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NONSTATIONARY NORMAL FORMS AND RIGIDITY OF GROUP ACTIONS

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Abstract. We develop a proper "nonstationary" generalization of the classical theory of normal forms for local contractions. In particular, it is shown under some assumptions that the centralizer of a contraction in an extension is a particular Lie group, determined by the spectrum of the linear part of the contractions. We show that most homogeneous Anosov actions of higher rank abelian groups are locally C^∞ rigid (up to an automorphism). This result is the main part in the proof of local C^∞ rigidity for two very different types of algebraic actions of irreducible lattices in higher rank semisimple Lie groups: (i) the actions of cocompact lattices on Furstenberg boundaries, in particular projective spaces, and (ii) the actions by automorphisms of tori and nilmanifolds. The main new technical ingredient in the proofs is the centralizer result mentioned above.

1. Normal forms for extensions of dynamical systems by contractions

In this section we announce results of [7] which generalize certain aspects of the classical theory of local normal forms [22, 1].

Let X be a compact metric space and $f : X \to X$ a homeomorphism (continuous dynamical system), V a vector bundle over X with projection $\pi : V \to X$, and $F : V \to V$ a continuous invertible linear extension of f. Let us fix a continuous family of Riemannian metrics in the fibers. An extension F of f is called *a contraction* if $\|DF\| < 1$. Consider the induced operator F^* in the Banach space of continuous sections of V provided with the uniform norm, i.e.

$$F^*v(x) = F(v(f^{-1}x)).$$

Under the very mild assumption that nonperiodic points of f are dense the spectrum of the operator F^* is the union of finitely many closed annuli centered at the origin. Hence the *characteristic set* of F,

$$\chi(F) = \{\lambda \in \mathbb{R}_+ : \exp \lambda \in sp\, F^*\},$$

is the union of finitely many intervals. We denote these intervals by $\Delta_1, \dots, \Delta_l$. Let $\Delta_i = [\lambda_i, \mu_i]$ and assume that the intervals are ordered in increasing order so that $\lambda_{i+1} > \mu_i$. Let us assume that the space X is connected or that the map f is

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topologically transitive. Then the bundle V splits into the direct sum of F-invariant subbundles $V_1, \dots, V_l$ such that $\chi(F) |_{V_i} = \Delta_i$, $i = 1, \dots, l$ (cf. [6]). Let m_i be the dimension of the subbundle V_i.

Definition 1. The extension F has *narrow band spectrum* if

$$\mu_i + \mu_l < \lambda_i$$

for $i = 1, \dots, l$.

From now on we assume that this condition is satisfied. Represent $\mathbb{R}^m$ as the direct sum of the spaces $\mathbb{R}^{m_1}, \dots, \mathbb{R}^{m_l}$ and let $(t_1, \dots, t_l)$ be the corresponding coordinate representation of a vector $t \in \mathbb{R}^m$. Let $P : \mathbb{R}^m \to \mathbb{R}^m$; $(t_1, \dots, t_l) \mapsto (P_1(t_1, \dots, t_l), \dots, P_l(t_1, \dots, t_l))$ be a polynomial map preserving the origin. We will say that the map P is of *subresonance type* if it has nonzero homogeneous terms in $P_i(t_1, \dots, t_l)$ with degree of homogeneity s_j in the coordinates of t_j, $i = 1, \dots, l$, only for

$$\lambda_i \le \sum_{j \ne i} s_j \mu_j. \qquad (*)$$

We remark that the notions of a homogeneous polynomial and degree of homogeneity in $\mathbb{R}^m$ are invariant under the action of the group $GL(m, \mathbb{R})$. Thus the notion of a map of subresonance type depends only on the decomposition $\mathbb{R}^m = \bigoplus_{i=1}^l \mathbb{R}^{m_i}$, but not on a choice of coordinates in each component of this decomposition.

We will call any inequality of type $(*)$ *a subresonance relation*. There are always subresonance relations of the form $\lambda_i \le \mu_j$ for $j = i, \dots, l$. They correspond to the linear terms of the polynomial. We will call such relations *trivial*. The narrow band condition guarantees that, for any nontrivial subresonance relation, $s_j = 0$ for $j = 1, \dots, i$.

Proposition 2. *Polynomial maps of the subresonance type with invertible derivative at the origin are invertible and form a group, which we will denote by* $G_{\lambda,\mu}$.

Let U be a neighborhood of the zero section in V. We consider an extension map $\mathcal{F} : U \to V$, which is continuous, smooth (usually C^∞) along the fibers and preserves the zero section. We will denote by $D\mathcal{F}_0$ the derivative of $\mathcal{F}$ in the fiber direction at the zero section. It is a linear extension of f. As before, fix a continuous family of Riemannian metrics in the fibers. The extension $\mathcal{F}$ is called *a contraction* if $D\mathcal{F}_0$ is a contraction.

Two extensions are conjugate if there exists a continuous family of local C^∞ diffeomorphisms of the fibers $V(x)$, preserving the origin, which transforms one extension into the other.

Theorem 3 (subresonance normal form)**.** *Suppose the extension* $\mathcal{F}$ *is a contraction and the linear extension* $D\mathcal{F}_0$ *has narrow band spectrum determined by the vectors* $\lambda = (\lambda_1, \dots, \lambda_l)$ *and* $\mu = (\mu_1, \dots, \mu_l)$.

Then there exists an extension $\tilde{\mathcal{F}}$ *equivalent to* $\mathcal{F}$ *such that for every* $x \in X$,

$$\tilde{\mathcal{F}} |_{V(x)} \colon \ \bigoplus_{i=1}^{l} V_i(x) \to \bigoplus_{i=1}^{l} V_i(f(x))$$

is given by a polynomial map of subresonance type, i.e an element of the group $G_{\lambda,\mu}$.

The proof of this theorem [7] follows one of the usual schemes in the normal form theory. In particular, it is parallel to the proof of the nonstationary normal form theorem [6, Theorem 2.1] for extensions of measure preserving transformations. It includes three steps:

Step 1. Finding a *formal* coordinate change, i.e. the Taylor series at the zero section for the desired coordinate change. Naturally, the coefficients should depend continuously on the base point. This is the most essential step.

Step 2. Constructing a continuous family of smooth (C^∞) coordinate changes in $V(x)$, $x \in X$, for which the formal power series found at Step 1 is the Taylor series at the zero section. This follows directly from a parametric version of [8, Proposition 6.6.3].

Step 3. The coordinate change constructed at Step 2 conjugates our extension with an extension which is C^∞ *tangent* to the derivative extension. We show that any two C^∞ tangent contracting extensions are conjugate via a C^∞ coordinate change C^∞ tangent to the identity. This is achieved via a parametric version of the "homotopy trick". See [8, Theorem 6.6.5].

Theorem 4 (Centralizer for subresonance maps)**.** *Suppose g is a homeomorphism of the space X commuting with f, and $\tilde{\mathcal{G}}$ is an extension of g by a C^∞ diffeomorphism of the fibers (not necessarily contraction) commuting with an extension $\tilde{\mathcal{F}}$ satisfying the assertion of Theorem 3. Then $\tilde{\mathcal{G}}$ has a similar form:*

$$\tilde{\mathcal{G}}\mid_{V(x)}\colon \bigoplus_{i=1}^{l} V_i(x) \to \bigoplus_{i=1}^{l} V_i(g(x))$$

is also a polynomial map of subresonance type.

The proof of this theorem [7] consists of two steps:

Step 1. Proving that the Taylor series of $\tilde{\mathcal{G}}$ at 0 are polynomial maps of subresonance type.

Step 2. Proving that the map $\tilde{\mathcal{G}}$ coincides with its Taylor series.

Combining these two theorems we see that a local action of an abelian group by extensions which contain a contraction with narrow band spectrum can be simultaneously brought to a normal form.

Corollary 5. *Let ρ be a continuous action of $\mathbb{R}^k$ on a compact connected metric space X. Let V be a vector bundle over X. Suppose that σ is a local action of $\mathbb{R}^k$ in a neighborhood of the zero section of V such that σ covers ρ, σ is differentiable along the leaves and each $\sigma(a)\mid_{V_x}$ depends continuously on the base point x in the C^∞ topology. Suppose further that for some $a \in \mathbb{R}^k$, $\sigma(a)$ is a contraction and that the induced linear operator on C^0 sections of V has narrow band spectrum. Then there exist C^∞ changes of coordinates in the fibers V_x, depending continuously on x, such that for all $b \in \mathbb{R}^k$, $\sigma(b)$ is a polynomial map of subresonance type.*

2. Rigidity of Anosov group actions: formulation of results

In this section and Section 4 we announce the results of [14].

2.1. **Anosov actions of $\mathbb{Z}^k$.** A C^∞ action ρ of a finitely generated discrete group Γ on a compact manifold M is called C^∞ *locally rigid* if any C^∞ action $\tilde{\rho}$ of Γ on M that is C^1 close to ρ on a fixed finite set of generators is C^∞ conjugate to ρ.

We recall that an action of a group G on a compact manifold is *Anosov* if some element $g \in G$ acts normally hyperbolically with respect to the orbit foliation (cf. [11] for more details). If G is discrete this simply means that the action contains an Anosov diffeomorphism. Since the only manifolds known to carry an Anosov diffeomorphism are infranilmanifolds (in particular, tori and nilmanifolds), Anosov actions of any discrete groups, including $\mathbb{Z}^k$ or lattices in semisimple Lie groups are also known to exist only on such manifolds. Any Anosov diffeomorphism g of an infranilmanifold is topologically conjugate to an automorphism f ([2, 16]) and hence has a fixed point. Any such map is a π_1-diffeomorphism in the sense of Franks [2]. Hence any commuting homeomorphism is unique in the homotopy class. Moreover, any automorphism of π_1 which commutes with f_* can be realized by another automorphism h commuting with f by Maltsev's theorems. The meaning of our next result is that, under certain assumptions on f and h, unless the conjugacy between g and f is smooth, the centralizer of g in the diffeomorphism group reduces to a finite extension of the powers of g. The assumptions on f and h are of two kinds. The first condition on f and h is essential and guarantees irreducibility of the $\mathbb{Z}^2$ action generated by f and h. The condition is that every nontrivial element of this $\mathbb{Z}^2$ is weak mixing. By a theorem of Parry, it is equivalent to an algebraic condition on the eigenvalues [17]. The second condition is that f and h are semisimple, which is only of technical nature.

We call an action of a group on an infranilmanifold *algebraic* or *affine* if the lift of every element of the action to the universal cover is affine w.r.t. the connection given by right invariant vector fields.

Theorem 6. *Let ρ be an algebraic Anosov action of $\mathbb{Z}^k$, $k \geq 2$, on an infranilmanifold M. Suppose that the linearization is semisimple and that no nontrivial element of the group has roots of unity as eigenvalues in the induced representation on $H_1(M, \mathbb{R})$. Then ρ is C^∞ locally rigid. Moreover, the conjugacy can be chosen C^1 close to the identity, and is unique amongst conjugacies close to the identity.*

This theorem is really a corollary of the corresponding result for $\mathbb{R}^k$ actions applying it to a suspension of the $\mathbb{Z}^k$ action (cf. Section 2.3 and [14]). The weak mixing condition then guarantees that ergodic components of any one-parameter subgroup of the suspension consist of entire nilmanifold fibers. This is sufficient to guarantee the technical hypothesis on ergodic components of our main result on $\mathbb{R}^k$ actions.

2.2. **Anosov actions of lattices in higher rank semisimple Lie groups.** Our results on smooth local rigidity of $\mathbb{Z}^k$ actions can be used to study Anosov actions of irreducible lattices in higher rank semisimple Lie groups of the noncompact type on tori and nilmanifolds by automorphisms. Such a lattice Γ always contains a $\mathbb{Z}^k$ subgroup, $k \geq 2$, a so-called *Cartan subgroup*, such that the restriction of the action to $\mathbb{Z}^k$ has semisimple linear part, and satisfies the eigenvalue condition of Theorem 6. The approach taken here uses R. Zimmer's cocycle superrigidity theory [24, 21] to provide us with a measurable Γ-equivariant framing, which is then shown to be smooth using the rigidity of the action of the Cartan subgroup. The first paper using this approach [10] was based on more limited results on the rigidity of abelian

Anosov actions from [9]. This approach was developed by N. Qian and C. Yue in [18, 19, 20]. With the help of our more powerful result, Theorem 6, we obtain the following definitive result.

Theorem 7. *Let G be a linear semisimple Lie group all of whose simple factors have real rank at least 2. Let Γ be an irreducible lattice in G. Then a sufficiently small (in the C^1 topology) perturbation of an Anosov action of Γ by affine maps on a nilmanifold is C^∞ conjugate to the original action by a conjugacy that is C^1 close to the identity.*

2.3. **Anosov actions of $\mathbb{R}^k$ and reductive groups.** For actions of a continuous group G, there are two aspects of local rigidity. First one can consider rigidity of the orbit foliation as a foliation, i.e. when a C^∞ small perturbation is not necessarily the orbit foliation of an action of the same group. Secondly, there is local rigidity of an action which in this case must allow a change of the action by an automorphism of the group G close to the identity. Local rigidity of such an action can be obtained from the foliation rigidity and certain rigidity results for cocycles over the action taking values in G (time changes).

We define algebraic $\mathbb{R}^k$ actions as follows. Suppose $\mathbb{R}^k \subset H$ is a subgroup of a connected Lie group H. Let $\mathbb{R}^k$ act on a compact quotient H/Λ by left translations where Λ is a lattice in H. Suppose C is a compact subgroup of H which commutes with $\mathbb{R}^k$. Then the $\mathbb{R}^k$ action on H/Λ descends to an action on $C \setminus H/\Lambda$. The general algebraic $\mathbb{R}^k$ action ρ is a finite factor of such an action. Let $\mathfrak{c}$ be the Lie algebra of C. The *linear part* of ρ is the representation of $\mathbb{R}^k$ on $\mathfrak{c} \setminus \mathfrak{h}$ induced by the adjoint representation of $\mathbb{R}^k$ on the Lie algebra $\mathfrak{h}$ of H.

Let us note that the suspension of an affine $\mathbb{Z}^k$ action is an algebraic $\mathbb{R}^k$ action (cf. [11, 2.2]).

The next theorem is our principal technical result for algebraic Anosov $\mathbb{R}^k$ actions. We denote the strong stable foliation of $a \in \mathbb{R}^k$ by $\mathcal{W}_a^-$, the strong stable distribution by E_a^-, and the 0-Lyapunov space by E_a^0. Note that E_a^0 is always integrable for algebraic actions. Denote the corresponding foliation by $\mathcal{W}_a^0$. Also note that any algebraic action leaves a Haar measure μ on the quotient invariant. Given a collection of subspaces of a vector space, we call a nontrivial intersection *maximal* if it does not contain any other nontrivial intersection of these subspaces.

Theorem 8. *Let ρ be an algebraic Anosov action of $\mathbb{R}^k$, for $k \geq 2$, such that the linear part of ρ is semisimple. Assume that for any maximal nontrivial intersection $\bigcap_{i=1\dots r} \mathcal{W}_{b_i}^-$ of stable manifolds of elements $b_1, \dots, b_r \in \mathbb{R}^k$ there is an element $a \in \mathbb{R}^k$ such that for a.e. $x \in M$, $\bigcap_{i=1\dots r} E_{b_i}^-(x) \subset E_a^0(x)$ and such that a.e. leaf of the intersection $\bigcap_{i=1\dots r} \mathcal{W}_{b_i}^-$ is contained in an ergodic component of the one-parameter subgroup ta of $\mathbb{R}^k$ (w.r.t. Haar measure). Then the orbit foliation of ρ is locally C^∞ rigid. In fact, the orbit equivalence can be chosen C^1 close to the identity. Moreover, the orbit equivalence is transversally unique, i.e. for any two different orbit equivalences close to the identity, the induced maps on the set of leaves agree.*

We immediately get the following corollary.

Corollary 9. *Let ρ be an algebraic Anosov action of $\mathbb{R}^k$, for $k \geq 2$, such that the linear part of ρ is semisimple. Assume that every one-parameter subgroup of $\mathbb{R}^k$ acts ergodically with respect to the Haar measure μ. Then the orbit foliation of ρ*

is locally C^∞ rigid. In fact, the orbit equivalence can be chosen C^1 close to the identity. Moreover, the orbit equivalence is transversally unique i.e. for any two different orbit equivalences close to the identity, the induced maps on the set of leaves agree.

All known weakly mixing Anosov $\mathbb{R}^k$ actions belong to and almost exhaust the list of *standard* $\mathbb{R}^k$ actions, introduced in [11]. They essentially consist of suspensions of actions by toral automorphisms, Weyl chamber flows, twisted Weyl chamber flows and some further extensions (cf. [11] for more details and for the definition of twisted Weyl chamber flows).

For the standard Anosov actions, we showed that every smooth cocycle is smoothly cohomologous to a constant cocycle [11, Theorem 2.9]. As a consequence, all smooth time changes are smoothly conjugate to the original action (possibly composed with an automorphism of $\mathbb{R}^k$). Combining this with Theorem 8, we obtain the following corollary.

Corollary 10. *The standard algebraic Anosov actions of $\mathbb{R}^k$ for $k \geq 2$ with semisimple linear part are locally C^∞ rigid. Moreover, the C^∞ conjugacy ϕ between the action composed with an automorphism ρ and a perturbation can be chosen C^1 close to the identity. The automorphism ρ is unique and also close to the identity. Finally, ϕ is unique amongst conjugacies close to the identity modulo translations in the acting group.*

In Section 4 we discuss local rigidity results for projective lattice actions. Those results are based on similar foliation rigidity results for Anosov actions of certain reductive groups. Let G be a connected semisimple Lie group with finite center and without compact factors. Let $\Gamma \subset G$ be an irreducible cocompact lattice. Let P be a parabolic subgroup of G, and let H be its Levi subgroup. Thus $P = H\,U^+$, where U^+ is the unipotent radical of P. Then H acts on G/Γ by left translations. These actions are Anosov.

Theorem 11. *If the real rank of G is at least 2, then the orbit foliation $\mathcal{O}$ of H is C^∞ locally rigid. Moreover, the orbit equivalence can be chosen C^1 close to the identity.*

We will actually use the following corollary of the proof of the theorem. Let $P = L\,C\,U^+$ be the Langlands decomposition of P (with respect to some Iwasawa decomposition $G = K\,A\,N$ of G), and let M_P be the centralizer of C in K. Then the orbit foliation of H on $\Gamma \setminus G$ descends to a foliation $\mathcal{R}$ on $\Gamma \setminus G/M_P$.

Corollary 12. *If the real rank of G is at least 2, then the foliation $\mathcal{R}$ on $\Gamma \setminus G/M_P$ is C^∞ locally rigid. Moreover, the orbit equivalence can be chosen C^1 close to the identity.*

3. Anosov actions: sketches of proofs

3.1. **Rigidity of orbit foliations of $\mathbb{R}^k$ actions via normal forms.** We present an outline of the proof of Theorem 8, emphasizing the crucial new step involving Corollary 5 from the theory of normal forms. Theorem 6 follows immediately from Theorem 8 applied to the suspension of the $\mathbb{Z}^k$ action once the technical condition on ergodic components is checked. The proofs of Theorem 11 and its corollary are obtained by minor modifications of this approach.

Consider an algebraic $\mathbb{R}^k$ action ρ with semisimple linear part σ. Define the *Lyapunov exponents* of ρ as the logarithms of the absolute values of the eigenvalues of σ. We get linear functionals $\chi : \mathbb{R}^k \to \mathbb{R}$. There is a splitting of the tangent bundle into $\mathbb{R}^k$-invariant subbundles $TM = \bigoplus_\chi E_\chi$ such that the Lyapunov exponent of $v \in E_\chi$ with respect to $\rho(a)$ is given by $\chi(a)$. We call E_χ a *Lyapunov space* or *Lyapunov distribution* for the action. Then the strong stable distribution E_a^- of $a \in \mathbb{R}^k$ is given by $E_a^- = \sum_{\chi(a)<0} E_\chi$. The sum $E^\chi = \bigoplus E_\lambda$, where λ ranges over all Lyapunov functionals which are positive multiples of a given Lyapunov functional χ, is always integrable. In fact, set $H = \{a \in \mathbb{R}^k \mid \chi(a) \leq 0\}$, and call it a *Lyapunov half-space*. Denote E^χ by E_H. Then we get a decomposition

$$TM = \bigoplus E_H \oplus T\mathcal{O},$$

where H runs over all Lyapunov half-spaces and $T\mathcal{O}$ is the tangent bundle to the orbits of the action. We call this decomposition the *coarse Lyapunov decomposition* of TM. Denote by $\mathcal{W}_H$ the integral foliation of the distribution E_H.

Now we start with the description of the proof of Theorem 8 proper. Let $\mathcal{F}$ and $\tilde{\mathcal{F}}$ denote the orbit foliation and a C^1 small perturbation. First, by Hirsch-Pugh-Shub's structural stability theorem, there exists a Hölder orbit equivalence ϕ between $\mathcal{F}$ and $\tilde{\mathcal{F}}$ which is close to the identity and transversally unique [4]. It can be chosen C^∞ along $\mathcal{F}$. Thus, it is sufficient to show that ϕ is C^∞ transversally. This in turn follows from standard elliptic theory once we establish the smoothness of ϕ along each foliation $\mathcal{W}_H$ via standard elliptic theory. This is the central part of the proof.

To prepare for the main argument, we consider the continuous action $\tilde{\rho}$ of $\mathbb{R}^k$ on M obtained by conjugating ρ by ϕ. Structural stability also implies that ϕ carries over the hull of the $\mathcal{W}_H$-foliation and $\mathcal{F}$ to a Hölder foliation $\tilde{\mathcal{U}}_H$ with smooth leaves, which is thus invariant under the holonomy of $\tilde{\mathcal{F}}$. This allows us to construct a natural extension of $\tilde{\rho}$ via holonomy in $\tilde{\mathcal{U}}_H$, smooth along the leaves. Since ρ is Anosov, this extension includes a contraction. Since the algebraic action has point spectrum, it follows that this natural extension satisfies the narrow band condition. Thus we can apply the results of Section 1, and in particular Corollary 5.

The conjugacy ϕ induces conjugacies ψ_x between the action of ρ on $\mathcal{W}_H(x)$ and the fibers T_x of the natural extension of $\tilde{\rho}$ at x described above. Smoothness of ψ_x will follow once we see that ψ_x intertwines smooth transitive actions of a certain Lie group G on the fibers.

The action of the group G on $\mathcal{W}_H(x)$ is obtained by taking limits of returns restricted to $\mathcal{W}_H(x)$ of some one-parameter subgroup a_t on the boundary of H to points in $\mathcal{W}_H(x)$. The semisimplicity of ρ implies that these returns are isometries along $\mathcal{W}_H$, and thus converge to an isometry. Our condition on ergodic components of a_t shows that this group of limits is transitive for generic x. A similar device was used in [12] to study invariant measures for Anosov actions of higher rank abelian groups.

The action of G on T_x is obtained by conjugating the returns of a_t by ϕ. Since the ψ_x form a uniformly continuous family of homeomorphisms, we get convergence in the C^0 topology of the conjugated returns any time that the returns converge themselves. The key point in the argument is that these conjugated returns belong to a Lie group by Corollary 5. Hence C^0 convergence implies C^∞ convergence.

3.2. **From rigidity of $\mathbb{Z}^k$ actions and cocycle superrigidity to rigidity of lattice actions.** Our proof of Theorem 7 is based on the local rigidity of the action of a Cartan subgroup Δ of Γ and Zimmer's measurable cocycle superrigidity. The former provides a smooth equivariant framing τ for the perturbed action of Δ, and the latter an a priori measurable framing, equivariant modulo a commuting compact group, for the extension of the perturbed action of Γ to a certain finite measurable cover of M. The main work consists in showing that the measurable framing can be projected back to M, and in fact is a constant translate of τ. The difference with previous approaches along these lines is that before, e.g. in [10], the smooth framing was moved around by elements of the perturbed action to produce enough smooth data to force the superrigidity framing to conform with a smooth one. This is in the spirit of topological superrigidity and requires more detailed structural information about the lattice and its representations than is available in the present setup.

Here instead we directly compare the measurable superrigidity framing with τ. Their difference produces certain cocycles, which are shown to be constant modulo a compact group by ergodicity type arguments. Along the way, to get rid of the finite cover, we consider the superrigidity framing on this cover as a multivalued framing on M, and apply the ergodicity arguments to it.

Therefore both actions transform a translation invariant framing according to a homomorphism (modulo a commuting compact group). Hence stable and unstable foliations of the original and the perturbed action are homogeneous, and also C^0 close. Lift these foliations to the universal cover. Since two distinct closed subgroups of a nilpotent group cannot stay a bounded distance apart, it follows that the stable (and unstable) foliations of the original and the perturbed action coincide. It follows quickly that the actions coincide on a subgroup of finite index since the actions agree on intersections of stable and unstable manifolds of suitable periodic points.

Finally, a detailed analysis of the previous arguments shows that we only get local rigidity on the subgroup of Γ generated by Δ and all its conjugates. This subgroup has finite index in Γ by Margulis' finiteness theorem. To obtain local rigidity for Γ itself, we again use our analysis of superrigidity framings and the fine structure of linear representations of arithmetic lattices discovered by Margulis.

4. Rigidity of lattice actions on Furstenberg boundaries

A class of algebraic actions of lattices on compact manifolds very different from the affine actions discussed in Section 2.2 are given by the "projective actions" on Furstenberg boundaries of the ambient Lie group. These actions include the classical actions on projective spaces and Grassmannians. These actions do not preserve any measure, are highly dissipative and do not possess the robust orbit structures of Anosov diffeomorphisms. In fact, while many elements of these actions are structurally stable (they are Morse-Smale), the conjugacies are highly nonunique and thus do not lend themselves to provide a common conjugacy for the whole group. However, these projective actions are "dual" to the actions of reductive groups discussed in Section 2.3. We use the local foliation rigidity of the latter actions to obtain rigidity of the projective actions. This approach is not specific to the higher rank case. E. Ghys first used this duality for smooth classification of boundary actions of Fuchsian groups [3]. Later, C. Yue obtained partial results of this nature for rank 1 symmetric spaces [23].

Theorem 13. *Let G be a connected semisimple Lie group with finite center and without compact factors. Suppose that the real rank of G is at least 2. Let $\Gamma \subset G$ be a cocompact irreducible lattice and P a parabolic subgroup of G. Then the action of Γ on G/P by left translations is locally C^∞ rigid.*

The boundary G/P can be thought of as a transversal to the weak stable foliation $\mathcal{W}^+$ of the action of a certain element $c \in P$ on $M_P \setminus G/\Gamma$, where M_P is a suitable compact subgroup of P. Then the action ρ of the lattice Γ on the boundary G/P is the holonomy of $\mathcal{W}^+$. A C^1 close smooth perturbation $\tilde{\rho}$ of the boundary Γ action is a perturbation of this holonomy, which in fact appears as the holonomy of a perturbed foliation $\tilde{\mathcal{W}}^+$ on $M_P \setminus G/\Gamma$. Similarly, the holonomy of the weak unstable foliation $\mathcal{W}^-$ of c is also given by the Γ action on G/P. Again, $\tilde{\rho}$ gives rise to a perturbed foliation $\tilde{\mathcal{W}}^-$ on $M_P \setminus G/\Gamma$. Then the intersection of $\tilde{\mathcal{W}}^-$ with $\tilde{\mathcal{W}}^+$ is a perturbation of the neutral foliation of c which is actually the orbit foliation of a suitable reductive subgroup of P on $M_P \setminus G/\Gamma$. Now Corollary 12 provides a C^∞ orbit equivalence of this orbit foliation and its perturbation, which in turn yields a C^∞ conjugacy between ρ and $\tilde{\rho}$.

Remark. This approach does not directly apply to projective actions of nonuniform lattices since the duality breaks down due to noncompactness. Thus the local rigidity of projective actions of nonuniform lattices remains an open problem.

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