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Non-uniform measure rigidity for $\mathbb{Z}^k$ actions of symplectic type

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Abstract. We make a modest progress in the nonuniform measure rigidity program started in 2007 and its applications to the Zimmer program. The principal innovation is in establishing rigidity of large measures for actions of $\mathbb{Z}^k$, $k \geq 2$ with pairs of negatively proportional Lyapunov exponents which translates to applicability of our results to actions of lattices in higher rank semisimple Lie groups other than $SL(n, \mathbb{R})$, namely, $Sp(2n, \mathbb{Z})$ and $SO(n, n; \mathbb{Z})$.

1. Introduction

This paper is a part of the non-uniform measure rigidity program that started in [**6**] and continued in [**8**, **10**, **11**]. While we refer the reader to those papers for the motivation and the general outline of the program, several comments are in order.

In the present paper, for the fist time in the non-uniform setting, we are able to deal with the situation where negatively proportional Lyapunov exponents appear. Presence of those exponents constitutes a fundamental difficulty in carrying out the central recurrence argument. In the previous work on the algebraic actions this was handled either by the methods that essentially rely on non-commutativity of stable and unstable foliations (first mentioned in [**13**] and developed in [**2**] and in later papers) that are not applicable in the torus situation, or by using specific Diophantine properties of eigenspaces for algebraic actions on the torus [**3**] that do not extend to the non-algebraic situation. In [**12**] non-uniform measure rigidity was applied to the study of actions of "large" groups. Specifically, we considered actions of finite index subgroups of $SL(n, \mathbb{Z})$ on the torus $\mathbb{T}^n$ with the standard homotopy data, i.e. inducing the same action on the first homology group $\mathbb{Z}^n$ as the standard action by automorphisms. The nonuniform measure rigidity results needed for that concerned actions of Cartan (i.e maximal rank semisimple abelian) subgroups of $SL(n, \mathbb{Z})$ and were taken from [**6**, **10**]. The method is to first consider the restriction of the action to a maximal split Cartan subgroup and to use our results for that action. However the conditions that appeared in our previous papers are too restrictive and essentially applicable only to certain actions of $SL(n, \mathbb{Z})$ and its finite index subgroups. Results of this paper significantly extend applicability

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of our methods to actions of other lattices in higher rank semisimple Lie groups, e.g. $SO(n,n;\mathbb{Z})$ and $Sp(2n,\mathbb{Z})$.

We extensively use terminology and notations from [**6**, **8**, **11**] with proper reminders, and refer to various results form those papers. The reader should keep in mind that some of those results may nominally refer to more restrictive situations than that of the present paper but if slightest modifications of the arguments are needed we provide appropriate explanations. Otherwise the arguments we refer to are directly applicable to the situations at hand.

2. Preliminary results on behavior of semi-conjugacies

2.1. Semi-conjugacies for maps homotopic to infranilmanifold Anosov diffeomorphisms. For reader's convenience we recall the setting from [**11**, Section 2.2]. Let N be a simply connected nilpotent Lie group and A a group of affine transformations of N acting freely that contains a finite index subgroup Γ of translations that is a lattice in N. Then the orbit space $M = N/A$ is a compact manifold that is called an *infranilmanifold*. An automorphism of N that maps orbits of A onto orbits of A generates a diffeomorphism of N/A that is called an infranilmanifold automorphism. If N is abelian i.e. $N = \mathbb{R}^m$, the infranilmanifold N/A is called an *infratorus*.

An action α_0 of $\mathbb{Z}^k$ by automorphisms of an infranilmanifold M is an Anosov action if induced linear action on the Lie algebra $\mathfrak{N}$ of N has all Lyapunov exponents non-zero, or equivalently there is one element of the linear action that is an Anosov automorphism.

Now let α be an action of $\mathbb{Z}^k$ by diffeomorphisms of M such that its elements are homotopic to elements of an Anosov action by automorphisms. We will say that α has *homotopy data* α_0. There may exist *affine* actions with homotopy data α_0 that are not isomorphic to α_0. This happens when α_0 has more than one fixed point and affine action interchanges those fixed points. Notice any affine action with homotopy data α_0 coincides with α_0 on a finite index subgroup $A \subset \mathbb{Z}^k$. There exists an affine action $\tilde{\alpha}$ with homotopy data α_0 and a continuous map $h : M \to M$ homotopic to identity such that

$$h \circ \alpha = \tilde{\alpha} \circ h. \tag{2.1}$$

and hence for $\gamma \in A$

$$h \circ \alpha(\gamma) = \alpha_0(\gamma) \circ h. \tag{2.2}$$

See [**4**, **11**]. The map h is customarily called a *semi-conjugacy* between α and $\tilde{\alpha}$. There are finitely many semi-conjugacies that differ by some translations by elements of the fixed point group of $\tilde{\alpha}$.

2.2. Ledrappier-Young entropy formula and its extensions. Let $g : N \to N$ be a $C^{1+\text{Hölder}}$ diffeomorphism preserving an ergodic measure ν. Let $\chi_1 > \cdots > \chi_s > 0$ be the positive Lyapunov exponents of g w.r.t. ν with associated Oseledets splitting of the unstable distribution $E^u_g = E_1 \oplus \cdots \oplus E_s$.

For $1 \le i \le u$ let us define

$$V^i(x) = \Big\{ y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d\big(g^{-n}(x),\, g^{-n}(y)\big) \le -\chi_i \Big\}.$$

For ν-a.e. x, $V^i(x)$ is a smooth manifold tangent to $\bigoplus_{j\le i} E_j$ and we thus have the flag $V^1 \subset V^2 \subset \cdots \subset V^s$ with $V^s = W^u$, the unstable manifold. In [**16**, Section 9] a class of increasing partitions ξ^i subordinate to V^i is constructed, i.e. $\xi^i(x) \subset V^i(x)$ and $\xi^i(x)$ is a bounded neighborhood of x in $V^i(x)$ for ν-a.e. x. Consider conditional measures ν_x^i associated with those partitions. Let $B^i(x,\varepsilon)$ be the ε ball in $V^i(x)$ centered in x with respect to the induced Riemannian metric. Then

$$\delta_i = \delta_i(g) = \lim_{\varepsilon\to 0} \frac{\log \nu_x^i B^i(x,\varepsilon)}{\log \varepsilon}$$

exists a.e. and does not depend on x. Moreover, writing $\gamma_i = \gamma_i(g) = \delta_i - \delta_{i-1}$ we have the Ledrappier–Young entropy formula (see [**16**, Theorem C])

$$h_\nu(g) = \sum_{1\le j\le s} \gamma_j \chi_j. \tag{2.3}$$

If $h_\nu(g,\xi)$ denotes the entropy w.r.t. the partition ξ, then [**16**] also gives the formula

$$h_\nu(g,\xi^i) = \sum_{1\le j\le i} \gamma_j \chi_j.$$

Thus the entropy does not depend of a particular choice of partition as long as it is increasing and subordinated to V^i.

The following corollary of the result by F. Ledrappier and Jian-Sheng Xie [**14**] provides the following consequence of the vanishing of the leading coefficient γ_s.

PROPOSITION 2.1. *If $\gamma_s = 0$, i.e. $h_\nu(g) = h_\nu(g,\xi^{s-1})$ for some partition ξ^{s-1} subordinated to V^{s-1}, then the conditional measure of ν on almost every leaf of $V^s = W^u$ is supported on a single leaf of V^{s-1}.*

2.3. Non-collapsing of Lyapunov directions under semiconjugacies. Let $f: M \to M$ be a $C^{1+\text{Hölder}}$ diffeomorphism. Assume that g is a factor of f via a continuous surjective map $h: M\to N$ such that $h\circ f = g\circ h$. Let μ be an ergodic invariant measure for f such that $h_*\mu = \nu$. It is important that no assumption on hyperbolicity of the measures be made since we will apply the results below in the setting without any a priori information on the measure μ.

PROPOSITION 2.2. *If $h(W_f^u(x)) \subset V_g^{s-1}(h(x))$ for μ-a.e. x then $\gamma_s^g = 0$ and hence the conditional measure ν^u of ν along W_g^u, is supported on a single leaf of V_g^{s-1}.*

As a corollary we get the following

COROLLARY 2.3. *Let f, g and h be as above. Assume that conditional measures of ν along W_g^u are absolutely continuous w.r.t. Lebesgue on W_g^u then for μ-a.e. x, $h(W_f^u(x))$ is not contained in $V_g^{s-1}(h(x))$. In particular this holds whenever ν is a measure absolutely continuous w.r.t. Lebesgue measure.*

To prove Proposition 2.2 we will use Proposition 2.1 and the following fact from abstract ergodic theory. Here for a partition ξ, we denote $\xi_S = \bigvee_{n\in\mathbb{Z}} S^n\xi$.

PROPOSITION 2.4. [**11**, Proposition 6.2] *Let $T\colon (X,\mu)\to(X,\mu)$ and $S\colon (Y,\nu)\to(Y,\nu)$, be measure preserving transformations and assume that S is a factor of T via a measure-preserving map $p\colon (X,\mu)\to(Y,\nu)$. Let η be a full-entropy partition for T (i.e. $h_\mu(T) = h_\mu(T,\eta)$) and ξ a generating partition for S, i.e. $\xi_S = \varepsilon$ the partition into points and $p^{-1}\xi < \eta$. Then ξ is a full-entropy partition for S, i.e. $h_\nu(S) = h_\nu(S,\xi)$.*

Proof of Proposition 2.2. We shall proof that for some partition ξ^{s-1} subordinated to V^{s-1}, $h_\nu(g,\xi^{s-1}) = h_\nu(g)$ and then the results will follow from Proposition 2.1. We can build as in [**16**] an increasing partition ξ^{s-1} subordinated to V_g^{s-1}. Moreover, we can build again as in [**16**] a partition η subordinated to W_f^u such that $h^{-1}\xi^{u-1} < \eta$. This can be done since $h^{-1}(V^{u-1}(h(x))) \supset W^u(x)$ for μ-a.e. x and h is continuous. Since ξ^{s-1} is subordinated to V^{s-1} we have that $\xi_g^{s-1} = \varepsilon$. Since η is an increasing partition subordinated to W_f^u we have that $h_\mu(f) = h_\mu(f,\eta)$. Hence Proposition 2.4 implies that $h_\nu(g) = h_\nu(g,\xi^{s-1})$. □

3. Formulation of results

3.1. Actions of higher rank abelian groups. We consider an action of $\mathbb{Z}^k$, $k \geq 2$ by diffeomorphisms of a compact manifold. We say that an action has *simple coarse Lyapunov spectrum* if Lyapunov exponents are simple and no pair of Lyapunov exponents are positively proportional.

Let α_0 be an Anosov action of $\mathbb{Z}^k$ by automorphisms of an infranilmanifold with simple coarse Lyapunov spectrum and let α be a smooth action with homotopy data α_0.

Let us call an α-invariant Borel probability measure μ *large* if the push-forward $h_*\mu$ is Haar measure.

Our first result is a generalization of Theorem 2.5 from [**11**]. Here we assume the same regularity as in the previous section, i.e. $C^{1+\beta}, \beta > 0$.

Theorem 3.1. *Let α_0 be a $\mathbb{Z}^k$, $k \geq 2$, Anosov action on an infranilmanifold M with simple coarse Lyapunov spectrum. Let μ be an ergodic large invariant measure for an action α with homotopy data α_0. Then*

(1) μ is absolutely continuous;

(2) Lyapunov characteristic exponents of the action α with respect to μ are equal to the Lyapunov characteristic exponents of the action α_0.

Recall that a *resonance* is a relation between different Lyapunov exponents $\chi_1, \dots, \chi_l$ of a $\mathbb{Z}^k$ action of the form $\chi_1 = \sum_{i=2}^{l} m_i \chi_i$ where $m_2, \dots m_l$ are positive integers.

The following theorem generalizes to the case of simple coarse Lyapunov spectrum Theorems 2.6 and 2.7 and Proposition 7.2 from [**11**].

Theorem 3.2. *Let α_0 be a hyperbolic linear $\mathbb{Z}^k$ action with simple coarse Lyapunov spectrum and without resonances on an infratorus N. Let α be a C^∞ action with homotopy data α_0. Then*

(1) α has a unique large invariant measure μ;

(2) the semiconjugacy p is bijective μ-a.e. and effects a measurable isomorphism between (α,μ) and an affine action $\tilde{\alpha}$ with homotopy data α_0 with Haar measure;

(3) the semiconjugacy is smooth and bijective on μ a.e. stable manifold of any action element;

(4) The semiconjugacy is smooth in Whitney sense on sets of μ measure arbitrary close to one.

Once absolute continuity of μ and smoothness of the semiconjugacy along the Lyapunov foliations is established, all statements of Theorem 3.2 are deduced exactly as in [**11**, Section 7]. In particular, in order to get uniqueness of the large invariant measure we will use smoothness of the semiconjugacy along stable manifolds. In [**11**, Section 7.1] smoothness is deduced from the existence of invariant affine structures and to get those we need to assume high regularity (although not necessarily C^∞) and absence of resonances.

REMARK 1. We think that using higher order normal forms it is also possible to get smoothness of the semiconjugacy and hence allow resonances.

Next we will show existence of proper periodic orbits generalizing to our setting [**10**, Theorem 3.1].

THEOREM 3.3. *Let α be an action as in Theorem 3.2 with μ the large measure. Then there is a periodic point $p \in \operatorname{supp}(\mu)$ such that the semiconjugacy is a diffeomorphism when restricted to the stable and unstable manifolds of p for some element of the action. Moreover, the stable and unstable manifold for that element of the action are in the support of μ, i.e. $\mathcal{W}^s(p) \cup \mathcal{W}^u(p) \subset \operatorname{supp}(\mu)$.*

COROLLARY 3.4. *Derivative of α at p is conjugate to the linear part of α_0.*

3.2. Actions of lattices in higher rank Lie groups on the torus. Let G be a simply connected semisimple Lie group with no compact factors, finite center of $\mathbb{R}$-rank greater than one, and let Γ be a lattice in G. Let ρ be an action of Γ on a torus $\mathbb{T}^m$. Induced action ρ_* on the first homology group can be viewed as an embedding $\Gamma \to GL(n, Z)$ that determines an action ρ_0 by automorphisms of $\mathbb{T}^m$ that, similarly to the abelian case, we call the *homotopy data* of ρ. By Margulis Normal Subgroup Theorem [**17**] either the image of ρ_* if finite or ρ_* has finite kernel.

We will assume that the second alternative takes place. By Margulis Superrigidity Theorem [**17**] the restriction of ρ_* to a finite index subgroup $\Gamma_0 \subset \Gamma$ can be extended to a homomorphism $\tilde{\rho}_* : G \to SL(n, \mathbb{R})$ with discrete kernel. Recall that weights of $\tilde{\rho}_*$ are eigenvalues of the restriction of $\tilde{\rho}$ to a maximal split Cartan subgroup of G. Since there is a maximal split Cartan subgroup of G that intersects $\rho_*(\Gamma_0)$ by a lattice L we may speak about weights of ρ_* that are essentially the exponentials of the Lyapunov characteristic exponents of the action ρ_0 of L by automorphisms of the torus $\mathbb{T}^m$. Thus we can define resonances between weights.

Notice that while individual elements of actions ρ_0 and ρ are homotopic, the actions may not be.

Let us call a compact set $K \subset \mathbb{T}^m$ *large* if there is no continuous map p on the torus homotopic to the identity such that $p(K)$ is a point.

THEOREM 3.5. *Let ρ be a C^ω (real-analytic) action of a lattice Γ in a simply connected semisimple Lie group with no compact factors finite center and of $\mathbb{R}$-rank greater than one on the torus $\mathbb{T}^{2n}$ such that induced action ρ_* on the first homology group has finite kernel. Assume that*

(1) ρ preserves an ergodic measure μ whose support is large;
(2) ρ_* has no zero weights, no multiple weights, no positively proportional weights and no resonances between weights.

Then there is a finite index subgroup $\Gamma' \subset \Gamma$, a finite ρ_0-invariant set F and a bijective real-analytic map

$$H : \mathbb{T}^m \setminus F \to D$$

where D is a dense subset of $\operatorname{supp} \mu$, such that for every $\gamma \in \Gamma'$,

$$H \circ \rho_0(\gamma) = \rho(\gamma) \circ H.$$

Here are two cases where Theorem 3.5 applies. Consider the standard inclusion of $Sp(2n,\mathbb{Z}) \subset SL(2n,\mathbb{Z})$ and the standard inclusion of $SO(n,n;\mathbb{Z}) \subset SL(2n,\mathbb{Z})$ Let Γ be any finite index subgroup of $Sp(2n,\mathbb{Z})$ or $SO(n,n;\mathbb{Z})$ and consider the standard action ρ_0 of Γ on $\mathbb{T}^{2n}$.

Corollary 3.6. *Let $\Gamma \subset Sp(2n,\mathbb{Z})$, $n \geq 2$ or $\Gamma \subset SO(n,n;\mathbb{Z})$, $n \geq 2$ be as above. Let ρ be a real-analytic action of Γ on $\mathbb{T}^{2n}$ with homotopy data ρ_0 preserving an ergodic measure μ whose support is a large set.*

Then there is a finite index subgroup $\Gamma' \subset \Gamma$, a finite ρ_0-invariant set F and a bijective real-analytic map

$$H : \mathbb{T}^m \setminus F \to D$$

where D is a dense subset of $\operatorname{supp}\mu$, such that for every $\gamma \in \Gamma'$,

$$H \circ \rho_0(\gamma) = \rho(\gamma) \circ H.$$

Notice that since Weyl groups of $Sp(2n)$ and $SO(n,n)$ contain the central symmetry, for every finite-dimensional representation of those groups weights come in pairs of opposite sign. Thus the present extension of nonuniform measure rigidity results that allows pairs of simple negatively proportional Lyapunov exponents is crucial for applications to any actions of lattices in those groups. In the standard cases of Corollary 3.6 the weights are simple, there are no positively proportional ones but they still appear in pairs of opposite sign. Notice however that those may be the only new examples compared to the action of finite index subgroups. $SL(n,\mathbb{Z})$ on $\mathbb{T}^n$ with standard homotopy data considered in [**12**]. For example, no new examples appear for $SL(3,\mathbb{Z})$ actions.[1]

The proof of Theorem 3.5 repeats almost verbatim the proof of [**12**, Theorem 1.1] with Theorem 3.2 replacing [**10**, Corollary 2.2] and Theorem 3.3 taking place of [**10**, Theorem 3.1] (that is restated as Theorem 2.4 in [**12**]). The only difference is that in our setting we cannot assert that pre-image of any point under the semi-conjugacy is the intersection of boxes ([**10**, Theorem 2.1]). For reader's convenience in Section 6 we describe all steps of the proof and explain in detail how the above minor difference is handled.

The crucial step where analyticity is used is in the application os the Cairns-Ghys local linearization [**1**]. For smooth actions we have a weaker result parallel to Theorem 3.2 that follows from the initial steps of the proof of Theorem 3.5. We formulate it for the C^∞ case.

Theorem 3.7. *Let ρ be a C^∞ action satisfying the rest of the assumptions of Theorem 3.5, then there exists a finite index subgroup $\Gamma' \subset \Gamma$ and a continuous map $h : \operatorname{supp}\mu \to \mathbb{T}^m$ such that*

(1) for $\gamma \in \Gamma'$, $\rho_0(\gamma) \circ h = h \circ \rho(\gamma)$;
(2) the map h is bijective μ-a.e. and effects a measurable isomorphism between (ρ,μ) and ρ_0 with Lebesgue measure;
(3) the map h is smooth and bijective on μ a.e. stable manifold of any action element;
(4) The map h is smooth in Whitney sense on sets of μ measure arbitrary close to one.

[1] We thank A. Gorodnik for showing that.

REMARK 2. In [**12**] we actually prove a stronger statement in the case of an action of a finite index subgroup Γ of $SL(n, \mathbb{Z})$ on $\mathbb{T}^n$ with standard homotopy data. Namely we show that h above extends to a continuous map $\mathbb{T}^n \to \mathbb{T}^n$ homotopic to identity; in fact it coincides with a semiconjugacy for restriction of ρ to *any* any Cartan subgroup of Γ; see Step 4 in the proof of [**12**, Proposition 2.1].

Notice that in the proof of Theorems 3.5 and 3.7 we restrict the action of Γ to a maximal rank semisimple abelian subgroup and apply to this action our results using only the fact that its homotopy data has simple coarse Lyapunov spectrum and no resonances. However, the fact that this abelian action appears as a restriction of a lattice action implies stronger properties that follow from cocycle super-rigidity. Systematic use of this additional information allows to obtain nonuniform measure rigidity for abelian actions that appear as restrictions of lattice actions under considerably more general assumptions on Lyapunov exponents than those of Theorem 3.2 and hence to extend the assertions of that theorem for those situations.

4. Absolute continuity of μ

While we shall follow here the scheme of proof of previous papers c.f. [**6**, **11**] this section contains new arguments that allow us to deal with negatively proportional Lyapunov exponents.

4.1. Matching of Lyapunov half-spaces. The first step is to prove matching of Lyapunov half-spaces for α and α_0.

We need to prove that the semiconjugacy does not collapse the unstable manifold. This a general result that does not need any assumption on the action α_0:

LEMMA 4.1. [**11**, Lemma 6.3] *If L is a Lyapunov hyperplane (the kernel of a Lyapunov exponent) for α_0 then L is a Lyapunov hyperplane for α and Lyapunov half-spaces match.*

Since the proof of the matching of Lyapunov half-spaces was not completely written in [**11**] we will add it here.

PROOF. Take L a Lyapunov hyperplane for α_0. We have, as proven in [**11**, Lemma 6.3], that it is a Lyapunov hyperplane for α also. We will prove that if χ is a Lyapunov exponent for α_0 such that $\ker \chi = L$ then there is a Lyapunov exponent for α positively proportional to χ. Assume this is not the case. Then any Lyapunov exponent $\tilde{\chi}$ for α with $\ker \tilde{\chi} = L$ is negatively proportional to χ.

Assume χ is such that all α_0 Lyapunov exponents proportional to χ has rate of proportionality smaller than one (i.e. χ is the largest Lyapunov exponent among the positively proportional to it). Take $\mathbf{t} \in L$ such that $\mathbf{t}$ is not in $\ker \hat{\chi}$ for any Lyapunov exponent $\hat{\chi}$ (for α or α_0) not proportional to χ. Take $\mathbf{n}$ a regular element for α_0 close to $\mathbf{t}$ such that $\chi(\mathbf{n}) > 0$ but smaller that any other non-proportional positive α_0-Lyapunov exponent. Then $\tilde{\chi}(\mathbf{n}) < 0$ for any α-Lyapunov exponent $\tilde{\chi}$ proportional to χ. Hence we have that $\mathcal{W}^u_{\alpha(\mathbf{n})}(x) = \mathcal{W}^u_{\alpha(\mathbf{t})}(x)$ for μ a.e. x but $\mathcal{W}^u_{\alpha_0(\mathbf{t})}(y) \subsetneq \mathcal{W}^u_{\alpha_0(\mathbf{n})}(y)$ for every y. Moreover $\mathcal{W}^u_{\alpha_0(\mathbf{t})}$ is inside the fast expanding direction of $\alpha_0(\mathbf{n})$ ($V^{u-1}_{\alpha_0(\mathbf{n})}$ in the notation of Section 2) by the choice of $\mathbf{n}$. So we have that

$$h(\mathcal{W}^u_{\alpha(\mathbf{n})}(x)) = h(\mathcal{W}^u_{\alpha(\mathbf{t})}(x)) \subset \mathcal{W}^u_{\alpha_0(\mathbf{t})}(h(x)) \subset V^{u-1}_{\alpha_0(\mathbf{n})}(h(x))$$

and Corollary 2.3 gives a contradiction with the fact that λ is Lebesgue measure. $\square$

To end with the matching of Lyapunov half-spaces we need to prove the following

PROPOSITION 4.2. *If* α_0 *has simple coarse Lyapunov spectrum then* $\dim M \geq \dim N$ *and if* $\dim M = \dim N$ *then Lyapunov hyperplanes and half-spaces for* α *and* α_0 *coincide. In particular* (α, μ) *has also simple Lyapunov spectrum.*

PROOF. The Proposition is true by counting Lyapunov exponents. Since α_0 has simple coarse Lyapunov spectrum we can list the Lyapunov exponents of α_0 as $\chi_1, \dots, \chi_n$, $n = \dim N$, in such a way that for some $l \geq n/2$, χ_{l+i} is negatively proportional to χ_i for $i = 1, \dots, n-l$ (here we use the convention that $l > n$ if there are no negatively proportional exponents). Lemma 4.1 says that for every χ_i there is an α-Lyapunov exponent $\tilde{\chi}_i$ positively proportional to χ_i, this implies that $\dim M \geq n$. If dimension of M is n, then there cannot be other α Lyapunov exponents and we get the Proposition. $\square$

4.2. A dichotomy for the conditionals along Lyapunov foliations. Let us fix a Lyapunov hyperplane $L = \ker \chi$ for α with associated Lyapunov exponent χ. Take a generic singular element $\mathbf{t} \in \ker \chi$ of the action i.e. such that $\chi(\mathbf{t}) = 0$ and $\chi_j(\mathbf{t}) \neq 0$ for any non-proportional Lyapunov exponent and call $f = \alpha(\mathbf{t})$. Let $\mathcal{W}$ be the Lyapunov foliation associated to χ, put $E_\chi(x) = T_x\mathcal{W}(x)$ and call E the α_0-Lyapunov direction corresponding to the α_0-Lyapunov exponent proportional to χ. In this subsection we shall proof that $\mu_x^{\mathcal{W}}$ is absolutely continuous w.r.t. Lebesgue for μ-a.e. x.

In the sequel η will denote an element of the ergodic decomposition of μ w.r.t. f and $\eta_x^{\mathcal{W}}$ its conditional measures w.r.t. to $\mathcal{W}$. The proof of the next proposition follows, with minor changes, exactly along the lines of section 3 in [**6**]. Observe that π-partition argument is used in [**6**] to get recurrence to the $\mathcal{W}$ leaf (i.e. $\mathcal{W}$ is inside an element of the ergodic partition), instead here we use the ergodicity of η w.r.t. f.

PROPOSITION 4.3. *One of the following holds:*

(1) $\eta_x^{\mathcal{W}}$ *is absolutely continuous w.r.t. Lebesgue for almost every ergodic component* η *and for* η*-a.e.* x *or,*
(2) $\eta_x^{\mathcal{W}}$ *is atomic for almost every ergodic component* η *and for* η*-a.e.* x.

For the proof of Proposition 4.3 let us recall Proposition 3.1 in [**6**]. An affine structure on a manifold is an atlas whose change of variables are affine maps.

PROPOSITION 4.4. [**6**, Proposition 3.1] *There exists a unique measurable family of* $C^{1+\varepsilon}$ *smooth* α*-invariant affine structures on the leaves* $\mathcal{W}(x)$*. Moreover, within a given Pesin set they depend Hölder continuously in the* $C^{1+\varepsilon}$ *topology.*

Once we have affine structure we need to freeze the the dynamics along $\mathcal{W}$ when we iterate f (remember that $f = \alpha(\mathbf{t})$ where $\mathbf{t}$ is in the Weyl chamber wall $L = \ker \chi$). We will prove as in [**6**], the following

LEMMA 4.5. *For* μ *a.e. ergodic component* η*, for every Pesin set* Λ *and for every* $\varepsilon > 0$ *there is a set* $\Lambda_\varepsilon \subset \Lambda$ *and* $K > 0$ *such that* $\eta(\Lambda \setminus \Lambda_\varepsilon) < \varepsilon$ *and*

$$K^{-1} \leq \|Df^k|E_\chi(x)\| \leq K$$

if both x *and* $f^k(x)$ *are in* Λ_ε.

Denote with $B_r^{\mathcal{W}}(x)$ the ball inside $\mathcal{W}(x)$ centered in x and of radius r and denote by $l(A)$ the length of an interval in E or any translate of E. To prove Lemma 4.5 we need to prove first the following:

LEMMA 4.6. *For μ a.e. ergodic component η, for every Pesin set Λ and for every $\varepsilon > 0$ and $r > 0$ there is a set $\Lambda_\varepsilon \subset \Lambda$ and $m > 0$ such that $\eta(\Lambda \setminus \Lambda_\varepsilon) < \varepsilon$ and $l(h(B_r^{\mathcal{W}}(x))) \geq m$.*

PROOF. Let us show first that for μ-a.e. x, length of $h(\mathcal{W}(x))$ is different from 0, i.e. $h(\mathcal{W}(x)) \neq h(x)$. By ergodicity of μ w.r.t. α we get that either for μ-a.e. x, $h(\mathcal{W}(x)) \neq h(x)$ or for μ-a.e. x, $h(\mathcal{W}(x)) = h(x)$. Let us assume by contradiction this latter is the case. Take an element $\mathbf{n}$ close to $\mathbf{t} \in \ker \chi$ such that $\chi(\mathbf{n}) > 0$ but it is still the smallest positive Lyapunov exponent. We have on one hand that $E^u_{\alpha(\mathbf{n})} = E^u_{\alpha(\mathbf{t})} \oplus E_\chi$ and $E^u_{\alpha_0(\mathbf{n})} = E^u_{\alpha_0(\mathbf{t})} \oplus E$. On the other hand, since $h(\mathcal{W}(x)) = h(x)$ for μ-a.e. x by our contradiction assumption, and $\mathcal{W}^u_{\alpha(n)}(x) = \bigcup_{z \in \mathcal{W}(x)} \mathcal{W}^u_{\alpha(t)}(z)$ we get that

$$h(\mathcal{W}^u_{\alpha(n)}(x)) = \bigcup_{z \in \mathcal{W}(x)} h(\mathcal{W}^u_{\alpha(t)}(z)) \subset \bigcup_{z \in \mathcal{W}(x)} \mathcal{W}^u_{\alpha_0(\mathbf{t})}(h(z)) = \mathcal{W}^u_{\alpha_0(\mathbf{t})}(h(x))$$

By the choice of $\mathbf{n}$ we have that $\mathcal{W}^u_{\alpha_0(\mathbf{t})} = V^{u-1}_{\alpha_0(\mathbf{n})}$ (recall that $V^{u-1}_{\alpha_0(\mathbf{n})}$ is the direction of fast expansion as defined in Section 2). Then Corollary 2.3 gives a contradiction with the fact that λ is Lebesgue measure.

So we get that for μ-a.e. x, length of $h(\mathcal{W}(x))$ is different from 0. Taking again $\mathbf{n}$ such that $\chi(n) > 0$ we have that the corresponding α_0-Lyapunov exponent for the linear is also positive and hence $\alpha_0(n)$ expands the E direction. Hence, α_0-invariance of $h(\mathcal{W}(x))$ implies that ones the length is nontrivial it has to be ∞. Finally, take $r > 0$, then if $h(B_r^{\mathcal{W}}(x)) = h(x)$ for a set of positive μ-measure then ergodicity of μ, expansion of $\mathcal{W}$ by $\alpha(\mathbf{n})$ and expansion of the E direction by $\alpha_0(\mathbf{n})$ would imply that $h(\mathcal{W}(x)) = h(x)$ for μ-a.e. x, which is a contradiction. Then, we have that for every $r > 0$, $l(h(B_r^{\mathcal{W}}(x))) > 0$ for μ-a.e. x. Then for μ a.e. ergodic component η and for η a.e. point x, $l(h(B_r^{\mathcal{W}}(x))) > 0$. So, given η, a Pesin set Λ and $\varepsilon > 0$, for m small enough, the set, Λ_ε of $x \in \Lambda$ such that $l(h(B_r^{\mathcal{W}}(x))) \geq m$ has measure $\eta(\Lambda \setminus \Lambda_\varepsilon) < \varepsilon$. ☐

Once we have that h does not collapse the $\mathcal{W}$ "foliation" the proof of Lemma 4.5 follows as in [**6**].

PROOF OF LEMMA 4.5. We shall proof the first inequality, the second one follows taking the inverse. Take r small and Λ_ε as in Lemma 4.6. Take $x \in \Lambda_\varepsilon$ and k such that $f^k(x) \in \Lambda_\varepsilon$. Then we have that

$$l(h(f^k(B_r^{\mathcal{W}}(x)))) = l(\alpha_0(k\mathbf{t})(h(B_r^{\mathcal{W}}(x)))) = l(h(B_r^{\mathcal{W}}(x))) \geq m$$

since $\alpha_0(k\mathbf{t})$ is an isometry along the E direction. Hence, since h is continuous there is $\delta > 0$ that only depends on m and there is $y \in f^k(B_r^{\mathcal{W}}(x))$ such that $d(y, f^k(x)) \geq \delta$, so, if $z = f^{-k}(y)$ we have that $d(x,z) < r$ and $d(f^k(z), f^k(x)) \geq \delta$. Finally, using the affine coordinates and that they vary continuously over Pesin sets we get that $\|Df^k|E_\chi(x)\| \geq \delta C^{-2}/r = K^{-1}$, where C is a bound for the derivative of the affine structure over the Pesin set Λ. ☐

Finally, to end the proof of Proposition 4.3, let us define for η a.e. x the group

$$G_x = \{L : E_\chi(x) \to E_\chi(x) \ \text{affine, s.t.} \ L_*\eta_x = c\eta_x \ \text{for some} \ c > 0\}.$$

Here we identify $E_\chi(x)$ with $\mathcal{W}(x)$ using the affine structure. G_x is a closed subgroup of the affine maps of the line. If G_x is not a discrete group the η_x is absolutely continuous w.r.t. Lebesgue. Using Lemma 4.5 and ergodicity of η we get that $\eta_x(G_x(x)) > 0$ ($G_x(x) = \{L(x) : L \in G_x\}$). So if η_x is not atomic, then G_x is not discrete and hence η_x is absolutely continuous w.r.t. Lebesgue.

4.3. Absolute continuity of conditionals along Lyapunov foliations. In case (1) of Proposition 4.3 we get that $\mu_x^{\mathcal{W}}$ is absolutely continuous w.r.t. Lebesgue for μ-a.e. x. So let us assume by contradiction that we are in case (2).

Let us take $\mathbf{s}$ an element of the action close to $\mathbf{t}$ such that $\chi(\mathbf{s}) < 0$ but still $\chi(\mathbf{s})$ is the closest to zero among the negative Lyapunov exponents of $\alpha(\mathbf{s})$ and no other Lyapunov exponent change sign. We have then that $\mathcal{W}^s_{\alpha(\mathbf{t})} \subset \mathcal{W}^s_{\alpha(\mathbf{s})}$ and in fact $E^s_{\alpha(\mathbf{s})} = E^s_{\alpha(\mathbf{t})} \oplus E_\chi$. The next proposition says that conditional measures of η along$\mathcal{W}^s_{\alpha(\mathbf{s})}$ sits inside $\mathcal{W}^s_{\alpha(\mathbf{t})}$.

PROPOSITION 4.7. $\eta^{\mathcal{W}^s_{\alpha(\mathbf{s})}} = \eta^{\mathcal{W}^s_{\alpha(\mathbf{t})}}$ *for almost every ergodic component* η.

PROOF. Here we follow the proof of Proposition 4.2. in [**11**] but instead of using time change to freeze the dynamics along $\mathcal{W}$ we use Lemma 4.5. □

Let us reach to a contradiction then. Recall that λ(=Lebesgue measure) is an ergodic measure for $\alpha_0(\mathbf{t})$, hence, since μ projects into Lebesgue, i.e. $h_*\mu = \lambda$ we have that almost every measure η in the ergodic decomposition of μ also projects into Lebesgue, that is $h_*\eta = \lambda$.

Now we shall argue as in Lemma 2.3. in [**6**]. Let x_0 be a point in a Pesin set and let Λ be a neighborhood of x_0 in this Pesin set. Let us consider $R = \bigcup_{x \in \mathcal{W}^u_{\alpha(\mathbf{s})}(x_0)} \mathcal{W}^s_{\alpha(\mathbf{t})}(x) \cap \Lambda$. Since $\eta^{\mathcal{W}^s_{\alpha(\mathbf{s})}} = \eta^{\mathcal{W}^s_{\alpha(\mathbf{t})}}$, we have that $\eta(R) = \eta(\bigcup_{x \in \mathcal{W}^u_{\alpha(\mathbf{s})}(x_0)} \mathcal{W}^s_{\alpha(\mathbf{s})}(x) \cap \Lambda) > 0$. Hence, since $h_*\eta = \lambda$ we get that $\lambda(h(R)) \geq \eta(R) > 0$. On the other hand, $h(R) \subset h(x_0)(W^u_{\alpha_0(\mathbf{s})} \oplus W^s_{\alpha_0(\mathbf{t})})$ which has dimension less than n and hence has zero Lebesgue measure. It has less dimension since Lyapunov hyperplanes match.

Hence we get a contradiction and $\mu^{\mathcal{W}}$ is absolutely continuous w.r.t. Lebesgue measure. Once conditional measures of μ along all Lyapunov foliations are absolutely continuous w.r.t. Lebesgue, absolute continuity of μ follows by the following theorem:

THEOREM 4.8. [**11**, Theorem 5.2]
Let $f\colon M \to M$ be a $C^{1+\theta}$ diffeomorphism preserving an ergodic measure μ. Let $TM = E^u \oplus E^c \oplus E^s$ be the Oseledets splitting associated to μ. Let us assume that:

(1) E^c is tangent to a smooth foliation $\mathcal{O}$, $Df|E^c$ is an isometry with respect to to the standard metric in M, and conditional measures along $\mathcal{O}$ are Lebesgue measure;
(2) $E^u = E_1 \oplus \cdots \oplus E_u$ and $E^s = E_s \oplus \cdots \oplus E_r$, where $\chi_i < \chi_j$ if $i < j$;
(3) each E_i is tangent to an absolutely continuous Lyapunov foliation $\mathcal{W}^i$ and the conditional measures along $\mathcal{W}^i$ are absolutely continuous with respect to Lebesgue measure for a.e. point.

As a corollary of the proof we get the smoothness of the semiconjugacy along the one dimensional $\mathcal{W}$ and the matching of the Lyapunov exponents.

PROPOSITION 4.9. *For every Lyapunov foliation $\mathcal{W}$, the semiconjugacy h restricted to $\mathcal{W}(x)$ is a smooth diffeomorphism for μ-a.e. x and the Lyapunov exponents for the linear and the nonlinear actions match.*

PROOF. As in [**6**] we have here that h intertwines the affine structure along $\mathcal{W}$ for α and the one of α_0. $\square$

4.4. Uniqueness of large invariant measure. The proof of Theorem 3.2 follows from the proof in Section 7 in [**11**] which works without any change allowing existence of negatively proportional exponents.

REMARK 3. It is in the use of Lemma 7.5. in [**11**] (see also Lemma 5.1 in the next section) where we use that the universal covering of M is contractible, i.e. M is a $K(\pi, 1)$ manifold. This Lemma is false without the $K(\pi, 1)$ assumption and examples can be constructed along the lines in Section 4 in [**10**] gluing two or more tori and presenting more than one large measure.

5. Existence of proper periodic orbits

PROOF OF THEOREM 3.3. The first issue is to get smoothness of h along stable and unstable manifolds of different elements of the action. To this end we will use that there are no resonances in the linear action. This follows as in [**11**, Section 7].

Take a hyperbolic element $\alpha(\mathbf{n})$ of the action. Consider a Pesin set Λ. It follows from the proof of the Main Lemma in [**7**] that for a.e. $x \in \Lambda$ there is a periodic point p for $\alpha(\mathbf{n})$ such that the local stable and unstable manifolds of p are approached by admissible (w.r.t. Λ) local stable and unstable manifolds of a sequence of points z_k that may be taken in a fixed Pesin set $\Lambda' \supset \Lambda$. Moreover this points z_k lie in the intersection of (global) stable and unstable manifolds of the orbit of x and hence are in the support of μ since conditional measures of μ along stable and unstable manifolds are equivalent to Lebesgue measure. Hence taking x such that the restriction of the semiconjugacy h to $\mathcal{W}^s_{\alpha(\mathbf{n})}(x)$ and $\mathcal{W}^u_{\alpha(\mathbf{n})}(x)$ be affine w.r.t. the corresponding affine structures and using the fact that affine structures varies Hölder continuously along Pesin sets we get that the semiconjugacy h is a local smooth diffeomorphism along the local stable and unstable manifolds of p. By invariance we get that the semiconjugacy is a smooth diffeomorphism along the global stable and unstable manifold of p for $\alpha(\mathbf{n})$.

Let us see that p has finite α-orbit. Indeed we will see that $\alpha(\mathbf{m})(p) = p$ if and only if $\alpha_0(\mathbf{m})(h(p)) = h(p)$. Of course one implication is obvious, let us see the other one. To this end we shall make use of the following Lemma that appeared as Lemma 7.5 in [**11**],

LEMMA 5.1. *Let $E^i \subset \mathbb{R}^n$, $i = 1, 2$ be two subspaces such that $E^1 \oplus E^2 = \mathbb{R}^n$. Let $p_i : E^i \to \mathbb{R}^n$, $i = 1, 2$ be two proper embeddings at a bounded distance from inclusion. Call $p_i(E^i) = W_i$, $i = 1, 2$. Then $W_1 \cap W_2 \neq \emptyset$.*

Assume $\alpha_0(\mathbf{m})(h(p)) = h(p)$, then $\alpha(\mathbf{m})(p)$ is a periodic point for $\alpha(\mathbf{n})$ and $\mathcal{W}^s_{\alpha(\mathbf{n})}(\alpha(\mathbf{m})(p)) = \alpha(\mathbf{m})(\mathcal{W}^s_{\alpha(\mathbf{n})}(p))$. But then, by Lemma 5.1 we have that $\mathcal{W}^s_{\alpha(\mathbf{n})}(\alpha(\mathbf{m})(p)) \cap \mathcal{W}^u_{\alpha(\mathbf{n})}(p) \neq \emptyset$, call z a point in this intersection. Since stable manifolds are sent into stable manifolds by h and unstable manifolds are sent into unstable manifolds, it follows that $h(z) = h(p)$ and hence $p = z$ since h restricted to the unstable manifold of p is a diffeomorphism, but this clearly implies that $\alpha(\mathbf{m})(p) = p$. This and the fact that h is a diffeomorphism when restricted to

$\mathcal{W}^s_{\alpha(\mathbf{n})}(p)$ and $\mathcal{W}^u_{\alpha(\mathbf{n})}(p)$ implies that $D_p\alpha(\mathbf{m})$ is conjugated to $\alpha_0(\mathbf{m})$ whenever $\alpha(\mathbf{m})(p) = p$. □

Again is in the use of Lemma 5.1 that we need the ambient manifold to be a $K(\pi, 1)$ space.

6. Differentiable rigidity of real analytic actions.

For the proof of Theorem 3.5 we will use the following consequence of Zimmer's cocycle superrigidity as we did in [**12**], see also [**5, 18**].

PROPOSITION 6.1. *Let $M, N, \Gamma, \alpha, \alpha_0, p_o, \hat{p}$ and μ be as above. Then there is a finite index subgroup $\Gamma' \subset \Gamma$ and a measurable map $\eta : M \to \mathcal{N}$ defined μ a.e. such that if we define the μ-a.e. defined map $p : M \to N$, $p(x) = \eta(x)\hat{p}$ then $p \circ \alpha(\gamma)(x) = \alpha_0(\gamma) \circ p(x)$ for μ-a.e. $x \in M$ and for every $\gamma \in \Gamma'$. Moreover, if α_0 is weakly-hyperbolic as defined in* [**18**] *(in particular if there is an Anosov element for α_0) then η and hence p extends to a continuous map $p : M \to N$ which is a semiconjugacy on* $\operatorname{supp}(\mu)$ *but may fail to be a semiconjugacy for the action outside* $\operatorname{supp}(\mu)$.

PROOF OF THEOREM 3.5. Take Γ, α, α_0 and μ in the hypothesis of Theorem 3.5. By Proposition 6.1 we have a continuous map $P : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ homotopic to identity such that $P \circ \alpha(\gamma)(x) = \alpha_0(\gamma) \circ P(x)$ for $x \in \operatorname{supp}(\mu)$ and for $\gamma \in \Gamma' \subset \Gamma$ a finite index subgroup. Hence $P(\operatorname{supp}(\mu))$ is compact and α_0-invariant and so equals either $\mathbb{T}^{2n}$ or a point. Assuming the support of the measure μ is large, $P(\operatorname{supp}(\mu)) = \mathbb{T}^{2n}$. If we take C a Cartan subgroup of Γ' then this Cartan subgroup is a full abelian symplectic subgroup. Hence its action on $\mathbb{T}^{2n}$ has simple coarse Lyapunov spectrum and is without resonances and so it is in the hypothesis of Theorems 3.1 and 3.2. And hence μ is absolutely continuous w.r.t. Lebesgue measure. Moreover, by Theorem 3.3 there is a periodic point $q \in \operatorname{supp}(\mu)$ for the restriction to the Cartan action C such that $\mathcal{W}^s(q) \cup \mathcal{W}^u(q) \subset \operatorname{supp}(\mu)$ and P restricted to $\mathcal{W}^s(q)$ and $\mathcal{W}^u(q)$ is a smooth diffeomorphism onto $P(q) + E^s$ and $P(q) + E^u$ respectively.

Let us prove that q is a periodic point for the complete action of α restricted to Γ'. $P(q)$ is a rational point since it is a periodic point for the action α_0 restricted to C. Hence $P(q)$ is a periodic orbit for the action α_0 restricted to Γ'. Take any $\gamma \in \Gamma'$, then $\alpha(\gamma)(q)$ is a periodic orbit for α restricted to $\gamma C \gamma^{-1}$. Take $\mathbf{n} \in C$, we have that $\alpha_0(\gamma)E^s_{\alpha_0(\mathbf{n})} \cap E^u_{\alpha_0(\mathbf{n})} = \{0\}$ and hence $\alpha_0(\gamma)E^s_{\alpha_0(\mathbf{n})} \oplus E^u_{\alpha_0(\mathbf{n})} = \mathbb{R}^n$. Moreover, $\mathcal{W}^s_{\alpha(\gamma\mathbf{n}\gamma^{-1})}(\alpha(\gamma)(q)) = \alpha(\gamma)(\mathcal{W}^s_{\alpha(\mathbf{n})}(q))$ and hence $P|\mathcal{W}^s_{\alpha(\gamma\mathbf{n}\gamma^{-1})}(\alpha(\gamma)(q))$ is a diffeomorphism onto $\alpha_0(\gamma)E^s_{\alpha_0(\mathbf{n})} = E^s_{\alpha_0(\gamma\mathbf{n}\gamma^{-1})}$. Using Lemma 5.1 we have that

$$\mathcal{W}^s_{\alpha(\gamma\mathbf{n}\gamma^{-1})}(\alpha(\gamma)(q)) \cap \mathcal{W}^u_{\alpha(\mathbf{n})}(q) \neq \emptyset.$$

Now, $P(\alpha(\gamma)(q)) = \alpha_0(\gamma)(P(q)) = P(q)$ since $\gamma \in \Gamma'$ and $P(q)$ is periodic for α_0 restricted to Γ'. Then we have that

$$P(\mathcal{W}^s_{\alpha(\gamma\mathbf{n}\gamma^{-1})}(\alpha(\gamma)(q)) \cap \mathcal{W}^u_{\alpha(\mathbf{n})}(q)) = P(q).$$

So, if z is in

$$\mathcal{W}^s_{\alpha(\gamma\mathbf{n}\gamma^{-1})}(\alpha(\gamma)(q)) \cap \mathcal{W}^u_{\alpha(\mathbf{n})}(q)$$

then $P(z) = P(q)$ and hence $z = q$ since P is injective when restricted to $\mathcal{W}^u_{\alpha(\mathbf{n})}(q)$ and also $z = \alpha(\gamma)(q)$ because P is injective when restricted to $\mathcal{W}^s_{\alpha(\gamma^{-1}\mathbf{n}\gamma)}(\alpha(\gamma)(q))$.

So $\alpha(\gamma)(q) = q$ for any $\gamma \in \Gamma'$ and hence is periodic. Also, by smoothness of P along the invariant manifolds through q we have that $D_q\alpha$ is conjugated to α_0. Hence the rest of the proof follows as in [**12**] using local linearization near fix points of real analytic actions proved by Cairns and Ghys [**1**]. □

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