ is a diffeomorphism. Hence we have a well defined skeleton

$$Sk = h^{-1}(\widehat{Sk}) = \bigcup_i h^{-1}(\partial C_i).$$

Since $C_i \subset \alpha_0(\mathbf{n})(\hat{B})$ for some $\mathbf{n} \in K$ we have that $h^{-1}(\partial C_i) \subset \alpha(\mathbf{n})(B)$ for some $\mathbf{n} \in K$. Hence $h^{-1} : \partial C_i \to \alpha(\mathbf{n})(B)$ is an embedding of the $(m-1)$-dimensional sphere into the an m-dimensional cube $\alpha(\mathbf{n})(B)$. Now, Schönflies Theorem and Alexander trick gives that h^{-1} extends to a homeomorphism. The differentiability part follows from the smooth Schönflies Theorem, valid for $m-1 \neq 3$, plus the nonexistence of exotic embeddings for the given dimensions. □

Lemma 6.6 and the fact that $int\, L_\epsilon$ is diffeomorphic to $L \setminus F$ completes the proof of Theorem 2.

APPENDIX A. NORMAL FORMS

THEOREM 4. *Let $f : M \to M$ be a C^k diffeomorphism $k \geq 2$ preserving a measure μ. Assume that the negative Lyapunov exponents are between $\log \sigma_\mu < \log \lambda_\mu < 0$ and that $\lambda_\mu^2 < \sigma_\mu$. Let s be the dimension of the stable space. Then there is a measurable family of C^k embeddings $H_x : \mathbb{R}^s \to M$ such that for μ a.e. x*

(1) $H_x(\mathbb{R}^s) = W^s(x)$,

(2) $H_x(0) = x$,

(3) $H_{f(x)} \circ L_x = f \circ H_x$ where $L_x = (D_0 H_{f(x)})^{-1} \circ D_x f \circ D_0 H_x$.

Moreover, such a measurable family is essentially unique in the sense that if $\hat{H}_x : \mathbb{R}^s \to M$ is another family of embeddings with properties (1), (2) and (3) then $(H_x)^{-1} \circ \hat{H}_x \in GL(s, \mathbb{R})$.

ADDENDUM 1. *In the case $k = 1 + \alpha$ with $0 < \alpha < 1$, Theorem 4 holds with the family of embeddings $H_x : \mathbb{R}^s \to M$ being $C^{1+\alpha}$ if we assume instead of $\lambda_\mu^2 < \sigma_\mu$ that $\lambda_\mu^{1+\alpha} < \sigma_\mu$.*

Theorem 4 follows from Theorem 5 and Proposition A.2 in Section A.1 and Theorem 6 in Section A.2. For the proof of the Addendum 1 one uses Addendum 2 instead of Theorem 6.

A.1. **Reduction.** In this section we introduce some general facts from Pesin theory that can be found in [3]. Let $f : M \to M$ be a C^k diffeomorphism with an ergodic invariant measure μ. Assume that the negative exponents of f w.r.t. μ are between $\log \sigma_\mu < \log \lambda_\mu < 0$. Let $B_r^s(0)$ be the ball centered at 0 of radius r in $E^s(x)$.

THEOREM 5 (Stable manifold theorem)**.** *For $k > 1$, $k \in \mathbb{R}$, given $\epsilon > 0$ such that $\lambda_\mu e^{20\epsilon} < 1$, there are measurable maps $r_\epsilon : M \to (0, \infty)$, $D_\epsilon : M \to (0, \infty)$ and a measurable family of C^k maps $\gamma_x : B^s_{r(x)}(0) \to (E^s(x))^\perp$ such that*

(1) $r_\epsilon(f^{\pm 1}(x)) \geq e^{-\epsilon} r_\epsilon(x)$ and $D_\epsilon(f^{\pm 1}(x)) \leq e^\epsilon D_\epsilon(x)$,

(2) $\exp(t + \gamma_x(t)) \in W^s(x)$ for $t \in B^s_{r(x)}(0)$,

(3) $\gamma_x(0) = 0$,

(4) $\|\gamma_x\|_{C^k(B^s_{r(x)}(x))} \leq D_\epsilon(x)$.

Moreover, if we define $W_r^s(x) = \{\exp(t + \gamma_x(t)) : t \in B_r^s(x)\}$ then

(5) $f(W_r^s(x)) \subset W^s_{r\lambda_\mu e^\epsilon}(f(x)) \subset W^s_{r(f(x))}(f(x))$ for $0 < r < r(x)$.

Given a linear map A between normed vector spaces we define

$$m(A) = \min_{\|v\|=1} \|Av\|.$$

The existence of Lyapunov adapted norms gives the following lemma. Here we also denote by $B^s_r(0) \subset \mathbb{R}^s$ the ball centered at 0 of radius r and by s the dimension of the stable space for f.

LEMMA A.1. *There is a measurable family of linear maps $A_x : \mathbb{R}^s \to E^s(x)$ such that if we define $L_x = A^{-1}_{f(x)} \circ D_x f \circ A_x$ then*

(1) $\sigma_\mu e^{-\epsilon} \le m(L_x) \le \|L_x\| \le \lambda_\mu e^{\epsilon}$,
(2) $A_x(B^s_2(0)) = B^s_{r(x)}(0)$,
(3) $\frac{1}{D_\epsilon(x)} \le m(A_x) \le \|A_x\| \le D_\epsilon(x)$.

Let us define the measurable family of embeddings $\hat{H}_x : B^s_2(0) \to W^s(x) \subset M$ by

$$\hat{H}_x(t) = \exp\left(A_x(t) + \gamma_x(A_x(t))\right)$$

and define $\hat{F}_x : B^s_2(0) \to B^s_2(0)$ by $\hat{F}_x = \hat{H}^{-1}_{f(x)} \circ f \circ \hat{H}_x$. Observe that $L_x = D_0 \hat{F}_x$ and let us extend $\hat{F}_x$ to a diffeomorphism $F_x : \mathbb{R}^s \to \mathbb{R}^s$ in such a way that $F_x \restriction_{B_1(0)} = \hat{F}_x$ and $F_x \restriction_{(B_2(0))^c} = L_x$. It is not hard to see that one can do it in a measurable way and without distorting too much the C^k norm of $\hat{F}_x$. Moreover, after conjugating with an appropriate family of homotheties measurable on x we can deduce the following:

PROPOSITION A.2. *There is a measurable map $C_\epsilon : M \to (0,\infty)$ such that for a.e. x*

(1) $C_\epsilon(f^{\pm 1}(x)) \le e^{10\epsilon} C_\epsilon(x)$,
(2) $F_x(0) = 0$,
(3) $F_x(t) = L_x + R_x$ *where* $R_x : \mathbb{R}^s \to \mathbb{R}^s$ *has* $R_x(0) = D_0 R_x = 0$,
(4) $\sigma_\mu e^{-\epsilon} \le m(L_x) \le \|L_x\| \le \|F_x\|_1 \le \lambda_\mu e^{\epsilon}$,
(5) $\|R_x\|_k \le C_\epsilon(x)$ *for a.e.* x.

It is to obtain the last inequality in item (4) that we use the homotheties.

A.2. **Main technical result on normal forms.** Let $f : \Omega \to \Omega$ preserve a measure μ and let $F : \Omega \to \mathit{Diff}^k(\mathbb{R}^s, 0)$ be a measurable map. Let us denote $F_x = F(x)$ and assume that $F_x(t) = L_x(t) + R_x(t)$, where $R_x(0) = D_0 R_x = 0$ and $L_x \in GL(d, \mathbb{R})$.

THEOREM 6 (Normal form)**.** *Let f and F be as in the previous paragraph. Assume $k \ge 2$ and $\epsilon > 0$ is small enough. Assume that there is a measurable function $C_\epsilon : \Omega \to (0,\infty)$ such that $C_\epsilon(f^{\pm 1}(x)) \le e^{\epsilon} C_\epsilon(x)$ for μ-a.e. x and that for μ-a.e. x,*

(1) $0 < \sigma \le m(L_x) \le \|L_x\| \le \|F_x\|_1 \le \lambda < 1$,
(2) $\|R_x\|_k \le C_\epsilon(x)$,
(3) $supp(R_x) \subset B_1(0)$,
(4) $\lambda^2 < \sigma$.

Then there is a measurable family $H_x : \mathbb{R}^s \to \mathbb{R}^s$ of C^k diffeomorphisms such that

(i) $H_x(0) = 0$ *and* $DH_x(0) = id$ *for a.e.* x,
(ii) $H_{f(x)} \circ F_x = L_x \circ H_x$.

Moreover, H_x is unique with properties (i) and (ii).

ADDENDUM 2. *In case $k = 1+\alpha$, $0 < \alpha < 1$ in Theorem 6 then the result still holds with $H_x \in C^{1+\alpha}$, if we assume instead of (4) that $\lambda^{1+\alpha} < \sigma$.*

Proof of Theorem 6. Let $F_x^{(n)} = F_{f^{n-1}(x)} \circ \cdots \circ F_x$,

$$F_x^{(n)}(t) = L_x^{(n)}(t) + R_x^{(n)}(t).$$

Then $L_x^{(n)} = L_{f^{n-1}(x)} \circ \cdots \circ L_x$ and

$$R_x^{(n+1)} = \sum_{k=0}^{n} L_{f^{k+1}(x)}^{(n-k)} \circ R_{f^k(x)} \circ F_x^{(k)}.$$

We shall prove that for a.e. x the sequence $(L_x^{(n)})^{-1} \circ F_x^{(n)}$ is a Cauchy sequence in C^k-topology. Note that

$$\begin{aligned}(L_x^{(n)})^{-1} \circ F_x^{(n)} - (L_x^{(n+1)})^{-1} \circ F_x^{(n+1)} &= (L_x^{(n+1)})^{-1} \circ [R_x^{(n+1)} - L_{f^n(x)} \circ R_x^{(n)}] \\ &= (L_x^{(n+1)})^{-1} \circ R_{f^n(x)} \circ F_x^{(n)} \\ &= (L_x^{(n)})^{-1} \circ \left((L_{f^n(x)})^{-1} R_{f^n(x)}\right) \circ F_x^{(n)}.\end{aligned}$$

Hence, applying Proposition A.3 below with $r = 2$, $L_n = L_{f^n(x)}, E_n = R_{f^n(x)}$ and $S_n = L_n^{-1} E_n$ we obtain, since $\|S_n\|_k \le \lambda C_\epsilon(x) e^{\epsilon n}$, that

$$\begin{aligned}\|(L_x^{(n)})^{-1} \circ F_x^{(n)} - (L_x^{(n+1)})^{-1} \circ F_x^{(n+1)}\|_k &= \|(L_x^{(n)})^{-1} \circ S_n \circ F_x^{(n)}\|_k \\ &\le K_{d,r,k} B_\epsilon^{rQ} n^{rQ} \left(\frac{(\lambda e^{Q\epsilon})^2}{\sigma}\right)^n \|S_n\|_k \\ &\le K_x \left(\frac{(\lambda e^{(Q+2)\epsilon})^2}{\sigma}\right)^n.\end{aligned}$$

Taking ϵ small enough we get that $\frac{(\lambda e^{(Q+2)\epsilon})^2}{\sigma} < 1$ and hence the sequence is Cauchy.

Since the space of C^k functions is complete, we obtain that the sequence is convergent. Measurability survives pointwise convergence hence

$$H_x = \lim_{n\to+\infty} (L_x^{(n)})^{-1} \circ F_x^{(n)}$$

is a measurable family of C^k maps. That it satisfies properties (i) and (ii) is straightforward, and uniqueness is routine. □

Proof of Addendum 2. Instead of using Proposition A.3 in the previous proof, we use Addendum 3 and we get this case as well. □

A.3. **Proposition A.3.** Given a sequence of diffeomorphisms ϕ_n, $n \ge 0$, we denote by $\phi^{(n)}$ their composition

$$\phi^{(n)} = \phi_{n-1} \circ \cdots \circ \phi_0.$$

We use the notation $m(L) = \|L^{-1}\|^{-1}$ for a linear operator.

PROPOSITION A.3. *Let $1 \le r \le k$, $k \ge 2$ and let $L_n \in GL(d,\mathbb{R})$, $E_n : \mathbb{R}^d \to \mathbb{R}^d$ be C^k, $\phi_n : \mathbb{R}^d \to \mathbb{R}^d$ given by $\phi_n = L_n + E_n$ be diffeomorphisms and $S_n : \mathbb{R}^d \to \mathbb{R}^d$, C^k be such that*

1. $0 < \sigma \le m(L_n) \le \|L_n\| \le \|\phi_n\|_1 \le \lambda < 1$,
2. $S_n(0) = D_0 S_n = D_0^i S_n = 0$ *for every* $0 \le i \le r-1$,
3. $\|E_n\|_k \le B_\epsilon e^{\epsilon n}$ *where* $B_\epsilon \ge 1$ *and* $\lambda e^{Q\epsilon} < 1$, *$Q$ from Lemma A.4.*

Then there is a constant $K_{d,r,k}$ depending on d, r, and k such that

$$\|(L^{(n)})^{-1} \circ S_n \circ \phi^{(n)}\|_{C^k} \le K_{d,r,k} B_\epsilon^{rQ} n^{rQ} \left(\frac{(\lambda e^{Q\epsilon})^r}{\sigma} \right)^n \|S_n\|_k.$$

ADDENDUM 3. *If in Proposition A.3 we assume $k = 1+\alpha$, $0 < \alpha < 1$ and $r = 2$, then we get that there are constant $Q > 0$ and $K_{d,k} > 0$ such that*

$$\|(L^{(n)})^{-1} \circ S_n \circ \phi^{(n)}\|_{C^k} \le K_{d,k} B_\epsilon^{(1+\alpha)Q} n^{rQ} \left(\frac{(\lambda e^{Q\epsilon})^{1+\alpha}}{\sigma} \right)^n \|S_n\|_k.$$

We will use the following Lemma 6.4 from [7]. Given a sequence of C^k diffeomorphisms of $\mathbb{R}^d$, ϕ_i, $0 \le i \le n-1$, let $N_l = \max_{0 \le i \le n-1} \|\phi_i\|_l$, $l = 1, k$.

The way the lemma is stated in [7] is not accurate, so we are rewriting it with the appropriate correction. The issue is that what appears in [7] is as if $N_1 \ge 1$ (in which case their statement is perfectly correct), but in case $N_1 < 1$ then one needs to change the N_1^k by a $\max\{N_1, N_1^k\}$. We are also majorizing their polynomial by their highest order term times a large constant.

LEMMA A.4 (Lemma 6.4 in [7])**.** *Given $k \ge 1$ and $d \ge 1$ there are constants $C > 0$ and $Q \ge 0$ such that if $\phi_0, \dots \phi_{n-1}$ are in $C^k(\mathbb{R}^d)$ then*

$$\|\phi^{(n)}\|_k = \|\phi_{n-1} \circ \dots \circ \phi_0\|_{C^k} \le C(\max\{N_1, N_1^k\})^n (nN_k)^Q.$$

For $k = 1$, $Q = 0$.

ADDENDUM 4. *If in Lemma A.4 we assume $k = 1+\alpha$, $0 < \alpha < 1$ then there is a constant $C > 0$ such that for n large enough,*

$$\|\phi^{(n)}\|_k = \|\phi_{n-1} \circ \dots \circ \phi_0\|_{C^k} \le C(\max\{N_1, N_1^k\})^n nN_k.$$

Proof of Addendum 4. We will estimate by using induction on n

$$\|\phi^{(n)}\|_k = \max\{\|\phi^{(n)}\|_1, [D\phi^{(n)}]_\alpha\},$$

where $[u]_\alpha = \max_{x \ne y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$. The $\|\phi^{(n)}\|_1$ term is trivial from chain rule. For the other term, assume by induction that we can control it for $n-1$ and let us see it for n.

We shall use that $[u\cdot v]_\alpha \le \hat{C}([u]_\alpha \|v\|_0 + [v]_\alpha \|u\|_0)$ for any α-Hölder functions u and v and that $[u\circ v]_\alpha \le [u]_\alpha \|Dv\|_0^\alpha$.

$$\begin{aligned}
[D\phi^{(n)}]_\alpha &= [(D\phi_{n-1})\circ\phi^{(n-1)}D\phi^{(n-1)}]_\alpha \\
&\le \hat{C}([(D\phi_{n-1})\circ\phi^{(n-1)}]_\alpha \|D\phi^{(n-1)}\|_0 + [D\phi^{(n-1)}]_\alpha \|(D\phi_{n-1})\circ\phi^{(n-1)}\|_0) \\
&\le \hat{C}(N_k \|\phi^{(n-1)}\|_1^\alpha \|\phi^{(n-1)}\|_1) + [D\phi^{(n-1)}]_\alpha N_1 \\
&\le \hat{C}(N_k N_1^{(1+\alpha)(n-1)}) + [D\phi^{(n-1)}]_\alpha N_1 \\
&= \hat{C}(N_k N_1^{k(n-1)}) + [D\phi^{(n-1)}]_\alpha N_1.
\end{aligned}$$

Considering the sequence $B_n = \frac{[D\phi^{(n)}]_\alpha}{N_1^{kn}}$ we get that

$$B_n \le \frac{1}{N_1^{k-1}} B_{n-1} + \hat{C}\frac{N_k}{N_1^k},$$

where we can assume that $B_0 = 0$. If $N_1 \le 1$, then we have that

$$B_n \le C\frac{N_k}{N_1^k}\frac{n}{N_1^{(k-1)n}}$$

and hence if n is large enough,

$$[D\phi^{(n)}]_\alpha \le CnN_kN_1^n.$$

If $N_1 \ge 1$, then

$$B_n \le C\frac{N_k}{N_1^k}n$$

and hence if n is large enough

$$[D\phi^{(n)}]_\alpha \le CnN_kN_1^{kn}. \qquad \square$$

LEMMA A.5. *Given $k \ge 1$ let $u:\mathbb{R}^d \to \mathbb{R}^d$ be a C^k function and let $\phi:\mathbb{R}^d \to \mathbb{R}^d$ be a C^k diffeomorphism with $\phi(B_1) \subset B_1$. Then there is a constant $C_{d,k} \ge 1$, depending only on d and k such that*

$$\|u\circ\phi\|_k \le C_{d,k}\|u\|_k(\max\{1, \|\phi\|_k^k\}).$$

ADDENDUM 5. *Lemma A.5 still holds in case $k = 1+\alpha$, $0<\alpha<1$.*

Proof of Lemma A.5. We argue by induction on k. For $k=1$, just use the chain rule. Let us assume it is known for $k-1$ and let us prove it for k.

$$\|u\circ\phi\|_k = \max\{\|u\circ\phi\|_0, \|((Du)\circ\phi)D\phi\|_{k-1}\}$$

so we need to estimate both terms,

$$\|u\circ\phi\|_0 \le \|u\|_0 \le \|u\|_k$$

and

$$\begin{aligned}\|((Du)\circ\phi)D\phi\|_{k-1} &\le L_{d,k-1}\|(Du)\circ\phi\|_{k-1}\|D\phi\|_{k-1}\\ &\le L_{d,k-1}C_{d,k-1}\|Du\|_{k-1}(\max\{1,\|\phi\|_{k-1}^{k-1}\})\|D\phi\|_{k-1}\\ &\le L_{d,k-1}C_{d,k-1}\|u\|_k(\max\{1,\|\phi\|_k^{k-1}\})\|\phi\|_k\\ &\le L_{d,k-1}C_{d,k-1}\|u\|_k(\max\{1,\|\phi\|_k^{k}\})\end{aligned}$$

where the $L_{d,k-1}\ge 1$ in the first inequality follows from Leibniz rule. In the second inequality we applied induction and in the last inequality we observe that $\max\{1,x^{k-1}\}x\le\max\{1,x^k\}$. Take $C_{d,k}=L_{d,k-1}C_{d,k-1}$. □

The same proof works for Addendum 5. Here is an improvement on the previous lemma.

LEMMA A.6. *Given $1\le r\le k$, let $u:\mathbb{R}^d\to\mathbb{R}^d$ be a C^k function with all derivatives of order less that or equal to $r-1$ vanishing at 0. Let $\phi:\mathbb{R}^d\to\mathbb{R}^d$ be a C^k diffeomorphism with $\phi(B_1)\subset B_1$, $\phi(0)=0$. Then there is a constant $C_{d,r,k}\ge 1$, depending only on d, r and k such that*

$$\|u\circ\phi\|_k\le C_{d,r,k}(\max\{1,\|\phi\|_k^k\})\|\phi\|_k^r\|u\|_k.$$

ADDENDUM 6. *Lemma A.6 still holds in case $k=1+\alpha$, $0<\alpha<1$, $r=2$, with the outcome that there is a constant $C_{d,\alpha}>0$ such that*

$$\|u\circ\phi\|_k\le C_{d,\alpha}(\max\{1,\|\phi\|_k^k\})\|\phi\|_k^{1+\alpha}\|u\|_k.$$

Proof of Lemma A.6. We argue as before and by induction on k. For $k=1$ it follows by chain rule. Assume it holds for $k-1$ and every $1\le r\le k-1$ and let us prove it for k and any $1\le r\le k$.

$$\|u\circ\phi\|_k=\max\{\|u\circ\phi\|_0,\|((Du)\circ\phi)D\phi\|_{k-1}\}$$

so we need to estimate both terms.

$$\|u\circ\phi\|_0\le\|u\|_r\|\phi\|_0^r\le\|u\|_k\|\phi\|_k^r\le(\max\{1,\|\phi\|_k^k\})\|\phi\|_k^r\|u\|_k$$

since all derivatives of u up to order $r-1$ vanishes at 0.

In case $r\ge 2$,

$$\begin{aligned}\|((Du)\circ\phi)D\phi\|_{k-1} &\le L_{d,k-1}\|(Du)\circ\phi\|_{k-1}\|D\phi\|_{k-1}\\ &\le L_{d,k-1}C_{d,r-1,k-1}\|Du\|_{k-1}(\max\{1,\|\phi\|_{k-1}^{k-1}\})\|\phi\|_{k-1}^{r-1}\|D\phi\|_{k-1}\\ &\le L_{d,k-1}C_{d,r-1,k-1}(\max\{1,\|\phi\|_k^{k-1}\})\|\phi\|_k^r\|u\|_k\\ &\le L_{d,k-1}C_{d,r-1,k-1}(\max\{1,\|\phi\|_k^{k}\})\|\phi\|_k^r\|u\|_k.\end{aligned}$$

In the second inequality we applied induction and the fact that Du has all derivatives vanishing up to order $r-2$. In the last inequality we used that $\max\{1,x^{k-1}\}\le\max\{1,x^k\}$ for every x.

When $r = 1$,

$$\begin{aligned}\|((Du)\circ\phi)D\phi\|_{k-1} &\le L_{d,k-1}\|(Du)\circ\phi\|_{k-1}\|D\phi\|_{k-1}\\ &\le L_{d,k-1}C_{d,k-1}\|Du\|_{k-1}(\max\{1,\|\phi\|_{k-1}^{k-1}\})\|D\phi\|_{k-1}\\ &\le L_{d,k-1}C_{d,k-1}(\max\{1,\|\phi\|_{k}^{k-1}\})\|\phi\|_k\|u\|_k\\ &\le L_{d,k-1}C_{d,k-1}(\max\{1,\|\phi\|_{k}^{k}\})\|\phi\|_k\|u\|_k.\end{aligned}$$

We use Lemma A.5 in the second inequality and we use $\max\{1, x^{k-1}\} \le \max\{1, x^k\}$ for every x again in the last one.

Taking $C_{d,r,k} = \max\{1, L_{d,k-1}C_{d,r-1,k-1}\}$ for $2 \le r \le k$ and then taking $C_{d,1,k} = \max\{1, L_{d,k-1}C_{d,k-1}\}$ we obtain the lemma. □

The same proof works for Addendum 6.

Proof of Proposition A.3. Observe that item (3) of Proposition A.3 implies that $\|\phi_i\|_k \le B_\epsilon e^{\epsilon i}$ for every i. By Lemma A.4,

$$\|\phi^{(n)}\|_k \le C\lambda^n (nB_\epsilon e^{\epsilon n})^Q = CB_\epsilon^Q n^Q(\lambda e^{Q\epsilon})^n < 1$$

for n large enough.

By Lemma A.6, if n is large enough so that $\|\phi^{(n)}\|_k < 1$, then

$$\begin{aligned}\|S_n\circ\phi^{(n)}\|_k &\le C_{d,r,k}\|S_n\|_k(\max\{1,\|\phi^{(n)}\|_k^k\})\|\phi^{(n)}\|_k^r\\ &= C_{d,r,k}\|\phi^{(n)}\|_k^r\|S_n\|_k \le C_{d,r,k}(CB_\epsilon^Q n^Q(\lambda e^{Q\epsilon})^n)^r\|S_n\|_k\\ &= (C_{d,r,k}C^r)B_\epsilon^{rQ}n^{rQ}((\lambda e^{Q\epsilon})^r)^n\|S_n\|_k.\end{aligned}$$ □

Proof of Addendum 3. The same proof as for Proposition A.3 works, using Addenda 4 and 6 instead of Lemmas A.4 and A.6. □

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