CHAPTER 11

Spectral Properties and Combinatorial Constructions in Ergodic Theory

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This survey primarily deals with certain aspects of ergodic theory, i.e. the study of groups of measure preserving transformations of a probability (Lebesgue) space up to a metric isomorphism [8, Section 3.4a]. General introduction to ergodic theory is presented in [8, Section 3]. Most of that section may serve as a preview and background to the present work. Accordingly we will often refer to definitions, results and examples discussed there. For the sake of convenience we reproduce some of the basic material here as need arises.

Here we will deal exclusively with actions of Abelian groups; for a general introduction to ergodic theory of locally compact groups as well as in-depth discussion of phenomena peculiar to certain classes of non-Abelian groups see [4]. Furthermore, we mostly concentrate on the classical case of cyclic systems, i.e. actions of $\mathbb{Z}$ and $\mathbb{R}$. Differences between those cases and the higher-rank situations (basically $\mathbb{Z}^k$ and $\mathbb{R}^k$ for $k \geqslant 2$) appear already at the measurable level but are particularly pronounced when one takes into account additional structures (e.g., smoothness).

Expository work on the topics directly related to those of the present survey includes the books by Cornfeld, Fomin and Sinai [29], Parry [124], Nadkarni [114], Queffelec [128], and the first author [78] and surveys by Lemańczyk [104] and Goodson [64]. Our bibliography is far from comprehensive. Its primary aim is to provide convenient references where proofs of results stated or outlined in the text could be found and the topics we mention are developed to a greater depth. So we do not make much distinction between original and expository sources. Accordingly our references omit original sources in many instances. We make comments about historical development of the methods and ideas described only occasionally. These deficiencies may be partially redeemed by looking into expository sources mentioned above. We recommend Nadkarni's book and Goodson's survey in particular for many references which are not included to our bibliography. Goodson's article also contains many valuable historical remarks.

1. Spectral theory for Abelian groups of unitary operators

1.1. *Preliminaries*

1.1.1. *Spectral vs. metric isomorphism.* Any measure preserving action Φ of a group G on a measure space (X, μ) generates a unitary representation of G in the Hilbert space $L^2(X, \mu)$ by $U_g : \varphi \mapsto \varphi \circ \Phi^{g^{-1}}$. For an action of $\mathbb{Z}$ generated by $T : X \to X$ the notation U_T for the operator U_1 is commonly used; often this operator is called Koopman operator since this connection was first observed in [95]. If two actions are isomorphic then the corresponding unitary representations in L^2 are unitarily equivalent, hence any invariant of unitary equivalence of such operators defines an invariant of isomorphism. Such invariants are said to be *spectral invariants* or *spectral properties*. Actions for which the corresponding unitary representations are unitarily equivalent are usually called *spectrally isomorphic*. We will use terms "unitarily equivalent" and "unitarily isomorphic" interchangeably.

Let us quickly describe the difference between the spectral and metric isomorphism for groups of unitary operators generated by measure preserving actions. In addition to the structure of Hilbert space which is preserved by any unitary operator, the space $L^2(X, \mu)$

has an extra multiplicative structure. There is a certain subtlety in describing this structure in purely algebraic terms since the product of two functions from $L^2(X,\mu)$ may not be an L^2 function so the whole space is not a ring with respect to addition and multiplication. There are however various dense subsets (e.g., bounded functions) for which multiplication is always defined; a proper abstract description leads to the notion of *unitary ring* [136].

An easier way to capture the essential part of the multiplicative structure which avoids many technical complications is as follows. First there is preferred element, the constant function equal to one which is the multiplicative unity. Second, there are *the idempotents* characterized by the equation $f^2 = f$ which evidently correspond to characteristic functions. Products of characteristic functions correspond to intersection of the sets: $\chi_{A_1} \cdot \chi_{A_2} = \chi_{A_1 \cap A_2}$ and hence the union is also recovered: $\chi_{A_1 \cup A_2} = \chi_{A_1} + \chi_{A_2} - \chi_{A_1 \cap A_2} = \chi_{A_1} + \chi_{A_2} - \chi_{A_1} \cdot \chi_{A_2}$.

Now let us call a unitary operator $U : L^2(X,\mu) \to L^2(Y,\nu)$ *multiplicative* if it takes idempotents into idempotents and preserves the product of such elements. Assuming that (X,μ) and (Y,ν) are Lebesgue spaces [8, Section 3.2b], [141,86] such an operator is generated by an isomorphism of measure spaces, $h : (Y,\nu) \to (X,\mu)$, i.e. $U(f) = f \circ h$. Naturally, the Koopman operator generated by a measure preserving transformation of a Lebesgue space is multiplicative.

This can summarized as follows:

PROPOSITION 1.1. *Unitary representations generated by measure preserving actions of a group G are metrically isomorphic if and only is they are unitarily equivalent via a multiplicative operator.*

A closed subspace $H \subset L^2(X,\mu)$ is called a *unitary $*$-subalgebra* if H is invariant under complex conjugation, bounded functions are dense in H and product of any two bounded functions from H is again in H. In this case characteristic functions generate H and H defines a *measurable partition* ξ of the space X in the following way.

PROPOSITION 1.2. *Any unitary $*$-subalgebra consists of all functions in $L^2(X,\mu)$ which are constant* mod 0 *on elements of a measurable partition. If a unitary $*$-subalgebra is U_T invariant then the corresponding measurable partition is T invariant and defines a factor of the measure preserving transformation T.*

For a more detailed description see [21, Section 5], [141].

For a general discussion of spectral properties for groups of measure preserving transformations see [4]. In the remainder of this section we will discuss the case of locally compact Abelian groups. In the rest of the survey we will restrict our considerations to the classical cases of automorphisms and flows, i.e. actions of $\mathbb{Z}$ and $\mathbb{R}$ correspondingly (and primarily the former) with only occasional comments related to actions of other groups.

1.1.2. *Duality for locally compact Abelian groups* [126, Chapter 6], [115, Section 31]. Let G be a locally compact second countable topological Abelian group. A *character* of G is a continuous homomorphism $\chi : G \to S^1$. Characters form a group which is often called *the dual group* of G and is denoted by G^*. There is a natural locally compact topology on G^*. It can be described as topology of uniform convergence on compact sets or, equivalently, as the weakest topology which makes any evaluation map $e_g : \chi \mapsto \chi(g)$ continuous. Obviously $e_g : G^* \to S^1$ thus defined is a continuous character of G^*. The Pontrjagin Duality Theorem asserts that any continuous character of G^* has the form e_g and that element $g \in G$ is uniquely defined [126, Section 40], [115, Section 31.6]. This is usually expressed in an attractive compact form

$$G^{**} = G.$$

A useful addition to the Pontrjagin Duality is the observation that G^* is compact if and only if G is discrete. In what follows the group G will be assumed not compact, but it may be discrete or continuous.

There are natural functorial properties of the duality, all easily derived from the fact that arrows in natural homomorphisms get reversed. For example, the dual to the direct sum of finitely many groups is the direct sum of their duals, the dual to the direct sum of countably many groups is the direct product of the duals, and there is a natural duality between subgroups and factors, and between direct and inverse limits.

EXAMPLE 1.3. $\mathbb{Z}^* = S^1 = \mathbb{R}/\mathbb{Z}$, $(\mathbb{Z}^k)^* = \mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, $(\mathbb{R}^k)^* = \mathbb{R}^k$, Furthermore, $(\mathbb{Z}^\infty)^* = \mathbb{T}^\infty$, where $\mathbb{Z}^\infty$ is the discrete direct sum of countably many copies of $\mathbb{Z}$ and $\mathbb{T}^\infty$ is the compact direct product of countably many copies of S^1.

EXAMPLE 1.4. The multiplicative group of roots of unity of degrees 2^n, $n = 1, 2, \ldots$, with discrete topology is the direct limit of cyclic groups of order 2^n, $n = 1, 2, \ldots$. Its dual is the compact additive group $\mathbb{Z}_2$ of dyadic integers, which is the inverse limit of such cyclic groups. By replacing 2 with a natural number m one gets roots of unity of degrees m^n, $n = 1, 2, \ldots$, and the m-adic integers correspondingly.

Using the duality between direct sums and direct products one sees that the dual to the group of all roots of unity is the direct product of the p-adic integers $\prod \mathbb{Z}_p$ over all prime numbers p.

Here is another example of the duality between direct and inverse limits.

EXAMPLE 1.5. The dual to the group $\mathbb{Z}[1/2]$ of rational numbers whose denominators are powers of 2 (which is a direct limit of free cyclic groups) is the dyadic solenoid

$$\mathcal{S}_2 \stackrel{\text{def}}{=} \bigl\{(z_1, z_2, \ldots)\colon z_1 \in S^1, z_{n+1}^2 = z_n, n = 1, 2, \ldots\bigr\}.$$

1.2. *The spectral theorem*

1.2.1. *Formulation in the general case.* A character χ can be viewed as a one-dimensional unitary representation of the group, namely the element $g \in G$ acts on $\mathbb{C}$ by the multiplication by $\chi(g)$. Every irreducible unitary representation of an Abelian group is one-dimensional (see, e.g., [115, Section 31.7]). The spectral theorem states essentially that every unitary representation of such a group in a separable Hilbert allows a canonical decomposition into a (in general, continuous) direct sum (i.e. direct integral) of characters. In this the spectral theorem represents a special case of the general theorem about the decomposition of a unitary group representation into irreducible representations [4, Theorem 3.1.3] (see [38] for a proof), but since in the Abelian case the structure of irreducible representations is simple and well understood it is considerably more specific than the general case.

Thus in the case of Abelian groups the spectral theorem gives a full collection of models for all unitary representations together with a necessary and sufficient condition for equivalence of such models.

Let G be a locally compact second countable Abelian group, ν be a σ-finite Borel measure on the dual group G^* and m be a ν-measurable function on G^* with values in $\mathbb{N} \cup \infty$. Let $H_{\nu,m}$ be the subspace of the ν-measurable square integrable functions $\varphi : G^* \to l^2$ such that at a point $\chi \in G^*$ all but the first $m(\chi)$ coordinates of $\varphi(\chi)$ vanish. The space $H_{\nu,m}$ is a separable Hilbert space with respect to the scalar product

$$\langle \varphi, \psi \rangle = \int_{G^*} \big(\varphi(\chi), \psi(\chi)\big)_{l^2}\, d\nu.$$

The group G acts unitarily on the space $H_{\nu,m}$ by the natural scalar multiplications:

$$U_g^{\nu,m} \varphi(\chi) = \chi(g)\varphi.$$

THEOREM 1.6 (The spectral theorem). *Any continuous in the strong operator topology unitary representation of G in a separable Hilbert space is unitarily equivalent to a representation $U^{\nu,m}$.*

Furthermore, representations U^{ν_1,m_1} and U^{ν_2,m_2} are unitarily equivalent if and only if measures ν_1 and ν_2 are equivalent (i.e. have the same null-sets) and $m_1 = m_2$ almost everywhere.

REMARK. Since every σ-finite measure is equivalent to a finite measure, one can assume without loss of generality that in the spectral theorem the measure ν is finite. If the group G is discrete (and hence G^* is compact) this is a customary assumption. However, in the case of a continuous group, such as $\mathbb{R}$, the most natural measure on the dual group, the Haar measure, is not finite. Accordingly, in the spectral theorem instead of finiteness of ν one assumes only local finiteness.

1.2.2. *Sketch of proof for single operator.* We outline a proof of the spectral theorem in the particular case of the action of a single operator U on a Hilbert space H.

DEFINITION 1.7. Consider a unitary operator U acting on a Hilbert space H. Let H_f be the norm closure of the linear span of the $U^n f$, $n \in \mathbb{Z}$. The space H_f is called *the cyclic subspace* generated by f.

Let us denote the scalar product in H by $\langle \cdot, \cdot \rangle$ and let θ be the natural cyclic coordinate on S^1.

THEOREM 1.8. *There exists a positive measure ν_f on $S^1 = \mathbb{R}/\mathbb{Z}$ with total mass $\|f\|^2$ such that for the unitary operator*

$$M : L^2\big(S^1, \nu_f\big) \to L^2\big(S^1, \nu_f\big), \quad g \mapsto e^{2\pi i\theta} g,$$

there exists an isometry V between H_f and $L^2(S^1, \nu_f)$ which conjugates the restriction of U to H_f and M (i.e. $VU = MV$), such that $Vf = 1$ (the constant function on S^1) and

$$\langle f, U^n f\rangle = \hat{\nu}_f(n), \quad n \in \mathbb{Z}. \tag{1.1}$$

PROOF. If (1.1) holds then the correspondence $U^n f \to e^{2\pi i n\theta}$, $n \in \mathbb{Z}$, extends to the isometry V with desirable properties. Thus it is sufficient to prove (1.1), i.e. to show that the *correlation coefficients* $\langle f, U^n f\rangle$ are Fourier coefficients of a measure. For that consider the following sequence of positive measures:

$$\nu_{N,f} = \frac{\big\|\sum_{n=1}^{N} e^{2\pi i n\theta} U^n f\big\|^2}{N}\, d\theta.$$

One can calculate the Fourier coefficients of these measures directly. In particular, if $|k| \leqslant N$, then

$$\begin{aligned}\hat{\nu}_{N,f}(k) &= \frac{1}{N}\int_{S^1} e^{-2\pi i k\theta} \sum_{1\leqslant m,n\leqslant N} \big\langle e^{2\pi i m\theta} U^m f, e^{2\pi i n\theta} U^n f\big\rangle\, d\theta \\ &= \frac{1}{N}\int_{S^1} \sum_{1\leqslant m,n\leqslant N} e^{2\pi i(m-n-k)\theta} \langle U^{m-n} f, f\rangle\, d\theta = \frac{N-|k|}{N}\langle f, U^k f\rangle.\end{aligned}$$

This equality for $k = 0$ means that the total mass of $\nu_{N,f}$ is constant, $\nu_{N,f}(S^1) = \|f\|^2$. Since for any $k \in \mathbb{Z}$, the Fourier coefficients $\hat{\nu}_{N,f}(k)$ converge to $\langle f, U^k f\rangle$, this implies that $\nu_{N,f}$ converge weakly to a measure ν_f on S^1 satisfying (1.1). □

The measure ν_f is called *the spectral measure* associated to f. If U_T is the Koopman operator acting on $L^2(X, \mu)$ and $f \in L^2(X, \mu)$ is a real-valued function, then the measure ν_f is symmetric with respect to the real axis.

We now state an important lemma, due to Wiener, which identifies all the invariant subspaces for the action of the operator $M : g \mapsto e^{2\pi i\theta} g$ in $L^2(S^1, \nu)$.

LEMMA 1.9. *If ν is a positive finite measure on S^1 and K is a closed M-invariant subspace of $L^2(S^1, \nu)$ then there exists a measurable set $E \subset S^1$ such that $K = \{f \in L^2(S^1, \nu)\colon f = 0 \text{ on } E^c\}$.*

PROOF. The projection of the constant function 1 on K, $\mathcal{P}_K 1$, is a characteristic function since if for every $n \in \mathbb{Z}$,

$$\int (1 - \mathcal{P}_K 1) e^{2\pi i n\theta} \overline{\mathcal{P}_K 1}\, d\nu = 0$$

then $\overline{\mathcal{P}_K 1}(1 - \mathcal{P}_K 1) = 0$, ν almost everywhere. This implies existence of a measurable set E such that $\mathcal{P}_K 1 = \chi_E$ and $1 - \mathcal{P}_K 1 = \chi_{E^c}$. □

As an immediate corollary one obtains

THEOREM 1.10. *Let U be a unitary operator acting on H. Let g_1 and g_2 be two elements in H such that the measures ν_{g_1} and ν_{g_2} are mutually singular. Then*

$$H_{g_1} \perp H_{g_2}.$$

Furthermore

$$H_{g_1+g_2} = H_{g_1} \oplus H_{g_2}.$$

Finally, if there exists $f \in H$ such that $H_{f_1} \subset H_f$, $H_{f_2} \subset H_f$ and $H_{f_1} \perp H_{f_2}$, then the measures ν_{g_1} and ν_{g_2} are mutually singular.

PROOF. This is an easy consequence in the circle model, constructed in Theorem 1.8, for the action of U on a cyclic subspace. For, since invariant subspaces are entirely characterized by subsets of the circle, we see that two such subspaces are orthogonal if and only if the corresponding sets are disjoint. In particular, a vector whose spectral measure has full support is cyclic. □

DEFINITION 1.11. Let U act on H as before. The *maximal spectral type* ν_U of the operator U is a positive measure on S^1 (which is defined up to equivalence) such that for every $f \in H$ the measure ν_f is absolutely continuous with respect to ν_U and no measure absolutely continuous with respect to ν_U but not equivalent to ν_U has the same property.

In the case of an action of $\mathbb{Z}$ the Spectral Theorem 1.6 which gives a complete set of invariants for a unitary operator, takes the following form.

THEOREM 1.12. *Let the unitary operator U act on H. There exists a family of positive measures on S^1, uniquely defined up to equivalence,*

$$\nu_1 \geqslant \nu_2 \geqslant \nu_3 \geqslant \cdots \geqslant \nu_n \geqslant \cdots,$$

where ν_1 is the maximal spectral type ν_U, such that the action of U on H is unitarily isomorphic to the action of M (the multiplication by $e^{2\pi i\theta}$) on the orthogonal sum

$$\bigoplus_{i \geqslant 1} L^2(S^1, \nu_i).$$

SKETCH OF PROOF. The theorem follows from the observation that if f and g in H have the property that $\nu_f \sim \nu_g$, then the two actions of U on $H_f^\perp$ and $H_g^\perp$ are unitarily equivalent. This can be seen as it suffices to check that the restrictions of U to the invariant spaces $H_f^{\perp} \cap H_{f,g}$ and $H_g^{\perp} \cap H_{f,g}$ are equivalent. Here $H_{f,g}$ denotes the invariant subspace generated by f and g. These two spaces are cyclic and it is easily checked that they have equivalent spectral measures. □

REMARK. Alternatively, one can take a sequence of ν_1 measurable sets in S^1, A_i, $i \geqslant 1$, $A_{i+1} \subset A_i$ such that $\nu_i = \nu_1 \cdot \chi_{A_i}$.

1.3. *Spectral representation and principal constructions*

One of the advantages of the spectral representation is that it behaves nicely under the natural functorial constructions.

1.3.1. *Restrictions*. For the representation $U^{\nu,m}$ all closed invariant subspaces can be described. We will denote by l_2^m the subspace of l_2 which consists of all vectors for which all but first m coordinates vanish. The following statement generalizes the Wiener Lemma 1.9.

THEOREM 1.13. *Any $U^{\nu,m}$ invariant closed subspace of $H_{\nu,m}$ is determined by a ν-measurable field of closed subspaces $L_\chi \subset l_2^{m(\chi)}$, where by definition $l_2^\infty = l_2$, and consists of all φ such that $\varphi(\chi) \in L_\chi$.*

PROOF. First, consider the case of a cyclic subspace for $U^{\nu,m}$ generated by $f \in H_{\nu,m}$. Since $U^{\nu,m}$ acts by scalar multiplications, the subspace H_f of all functions proportional to f on the set

$$S_f \stackrel{\text{def}}{=} \{\chi \in G^*\colon\ f(\chi) \neq 0\}$$

and vanishing on $G^* \setminus S_f$, is $U^{\nu,m}$ invariant. The maximal spectral type on the subspace H_f is the restriction of ν to the set S_f. But then f generates this subspace since by the Wiener Lemma 1.9 any invariant subspace of H_f consists of functions vanishing on a certain subset of S_f of positive ν-measure and hence it cannot contain f.

Now consider an arbitrary invariant subspace H. It is generated by a finite or countable set of functions $f_1, \ldots$. Every cyclic subspace H_{f_n} determines a subset S_{f_n} and a field $L_{n,\chi}$ of one-dimensional subspaces on S_{f_n}. The sum of those subspaces at each $\chi \in G^*$ forms a ν-measurable field of subspaces L_χ and since every function g with values in L_χ is the limit of linear combinations of functions with values in $L_{n,\chi}$, we conclude that $g \in H$. □

1.3.2. *Direct products*. Similarly it is easy to represent the Cartesian product of representations of the form $U^{\nu,m}$ in a similar form.

THEOREM 1.14. *The Cartesian product of representations $U^{\nu,m}$ and $U^{\nu',m'}$ is unitarily equivalent to the representation $U^{\nu+\nu',m+m'}$.*

1.3.3. *Tensor products*. The tensor product of representations $U^{\nu,m}$ and $U^{\nu',m'}$ can be described as follows. Take the group $G^* \times G^* = (G \times G)^*$ with the measure $\nu \times \nu'$. Let $m(\chi_1, \chi_2) = m_1(\chi_1) \cdot m_2(\chi_2)$. Consider the space $H_{\nu \times \nu', m}$. The group G acts diagonally on that space:

$$(U_g \varphi)(\chi_1, \chi_2) = \chi_1(g)\chi_2(g)\varphi(\chi_1, \chi_2).$$

This is a representation of the tensor product of $U^{\nu,m}$ and $U^{\nu',m'}$. From this representation the spectral representation of the tensor product can be deduced. We do this explicitly for the case of two Koopman operators in Section 4.1.3.

1.4. *Spectral invariants*

DEFINITION 1.15. For a given unitary representation of G the equivalence class of the measure ν in a unitarily equivalent representation $U^{\nu,m}$ is called *the maximal spectral type of the representation*. The function m is called *the multiplicity function*.

The maximal spectral type and the multiplicity function form a complete set of invariants for a unitary representation of a locally compact Abelian group.

1.4.1. *Maximal spectral type*. The maximal spectral type is an equivalence class of measures on the locally compact group G^*. In the two classical cases $G = \mathbb{Z}$ and $G = \mathbb{R}$ the maximal spectral type is a class of measures on the circle and the real line correspondingly. The first of those cases has already been discussed in Section 1.2.2, the second is summarized in Section 1.4.4 below.

The crudest distinction among measures is between atomic and continuous. Any measure uniquely decomposes into an atomic (discrete) and continuous part and this decomposition is invariant under equivalence of measures. Atoms in the maximal spectral type correspond to the eigenvectors for the representation: the characteristic function of such an atom is an eigenfunction.

If the maximal spectral type is atomic the representation is said to have *pure point spectrum*. There is a difference with the notion of discrete spectrum common in many areas of analysis. The spectrum may be pure point but the eigenvalues may be dense in G^* or, more generally the eigenvalues may not be isolated; in other words, the spectrum as a set does not have to be discrete. Since genuine discrete spectrum appears in ergodic theory only in some trivial situations the term “discrete spectrum” is sometimes used instead of “pure point”.

On the other hand, in our setting there is a special continuous measure on G^*, the Haar measure λ; any measure absolutely continuous with respect to Haar is called simply *absolutely continuous*. Any non-atomic measure singular with respect to the Haar measure is referred to as simply *singular*. Thus, an arbitrary measure on G^* allows a unique decomposition into atomic, absolutely continuous and singular parts invariant under equivalence. Representations whose maximal spectral type is atomic, absolutely continuous, or singular are referred to correspondingly as representations with *pure point*, *absolutely continuous*, or *singular spectrum*.

Theorems 1.6 and 1.13 easily imply

COROLLARY 1.16. *The maximal spectral type of the restriction of a representation to any closed invariant subspace is absolutely continuous with respect to the maximal spectral type of the representation*.

1.4.2. *Correlation coefficients*. Similarly to the case of a single operator (Definition 1.7) for a unitary representation U of G in a Hilbert space H with the scalar product $\langle\cdot,\cdot\rangle$, every element $v \in H$ determines the *cyclic space* H_v, the minimal closed U-invariant subspace which contains v.

Theorem 1.13 implies that an invariant subspace is cyclic if and only if for almost every with respect to the maximal spectral type $\chi \in G^*$ the space L_χ has dimension at most one. The maximal spectral type in the space H_v is represented by the measure ν_v, called *the spectral measure of* v, where

$$\langle v, U(g)v\rangle = \int_{G^*} \chi(g)\, d\nu_v.$$

Similarly to the case of the single operator (Section 1.2.2) these scalar products are often called *the correlation coefficients* of the element v. Notice that the spectral measure is always finite, since $\|v\|^2 = \nu_v(G^*)$.

REMARK. Correlation coefficients can be viewed as the Fourier transform of the measure. It is useful to remember that the Fourier transform is linear and that the product of Fourier transforms corresponds to the convolution of measures. In ergodic theory convolutions appears in connection with multiplication of functions in the study of Cartesian products (Section 4.1.3) as well as in situation when multiplicative structure is well related with the spectral picture, such as the pure point spectrum (see Section 3.5), Gaussian systems (Section 6.4.1), and Gaussian Kronecker systems (Section 6.4.3).

For the representation $U^{\nu,m}$ the cyclic space determined by a function φ consists of all functions whose values are proportional to those of φ. This implies that $\nu_\varphi = |\varphi|^2\nu$ and hence

COROLLARY 1.17. *For any finite measure* μ *absolutely continuous with respect to the maximal spectral type there exists* $v \in H$ *such that* $\nu_v = \mu$.

Recall that a set in a topological space is called *residual* if its complement is the union of countably many nowhere dense sets. In the space $H^{\nu,m}$ the set of elements which do not vanish is residual. Thus we obtain from Theorem 1.6.

COROLLARY 1.18. *The spectral measures for a residual set of elements belong to the maximal spectral type*.

1.4.3. *Multiplicity function*

DEFINITION 1.19. An *essential value* $n \in \mathbb{N} \cup \{\infty\}$ *of the spectral multiplicity* for a unitary representation of an Abelian group G is any number such that the multiplicity function m takes value n on a set of non-zero measure with respect to the maximal spectral type.

The *maximal multiplicity* is the supremum of essential values.

The representation is said to have a *homogeneous spectrum* if there is only one essential value of the spectral multiplicity. This value is then called *the multiplicity of the homogeneous spectrum*. If the only essential value is 1, the representation is said to have *simple spectrum*.

Simple spectrum is equivalent to cyclicity of the whole space: there exists a vector v such that the linear combinations of vectors $U_g(v)$, $g \in G$, are dense. Homogeneous spectrum of multiplicity m (finite or infinite) can be characterized as follows:

There exists a decomposition of the space H into an orthogonal sum of m cyclic subspaces such that the restrictions of the representation into all of those subspaces are unitarily equivalent.

The following closely related fact which follows immediately from Theorems 1.6 and 1.14, is often used in ergodic theory and is central in relating various symmetries with spectral properties (see Section 5.8).

COROLLARY 1.20. *Suppose U is a unitary representation of a locally compact Abelian group G in the Hilbert space H. Suppose that H decomposes into the orthogonal sum of $k \in \mathbb{N} \cup \{\infty\}$ invariant subspaces and the restrictions of U to all of those subspaces are unitarily equivalent. Then all essential values of the spectral multiplicity are multiples of k.*

Given a collection of elements $v_1, \dots, v_k \in H$, the subspace $H_{v_1,\dots,v_k}$ is defined as the minimal closed U-invariant subspace which contains $v_1, \dots, v_k$. It follows directly from Theorems 1.6 and 1.13 that the maximal multiplicity of a representation U is equal to the infimum of k such that $H = H_{v_1,\dots,v_k}$. In particular, if there is no such finite collection $v_1, \dots, v_k$ then the maximal multiplicity is infinite; this does not imply though that ∞ is an essential value.

In ergodic theory it often happens that one can construct sequences of cyclic subspaces, or, more generally, subspaces generated by a given number of vectors, which approximate every vector sufficiently well. Existence of such a sequence allows to estimate maximal spectral multiplicity from above thus improving the criterion above. For a $v \in H$ and a closed subspace $L \subset H$ let us denote as before by $\mathcal{P}_L v$ the orthogonal projection of the vector v onto L.

THEOREM 1.21. *Let U be a unitary operator on the Hilbert space H. If for every orthonormal m-tuple of vectors $v_1, \dots, v_m \in H$, there exists $w \in H$ such that the cyclic subspace generated by w, H_w satisfies*:

$$\sum_{i=1}^{i=m} \|\mathcal{P}_{H_w} v_i\|^2 > 1$$

then the maximal spectral multiplicity of U is $\leqslant m-1$.

In particular if $m=2$ the representation has simple spectrum.

PROOF. If the spectral multiplicity of U is $\geqslant m$, and if ν is the spectral measure of U we can find, by the spectral Theorem 1.12, a set $\mathcal{A}$ in S^1 such that there exists a U-invariant subspace K of H restricted to which U is isomorphic to the sum of m copies K_i, $i=1,\dots,m$, of the action of M on $L^2(S^1,\chi_{\mathcal{A}}\nu)$ (M as in Theorem 1.8). We choose for v_i, $i=1,2,\dots,m$, the functions $\chi_{\mathcal{A}}/\nu(\mathcal{A})^{1/2}$, $i=1,2,\dots,m$, of the above model. Then there exists $w_i\in K_i$, $i=1,2,\dots,m$, and w' orthogonal to K such that $w=\sum_{i=1}^{i=m}w_i+w'$. Let $\tilde{w}=\sum_{i=1}^{i=m}w_i$. Then $\mathcal{P}_{\mathcal{H}_{\tilde{w}}}v_i=\mathcal{P}_{\mathcal{H}_w}v_i$.

Now, if W is an invariant subspace of $\bigoplus_{i=1}^{i=m}L^2(S^1,\chi_{\mathcal{A}}\nu)=\mathcal{H}$, exactly the same proof as the one which was used for the Wiener lemma gives: $(v_i(x),\mathcal{P}_W v_i(x))=\|\mathcal{P}_W v_i(x)\|^2$ (the scalar product is taken in $\mathcal{H}$). This says that the restriction of $\mathcal{P}_W$ to the fiber at x is a projection on a subspace W_x. Therefore, $\sum_{i=1}^{i=m}\|\mathcal{P}_W v(x)\|^2=\dim W_x$. In our case, $\dim\mathcal{H}_w(x)=1$ for ν almost all x, and we get, if B is the support of the spectral measure of w, $\sum_{i=1}^{i=m}\|\mathcal{P}_{\mathcal{H}_w}v_i\|^2=\nu(B)/\nu(A)\leqslant 1$. □

1.4.4. *Spectral theorem and spectral invariants for one-parameter groups of unitary operators.* By the Stone Theorem any continuous one-parameter groups of unitary operators U_t: $t\in\mathbb{R}$, $U_{t+s}=U_t\cdot U_s$ in a Hilbert space H has the form $U_t=\exp itA$ where A is a Hermitian operator ($A^*=A$). Notice that A is not necessarily bounded. Nevertheless A is uniquely defined on a dense subset as $-i\frac{dU_t}{dt}|_{t=0}$. The operator A or sometimes the skew-Hermitian operator $-iA$ is called the (*infinitesimal*) *generator* of the group U_t. The spectral theorem for one-parameter groups of unitary operators takes the following form.

THEOREM 1.22. *Let $U_t=\exp itA$ be a one-parameter groups of unitary operators in a Hilbert space on H continuous in the strong operator topology. There exists a family of locally finite positive measures on $\mathbb{R}$ uniquely defined up to equivalence* (*and called the spectral types for the group*)

$$\nu_1\geqslant\nu_2\geqslant\nu_3\geqslant\cdots\geqslant\nu_n\geqslant\cdots$$

such that the action of U_t on H is unitarily isomorphic to the multiplication by $e^{2\pi itx}$) on the orthogonal sum

$$\bigoplus_{i\geqslant 1}L^2(\mathbb{R},\nu_i).$$

Accordingly the Hermitian infinitesimal generator A acts as multiplication by the independent variable x in each $L^2(\mathbb{R},\nu_i)$.

Notice that in this case the maximal spectral type and multiplicity function for each individual operator U_t are defined on the circle: they are obtained from the spectral types of the group via the standard projection $\pi_t:\mathbb{R}\to S^1$, $\pi_t(s)=\exp its$. Thus spectral multiplicity of individual operators tend to be greater than for the group. A typical example is the case of Lebesgue spectrum: every non-identity elements of a one-parameter group of unitary operators with simple Lebesgue spectrum has countable Lebesgue spectrum.

Here is a simple but useful criterion of Lebesgue spectrum for one-parameter groups:

PROPOSITION 1.23. *If the one-parameter group of unitary operators U_t is unitarily equivalent to the renormalized group U_{st} for any $s > 0$, then U_t has homogeneous Lebesgue spectrum.*

PROOF. It follows from the assumption that the infinitesimal generator A of U_t is unitarily equivalent to sA. But the spectral measures of sA are obtained from those of A by applying the multiplication by s on the real line. Hence the spectral measures are invariant under these multiplications and Lebesgue is the only type satisfying this property. □

2. Spectral properties and typical behavior in ergodic theory

Now we will consider a single unitary operator $U : H \to H$, or, equivalently, a unitary representation of the group $\mathbb{Z}$. The spectral measures in this case are measures on the circle S^1 (see Section 1.2.2). We will always assume that all measures we are considering are finite. Most of the discussion below can be extended straightforwardly to the case of discrete Abelian groups while in the continuous case certain subtle points appear. We will address some of these points for the case of one-parameter groups of operators, i.e. representations of $\mathbb{R}$.

A NOTE ON TERMINOLOGY. We will apply the spectral notions discussed below for unitary operators to measure preserving transformations if the Koopman operator in the orthogonal complement to the constants possesses the corresponding property. Thus we will speak about transformations with Lebesgue spectrum, mixing, mildly mixing, rigid, and so on. From now on, the scalar product will usually be denoted by $(\cdot, \cdot)$.

2.1. *Lebesgue spectrum*

2.1.1. *Correlation decay.* The maximal spectral type in a cyclic subspace $L \subset H$ is Lebesgue if and only if there exists $v \in L$ such that the iterates $U^n v$, $n \in \mathbb{Z}$, form an orthogonal basis in L. There are natural sufficient conditions for absolute continuity of the spectral measure, e.g., a certain decay rate for the correlation coefficients, such as l^2, but non of such conditions is necessary since an L^1 function on the circle may have very slowly decaying Fourier coefficients. The most general decay condition sufficient to guarantee that the spectral measure is actually equivalent to Lebesgue is an exponential decay

$$(v, U^n v) \leqslant c \exp\bigl(-\beta |n|\bigr)$$

for some positive numbers c and β. For, in this case the Fourier transform of the sequence $(v, U^n v)$ is a real-analytic function on the circle; it is nonnegative since it is a density of a measure and by analyticity it can only have finitely many zeroes.

The corresponding condition in the continuous time case is particularly useful because in that case there is no convenient direct counterpart of the orthogonality condition above.

2.1.2. *Countable Lebesgue spectrum in ergodic theory.* A particular type of spectrum which is ubiquitous in ergodic theory is *countable Lebesgue spectrum*, i.e. the Lebesgue maximal spectral type with the multiplicity function identically equal to ∞. The following criterion is evident from the definition.

A unitary operator $U : H \to H$ has countable Lebesgue spectrum if and only if there exists an infinite-dimensional closed subspace $H_0 \subset H$ such that

(i) *H_0 and $U^n H_0$ are orthogonal for $n > 0$ (or, equivalently for $n \neq 0$), and*

(ii) $H = \bigoplus_{n\in\mathbb{Z}} U^n H_0$.

As we already mentioned in the case of one-parameter group of operators if the infinitesimal generator A of the group has simple Lebesgue spectrum (i.e. the maximal spectral type is the type of Lebesgue measure on the line) then the unitary operators $\exp(-itA)$ have countable Lebesgue spectrum for every $t \neq 0$. Still the term "countable Lebesgue spectrum" is reserved for the case where the generator has Lebesgue spectrum of infinite multiplicity.

Here is a good illustration of how countable Lebesgue spectrum appears in ergodic theory.

EXAMPLE 2.1. Consider an automorphism A of a compact Abelian group G. It preserves Haar measure χ and the Koopman operator maps characters into characters. The characters form an orthonormal basis in $L^2(G, \chi)$. The cyclic subspace of each character is either finite-dimensional (and hence the spectral measure is atomic and the eigenvalues are roots of unity) or Lebesgue where the orbit of the character is infinite. Thus the spectrum of U_A in $L_0^2(G, \chi)$ is in general a combination of pure point and Lebesgue. If A is ergodic (see Section 3.3) the first case does not appear and the spectrum is Lebesgue. It is not difficult to show that the number of orbits in the dual group is always infinite so Lebesgue spectrum is always countable.

This conclusion extends with a slight modification to a more general class of *affine maps* on compact Abelian groups. Such a map is a product (composition) of an automorphism and a translation. In this case again the spectrum in general is a combination of pure point and countable Lebesgue, however it can be mixed even in the ergodic case, see Examples 3.17 and 3.18.

Other standard examples of transformations with countable Lebesgue spectrum are Bernoulli shifts introduced in Example 3.10 (see also [8, Section 3.3e]) and, more generally, transitive Markov shifts [8, Section 3.3f].

2.1.3. *Hyperbolic and parabolic paradigms*

Positive entropy, K-property, hyperbolic behavior. The main source of the presence of countable Lebesgue *part* in the spectrum is positivity of entropy [8, Theorem 3.7.13], [12]; in particular, the completely positive entropy (the K-property) implies that the spectrum in the orthogonal complement L_0^2 to the constants is countable Lebesgue [8, Theorem 3.6.9], [12].

This kind of behavior appears in systems with *hyperbolic* and *partially hyperbolic* behavior [8, Section 6], [7,1,9]. Example 2.1 in the case when the group G is a torus $\mathbb{T}^n$ provides simple particular cases for both hyperbolic and partially hyperbolic situations. For, in this case the automorphism is determined by an integer $n \times n$ matrix with determinant ± 1. The hyperbolic case corresponds to the situation when the matrix has no eigenvalues of absolute value one; partially hyperbolic case appears when there are some such eigenvalues but no roots of unity among them. See [8, Sections 5.1h and 6.5a].

Zero entropy; parabolic behavior. Countable Lebesgue spectrum also appears in many zero entropy systems, sometimes accompanied by a pure-point part. This is typical for the *parabolic* paradigm [8, Section 8] which appears in particular in many systems of algebraic origin and their modifications. See Examples 3.17 and 3.18, Section 6.2.2 and [10] (especially Section 2.3a).

Horocycle flows. Now we will describe a particularly characteristic example of parabolic system which show how Lebesgue spectrum follow from a renormalization arguments.

Let X be the manifold $X = SL(2,\mathbb{R})/\Gamma$ where Γ is a discrete subgroup of finite covolume in $SL(2,\mathbb{R})$. Consider the following one-parameter subgroup of $SL(2,\mathbb{R})$: $H_t = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, $t \in \mathbb{R}$.

The action of H_t by left translations on the right homogeneous space X preserves the measure m on X induced by the Haar volume in $SL(2,\mathbb{R})$. Let us denote this action h_t; it is called the *horocycle flow*.

PROPOSITION 2.2. *Every transformation h_t, $t \neq 0$ has countable Lebesgue spectrum.*

PROOF. Consider the one-parameter diagonal subgroup of $SL(2,\mathbb{R})$; $G_t = \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$ and the corresponding left action of G_t on X by g_t; the latter is called the *geodesic flow*.[1]

Direct calculation shows that the commutation relation $G_t H_s G_{-t} = H_{se^t}$ holds and hence $g_t h_s g_{-t} = h_{se^t}$. Thus the flows h_t and h_{st} for any positive s are metrically and hence spectrally isomorphic. Hence by Proposition 1.23 the horocycle flow has homogeneous Lebesgue spectrum and each transformation h_t, $t \neq 0$ has countable Lebesgue spectrum. □

In fact, it is also true that the horocycle flow (i.e. its infinitesimal generator) has countable Lebesgue spectrum. For this it is enough to show that there are countably many mutually orthogonal subspaces in $L^2(X, m)$ simultaneously invariant under the geodesic and horocycle flows. Then the above argument can be applied separately to each of those subspaces producing Lebesgue spectrum there.

To find such subspaces one can use elements of theory of unitary representations for semisimple Lie groups; in this case $SL(2,\mathbb{R})$. Namely, notice that the whole group $SL(2,\mathbb{R})$

[1]The geometric terminology came from the interpretation of $SL(2,\mathbb{R})$ as the unit tangent bundle to the hyperbolic plane $\mathbb{H}^2$ which can be identified with the symmetric space $SO(2)\backslash SL(2,\mathbb{R})$, see, e.g., [79, Section 17.5].

acts by left translations on X and the corresponding Koopman operators produce a unitary representation of $SL(2,\mathbb{R})$ in $L^2(X,m)$. Consider the compact subgroup of rotations $SO(2) \subset SL(2,\mathbb{R})$. One sees easily that the action of that group decomposes into eigenspaces corresponding all the characters. Each such eigenspace is invariant under the whole group $SL(2,\mathbb{R})$.

2.2. *Mixing and recurrence*

2.2.1. *Mixing*. (See also [8, Section 3.6h].) A measure μ on the circle is called *mixing* (or sometimes *Rajchman*) if its Fourier coefficients (correlation coefficients) $\hat{\mu}_n = \int_{S^1} z^n \, d\mu(z)$ converge to 0 as $n \to \pm\infty$.

By the Riemann–Lebesgue lemma any absolutely continuous measure is mixing. However there are many mixing singular measures as well. To see this notice that the correspondence between taking convolutions and multiplication of Fourier coefficients implies the following

PROPOSITION 2.3. *Convolution of two mixing measures is mixing. If for a measure μ and for some m the* mth *convolution $\mu^{(m)} = \mu * \cdots * \mu$ of μ with itself is mixing, then μ is mixing.*

EXAMPLE 2.4. Let C be the projection of the standard (ternary) Cantor set on the unit interval to the circle. Construct the "uniform" measure μ on C by assigning the measures $1/2^n$ to the intersection of C with the intervals of nth order.[2] This measure is obviously singular. It is however mixing. This can be seen by looking at the convolution $\mu * \mu$ of μ with itself. The convolution is absolutely continuous, and hence mixing (its density is easy to calculate). Thus the Fourier coefficients of μ which are square roots of Fourier coefficients of $\mu * \mu$ also vanish at infinity.

PROPOSITION 2.5. *Any measure absolutely continuous with respect to a mixing measure is mixing.*

PROOF. Let μ be a mixing measure and $\rho \in L^1(S^1,\mu)$. We need to show that $\rho\mu$ is a mixing measure. We will prove decay of correlation coefficients without assuming nonnegativity of ρ. First, notice that multiplication by the independent variable correspond to the shift in Fourier coefficients and hence preserves the decay of correlation coefficients at infinity. Second, this decay is a linear property. Thus for any trigonometric polynomial p the correlation coefficients of the complex measure $p\mu$ decay at infinity. Since trigonometric polynomials are dense in $L^1(S^1,\mu)$, the same property holds for $\rho\mu$. □

REMARK. The above argument naturally can be applied to the case of Lebesgue measure and thus it gives a proof of the Riemann–Lebesgue Lemma.

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[2]This is the Hausdorff measure corresponding to the exponent $\frac{\log 2}{\log 3}$ which is equal to the Hausdorff dimension of C.

Mixing measures can be characterized in a geometric way as being asymptotically uniformly distributed. Let $E_n : S^1 \to S^1$ be the nth power map: $E_n(z) = z^n$. The pull-back $f^*\mu$ of the measure μ under a transformation f is defined by $f^*\mu(A) = \mu(f^{-1}A)$ for any μ-measurable set A.

PROPOSITION 2.6. *A measure μ on the circle is mixing if and only if the sequence $(E_n)^*\mu$ weakly converges to Lebesgue measure as $n \to \pm\infty$.*

PROOF. Since the mth Fourier coefficient of the measure $(E_n)^*\mu$ is equal to $\hat{\mu}_{mn}$, mixing implies that every non-zero Fourier coefficient of $(E_n)^*\mu$ converges to 0 as $n \to \pm\infty$ while the zero Fourier coefficients of all those measures are equal to one. Convergence of Fourier coefficients for probability measures on the circle is equivalent to weak convergence. This proves the "only if" part.

Conversely, weak convergence implies that the first Fourier coefficients of $(E_n)^*\mu$ which are equal to $\hat{\mu}_n$ converge to zero as $n \to \pm\infty$ implying mixing. □

By Proposition 2.5 mixing is a property of an equivalence class of measures. This justifies the following definition.

DEFINITION 2.7. A unitary operator is called *mixing* if some (and hence any) measure of maximal spectral type is mixing.

In fact, mixing can characterized directly:

PROPOSITION 2.8. *A unitary operator U is mixing if and only if U^n converges to* 0 *in the weak operator topology as $n \to \infty$.*

2.2.2. *Rigidity and pure point spectrum*. (See also [8, Section 3.6e].) Rigidity is a property of spectral measures which is opposite to mixing in a natural way.

DEFINITION 2.9. A measure μ on the circle is called *rigid* (or sometimes a *Dirichlet* measure) if $\hat{\mu}_{n_k} \to \mu(S^1)$ for some sequence $n_k \to \infty$.

The contrast between rigidity and mixing is seen from the following geometric characterization.

PROPOSITION 2.10. *The measure μ is rigid if and only if for certain sequence $n_k \to \infty$ the sequence of measures $(E_{n_k})^*\mu$ weakly converges to a δ-measure.*

LEMMA 2.11. *If for a certain sequence $n_k \to \infty$ $\hat{\mu}_{n_k} \to \alpha\mu(S^1)$ where $|\alpha| = 1$, then for any $m \in \mathbb{Z}$, $\hat{\mu}_{mn_k} \to \alpha^m \mu(S^1)$.*

PROOF. Fixing m, for every ε, there exists k_0 such that for $k > k_0$ if $A_k = \{\theta \in S^1$: $|e^{2\pi i n_k \theta} - e^{2\pi i \alpha}| > \varepsilon\}$, then $m(A_k) < \varepsilon^2$. The conclusion now follows from the fact that, for complex numbers z_1 and z_2 such that $|z_1| = |z_2| = 1$, $|z_1^m - z_2^m| \leqslant m|z_1 - z_2|$. □

PROOF OF THE PROPOSITION. If $(E_{n_k})^*\mu \to \delta_\alpha$, then by Lemma 2.11 for any natural number p, $(E_{pn_k})^*\mu \to \delta_{\alpha^p}$ and we are able to produce a sequence $l_k = p_k n_k$ such that $(E_{l_k})^*\mu \to \delta_1$, which is just rigidity since the first Fourier coefficient of the measure $(E_{l_k})^*\mu)$ is equal to $\hat{\mu}_{l_k}$. This also gives the converse. □

COROLLARY 2.12. *Any measure absolutely continuous with respect to a rigid measure is rigid.*

Thus rigidity like mixing is also a property of an equivalence class of measures and hence one can speak about *rigid* unitary operators.

PROPOSITION 2.13. *Any atomic measure on S^1 is rigid.*

PROOF. For any atomic measure all but arbitrary small measure is concentrated on a finite set. But for any finite set $\lambda_1, \ldots, \lambda_n \in S^1$ one can find a sequence $n_k \to \infty$ such that $\lambda_i^{n_k} \to 1$, $i = 1, \ldots, n$. □

For a given unitary operator U the closure of powers U^n, $n \in \mathbb{Z}$ in the strong operator topology is a useful object whose structure is related to the spectral properties of U. First, all of its elements are unitary operators, and it forms a Polish Abelian group under composition. Let us denote this group by $\mathcal{G}(U)$.

It follows from the definition of rigidity that the operator U is rigid if and only if the group $\mathcal{G}(U)$ is perfect.

Notice that the group of Koopman operators is a closed subgroup of the group of all unitary operators in the strong operator topology (this is not true in weak topology). Thus we have the following useful corollary.

COROLLARY 2.14. *Any rigid measure preserving transformation T of Lebesgue space has an uncountable centralizer, i.e. there are uncountably many measure preserving transformations commuting with T.*

In fact, unitary operators with pure point spectrum (i.e. operators whose maximal spectral type is atomic) can be characterized by a property stronger than rigidity.

PROPOSITION 2.15. *A unitary operator U has pure point spectrum if and only if the group $\mathcal{G}(U)$ is compact.*

Thus, for any transformation with pure point spectrum a certain compact Abelian group can be associated. It is not surprising then that such transformations can be represented as shifts on compact Abelian groups. See Section 3.4.3 for a detailed discussion. At the moment we just notice that given a compact Abelian group G any translation on G preserves Haar measure χ and has pure point spectrum since characters are eigenfunctions for it and characters form an orthonormal basis in $L^2(G, \chi)$. Let us illustrate this by some concrete examples. Recall that a measure preserving transformation is *ergodic* if any invariant measurable set is either a null-set or has null-set complement.

EXAMPLE 2.16.

(1) *Circle rotation*. Let for $\alpha \in \mathbb{R}$

$$R_\alpha : S^1 \to S^1, \quad R_\alpha x = x + \alpha \pmod 1.$$

This rotation is ergodic if and only if α is irrational.

(2) *Translation on the torus*. For a vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ the translation T_α of the n-torus

$$T_\alpha x = x + \alpha \pmod 1$$

is ergodic with respect to Haar measure is and only if $\alpha_1, \dots, \alpha_n$ and 1 are independent over rationals.

The translation vector α is sometimes called the vector of *frequencies* and the rational relations between its components are called *resonances*. Even if there are no resonances there may be near resonances which play important role in causing complications when the translation is modified in some way.

(3) *Adding machine* or *odometer*. An *adding machine* is an ergodic transformation with pure point spectrum all of whose eigenvalues are roots of unity. In other words, it is an ergodic shift on the dual to a subgroup of $\mathbb{Q}/\mathbb{Z}$, or by duality on a factor of the group of ideles (Example 1.4). It can also can be characterized as the inverse limit of cyclic permutations.

For example, a translation T_{x_0} on the group $\mathbb{Z}_p$ of p-adic integers (p prime) is ergodic is and only if x_0 is not divisible by p.

(4) *Shifts on solenoids*. A solenoid is the inverse limit of tori of the same dimension.

2.2.3. *Mild and weak mixing*. (See also [8, Sections 3.6f,g].)

DEFINITION 2.17. A measure μ on the circle is called *mildly mixing* if no measure absolutely continuous with respect to μ is rigid.

Notice that given a sequence $n_k \to \infty$, the space of all functions $f \in L^2(X, \mu)$ for which $U_T^{n_k} f \to f$ is a unitary $*$-subalgebra. Hence by Proposition 1.2 if U_T is not mildly mixing, T has a rigid factor. Thus

T is mildly mixing if and only if it has no non-trivial rigid factors.

Proposition 2.13 implies that any mildly mixing measure is continuous (non-atomic). The following characterization justifies calling non-atomic measures *weak* (*or weakly*) *mixing*.

Recall that a subset $S \subset \mathbb{Z}$ is called a *set of full density* if

$$\lim_{n\to\infty} \frac{S \cap [-n, n]}{2n+1} = 1.$$

PROPOSITION 2.18. *A measure μ on the circle is non-atomic if and only if for a set S of full density*

$$\lim_{n\in S, n\to\pm\infty} \hat{\mu}_n = 0.$$

SKETCH OF PROOF. Let Δ be the diagonal of $S^1 \times S^1$. Fubini's theorem implies that $(\mu \times \mu)(\Delta) = \sum |\hat{\mu}(\{\lambda\})|^2$, where the summation is taken over the atoms of μ. Now we have

$$\int \frac{1}{N} \sum_{n=1}^{N} \exp\bigl(2i\pi n(x-y)\bigr)\, d(\mu \times \mu) = \frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}_n|^2.$$

By Lebesgue theorem the left-hand side of this last equality converges, when $N \to +\infty$, to $\mu \times \mu(\Delta)$. A simple calculation shows the equivalence between $\lim_{N\to\infty} \frac{1}{N} \sum_1^N |\hat{\mu}_n|^2 = 0$ and

$$\lim_{n\in S, n\to\pm\infty} \hat{\mu}_n = 0$$

for a set S of full density. □

While checking convergence along a sequence of full density may present problems there is an alternative criterion which is often convenient in the context of ergodic theory.

PROPOSITION 2.19. *An equivalence class of measures on S^1 is non-atomic if and only if there exists a sequence $n_k \to \infty$ such that for any measure μ from this class* (*or, equivalently, for an L^1 dense set of such measures*) $\hat{\mu}_{n_k} \to 0$.

2.2.4. *An elliptic paradigm*

Diophantine and Liouvillean behavior. Simple rigid spectrum, whether atomic, mixed or continuous, is the second type (after countable Lebesgue spectrum) ubiquitous in ergodic theory and other branches of dynamics. These spectral properties are associated with the *elliptic* behavior [8, Section 7] in its two manifestations, Diophantine and Liouvillean [47]. Simplicity of the spectrum relies on criteria like Theorem 1.21, rigidity on Proposition 2.10.

Diophantine paradigm is associated with rather simple and fully understood type of behavior: pure point spectrum with frequency vector which avoids too close near resonances, see Example 2.16(2); it is of great importance in classical mechanics due to KAM theory [30].

Liouvillean behavior is associated with simple singular (and usually continuous) rigid spectrum and with a very fast periodic approximation, see Proposition 5.39, and for more details, [81,78]; it is typical in the weak topology in the space of measure preserving transformations and various other spaces of dynamical systems, see Theorems 5.47 and 5.49, [47,78]. Although more exceptional from the point of view of classical and Hamiltonian mechanics, it is still unavoidable in typical perturbations of completely integrable systems, twist maps and so on [70].

Time change in linear flows on $\mathbb{T}^2$. We will present now the most classical and very simple situation where non-trivial Liouvillean behavior appears.

We begin with an irrational linear flow on the two-dimensional torus. We will denote cyclic coordinates on $\mathbb{T}^2$ by x and y. Given a vector $\gamma = (\gamma_1, \gamma_2)$ the flow $\{T_\gamma^t\}$ is generated by the constant vector field with coordinates γ_1, γ_2 and has the form

$$T_\gamma^t(x, y) = (x + t\gamma_1, y + t\gamma_2) \pmod 1.$$

We assume that the slope γ_1/γ_2 is irrational which is equivalent to minimality or ergodicity of the flow.

Now consider a time change of the flow. Namely take a positive C^∞ function ρ and consider the flow generated by the vector field $(\rho\gamma_1, \rho\gamma_2)$. Denote the new flow by $\{S^t\}$. This flow preserves the smooth measure $\rho^{-1}\lambda$. It is also rigid, and has simple spectrum. If the number γ_1/γ_2 is *Diophantine*, i.e. there exist positive numbers N and C such that for any integers p and q,

$$|\gamma_1/\gamma_2 - p/q| > C/q^N$$

then there exists a C^∞ diffeomorphism preserving the orbits which conjugates the flow $\{S^t\}$ to a linear flow and hence has pure point spectrum. This goes back to Kolmogorov [94], see [8, Section 7.3], [78, Section 11.2] for proofs and discussions. Thus in the Diophantine situation the orbit structure of time changes is rigid.

On the other hand, if the slope is not a Diophantine number then generically in the C^∞ topology for ρ the flow $\{S^t\}$ is weakly mixing [42] (see also [78, Section 13.3] for related results and historical discussion). Furthermore, for some special values of the slope one can find a real-analytic ρ for which the flow has mixed spectrum [48]. We will continue discussion of this and similar situations in Section 5.6.3.

2.3. *Homogeneous systems*

We briefly mention now a very important class of dynamical systems which is discussed in much greater detail in [10]; see also [4], especially Section 3.

Let G be a Lie group, $\Gamma \subset G$ a lattice, i.e. a discrete group with the factor of finite volume (compact or not). A *homogeneous dynamical system* is the action of a subgroup $H \subset G$ on the homogeneous space G/Γ by left translations. Both horocycle and geodesic flows are examples of homogeneous dynamical systems where H is a one-parameter subgroup. Even more basic examples are translations on the torus or one-parameter groups of such translations (linear flows).

Homogeneous systems possess large symmetry since any such system is a part of a transitive action of G by left translations. Due to this symmetry spectral analysis of homogeneous dynamical systems can be carried out with the help of the theory of unitary representations of Lie groups (see [10, Section 2.3]). While ergodic properties of homogeneous flows may be complicated and surprising (see Section 6.2.2) their spectral properties are rather simple.

If H is a one-parameter subgroup of G the left action by H is called a *homogeneous flow*.

THEOREM 2.20 [25]. *The spectrum of any homogeneous flow is the sum of pure point and countable Lebesgue.*

A similar conclusion holds for *homogeneous maps*, i.e. the homogeneous actions of $\mathbb{Z}$; this of course follows immediately from Theorem 2.20 for most homogeneous maps since such maps are parts of homogeneous flows.

This is similar to the case of automorphisms and affine maps on compact Abelian groups (Section 2.1.2).

3. General properties of spectra for measure preserving transformations and group actions

3.1. *The realization problem and the spectral isomorphism problem*

3.1.1. *Formulation of the problems.* Unitary operators which appear as Koopman operators associated with measure preserving transformations and, more generally, group actions, possess some special properties. The interface between the unitary operator theory (and the theory of unitary group representations), and ergodic theory centers on two general problems:

SPECTRAL REALIZATION PROBLEM. What are possible spectral properties for a Koopman operator or a group of such operators?

SPECTRAL ISOMORPHISM PROBLEM. Given two Koopman operators U_T and U_S (or groups) which are unitarily equivalent (i.e. have the same spectral invariants) what extra information is needed to conclude that the measure preserving transformations T and S (or the corresponding group actions) are isomorphic? More specifically, one is interested in the cases when such extra non-spectral invariants can be reasonably described, and, in particular, when they are not needed at all.

Both of those problems go back to the founding text of the modern ergodic theory, the 1932 article by John von Neumann [155]. Concerning the Spectral Realization Problem there are very few known restrictions, all of them quite general. The proofs are not difficult and all results in this direction will be presented in the rest of this section.

Still, many simply sounding questions are unanswered. Here is a famous example.

PROBLEM 3.1. Does there exist a measure preserving transformation whose Koopman operator has simple Lebesgue spectrum (or even Lebesgue spectrum of bounded multiplicity) in $L_0^2(X, \mu)$.

On the other hand, there is a large number of "positive" results asserting existence of measure preserving transformations with specific properties. This is achieved via a variety of specific constructions. Essentially the whole later part of this survey is dedicated to the development of these constructions and presenting specific examples, sometimes natural, sometimes "exotic".

3.1.2. *Elementary restrictions*

Invariance of constants. The most basic restriction on the spectral realization is presence of eigenvalue 1 in the spectrum, since constants are invariant functions. Thus the maximal spectral type of any Koopman operator always has an atom at 1. Due to this simple observation by spectral properties of a measure preserving transformation one usually means the corresponding properties of the operator U_T in the orthogonal complement to the space of constants, i.e. the space $L_0^2(X, \mu)$ of square-integrable functions with zero average. Sometimes, however, it is useful to remember presence of the atom at one which we will naturally denote by δ_1. Let μ_0 be a measure of maximal spectral type in the space $L_0^2(X, \mu)$. Then μ, the maximal spectral type in $L^2(X, \mu)$ can be represented by $\mu_0 + \delta_1$. Consider the convolution

$$\mu * \mu = (\mu_0 + \delta_1) * (\mu_0 + \delta_1) = \mu_0 * \mu_0 + \mu_0 + \delta_1 = \mu_0 * \mu_0 + \mu > \mu$$

(equality and inequality signs refer to measure types).

This simple fact can be expressed in a way useful for the discussion of the convolution problem (Section 3.5).

PROPOSITION 3.2. *The maximal spectral type μ of the Koopman operator in the whole space $L^2(X, \mu)$ is dominated by its convolution $\mu * \mu$.*

Symmetry of the spectrum. Another, almost equally basic, is the symmetry of both the maximal spectral type and the multiplicity function with respect to the involution $\chi \to \chi^{-1}$ of the dual group G^*. This immediately follows from the fact the Koopman operator preserves the complex conjugation in $L^2(X, \mu)$.

In particular this implies the following restriction of the spectral realization.

PROPOSITION 3.3. *Any Koopman operator is unitarily isomorphic to its inverse.*

3.2. *Rokhlin lemma and its consequences*

3.2.1. *The Rokhlin lemma.* Recall that an action of a group is called *free* if the stationary subgroup of almost every point in the space is trivial; for $\mathbb{Z}$ this means aperiodicity.

The Rokhlin lemma gives a way to produce an approximate section for a free action for certain kinds of discrete groups, and therefore to control large pieces of orbits on a large part of the space.

In general, this is related to the existence of sets in the group which tile it. A set A in a group G is said to *tile* G if there exists a family of elements of G, g_i, $i \in I$, such that $G = \bigcup_{i \in I} g_i A$ and the sets $g_i A$ are mutually disjoint. All countable Abelian groups can be endowed with Fölner sequences such that every set in the sequence tiles the group, and therefore a version of the Rokhlin lemma can be stated in this framework. The proper setting for the most general version of the Rokhlin lemma is in fact for actions of countable amenable group. In this case there need not be any set in a Fölner sequence which tiles the group. However the group can always be almost tiled by a finite number of elements in the Fölner sequence which furthermore can be chosen as invariant as one wishes. This gives rise to the Ornstein–Weiss version of the Rokhlin lemma for amenable groups which has many important applications and in particular is the first key step in extending the Ornstein isomorphism theory to actions of arbitrary amenable groups [121,122].

Rokhlin [137] considered only $\mathbb{Z}$ actions, see [12, Section 5] for a proof in that case. Here we consider a free measure preserving action of $\mathbb{Z}^d$ on a measure space (X, μ), see [28,84].

THEOREM 3.4. *Consider a free action of $\mathbb{Z}^d$ on (X, μ) generated by d commuting automorphisms $T_1, T_2, \ldots, T_d$. For every $\varepsilon > 0$, and an integer N, there exists a set $F \subset X$ such that the sets $T_1^{n_1} T_2^{n_2}, \ldots, T_d^{n_d} F$, $0 \leqslant n_1, n_2, \ldots, n_d \leqslant N - 1$, are mutually disjoint and their union has measure greater than $1 - \varepsilon$.*

REMARK. Notice that the assumption of freeness which is natural in the ergodic theory setting is very restrictive in other branches of dynamics such as topological dynamics [8, Section 2] or theory of smooth dynamical systems [8, Section 5] since periodic orbits form an important ingredient of the orbit structure in many cases. For example, for hyperbolic systems [8, Section 6] periodic orbits are dense.

PROOF. To simplify notations we consider the case $d = 2$, i.e. we consider a $\mathbb{Z}^2$ action on (X, μ) generated by two commuting measure preserving transformations S and T. Fix an integer $L > N^2/\varepsilon^2$. Since the action is free one can find, using the ergodic decomposition (Section 3.3), a measurable set A such that:

(1) the sets $S^i T^j(A)$, $-L < i, j < +L$ are pairwise disjoint

(2) $\mu_y(A) > 0$ for almost every y in the ergodic decomposition.

Thus $\bigcup_{m \geqslant 0, n \geqslant 0} S^m T^n A = X$ and there exists M' such that for all $M > M'$, $\mu(\bigcup_{0 \leqslant m, n \leqslant M} S^m T^n A) > 1 - \varepsilon^2$. For an element $x \in X$ its itinerary is an element ω in $\{0, 1\}^{\mathbb{Z} \times \mathbb{Z}}$ where $\omega_{i,j} = 1$ if $S^i T^j x$ is in A, $\omega_{i,j} = 0$ otherwise. We call M-itinerary the restriction of the previous itinerary to the values of (i, j) which lie inside the square $C_M = \{(i, j)\colon 0 \leqslant i, j \leqslant M - 1\}$. An itinerary ω being given, we consider $Y_\omega \subset \mathbb{Z}^2$ the union of these indices $(i, j) \in \mathbb{Z}^2$ such that $\omega_{i,j} = 1$. We call $(C_y, y \in Y_\omega)$ the tiling of $\mathbb{R}^2$ determined by the Voronoi cells

$$C_y = \bigl\{x\colon |x - y| < |x - y'| \text{ for all } y' \neq y \text{ in } Y_\omega\bigr\}.$$

For every such cell C_y ,we call y its center. We consider the partition P_M of A whose atoms are made of points which have the same M-itinerary. A cell C_y being given, we consider T_y

the union of the squares of the tiling of $\mathbb{Z}^2$ by squares of size N with base point y which lie entirely inside C_y. Each square c in T_y is of the form $(i_c, j_c \leqslant i, j \leqslant i_c+N-1, j_c+N-1)$; (i_c, j_c) is called the base of the square c. Assume that

(3) M is large enough (and in particular greater than M') so that the union of those y in A such that C_y is not contained in a square of size less than $\varepsilon^2 M$ occupies a fraction less that ε^2 of A.

For $p \in P_M$ with itinerary ω_p, we consider F_p the union of $S^{i_c}T^{j_c}(p)$ for all i_c, j_c which are the bases of squares in T_y for all $y \in Y_{\omega_p}$ such that C_y is entirely in C_M. Let $F = \bigcup_{p\in P_M} F_p$. Clearly the sets S^iT^jF, $(i, j) \in C_N$ are pairwise disjoint. (1), (2) and (3) imply that the measure of their union is greater than $1 - \varepsilon$. □

3.2.2. *Density of the maximal spectral type*. An important corollary of the Rokhlin lemma is the following restriction on the spectral realization.

THEOREM 3.5. *The support of the spectral measure of the Koopman operator for an aperiodic transformation is the whole circle S^1*.

PROOF. If λ is not in the support of the spectral measure of $U_{\mathbb{T}}$ then $U_T - \lambda \times \mathrm{Id}$ is invertible in $L^2(X, \mathcal{A}, m)$. However, for every ε, for every λ there exists $f \in L^2(X)$ such that $\|f\| = 1$ and $\|U_T f - \lambda f\| < \varepsilon$. This is sufficient to imply what we asserted. Given ε and λ, we construct f in the following way: take $n \ll 1/\varepsilon$, and take a set F, given by the Rokhlin lemma, such that the family of sets T^iF, $0 \leqslant i \leqslant n-1$, is a disjoint family and such that the measure of their union is $\geqslant 1-\varepsilon$. Define now f as taking the constant value λ^i on T^iF, $0 \leqslant i \leqslant n-1$, 1 on the complement of the union of the T^iF. □

3.2.3. *Combinatorial constructions*. Rokhlin lemma has an interesting "negative" aspect. It implies that all asymptotic behavior of a measure preserving transformation depends on sets of arbitrary small measure and hence can be altered in an arbitrary way by changing the action on such a set. In the case of a single transformation this can be rephrased by saying that if one defines the *uniform topology* by the metric

$$d_u(T, S) = \mu\{x\colon\ Tx \neq Sx\}$$

then

PROPOSITION 3.6. *Conjugates of any aperiodic transformation are dense in the uniform topology in the set of all aperiodic measure preserving transformations*.

PROOF. Fixing n and ε construct Rokhlin towers with given n and ε for two aperiodic transformations T and S. Thus the towers have the form T^iF and S^iF', $i = 0, 1, \ldots, n-1$, correspondingly. Without loss of generality we may assume that the *bases* F and F' of two towers have the same measure. Pick some measure preserving transformation $h : F \to F'$

and define H on $T^i F$ as $S^i \circ h \circ T^{-i}$ for every $i = 0, 1, \dots, n-1$. Complete H in an arbitrary way to a measure preserving transformation of X. Obviously

$$H \circ T \circ H^{-1} = S \quad \text{on} \bigcup_{i=0}^{n-1} S^i F',$$

hence

$$d_u\big(H \circ T \circ H^{-1}, S < \varepsilon\big). \qquad \square$$

This is somewhat deceptive however. Small sets determining asymptotic behavior become more and more complicated as their measure decreases.

A related fact is that the base F of a Rokhlin tower and its images although of small measure normally become "diffused" all over the space. The idea of looking at transformations for which the level sets of Rokhlin towers stay sufficiently "compact" leads to the notion of *rank* (Section 5.2.2) and the concept of *periodic approximation* (Section 5.4) as well as to the class of constructions known as *cutting and stacking* discussed in Section 5.2.

3.3. *Ergodicity and ergodic decomposition*

3.3.1. *Definitions*

DEFINITION 3.7. A measure preserving transformation $T : (X, \mu) \to (X, \mu)$ is *ergodic* if 1 is a simple eigenvalue of the Koopman operator U_T.

Equivalently, T is ergodic is any T-invariant measurable set A is either null ($\mu(A) = 0$) or co-null ($\mu(X \setminus A) = 0$).

For an arbitrary measure preserving transformation T consider the space I_T of invariant functions for the Koopman operator U_T. This space is generated by characteristic functions of invariant sets and by multiplicativity the product of U_T-invariant functions is also U_T invariant. Thus I_T is a unitary subalgebra of $L^2(X, \mu)$ (Proposition 1.2) and hence it defines a factor of T on which T obviously acts as the identity. Denote the measurable partition corresponding to that factor by η_T. The transformation T acts on elements of this partition preserving the system of conditional measures. Ergodic Decomposition Theorem [8, Theorem 3.4.3] states that for almost every $c \in \eta_T$ T acts ergodically with respect to the conditional measure μ_c. See [8, Sections 4.2d, 4.2e] for a more detailed discussion and references to detailed proofs.

DEFINITION 3.8. A measure preserving transformation is called *totally ergodic* if any of its non-zero powers is ergodic.

Total ergodicity is equivalent to the absence of roots of unity (other than 1 itself) among the eigenvalues. The inverse limit of totally ergodic transformations is totally ergodic.

Adding machines from Example 2.16(3) are interesting examples of ergodic but not totally ergodic transformations. This simple property is important in various non-spectral aspects of ergodic theory. A typical situation where total ergodicity plays a role is the following: when one considers the ergodic averages of an L^2 function taken at iterates which are perfect squares, there is convergence in L^2 and also almost everywhere (this is a deep theorem of Bourgain [23]); however the limit is the integral of the function only in the case when the transformation is totally ergodic.

3.3.2. *Ergodicity and spectrum*. Thus, the study of spectral properties of general measure preserving transformations can be separated into two questions: (i) finding ergodic decomposition, in particular establishing ergodicity, and (ii) studying spectral properties of the operators which appear on the ergodic components. Establishing ergodicity for a particular transformation or a class of transformation may be highly non-trivial. However in this survey we will primary (although not exclusively) discuss spectral and other closely related properties for ergodic measure preserving transformations. The argument for separating the study of ergodic decomposition from spectral analysis in the ergodic case may be illustrated by the following example which demonstrate that some properties of ergodic decomposition are non-spectral.

EXAMPLE 3.9. Let T and S be two ergodic measure preserving transformations on the measure spaces (X,μ) and (Y,ν) respectively. For any $0<t<1$ consider the space $X_t \stackrel{\text{def}}{=} X \cup (Y \times [0,t])$ with the probability measure $\mu_t \stackrel{\text{def}}{=} (1-t)\mu + \nu \times \lambda$, where λ is Lebesgue measure. Let T_t be defined on X_t as T on the X part and as $S \times \mathrm{Id}$ on the $Y \times [0,t]$. Obviously the spaces of ergodic components for X_t for different t are not isomorphic because this space contains exactly one atom of measure t. Hence T_t for different t are not isomorphic. However, they are spectrally isomorphic since they all have countable multiplicity for the eigenvalue one and the spectrum in the orthogonal complement to invariant functions is the union of the spectrum of U_T and the spectrum with the maximal spectral type of U_S and countable multiplicity.

3.3.3. *Difference between spectral and metric isomorphism in the ergodic case*

Entropy as an extra invariant. The following classical example shows that the Spectral Isomorphism Problem is non-trivial even in the ergodic situation.

EXAMPLE 3.10. Consider the Bernoulli shift σ_N on the space Ω_N of bi-infinite sequences of an alphabet N symbols provided with the product measure μ_p where $p = (p_0,\dots,p_{N-1})$ is a probability distribution on the alphabet.

The spectrum of this transformation is always countable Lebesgue. This can be readily seen as follows. Let for $n \in \mathbb{Z}$, H_n be the subspace of $L^2_0(\Omega_N,\mu_p)$ of all functions which depend only on coordinates ω_k of the sequence $\omega \in \Omega_N$ with $k \leqslant n$. By definition of the shift one has $U_{\sigma_N} H_n = H_{n+1}$. The spaces H_n generate $L^2_0(\Omega_N,\mu_p)$ since every function can be approximated by a function which depends only on finitely many coordinates. Similarly $\bigcap_{n\in\mathbb{Z}} H_n = \{0\}$. Now let G_n be the orthogonal complement to H_n in H_{n+1}. Obviously these spaces are infinite-dimensional; they are orthogonal to each other by definition, $U_{\sigma_N} G_n = G_{n+1}$ and $\bigoplus_{n\in\mathbb{Z}} G_n = L_0^2(\Omega_N, \mu_p)$.

However the *entropy* $-\sum_{i=0}^{N-1} p_i \log p_i$ is an invariant of metric isomorphism [8, Section 3.7] so there are uncountably many non-isomorphic measure preserving transformation with countable Lebesgue spectrum.

This example directly extends to the case of $\mathbb{Z}^k$ actions and less directly to the continuous-time case [12].

In the case of zero entropy extra invariants including Kushnirenko's sequence entropy [97] and *slow entropy* [8, Section 3.7l], [83] sometimes distinguish spectrally isomorphic systems; see [97] for a classical example of non-isomorphic flows with countable Lebesgue spectrum and zero entropy which are distinguished by sequence entropy.

Asymmetry of metric isomorphism. Entropy shares with the spectral invariants the property of being symmetric with respect to the reversal of time [8, Section 3.7i(4)] and thus never distinguishes a transformation from its inverse. However there are instances where T and T^{-1} are not metrically isomorphic. The earliest examples of that phenomena were found in 1968 by S. Malkin [110] and are not particularly exotic: the spectrum is simple and the transformation itself is a two-point extension of an irrational rotation R_α with only four discontinuity points. These transformations have zero entropy. An interesting criterion which helps to decide whether a transformation T is conjugate to its inverse is in [66]. It implies for example that is the square of the conjugating map S is ergodic then all essential values of the multiplicity function for T are even.

3.4. *Pure point spectrum and extensions*

3.4.1. *Multiplicative structure of eigenfunctions*. As we pointed out, ergodicity is a spectral invariant: it is equivalent to 1 being a simple eigenvalue.

The complex conjugate of an eigenfunction is also an eigenfunction with the complex conjugate eigenvalue.

Ergodicity implies that eigenfunctions have constant absolute value: if $U_T f = \lambda f$ then

$$U_T(f \cdot \bar{f}) = U_T(f) \cdot U_T(\bar{f}) = \lambda\bar{\lambda} f \bar{f} = f \bar{f},$$

hence $f\bar{f} \equiv \text{const}$. Furthermore, both the eigenfunctions and the eigenvalues for an ergodic transformation form a group invariant under complex conjugation. Consequently linear combinations of eigenfunctions form an $*$-algebra and hence their L^2 closure is an invariant unitary $*$-subalgebra of $L^2(X, \mu)$ which we will denote by $\mathcal{K}(T)$. Thus by Proposition 1.2 $\mathcal{K}(T)$ determines a factor of T called the *Kronecker factor* of T. We will denote this factor transformation by $T_{\mathcal{K}}$; it is the maximal factor with pure point spectrum [156, 29]. The measurable partition corresponding to the Kronecker factor will also be denoted by $\mathcal{K}(T)$.

3.4.2. *The isomorphism theorem*. In the case of pure point spectrum the Spectral Isomorphism Problem has a complete and optimal solution.

THEOREM 3.11 (von Neumann Discrete Spectrum Theorem). *Any two ergodic measure preserving transformations with pure point spectrum that are spectrally isomorphic* (*i.e. have the same groups of eigenvalues*) *are metrically isomorphic. A complete system of invariants is given by the countable subgroup $\Gamma \subset S^1$ of eigenvalues.*

SKETCH OF PROOF. Let $T : (X, \mu) \to (X, \mu)$ be an ergodic measure preserving transformation with pure point spectrum and let Γ be the group of eigenvalues for U_T. Let x_0 be a common Lebesgue point for all eigenfunctions of U_T. Denote for each eigenvalue $\gamma \in \Gamma$ by f_γ the unique eigenfunction for which the Lebesgue value at x_0 is 1. Then

$$f_{\gamma_1 \gamma_2} = f_{\gamma_1} f_{\gamma_2}. \tag{3.1}$$

Now identify Γ with the group of characters of the compact dual group Γ^* and denote the character on Γ^* corresponding to the evaluation at γ by χ_γ. Thus, we have orthonormal bases $\{f_\gamma\}_{\gamma \in \Gamma}$ and $\{\chi_\gamma\}_{\gamma \in \Gamma}$ in the Hilbert spaces $L^2(X, \mu)$ and $L^2(\Gamma^*, \lambda)$ correspondingly, where λ is the normalized Haar measure.

Now extend the correspondence $f_\gamma \to \chi_\gamma$ by linearity to a unitary operator $V : L^2(X, \mu) \to L^2(\Gamma^*, \lambda)$, which is multiplicative on the eigenfunctions by (3.1) and preserves complex conjugation. Their finite linear combinations are dense in $L^2(X, \mu)$, so V is generated by a measure preserving invertible transformation $H : (X, \mu) \to (\Gamma^*, \lambda)$. One immediately sees that $V U_T V^{-1} \chi_\gamma(s) = \gamma \chi_\gamma(s) = \chi_\gamma(s_0 s)$ for any $s \in \Gamma^*$, hence $H \circ T \circ H^{-1} = L_{s_0}$. □

For another proof see Section 4.1.2(6). See also [29, Section 12.2] for yet another proof and detailed discussion.

3.4.3. *Representation by compact Abelian groups translations*

THEOREM 3.12. *An ergodic transformation with pure point spectrum whose group of eigenvalues is Γ is metrically isomorphic to the translation on the compact group Γ^* of characters of Γ, considered as a discrete group, by the character s_0 that defines the inclusion $\Gamma \hookrightarrow S^1$. The invariant measure is Haar measure.*

Furthermore, every countable subgroup of the unit circle appears as the group of eigenvalues for an ergodic measure preserving transformations of a Lebesgue space with pure point spectrum.

Thus, translations on compact Abelian groups provide universal models for ergodic transformation with pure point spectrum. This justifies looking for criteria of ergodicity for such translations as well as considering characteristic examples.

PROPOSITION 3.13. *Translation T_{h_0} on a compact Abelian group H, $T_{h_0}(h) = h h_0$ is ergodic with respect to Haar* (*Lebesgue*) *measure if and only if for any character $\chi \in H^*$ $\chi(h_0) \neq 1$.*

Furthermore, ergodicity with respect to Haar measure is equivalent to topological transitivity, minimality and unique ergodicity.

Recall that the *weak topology* on the group of all measure preserving transformations of a Lebesgue space coincides with the strong operator topology for the Koopman operators.

PROPOSITION 3.14. *The centralizer of an ergodic translation T_{h_0} on a compact Abelian group H in the weak topology on the group of all Haar measure preserving transformations of H consists of all translations of H.*

This implies that ergodic transformations with pure point spectrum possess a certain kind of rigidity: Isomorphism and factor maps between such systems are rather limited.

Notice that the centralizer described in Proposition 3.14 coincides with the closure $\mathcal{G}(U_T)$ of the powers of U_T. By Proposition 2.15 if T has pure point spectrum then $\mathcal{G}(U_T)$ is a compact Abelian group. The multiplication by U_T is a translation on that group which preserves Haar measure χ. It follows from Theorems 1.6 and 3.11 that

PROPOSITION 3.15. *If a measure preserving transformation T has pure point spectrum than the multiplication by U_T on $(\mathcal{G}(U_T), \chi)$ is metrically isomorphic to T.*

3.4.4. *Invariance of the spectrum with respect to the discrete part.* By comparing the correlation coefficients for an arbitrary function $g \in L_0^2$ with those of the function $f \cdot g$ where f is an eigenfunction of absolute value one with the eigenvalue $\exp 2\pi i\alpha$ one sees that the spectral measure λ_{gf} is obtained from λ_g by rotation by α. The same argument applies to orthogonal functions with the same spectral measure. Hence we obtain the following general spectral property of measure preserving transformations.

THEOREM 3.16. *The maximal spectral type and the multiplicity function of the operator U_T induced by an ergodic measure preserving transformation T is invariant under multiplication by any eigenvalue.*

3.4.5. *The Kronecker factor.* By Theorem 3.12 the Kronecker factor defined in 3.4.1 is isomorphic to a particular translation on the dual to the group of eigenvalues. The Kronecker factor is the simplest example of a *characteristic factor* for an ergodic measure preserving transformation. Other examples include the maximal distal factor defined in the next subsection whose characteristic property appears in Proposition 4.5.

As was explained in Section 1.1.1 T itself is isomorphic to a skew product transformation over its Kronecker factor.

EXAMPLE 3.17 (*Affine twist on the torus*). An *affine* map of an Abelian group is a composition of an automorphism and a translation. Fix an irrational number α and consider the following affine map of $\mathbb{T}^2$:

$$A_\alpha(x, y) = (x + \alpha, x + y) \pmod 1.$$

This map has mixed spectrum. The Kronecker factor is the circle rotation R_α, the spectrum in the orthogonal complement to this factor is countable Lebesgue. This is the simplest example of a transformation with a *quasi-discrete spectrum* [13].

Transformations with quasi-discrete spectrum provide easiest examples of ergodic spectrally isomorphic transformations with zero entropy which are not metrically isomorphic. This possibility was mentioned in a different context in Section 3.3.3. Here is a simple example in the present context.

EXAMPLE 3.18. Consider the following affine map on $\mathbb{T}^3$,

$$B_\alpha(x, y, z) = (x + \alpha, x + y, y + z) \pmod 1.$$

The maps A_α and B_α are spectrally isomorphic. In both cases there is the same pure point part (the Kronecker factor is the rotation R_α) plus countable Lebesgue spectrum in the orthogonal complement. However A_α is a factor of B_α and a simple argument shows that any multiplicative correspondence must preserve this factor [13].

Proposition 3.14 provides for certain restrictions on isomorphisms between transformations with a pure point component in the spectrum. Such a transformation is an extension of its Kronecker factor. A particularly interesting case is those of a *finite extensions* when the measurable partition $\mathcal{K}(T)$ has finite elements. By ergodicity it follows that the number of elements is almost everywhere constant, say, equal to n, and hence such a transformation is metrically isomorphic to a skew product transformation on $H \times \{0, 1, \dots, n-1\}$ of the form

$$T(x, m) = (T_h x, \sigma_x m),$$

where $\sigma_x \in S_n$, the permutation group. We will briefly return to this subject in Section 3.6.3 and in more detail is Section 5.8.

3.4.6. *Distal systems*. Transformations with quasi-discrete spectrum and finite extensions are specimens of a more general class of systems which appears in many cases, in particular in the Furstenberg ergodic theoretical proof of the Szemeredi's theorem) [60,2].

DEFINITION 3.19. Consider an ergodic transformation $(Y, \mathcal{B}, \mu, S)$, a compact group G with a closed subgroup H and a measurable mapping $\phi: Y \to G$. Call the quotient G/H Z and equip Z with the Borel algebra $\mathcal{C}$ and the Haar measure ν. The transformation S_ϕ acting on $X = Y \times Z$ by $T_\phi(y, z) = (S(y), \phi(y)z)$ leaves the product measure $\mu \times \nu$ invariant. S_ϕ is called an *isometric extension* of S.

DEFINITION 3.20. A transformation T is said to be *distal* if there exists a countable family of T-invariant factor algebras indexed by ordinals $\mathcal{A}_\eta$, $\eta \leqslant \eta_0$, such that $\mathcal{A}_1 = \nu$ (the trivial algebra), $\mathcal{A}_{\eta_0} = \mathcal{A}$, for every $\xi < \eta$, $\mathcal{A}_\xi \subset \mathcal{A}_\eta$, T restricted to $\mathcal{A}_{\eta+1}$ is an isometric extension of its restriction to $\mathcal{A}_\eta$ and if ξ is a limit ordinal, $\mathcal{A}_\xi = \lim \uparrow \mathcal{A}_\eta$, $(\eta \uparrow \xi)$.

PROPOSITION 3.21. *Every ergodic measure preserving transformation has a unique maximal distal factor, i.e. a distal factor such that any other distal factor is contained in it.*

The distal factor contains Kronecker factor and is another example of a characteristic factor. It is trivial if and only if the transformation is weakly mixing. On the other hand, it may contain functions whose spectral type is mixing or even Lebesgue as in Examples 3.17 and 3.18.

Thus it is not defined in spectral terms.

3.5. *The convolution problem*

3.5.1. *Discrete and mixed spectrum.* In this section we will mean by the maximal spectral type of a transformation the maximal spectral type in the whole space L^2 including the atom δ_1 as was discussed in Section 3.1.2. Notice that the group property of the eigenvalues can be expressed equivalently as equivalence of the maximal spectral type of an ergodic transformation with pure point spectrum and its convolution. Thus we obtain the following statement which strengthens Proposition 3.2 in this case.

COROLLARY 3.22. *An atomic measure μ on the unit circle belongs to the maximal spectral type of the Koopman operator for an ergodic measure preserving transformation if and only if μ is equivalent to $\mu * \mu$.*

Furthermore, Theorem 3.16 is equivalent to the following statement.

COROLLARY 3.23. *If μ is a measure of the maximal spectral type for an ergodic measure preserving transformation and μ_d its atomic part then the convolution $\mu * \mu_d$ is equivalent to μ.*

3.5.2. *Continuous spectrum.* Observations above lead to a following question related to the general Spectral Realization Problem.

PROBLEM 3.24. What are connections between the maximal spectral type of an ergodic measure preserving transformation and its convolution with itself?

We will see below (Theorem 5.15, Propositions 5.43 and 5.44, and Theorem 5.49) that in general those measures are not directly connected. On the other hand, let us notice that for a weakly mixing transformation T the Cartesian powers $T \times T$, $T \times T \times T$, etc. including the infinite Cartesian power $T^{(\infty)}$ can be easily analyzed spectrally. In particular, if μ is the maximal spectral type of U_T in L_0^2 then for any $n \in \mathbb{N} \cup \infty$ the maximal spectral type of the n Cartesian power of T is equal to

$$\sum_{i=1}^{n} \mu^{(i)},$$

where $\mu^{(n)}$ is the convolution of n copies of μ. In particular, the measure $\sum_{n=1}^{\infty}\mu^{(n)}$, the maximal spectral type of $T^{(\infty)}$, is equivalent to its convolution. (See also Section 4.1.3.) This implies the following partial result related to the Spectral Realization Problem.

PROPOSITION 3.25. *If the class of a non-atomic measure μ appears as the maximal spectral type of an ergodic measure preserving transformation then for any $n \in \mathbb{N} \cup \{\infty\}$ the class of the measure $\sum_{i=1}^{n}\mu^{(i)}$ also appears a maximal spectral type of an ergodic measure preserving transformation.*

An effective method for realizing maximal spectral types is given by the construction of Gaussian dynamical systems (Section 6.4). It implies one of the few general results in the direction of realization of spectral types.

THEOREM 3.26. *Any non-atomic measure μ on the unit circle symmetric under the reflection in the real axis and equivalent to $\mu * \mu$ appears as a measure of maximal spectral type of an ergodic measure preserving transformation.*

This theorem follows directly from Proposition 6.12 by taking the Gaussian transformation T_μ.

3.6. *Summary*

3.6.1. *General restrictions*. In this section we have described all known general restrictions on the spectral properties of ergodic measure preserving transformations which then has to be taken into account in the discussion of the Spectral Realization Problem. For the sake of convenience let us summarize these restrictions:

Let T be an ergodic measure preserving transformation of a Lebesgue space. Then the Koopman operator U_T has the following properties:

(1) *One is always a simple eigenvalue of U_T.*
(2) *All eigenvalues are simple and form a finite or countable subgroup of the unit circle $S^1 \subset \mathbb{C}$.*
(3) *The maximal spectral type and the multiplicity function are symmetric under the reflection in the real axis.*
(4) *The maximal spectral type and the multiplicity function are invariant under multiplication by the eigenvalues.*
(5) *The support of the maximal spectral type is the whole circle.*

3.6.2. *Realization results*. Possibility of particular spectral properties for Koopman operators is proven by demonstrating pertinent examples which may either appear in the course of study of specific classes of systems or are constructed on demand. The state of our knowledge for the cases of the full spectral invariants or even just the maximal spectral type is much less advanced then for the case of the possible sets of values for the multiplicity function.

For the former problem there are very few results asserting that a given specific set of spectral data or even a given maximal spectral type can be realized. Theorem 3.26 is almost an exception in that respect. On the other hand, there are many examples showing possibility of realization of certain properties of the spectral type. An outstanding example is the possibility (see Theorem 5.15) and in fact genericity of the mutual singularity of the maximal spectral type in $L^2_0(X, \mu)$ and all its convolutions discussed in Section 5.4 and [78, Section 3.3], which demonstrates an extreme "negative" situation for the Problem 3.24. Another example is extreme "thinness" of the maximal spectral type for a generic measure preserving transformation which follows from very fast periodic approximation, cyclic (see Section 5.4), or, more generally, homogeneous [78, Section 5] which is a spectral property [78, Corollary 5.3].

In one considers the spectral multiplicity by itself, in other words, asks about what subsets of $\mathbb{N} \cup \infty$ appear as the sets of essential values of the multiplicity function for the Koopman operator in L^2_0, the constructive approach goes much further toward a definitive answer. No restrictions on the set of essential values are known and there is an impressive list of sets which do appear as well as certain technology which allows to add many new examples once some key cases have been constructed. Here is an incomplete list of cases when realization is possible:

(1) *If s subset* $S \subset \mathbb{N}$ *is realized then* $S \cup \{\infty\}$ *is realized.*
(2) *Any finite or infinite subset of* $\mathbb{N}$ *containing* 1 [65,100].
(3) *Any finite or infinite subset of even numbers containing* 2.
(4) $\{2, 3\}, \{3, 6\}$ [78].
(5) $\{n\}$ *for any* $n \in \mathbb{N}$ [17].

So one may venture to conjecture that no restriction on the set of essential values of spectral multiplicity in L^2_0 exist.

Let us mention that the notion of multiplicity makes sense also for the action of the Koopman operator in L^p. An open question is the following: Does every transformation have simple spectrum L^1? An equivalent way to formulate the question is to ask whether for every ergodic transformation T, there exists an L^1 function ϕ such that the L^1 closure of the linear span of the $T^n\phi$ is the whole of L^1. More generally, does there exist, for every $p < q$ a transformation whose Koopman operator has a cyclic vector in L^p but has no cyclic vector in L^q?

3.6.3. *Extra-spectral information*. Theorem 3.4.2 proved by von Neumann in [155] originally arose some hope that spectrum may serve as a basis of classification for measure preserving transformations up to metric isomorphism.

It later became apparent that for certain classes systems with non-trivial Kronecker factors such as finite or compact group extensions metric isomorphisms exhibit certain rigidity properties. The simplest of those is of course is Proposition 3.14, namely the fact that for an ergodic translation on a compact Abelian group measurable centralizer coincides with algebraic one (other translations) and hence every measurable isomorphism between two such translations is algebraic. Since the Kronecker factors of isomorphic transformations should match this restricts isomorphisms between extensions [19,13,110]. In some cases this allows a complete metric classification of extensions. Abramov's classification of transformations with quasi-discrete is a prime example [13]. In other situations classification depends on cohomology classes of certain cocycles which may or may not behave regularly. Rigidity phenomena also appear in certain weakly mixing transformations, for example for those where measurable centralizer is sufficiently small. The notion of self-joining discussed in Section 4.3 is a useful tool of studying rigidity properties beyond pure point spectrum and simple extensions.

There are some exceptional cases when continuous spectrum provides the complete metric invariant in analogy with the pure point spectrum case. The Kronecker Gaussian systems provide the prime example, see Section 6.4.3 [54]. In is not quite clear to what extent very thin continuous spectral measures with strong arithmetic properties (concentration around roots of unity of particular orders) may carry substantial information about metric isomorphism; this information is certainly not complete as [110] and similar examples with continuous spectrum show.

In general, natural non-spectral invariants do not match well with the spectrum. One example where classification of systems with a fixed spectral type looks hopeless is the case of countable Lebesgue spectrum. Recall that every K-system has countable Lebesgue spectrum. On the other hand, every ergodic transformation with positive entropy induces a K-automorphism on some subset, see Theorem 5.65 [120]. Thus any positive entropy class of Kakutani equivalent transformations contains a transformation with countable Lebesgue spectrum. But complete classification up to Kakutani equivalence does not seem more feasible than classification up to metric isomorphism. For basic information on Kakutani (monotone) equivalence see [75,118] and for a summary [12, Section 13].

4. Some aspects of theory of joinings

4.1. *Basic properties*

See [12, Section 3.1, 3.2]. Unlike the other parts of this survey in this section we will often indicate the σ-algebra of measurable sets in our description of dynamical systems. The reason is that we will consider several different invariant measures for the same transformation.

4.1.1. *Definitions*

DEFINITION 4.1. Given two dynamical systems (measure preserving transformations) T acting on $(X, \mathcal{A}, m)$ and S acting on $(Y, \mathcal{B}, \mu)$, a *joining* is a probability measure λ on the Cartesian product $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ which is $T \times S$ invariant and such that $\lambda(A \times Y) = m(A)$ for all A in $\mathcal{A}$ and $\lambda(X \times B) = \mu(B)$ for all B in $\mathcal{B}$.

Joining of several transformations are defined similarly. Joinings were introduced by H. Furstenberg [59]. It is a powerful tool in a great variety of questions in ergodic theory, both spectral and non-spectral. The survey [151] presents a compact treatment of the subject. The book [62] contains extensive information about joinings and in fact represents an attempt to develop the core part of ergodic theory around that notion. See also [6, Section 1.3] for interesting insights and especially for comparison of relevant measure-theoretic and topological concepts and results.

Note that the set of joinings is never empty since there is always the *independent joining* $\lambda = m \times \mu$ but this may be the only one, see Section 4.2.

4.1.2. *Principal constructions*. We list now several basic constructions related to joinings. We will restrict ourselves to the case of two transformations since multiple joinings usually are treated similarly.

(1) *Ergodic decomposition of a joining*. When two systems are ergodic, there always exists an ergodic joining between them. For, ergodic components of a joining measure between ergodic systems are joinings too.

(2) *Factors as joinings*. Considering two systems given as in the definition we call $\mathcal{V}$ and $\mathcal{H}$ the algebras $\mathcal{A} \times Y$ and $X \times \mathcal{B}$ respectively. If $\mathcal{H} \subset \mathcal{V}$ (λ) (by which we mean that for every set A in $\mathcal{H}$ there exist a set B in $\mathcal{V}$ such that $\lambda(A \Delta B) = 0$) then $(Y, \mathcal{B}, \mu, S)$ is a factor of $(X, \mathcal{A}, m, T)$. (For this we need that both measure spaces are Lebesgue.) Conversely, if ϕ is the factor map from X to Y, and $A \times B$ is a rectangle in $\mathcal{A} \otimes \mathcal{B}$, the joining defined by $\lambda(A \times B) = m(A \cap \phi^{-1}(B))$ satisfies the inclusion $\mathcal{H} \subset \mathcal{V}$.

(3) *Isomorphisms as joinings*. In the same way a joining λ such that $\mathcal{V} = \mathcal{H}$ (λ) defines an isomorphism between the two transformations, with the same converse as before: An isomorphism gives rise to a joining for which $\mathcal{V} = \mathcal{H}$.

    Weak isomorphism means that there exists two joinings λ_1 and λ_2 such that $\mathcal{H} \subset \mathcal{V}$ (λ_1) and $\mathcal{V} \subset \mathcal{H}$ (λ_2).

(4) *Relatively independent joining over a common factor.* If two transformations have isomorphic factors, a useful construction is the *relatively independent joining* above this common factor. If $\mathcal{A}_1$ and $\mathcal{B}_1$ are the two invariant subalgebras of $\mathcal{A}$ and $\mathcal{B}$ respectively such that T restricted to $\mathcal{A}_1$ is isomorphic to S restricted to $\mathcal{B}_1$, we extend the joining λ_1 between these two algebras given by the isomorphism as in (3) which identifies them (we call the global algebra of this object $\mathcal{C}$) to a joining $\lambda_{\mathcal{C}}$ of the whole product in such a way that $\mathcal{A}$ and $\mathcal{B}$ are relatively independent given $\mathcal{C}$. This is done by defining, for a product set $A \times B$ its $\lambda_{\mathcal{C}}$ measure by taking the integral for the measure λ_1 of the product of $E^{\mathcal{A}_1} 1_A \times E^{\mathcal{B}_1} 1_B$. This makes sense as λ_1 is a measure on $\mathcal{A}_1 \otimes \mathcal{B}_1$.

(5) *Topology in the set of joinings*. We introduce a topology in the set of joinings of $(X, \mathcal{A}, m, T)$ and $(Y, \mathcal{B}, \mu, S)$ in the following way: Take $A_n, n \geqslant 1$, and $B_n, n \geqslant 1$, two sequences of sets dense in $\mathcal{A}$ and $\mathcal{B}$ respectively (the density is for the topology associated to the distance between sets which is the measure of the symmetric difference). Given two joinings λ_1 and λ_2 define

$$\delta(\lambda_1, \lambda_2) = \sum_{m,n \geqslant 1} \frac{1}{2^{m+n}} \times \left|\lambda_1(A_m \times B_n) - \lambda_2(A_m \times B_n)\right|.$$

    δ is obviously a distance. The set of joinings is compact in the topology generated by this distance.

(6) *Proof of Theorem 3.4.2 via joinings*. There is a nice proof using joinings, due to Lemańczyk and Mentzen [106] of the von Neumann Isomorphism Theorem 3.4.2. We are going to show that any ergodic joining between two such transformations

(which always exists by (1)) is in fact an isomorphism. The L^2 space of both transformations is generated by the eigenfunctions (because they have discrete spectrum), and since the joining is ergodic, every eigenvalue in this joining must be simple. But by spectral isomorphism both transformations have the same eigenvalues and therefore the corresponding eigenfunctions for the ergodic joining are necessarily $L^2(\mathcal{V})$ and $L^2(\mathcal{H})$ measurable. This forces $\mathcal{V} = \mathcal{H}$ for our joining whence the announced isomorphism.

4.1.3. *Spectral analysis of Cartesian products*. Since there is always the independent joining between any two transformations it is appropriate now to describe the spectral properties of the Cartesian product of two measure preserving transformations with respect to the product measure. The Koopman operator of the Cartesian (direct) product $T \times S$ is isomorphic to the tensor product of U_T and U_S. This is a particular case of the tensor products of representations described in Section 1.3.3. Assume that the maximal spectral types of the two Koopman operators are represented by the measures μ and ν (including the δ measure at 1) with multiplicity functions m and m'.

PROPOSITION 4.2. *The maximal spectral type of $T \times S$ is represented by the convolution $\mu * \nu$.*

The multiplicity function m at the point $\lambda \in S^1$ is calculated as follows: take the product $\mu \times \nu$ on the two-dimensional torus $\mathbb{T}^2$ and consider the system of conditional measures with respect to the partition of $\mathbb{T}^2$ into the "diagonal" circles $\lambda_1 + \lambda_2 = c$.

(1) *If the conditional measure at $c = \lambda$ is not supported in the finite number of points then $m(\lambda) = \infty$.*
(2) *Otherwise, let the support of the conditional measure be the points $(\lambda_1^1, \lambda_2^1), \ldots, (\lambda_1^n, \lambda_2^n)$. Then*

$$m(\lambda) = \sum_{i=1}^{n} m_1(\lambda_1^i) \times m_2(\lambda_2^i).$$

A similar albeit more complicated description can be given in the case of Cartesian product of several transformations.

4.2. *Disjointness*

Joinings provide a good way to compare transformations; more precisely, how far is the isomorphism class of a transformation from that of another. We saw that when two transformations are isomorphic, there is a joining which identifies $\mathcal{V}$ and $\mathcal{H}$. At the opposite end, two transformations are said to be *disjoint* when the product joining is the only joining between them. (That is for every joining λ, $\mathcal{V} \perp \mathcal{H}(\lambda)$.) This notion was introduced by H. Furstenberg in his seminal paper [59]. One may say that disjoint transformations have as little is common as possible, e.g., no common factors since if there is one there is also the relatively independent joining over it.

Disjointness is also a tool, if we know that the restrictions of a transformation T to two invariant algebras are disjoint, to show independence of these algebras. The simple observation that the identity is disjoint from any ergodic transformation has shown surprising efficiency in various contexts.

PROPOSITION 4.3. *Two transformations whose spectral types are mutually singular are disjoint. In particular, rigid transformations are disjoint from mildly mixing transformations.*

PROOF. Assume that T_1 and T_2 are two transformations with spectral types ν_1 and ν_2 on the orthogonal complement of constants which are mutually singular. Consider a joining λ between them and let f and g be in $L^2(\mathcal{V})$ and $L^2(\mathcal{H})$ respectively, both with 0 integral. The projection of f in H_g (the cyclic subspace generated by g) for the joining λ has a spectral measure which is absolutely continuous both with respect to ν_1 and ν_2 and must therefore be 0. This says that $\int fg\,d\lambda = 0$, and λ is the product measure. □

PROPOSITION 4.4. *Distal transformations are disjoint from weakly mixing transformations.*

More generally, joinings allow to give a characterization of the maximal distal factor defined in 3.4.3–6.

Call a transformation *weakly mixing relative to a factor* if its relatively independent joining with itself above the given factor is ergodic.

PROPOSITION 4.5. *The maximal distal factor is the smallest factor algebra relative to which the transformation is weakly mixing.*

Since transformations with positive entropy have Bernoulli factors we see that

PROPOSITION 4.6. *Any two transformations with positive entropy are not disjoint.*

PROPOSITION 4.7. *K-automorphisms are disjoint from* 0*-entropy transformations.*

Here is a nice application of this last fact which goes back to the original paper of Furstenberg [59]. It is sometimes called the possibility of perfect filtering.

THEOREM 4.8. *Assume that are given two independent stationary processes (X_n) and (Y_n) such that (X_n) generates a K-automorphism and such that (Y_n) generates a zero entropy transformation. Assume furthermore that X_0 and Y_0 are both in L^2. Then (X_n) is measurable with respect to the $(X_n + Y_n)$ process. That is to say the (X_n) process can be recovered from the system perturbed by a random noise $(Z_n) = (X_n + Y_n)$.*

PROOF. Consider the relatively independent joining of (X_n, Z_n) with itself above (Z_n). This is a triple (X_n, Z_n, X_n') such that (X_n', Z_n) is a copy of (X_n, Z_n) and (X_n) and (X_n') are relatively independent over (Z_n). (In the previous constructions, we identify a process to the measure preserving transformation to which it gives rise.) $(Y_n') = (Z_n - X_n')$ is obviously isomorphic to (Y_n) and therefore K. $(X_n' + Y_n') = (X_n + Y_n)$. We compute $E(X_n - X_n')^2 = E(X_n - X_n')(Y_n' - Y_n)$. As (X_n) and (Y_n) are independent (as well as (X_n') and (Y_n) as a consequence of the disjointness of 0-entropy transformations with K-automorphisms), we get that the preceding expectation is 0 and therefore that $X_n = X_n'$ a.e. which is saying that (X_n) is measurable with respect to the (Z_n) process. □

Note that this can be considered an extension of the same statement obtained under the spectral hypothesis that the spectral measures of the two processes (X_n) and (Y_n) are mutually singular.

4.3. *Self-joinings*

4.3.1. *Basic properties*. Every transformation is isomorphic to itself which is reflected by the presence of the trivial diagonal joining: the measure Δ on $X \times X$ defined by

$$\Delta(A \times B) = m(A \cap B)$$

is a self-joining. Studying the collection of all joinings of a transformation with itself and the structure of such joinings provides deep insights into the orbit structure of the system. In particular presence of few joinings indicates a certain rigidity of the orbit structure while abundance of joinings indicates its richness and "plasticity". Thus, the family of self-joinings Δ_n, $n \geqslant 1$, defined by $\Delta_n = (\mathrm{Id} \times T^n)_* \Delta$ is quite interesting.

(1) T is mixing if and only if $\Delta_n \to m \times m$. Self-joinings of higher order are closely related to mixing of higher order.
(2) T is rigid if there exists a sequence n_i such that $\Delta_{n_i} \to \mathrm{Id}$.
(3) If S is an automorphism which commutes with T, then there is a joining $\Delta_S = (\mathrm{Id} \times S)_* \Delta$. As a consequence of Section 4.1.2(3) it is equivalent for a self-joining λ to be of this form, or to satisfy $\mathcal{V} = \mathcal{H}(\lambda)$.

Something analogous to Proposition 4.4 for weakly mixing but not mixing transformations follows from a recent work of F. Parreau (unpublished) who proved that if a transformation T is weakly mixing and not mixing, it possesses a non-trivial factor which is disjoint from all mixing transformations. A starting point for the construction of this factor is the consideration of a non-trivial limit for Δ_{n_i}.

It can be useful to consider joinings from a more functional analytic viewpoint [143]. Assume that we are given a linear operator $\phi : L^2(X, \mathcal{A}, m) \to L^2(Y, \mathcal{B}, \mu)$ satisfying the following properties: $U_T \phi = \phi U_S$, $\phi 1 = 1$, $\phi^* 1 = 1$, $\phi(f) \geqslant 0$ if $f \geqslant 0$.

Then the measure λ defined by $\lambda(A \times B) = \int_B \phi(1_A)$ gives a joining.

The converse is obvious: given a joining λ take for ϕ the conditional expectation with respect to $\mathcal{H}$ restricted to $L^2(\mathcal{V})$. As an application, we see that if λ is a self-joining of $(X, \mathcal{A}, m, T)$ with itself, and if T has simple spectrum then λ is $S \times S$ invariant for every automorphism S which commutes with T. Therefore λ is a self-joining for the S-action.

4.3.2. *Joinings and group extensions*. There is an important theorem, due to Veech, which contains many of the compactness arguments which appear in ergodic theory.

THEOREM 4.9. *Consider an ergodic transformation* $(X, \mathcal{A}, m, T)$ *together with a factor* (*a* T *invariant subalgebra*) $\mathcal{B}$. *The following statements are equivalent*:

(1) *Almost all ergodic components of the relatively independent joining of* $(X, \mathcal{A}, m, T)$ *with itself above* $\mathcal{B}$ *identify* $\mathcal{V}$ *and* $\mathcal{H}$.

(2) *There exists a compact group* G *and a measurable mapping* $\phi : (X, \mathcal{B}) \to G$ *such that* $(X, \mathcal{A}, m, T)$ *is isomorphic to the skew product on* $(X_{\mathcal{B}}, \mathcal{B}, m) \times (G, \mathcal{G}, \mu_G)$ (μ_G *the Haar measure on* G) *given by* $T(x, g) = (Tx, \phi(x)g)$ *by an isomorphism which is the identity restricted to* $\mathcal{B}$. (*This is to compare with isometric extensions which have been defined in* 3.4.9.)

4.3.3. *Minimal self-joinings*. D. Rudolph [140] introduced, for a transformation, the notion of minimal self-joinings, which basically says that a transformation has no other joinings with itself than the obvious ones, and proved existence of mixing transformations with that property.

DEFINITION 4.10. A weakly mixing transformation $(X, \mathcal{A}, m, T)$ has *minimal self-joinings* (*MSJ*) if the following is true: for all $n \geqslant 2$ any ergodic joining λ of n copies of $(X, \mathcal{A}, m, T)$ (λ is a probability measure on $\prod_1^n (X_i, \mathcal{A}_i)$ invariant under $\prod_{i=1}^n T_i$, which satisfies

$$\lambda\left(A_i \times \prod_{j \neq i} X_j\right) = m_i(A_i)$$

for all $1 \leqslant i \leqslant n$ and all $A_i \in \mathcal{A}_i$. $(X_i, \mathcal{A}_i, m_i, T_i)$, $1 \leqslant i \leqslant n$, is a copy of $(X, \mathcal{A}, m, T)$) satisfies the following: the set $[1, n]$ can be decomposed into a disjoint union of subsets E_k, $1 \leqslant k \leqslant r$, such that:

(1) The algebras

$$\mathcal{B}_k = \bigotimes_{i \in E_k} \mathcal{A}_i \times \prod_{j \in E_k^c} X_j, \quad 1 \leqslant k \leqslant r,$$

are λ-independent.

(2) For every $1 \leqslant k \leqslant r$ there exists integers

$$n_{i_1}, n_{i_2}, \ldots, n_{i_{s-1}} \quad (s = |E_k|)$$

such that λ restricted to $\mathcal{B}_k$ is exactly

$$\left(\mathrm{Id} \times T^{n_{i_1}} \times T^{n_{i_2}} \times \cdots \times T^{n_{i_{s-1}}}\right)_* \Delta.$$

Δ is the diagonal measure.

Since factors and commuting transformations other than powers produce joinings of types other than those described in the definition of MSJ as an immediate corollary of the definition we obtain

PROPOSITION 4.11. *Any MSJ transformation has no factors and its centralizer consists only of its powers.*

Weak mixing is not a restriction here since a pure point spectrum transformation has many self-joinings coming from the centralizer and presence of a non-trivial Kronecker factor provides for the independent joining over it. In fact one can show more.

PROPOSITION 4.12. *Any MSJ transformation is mildly mixing.*

PROOF. If T is MSJ and has a rigid factor the factor must be the whole T. But then by Proposition 2.14 T has an uncountable centralizer. □

We will see later that non-mixing MSJ transformations exist (Theorems 5.12 and 5.13).

4.3.4. *Minimal self-joinings for flows and simple transformations.* A transformation from a flow cannot have minimal self-joinings since it commutes not only with its powers but also with other transformations from the flow. This is taken into account in the definition of minimal self-joinings for flows. More generally, it turned out to be useful to have a notion which is somewhat weaker than minimal self-joining and which roughly speaking allows for joinings coming from non-trivial commuting transformations. This was done by Veech [152]. The class of simple transformations which he defined includes in particular transformations from flows with minimal self-joinings as well as certain rigid transformations.

DEFINITION 4.13. A weakly mixing transformation is *simple* if the following property is true:

For all $n \geqslant 2$ any ergodic joining λ of n copies of $(X, \mathcal{A}, m, T)$ satisfies the following: the set $[1, n]$ can be decomposed into a disjoint union of subsets E_k, $1 \leqslant k \leqslant r$, such that:

(1) The algebras

$$\mathcal{B}_k = \bigotimes_{i \in E_k} \mathcal{A}_i \times \prod_{j \in E_k^c} X_j, \quad 1 \leqslant k \leqslant r,$$

are λ-independent.

(2) For every $1 \leqslant k \leqslant r$ the $|E_k|$ algebras $\mathcal{A}_\rangle \times \prod_{j \neq i}$, $i \in E_k$, are λ-identical (which is the same as saying that there exists $|E_k|$ automorphisms commuting with T, S_j such that λ restricted to $\mathcal{B}_k$ is exactly $(\prod_{j \in E_k} S_j)_* \Delta$).

We note that we could have labeled these two definitions according to the number of copies which were used. But in fact a theorem of Glasner, Host and Rudolph [63] asserts that as soon as the definition is satisfied for a joining of three copies, it is satisfied for any number of copies. It is not known whether the definition for two copies only would suffice to imply that it is satisfied for three copies (and therefore for any number of copies).

It follows from Theorem 4.9 that if T is simple, it is a compact group extension of any of its non-trivial factors.

By Section 4.1(2) two ergodic transformations with isomorphic common factors are never disjoint. In general the converse is not true. However the following holds [34]:

THEOREM 4.14. *Two simple transformations with no isomorphic common factors are disjoint*.

4.3.5. *Mixing properties and joinings*. We saw that $\Delta_n \to m \times m$ is equivalent to mixing. The study of self-joinings of higher order is closely related to higher order mixing properties. The next definition is due to del Junco and Rudolph [34].

DEFINITION 4.15. An ergodic transformation $(X, \mathcal{A}, m, T)$ is said to be *pairwise independently determined* if the following is true: for every integer k a joining λ of k copies of $(X, \mathcal{A}, m, T)$ which is such that any two of the k factors of the product of the k copies are pairwise independent (λ) must be the product joining (which is the one for which the k factors are globally independent).

One immediate fact is the following: if a transformation is mixing and pairwise independently determined, it is mixing of all orders. B. Host [72] has proved the following important theorem:

THEOREM 4.16. *An ergodic transformation with singular spectral measure is pairwise independently determined*.

COROLLARY 4.17. *A mixing transformation with singular spectral measure is mixing of all orders*.

The last corollary is one of the few deep structural results in ergodic theory. It sheds light on a long-standing unsolved problem (Does mixing apply mixing of all orders?) by giving an affirmative answer in one of the "most suspicious" cases. See also Theorems 5.18 and 5.19.

Let us remark that a simple transformation is one which is 2-simple (that is every ergodic joining of two copies of the transformation is either product measure or identifies $\mathcal{V}$ and $\mathcal{H}$) and pairwise independently determined. In case of an $\mathbb{R}$ action, V. Ryzhikov [143] has proved that it is always true that 2-simplicity implies pairwise independently determined (and therefore simplicity).

There are no examples known of transformations which are weakly mixing, have 0 entropy, and which are not pairwise independently determined.

5. Combinatorial constructions and applications

5.1. *From Rokhlin lemma to approximation*

5.1.1. *Genericity in the weak and uniform topologies*. Let us recall definitions of the two principal topologies in the group of all measure preserving transformations of a Lebesgue space (X, μ) [68].

The uniform topology first mentioned in Section 3.2.3 is quite strong: it is defined by the metric

$$d_u(T, S) = \mu\{x\colon Tx \neq Sx\} \tag{5.1}$$

invariant by both left and right multiplications.

Notice however that it is weaker than the topology induced from the uniform operator topology on the Koopman operators which is simply discrete.

The weak topology which appeared in Section 3.4.3 is metrizable but no canonical two-side invariant metric similar to (5.1) is available to define it. One way to define a metric is to pick a countable dense collection of measurable sets $A_1, \ldots$ and define the distance as

$$d_w(T, S) = \sum_{n=1}^{\infty} \mu(TA_n \Delta SA_n). \tag{5.2}$$

This topology coincides with the topology induced from the strong operator topology on Koopman operators. Weak topology is weaker than uniform and aperiodic transformations are dense in weak topology. Hence the density of conjugates of any aperiodic transformation in all aperiodic transformations in uniform topology (Proposition 3.6) implies

PROPOSITION 5.1. *Conjugates of any aperiodic transformation are dense in the group of all measure preserving transformations in weak topology.*

Here is an immediate corollary which due to the Baire Category Theorem plays a great role in proving existence and abundance of measure preserving transformations with many interesting properties including spectral ones.

COROLLARY 5.2. *Any conjugacy invariant G_δ in weak topology set which does not contain transformations with sets of periodic points of positive measure is dense and hence residual.*

This fact is widely used in existence proofs.

Another related method is to establish a property via checking its approximate versions which can be shown to be satisfied on open dense sets. This works with properties which can be expressed by the behavior along an unspecified subsequence of iterates (e.g., ergodicity, rigidity, weak mixing) but not along the whole sequence (mixing, Lebesgue spectrum).

5.1.2. *Towers and cityscapes*

DEFINITION 5.3. An *n-tower* in a Lebesgue space (X, μ) is a collection of disjoint subsets $F_1, \ldots, F_n$ of equal measure together with measure preserving transformations $T_i : F_i \to F_{i+1}$, $i = 1, \ldots, n-1$.

The sets F_i are called the *levels* of the tower; in particular, the set F_1 is called the *base* of the tower and the set F_n the *roof* of the tower.

The union of all levels is called the *support* of the tower.

The number n is sometimes called the *height* of the tower. The quantity $n\mu(F_1)$, i.e. the measure of tower's support, is called the *measure* or the *size* of the tower.

We will say that the tower $\mathcal{T}$ *agrees* with a measure preserving transformation T if $T = T_i$ on the ith level of the tower.

The Rokhlin Theorem 3.4 says that for every aperiodic measure preserving transformation T, there exists arbitrarily high (or long) towers of measure arbitrary close to one which agrees with T. If the measure of a Rokhlin tower is greater than $\frac{n}{n+1}$ then the image of its roof must overlap with the base. And if the size is very close to one then most of the roof is mapped into the base. However the Rokhlin Theorem says nothing about how most of the roof is mapped into the base. Thus an approximation of a measure preserving transformation by a single tower does not say much about the asymptotic properties of the transformation apart from the crudest one, the aperiodicity.

DEFINITION 5.4. A *cityscape* is a union of disjoint towers, in general, of varying heights. The *measure* of a cityscape is defined as the sum of measures of towers comprising the cityscape.

A cityscape *agrees* with a measure preserving transformation T if every tower comprising it agrees with T.

5.1.3. *Uniform approximation*. In order to make certain conclusions from approximation of a measure preserving transformation by towers, or more generally, cityscapes the latter should in some sense be representative of the σ-algebra of all measurable sets. This of course makes sense only if one considers not a single approximation but a sequence of such approximations. A useful model to visualize the requirement of being representative is to think of X as a metric space and of the levels of the towers (or of towers comprising the cityscape) as sets of small diameter. In this situation every fixed measurable set can be approximated up to a set of small measure by a union of levels and combinatorics of transformations in towers approximates the dynamics of the map at sufficiently long time ranges.

This model is suggestive but restrictive in two ways: first, the appropriate topological structure is not always available, and second, even if it is, the levels need not really be sets of small diameter: only after throwing away a set of small measure the intersections of the levels with the remainder would have this property. Anyway, there is a purely measure theoretic way to formulate the property we have in mind as well as its variations.

Every measurable partition ξ of the space X generates the σ-algebra $\mathfrak{B}(\xi)$ of sets *measurable with respect to the partition*. For every set $A \in \mathfrak{B}(\xi)$ one can find another set A' which is the union of elements of ξ such that the symmetric difference of A and A' is a null-set.

To each cityscape $\mathcal{C}$ we associate partition $\xi(\mathcal{C})$ on the space whose elements are level of towers comprising the cityscape and the complement to the union of all such levels.

Recall that the sequence of measurable partitions $\eta_n \to \varepsilon$ as $n \to \infty$, if for every measurable set $A \subset X$ there exists a sequence of sets

$A_n \in \mathfrak{B}(\eta_n)$ such that $\mu(A \Delta A_n) \to 0$ as $n \to \infty$.

This notion can be reformulated as follows. We will say that partition ξ δ-*refines* partition η if for every $A \in \mathfrak{B}(\eta)$ there exists $A' \in \mathfrak{B}(\xi)$ such that $\mu(A \Delta A') < \delta$. Then $\xi_n \to \varepsilon$ if for any finite partition η and every $\delta > 0$ there exists $N = N(\eta, \delta)$ such that for $n \geqslant N$, ξ_n ε-refines η.

DEFINITION 5.5. A sequence $\mathcal{C}_n$ of cityscapes is called *exhaustive* if $\xi(\mathcal{C}_n) \to \varepsilon$ as $n \to \infty$.

DEFINITION 5.6. An exhaustive sequence of cityscapes which agrees with a measure preserving transformation T is called a *uniform approximation* of T.

It follows from the Rokhlin Theorem 3.4 that every measure preserving transformation allows a uniform approximation. To see that one needs to take a Rokhlin tower and split its base in such a way that the partition into levels of resulting towers would be a refinement of a given partition. However if one restricts the type of cityscapes (e.g., consider cityscapes consisting of a single tower or a fixed number of towers) existence of a uniform approximation becomes a restrictive property and implies interesting properties of T, see Section 5.2.2.

Uniform approximation and its variations are used to produce measure preserving transformations with interesting properties. We will consider three ways to produce such approximations: cutting and stacking, coding with respect to a given generating partition, and periodic approximation.

5.2. *Cutting and stacking and applications*

5.2.1. *The method of cutting and stacking.* (See also [138].) The cutting and stacking method is a particular way to produce inductively an exhaustive sequence of cityscapes which form a uniform approximation of a measure preserving transformation.

At nth step a cityscape $\mathcal{C}_n$ is defined. The transformation is thus defined everywhere except for the roofs of the towers from $\mathcal{C}_n$ and a certain set A_n which is the complement to the union of supports of the towers in the cityscape. Then each tower of $\mathcal{C}_n$ is divided into towers and new levels are added from A_n to some of the towers. Then the roofs of most of new towers are mapped into bases of other towers. This produces the cityscape $\mathcal{C}_{n+1}$ and the set $A_{n+1} \subset A_n$. Specifically, those parts of the bases of old towers which do not belong to the images of the roofs of extended old towers serve as bases of new towers. Each new tower is defined by an itinerary, namely a sequence of old towers which are visited in succession. This is why the construction is called cutting and stacking: bases of old towers are cut according to the itineraries and this new thin towers are stacked on top of each other.

The list of important examples constructed with the cutting and stacking method is quite large. Let us mention the "Chacon transformation" described below in Section 5.2.3, the rank one mixing transformations (Section 5.2.4), the first examples of Ornstein of K-automorphisms which are not Bernoulli later developed in [119] (as well as his counterexamples to the Pinsker conjecture), the Feldman examples of non-standard transformations with zero entropy [51]. To illustrate the usefulness of the method for other groups let us mention [83] where the cutting and stacking method is used to construct examples of $\mathbb{Z}^k$ and $\mathbb{R}^k$ actions with $k \geqslant 2$ where individual elements have zero entropy which cannot be realized by diffeomorphisms of compact manifolds with respect to any Borel measure. We will mention other specific constructions in due course.

5.2.2. *Approximations with towers or large size; rank*

DEFINITION 5.7 [117]. A measure preserving transformation T has *rank one* if it admits uniform approximation by single towers.

Equivalently, T is rank one if for every finite partition η and every $\delta > 0$ there is a tower $\mathcal{T}$ which agrees with T and such that the partition $\xi(\mathcal{T})$ into the levels of the tower and the complement to its support δ-refines the partition η.

Importance of the rank one property for the spectral theory of measure preserving transformations is based on the following fact.

PROPOSITION 5.8. *Any rank one transformation has simple spectrum and is hence ergodic.*

PROOF. Consider a tower $\mathcal{T}$ of height n approximating T with base F. The images of the characteristic function ξ_F under U_T^i, $i = 0, 1, \ldots, n-1$, are characteristic functions of the disjoint levels of the tower. Thus there is a cyclic subspace which contains all characteristic functions of the levels of the tower and their linear combinations. Consider these cyclic subspaces for an exhaustive sequence of towers. From the approximation property it follows that for any given $f \in L^2(X, \mu)$ projections to these cyclic subspaces converge to f. Hence by Theorem 1.21, U_T has simple spectrum. □

The spectral multiplicity estimates based on uniform approximation can be obtained under more general conditions than rank one.

DEFINITION 5.9. An ergodic transformation T is *locally rank one* if there exists $a > 0$ such that for every finite partition

$$\eta = (p_0, p_1, \ldots, p_l)$$

and for every $\varepsilon > 0$, there exists a tower $\mathcal{T}$ of size $\geqslant a$ and a partition

$$\bar{\eta} = (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_l)$$

of $\mathcal{T}$ whose elements are unions of levels such that

$$\sum_{s=0}^{l} m(\bar{p}_s \setminus p_s) < \varepsilon.$$

We call any number a satisfying the above definition an *order* of T.

REMARK. Property of local rank one of order a is equivalent to existence of a uniform approximation by cityscapes where one tower has measure at least a.

If the transformation T allows uniform approximation by cityscapes with k towers, the transformation is said to have *rank no greater than k*. Since in each cityscape at least one tower has measure at least $1/k$ any transformation of rank no greater than k is locally rank one of order at least $1/k$. The following theorem generalizes Proposition 5.8.

THEOREM 5.10. *If an ergodic transformation is locally rank one of order a, its spectral multiplicity is bounded by* $[1/a]$.

SKETCH OF PROOF. As before in the proof of Proposition 5.8 this easily follows form the definition and Theorem 1.21. Given $k = [1/a] + 1$ orthonormal functions $f_1, f_2, \dots, f_k$, we first approximate them in L^2 by finite valued functions. We call η the partition which makes all these finite valued functions measurable. We consider a tower $\mathcal{T}$ with base F which locally approximates this partition (as in the definition) and which is sufficiently long to have the ergodic theorem giving that the frequency of appearances of each set in η in the tower is close to its measure. If we take for H the cyclic space generated by ξ_F, we see that the conditions of Theorem 1.21 are satisfied. □

Simplicity of the spectrum does not force anything on the rank of the transformation, see [52,36,108] for examples of transformations with simple spectrum which are not locally rank one. The relations between rank and spectral multiplicity have been thoroughly studied by J. Kwiatkowski and Y. Lacroix [99].

Another interesting property of local rank one transformations is related to Kakutani equivalence theory.

PROPOSITION 5.11 [80]. *Any locally rank one transformation is standard* (*zero entropy loosely Bernoulli*, *sometimes also called loosely Kronecker*), *i.e. it is induced by any odometer and any irrational circle rotation and induces any of those transformations*.

Ferenczi [52] and De la Rue [36] (see Theorem 6.23) constructed transformations with simple spectrum which are not standard and therefore also not locally rank one.

5.2.3. *Chacon transformation* [26]. The Chacon Transformation which is a particular rank one transformation is one of the jewels of ergodic theory. As we shall see, it can be used as a source of examples with interesting, often exotic, properties. Its particular interest is that while it exhibits very moderate and rather regular pattern of orbit growth properties it does not fit into either of the three main paradigms of smooth ergodic theory: elliptic (Section 2.2.4), hyperbolic and parabolic (Section 2.1.3). Smooth realization of this map is unknown and seems to be beyond the reach of available methods.

The transformation is defined inductively on the unit interval equipped with Lebesgue measure I. At stage n, there are $h(n)$ intervals of equal length $I_1, I_2, \dots, I_{h(n)}$ and T maps I_k onto I_{k+1}, $1 \leqslant k \leqslant h(n) - 1$, by translations. T is not defined on $I_{h(n)}$. To go from stage

n to stage $n+1$, we divide the interval I_1 into three intervals of equal length, I_1^1, I_1^2, I_1^3, and therefore divide the tower

$$\tau_n = \bigcup_{i=0}^{h(n)-1} T^i I_1$$

into three columns

$$\tau_n^j = \bigcup_{i=0}^{h(n)-1} T^i I_1^j,$$

$1 \leqslant j \leqslant 3$. We now pick an interval J_n disjoint from τ_n with length equal to the length of I_1^1 and define τ_{n+1} mapping by translations $T^{h(n)-1} I_1^1$ onto I_1^2 then $T^{h(n)-1} I_1^2$ onto J_n and finally J_n onto I_1^3 (as all these intervals have the same width). The interval I_1^1 is thus the basis of a new tower τ_{n+1} of height $3h(n)+1$. It is easy to adjust the length of the interval at stage 0 ($h(0)=1$) in such a way that the limit transformation T will be defined on I. This transformation is rank one since the sequence of towers defines a refining sequence of partitions into intervals of length going to 0 which will generate the Lebesgue algebra.

THEOREM 5.12. *The Chacon transformation is weakly mixing but not mixing.*

PROOF. Absence of mixing is a consequence of the fact that any set A which is the union of intervals in τ_n satisfies

$$m\big(A \cap T^{h(n)} A\big) \geqslant \frac{1}{3} m(A).$$

Weak mixing comes from the fact that if f is an eigenfunction corresponding to the eigenvalue $\lambda, \lambda \neq 1$, then given $\varepsilon > 0$, there will be an n and a level J in τ_n on which f will not vary by more than ε on a fraction $9/10$ of J. Call a the value to which f is close on J. But $T^{h(n)} f$ will be close to $\lambda^n a$ on a third of J, and $T^{h(n)+1} f$ will be close to $\lambda^{n+1} a$ on another third of J, forcing

$$\big|\lambda^n a - \lambda^{n+1} a\big| < \varepsilon,$$

$|\lambda - 1| < \varepsilon$. As ε was arbitrary, we obtain a contradiction. □

The following theorem is due to del Junco, Rahe and Swanson [33].

THEOREM 5.13. *The Chacon transformation has minimal self-joinings.*

Note that on immediate consequence of the definition implies that a transformation with MSJ commutes only which its powers and has no non-trivial factors. Thus the Chacon transformation is not rigid since the centralizer of a rigid transformation contains its orbit closure which is perfect and hence uncountable and has no rigid factors. Hence

COROLLARY 5.14. *The Chacon transformation is mildly mixing but not mixing.*

The Chacon transformation can be used to give an answer to the convolution problem. In fact M. Lemańczyk first proved that if σ is the spectral measure of the Chacon transformation, then $\sigma * \sigma \perp \sigma$. This was extended by A. Prikhod'ko and V. Ryzhikov [127] to the following.

THEOREM 5.15. *Let σ be the spectral measure of the Chacon transformation. Then for every $n \neq m$, $\sigma^{*n} \perp \sigma^{*m}$.*

For other methods of proving singularity of convolutions see Propositions 5.43 and 5.44 and Theorem 5.49.

A transformation with minimal self-joinings can be used as a "building block" for a great variety of examples. D. Rudolph in [140] developed a useful unifying concept of "counterexample machine". Very roughly, the counterexample machine can be thought of as a functor from the category of permutations of the set of integers $\mathbb{N}$ to measure preserving transformations. The arrows in the first category are injections $\mathbb{N} \to \mathbb{N}$ which are such that together with the corresponding permutations, they make the diagram commutative. In this last category, for example, it is easy to see that weak isomorphism does not imply isomorphism.

An interesting open question is related with Kakutani equivalence. The Chacon transformation itself is standard by Proposition 5.11, but it is not known whether its Cartesian square is standard.

5.2.4. *Rank one mixing transformations.* There is a method, due to D. Ornstein [117], to construct "random" rank one transformations which almost surely show very interesting properties.

We are given two sequences of integers $p(n)$ and $t(n)$ and a family of integers

$$a_{n,i}, \quad 1 \leqslant i \leqslant p(n), \quad a_{n,i} \leqslant t(n).$$

The construction is as in the Chacon example, with a tower τ_n which is made of $h(n)$ intervals $I_1, I_2, \dots, I_{h(n)}$ of equal length such that

$$T I_k = I_{k+1}, \quad 1 \leqslant k \leqslant h(n) - 1,$$

and the map acts by translations. To go to τ_{n+1}, I_1 is divided this time in $p(n)$ intervals of equal length, producing $p(n)$ columns τ_n^i, $1 \leqslant i \leqslant p(n)$, and τ_{n+1} is constructed by stacking τ_n^{i+1} onto τ_n^i, $1 \leqslant i \leqslant p(n) - 1$, after the insertion, between the last level of τ_n^i and the basis of τ_n^{i+1} of $a_{n,i}$ intervals (which all have the same length); t_n is chosen so that $t_n \leqslant h_{n-1}$ and $t_n \to \infty$. These added intervals are called spacers. The randomness is on the $a_{n,i}$ which are chosen independently, such that for given n, all the $a_{n,i}$ take values on

$[1, t_n]$ which uniform probability $1/t(n)$. In this probability space, a point ω is the sequence of $a_{n,i}$, $1 \leqslant i \leqslant p(n)$, $n \geqslant 1$, and to every such ω corresponds a rank one transformation T_ω. D. Ornstein has proved:

THEOREM 5.16. *In the previous model, almost surely T_ω is mixing.*

We have seen (Proposition 5.8) that rank one transformations have simple spectrum, the fact that they could be mixing made them interesting candidates for examples with simple Lebesgue spectrum. However J. Bourgain [24] has proved

THEOREM 5.17. *In the previous model, almost surely the spectral measure of T_ω is singular with respect to Lebesgue measure.*

In fact it looks quite plausible that every rank one transformation has purely singular spectral measure. This is justified by the previous theorem which implies by Host's theorem that these transformations are almost surely mixing of all orders, and by the following result of S. Kalikow [73].

THEOREM 5.18. *A mixing rank one transformation is mixing of all orders.*

V. Ryzhikov [142] has extended the previous theorem to the following:

THEOREM 5.19. *A mixing finite rank transformation is mixing of all orders.*

It is not known whether the same holds for mixing locally rank one transformations.

As as consequence of the theorem of Kalikow, J. King proved

THEOREM 5.20. *A mixing rank one transformation has minimal self-joinings.*

This implies in particular that a mixing rank one transformation commutes only with its powers (this was proved in the original paper of Ornstein) and has no factors.

For a long time existence of mixing rank one transformations was only known through the construction of Ornstein. Much later T. Adams [14] gave an explicit construction of mixing rank one transformations (the staircase Smorodinsky's rank one where the spacers are added in such a way that they follow the shape of a staircase). And recently B. Fayad has constructed C^1 flows which are mixing and rank one (as flows) [45].

5.2.5. *Riesz products and spectra of rank one transformations.* Riesz products appear naturally as spectral measures in several natural examples in ergodic theory. For detailed definitions and extensive discussion of the subject see [114, Chapter 16]. Riesz products in the context of ergodic theory first appeared in the paper by Ledrappier [102], where a certain finite extension of a system with pure point spectrum is shown to have a component in its spectral measure which is a Riesz product. It is important and interesting because there exist in many cases explicit criteria which can determine whether the corresponding measures are singular or absolutely continuous [125]. Riesz products occur also explicitly as components of the spectral measure of many substitutions [128].

A revival of the use of Riesz product techniques arose from already mentioned result of Bourgain (Theorem 5.17) where he first gives an explicit formula for the spectral measure of the general rank one transformation. Such measures can be viewed as generalized Riesz products. In his proof, Bourgain produces, using the fact that for any ergodic transformation $(X, \mathcal{A}, m, T)$, for any function f in L^2, almost surely the sequence of measures

$$\frac{1}{N}\left|\sum_{k=1}^{k=N} f(T^k x)e^{2i\pi k\theta}\right|^2 d\theta$$

converges weakly to ν_f, the spectral measure ν_T of the general rank one transformation as a generalized Riesz product

$$\prod_{n=1}^{n=\infty} |P_n|^2.$$

This can be seen exactly in the same way as in the proof of Theorem 1.8. By the weak convergence we mean that the measures $\prod_{n=1}^{n=N} |P_n|^2\, d\theta$ converge weakly to the spectral measure ν_T. The polynomials $P_n(\theta)$ are equal to

$$\big(p(n)\big)^{-1/2}\sum_{k=0}^{p(n)-1} e^{2\pi i(kh(n)+\sum_{j=1}^{j=k} a(j,n)\theta)}.$$

This is obtained by applying the previous formula to the characteristic functions of the base of the tower τ_1. Then Bourgain shows that it is sufficient to prove singularity for a product of a subsequence of the previous polynomials which are dissociated and to which classical Riesz product techniques can be applied. Note that the mixing property can not be verified by the use of this formula.

The same ideas are present in the paper of Klemes [89] where he shows that the spectral measure of the Adams example [14] is singular. It is also with a proof in the same spirit that El Abdalaoui [40] has shown that if we endow the Cartesian product of the parameter space of the Ornstein example with the product measure, for almost every pair ω, ω', T_ω and $T_{\omega'}$ have mutually singular spectral measures (and are therefore disjoint by Proposition 4.3).

5.2.6. *Cutting and stacking and orbit growth*. In Sections 5.2.3 and 5.2.4 we described constructions of rank one transformations where interesting behavior is achieved by time delays in the return from a part of the roof of the single tower to its base. For the Chacon transformation this delay was by time one on one third of the tower and for rank one mixing transformations the delays were uniformly distributed in an appropriate sense. Thus non-trivial combinatorics was achieved by the distribution of the delay times.

These examples represent instances of intermediate orbit growth; not slow elliptic and not exponential hyperbolic or uniform polynomial parabolic like horocycle flows (Section 6.2.2). They probably are best described are being outside of three principal paradigms. This fits well with the fact that no smooth realization for Chacon transformation is known and for rank one mixing map realization has only been achieved in C^1 which is considered somewhat pathological in the smooth setting.

We should point out that interesting behavior (but not mixing or even mild mixing) may be achieved also in the context of fast cyclic approximation (elliptic behavior) (see Section 5.4.2) which can be interpreted as uniform approximation with single towers and direct return of most of the roof to the base. In this case there are spacers too but their effect becomes noticeable only after running the cycle for many times.

5.2.7. *Constructions with many towers*. At the other end of the spectrum of possibilities for cutting and stacking lie situations where the number of towers grows and the roofs of towers at an inductive step are mapped to the base in a complicated way. All positive entropy examples constructed by cutting and stacking necessarily have such structure as well as examples with subexponential but still substantial orbit growth such as transformations from the actions in [83]. The most straightforward way to carry out such constructions is to match the roofs to the bases more or less independently. This method allows to produce any desirable speed and regularity of orbit growth by controlling the number of towers in the approximating cityscapes.

In such constructions if spacers are not used at all (in other words, if at every step the cityscape $\mathcal{C}_n$ fills the whole space) the resulting transformation has an odometer (Example 2.16(3)) as a factor. In order to achieve weak mixing, not speaking of mixing or K-property, spacers are needed in addition to the distribution of roofs. Non-isomorphic K-automorphisms with the same entropy from [119] as well as non-loosely Bernoulli K-automorphisms from [51] are examples produced by cutting and stacking constructions of that type. The original Feldman example has been extended [118] to provide uncountably many zero entropy transformations which are pairwise not equivalent.

There are various types of cutting and stacking constructions: the ones we mentioned are based on the idea that a fixed pattern is repeated at every stage. Some others alternate two very different patterns. A typical one in that class is the Rothstein's construction of non-loosely Bernoulli transformation [138]: there is an alternation of stages where independent cutting and stacking is performed thereby creating so many names that most of them are far apart in the $\bar{f}$ distance (based on the Kakutani distance between string of symbols) and is next followed by a stage where names are just repeated twice, which has an effect of dropping the entropy without altering too much the separation of names previously created in $\bar{f}$ metric.

Very beautiful cutting and stacking constructions have been found by C. Hoffman [71]: he has developed a version of Rudoplh's counterexample machine (see Section 5.2.3) for K-automorphisms and in particular, produced two weakly isomorphic but not isomorphic K-automorphisms with finite entropy.

5.3. *Coding*

The coding constructions are very close to symbolic dynamics, see [8, Section 2.6] for an overview of that subject. For a comprehensive introduction and many interesting examples see [109]. In some of those constructions invariant measure is given as is the case for interval exchange transformations (Section 5.3.1) in others it is not fixed from the beginning but is constructed as the asymptotic distribution on the chosen names as for substitution dynamical systems discussed in Section 5.3.2.

To put the coding-based constructions into the general framework of the inductive combinatorial constructions we consider a space with a partition into "symbols" of an "alphabet" and define certain rules by which allowable words are produced. Similarly to the cutting and stacking (uniform approximation) constructions the coding method is very general since any ergodic finite entropy transformation allows a finite generator and hence a symbolic representation [96]. However when we speak of combinatorial constructions of coding type we mean certain recursive procedures which allow inductively to produce distributions of longer words from those of shorter ones.

Now we will consider several specific classes of such constructions.

5.3.1. *Interval exchange transformations*

Definition and parametrization. Consider $n \geqslant 2$ and π an irreducible permutation of $\{1, \ldots, n\}$. A permutation π is called *irreducible* if $\pi\{1, \ldots, d\} \neq \{1, \ldots, d\}$, $1 \leqslant d < n$. Let Δ be the simplex in R^n,

$$\lambda = (\lambda_i), \quad 1 \leqslant i \leqslant n, \lambda_i \geqslant 0, \sum_{i=1}^{i=n} \lambda_i = 1.$$

The unit interval $I = [0, 1)$ is divided into semi-open intervals $I_d = [\sum_{i<d} \lambda_i, \sum_{i \leqslant d} \lambda_i)$, $1 \leqslant d \leqslant n$.

The *interval exchange transformation* $T_{\pi,\lambda}$ acts on every I_d by a translation in such a way that the intervals are rearranged according to the permutation π. That is, on I_d, $T_{\pi,\lambda}$ is the translation by $\sum_{\pi(i)<\pi(d)} \lambda_i - \sum_{i<d} \lambda_i$.

Interval exchange transformations preserve Lebesgue measure. Sometimes more general transformations which change orientation on some of the intervals are also considered.

Interval exchange transformations are briefly mentioned in [8, Sections 4.3g and 8.4] and more thoroughly discussed in [11, Section 6]. For an elementary self-contained introduction to the subject see [79, Section 14.5]. The area has developed into a major subject of research with some of the deepest and most beautiful results and constructions in the whole of ergodic theory. Some of the recent work in the area is described in [5].

The parameter space for the set of exchanges of n intervals is the simplex of the lengths of the intervals multiplied by the finite set of irreducible permutations. Notice that dynamics obviously depend on the choice of parameters and is fairly simple in some cases. For example, if all λ's are rational all points are periodic albeit with different periods. This is of course similar to the case of translations on the torus. Another similarity with toral translations is prevalence of minimality.

THEOREM 5.21 ([79, Corollary 14.5.12], originally appeared in [85]). *If one excludes from the simplex of lengths intersections with countably many hyperplanes then every orbit* *of the interval exchange transformation corresponding to the remaining set of parameters is dense.*

The proof is based on the observation that unless there is a "saddle connection", i.e. an orbit segment beginning and ending in discontinuity points then all orbits are dense. For an irreducible interval exchange any type of saddle connection generates a rational relation between the lengths of the intervals.

However, ergodic properties of interval exchange transformations with respect to Lebesgue measure exhibit more complicated dependence of the parameters than is the case with toral translations. The same apply to the question of unique ergodicity.

Finiteness properties. The following three theorems summarize the basic distinctive properties of interval exchange transformations which do not depend on the choice of parameters and which can be described as something like "finiteness of dynamical complexity". The key observation here is that the transformation induced by an exchange on n intervals on any interval, however small, is again an exchange of at most $n+1$ intervals.

THEOREM 5.22 [76]. *An aperiodic interval exchange transformation on k intervals is of finite rank at most k. Furthermore, it is rank k by intervals: that is all the levels of the towers which appear in the definition of finite rank are intervals. Furthermore these towers fill the whole space.*

Unlike the general finite rank (or even rank one) property this kind of uniform approximation implies absence of mixing.

THEOREM 5.23 ([11, Theorem 6.10], originally proved in [76]). *An interval exchange transformation is never mixing.*

Another consequence of Theorem 5.22 is an estimate on the number of ergodic measures and the spectral multiplicity of any such measure.

THEOREM 5.24. *An aperiodic interval exchange transformation of n intervals has at most $n-1$ ergodic Borel probability invariant measures. Spectral multiplicity of the transformation with respect to any invariant measure (ergodic or not) does not exceed n.*

The estimate on the number of ergodic measures can be improved. The best estimate which depends only on the number of intervals is $n/2$ for n even (this includes unique ergodicity of irrational rotation for $n=2$) and $n-1/2$ for n odd. This estimate is sharp for the reverse permutation $\pi(i)=n-i$. On the other hand, there is a sharp estimate for any permutation which depends not only on the number of intervals but on the permutation π. For a "generic" combinatorics, the resulting estimate is slightly above $n/4$. This may sound mysterious but becomes transparent when one constructs for any interval exchange transformation an oriented surface with a flow for which the original transformation serves as a section map on a certain arc connecting two (not necessarily different) saddles. The sharp estimate for the number of ergodic invariant measures is the genus of the surface which depends only on the permutation π. See [79, Theorem 14.7.6] for the inequality and [145] for constructions of minimal examples with any number of ergodic measures between one and the genus. Satayev's method in [145] makes use of symmetry in a way somewhat similar to that used in the construction of transformations with given values of multiplicity in the spectrum which we discuss in Section 5.8.4.

Typical behavior in the parameter space: direct methods. Finiteness of the number of ergodic invariant measures implies in particular that Lebesgue measure has finite number of ergodic components. Hence one may ask when an interval exchange transformation is ergodic with respect to Lebesgue measure, or, which is even more natural in the present context when Lebesgue measure is the only invariant measure for an interval exchange transformation. This is one of the few places in this survey when we do not have ergodicity given *a priori* or following from a construction but discuss conditions for ergodicity instead.

The answer is easy and explicit for $n = 2$ and 3 because in those cases the surface discussed above is a torus.

In both cases the only irreducible permutations are reverse permutations. Exchange of two intervals of lengths λ and $1-\lambda$ becomes the circle rotation R_λ once the interval $[0, 1)$ is identified with the circle. Thus irrationality of λ is equivalent to unique ergodicity.

The situation is only slightly more complicated for the exchange of three intervals of lengths λ_1, λ_2 and $1 - \lambda_1 - \lambda_2$ in reverse order. Direct inspection shows that this transformation is identified with the transformation induced by the rotation $R_{\frac{1-\lambda_1}{1+\lambda_2}}$ on any interval of length $\frac{\lambda_2}{1+\lambda_2}$. Hence the interval exchange is ergodic with respect to Lebesgue measure if and only if it is uniquely ergodic and this happens exactly when the number $\frac{1-\lambda_1}{1+\lambda_2}$ is irrational.

For $n \geqslant 4$ the picture becomes considerably more complicated. First, there is no more dichotomy between periodicity and unique ergodicity. Necessary and sufficient conditions for ergodicity with respect to Lebesgue measure or unique ergodicity (which are also not equivalent anymore) are not available. However, the following fundamental result holds.

THEOREM 5.25. *Almost every with respect to Lebesgue measure on the simplex of length interval exchanges transformations is uniquely ergodic*.

This theorem was originally proved independently by Veech [153] and Masur [111] using advanced indirect methods which are discussed below. Shortly afterwards Boshernitzan [22] found a direct (albeit fairly complicated) proof based on the following sufficient criterion for unique ergodicity.

For a given interval exchange transformations let ξ be the partition into its intervals of continuity and $\xi_n = \bigvee_{k=0}^{n-1} T^k_{\pi,\lambda}\xi$ be the iterated partition. Notice that the number of elements in ξ_n grows linearly with n. Aperiodicity of the transformation is equivalent to fact that the maximal length of elements in ξ_n goes to zero as $n \to \infty$. Let m_n be the *minimal* length of an element in ξ_n. Given $\varepsilon > 0$ we will call positive integer n *ε-regular* if $m_n \geqslant \frac{\varepsilon}{n}$.

An interval exchange transformation satisfies *property P* if for any $l \geqslant 2$ there exists $\lambda(l)$ such that there are infinitely many sequences of ε-regular numbers of length l, $n_1, \ldots, n_l$ with $n_{i+1} > 2n_i$, $i = 1, \ldots, l-1$, and $n_l < \lambda n_1$. Any set of natural numbers which contains such sequences will be called *essential*.

THEOREM 5.26. *Any interval exchange transformation which satisfies property P is uniquely ergodic*.

Now consider a family of interval exchange transformations parametrized by a space Ω with a probability measure μ. For a given ε let $u(n, \varepsilon)$ be the measure of the set of parameters for which the number n is ε-regular for the corresponding interval exchange transformation.

A family of interval exchange transformations parametrized by Ω satisfies *collective property P* if for any $\varepsilon > 0$ one can find $\delta > 0$ and a single essential set $A(\varepsilon)$ such that $u(n, \delta) > 1 - \varepsilon$ for all $n \in A(\varepsilon)$.

PROPOSITION 5.27. *If a family satisfies collective property P then almost every element in the family satisfies property P.*

One can show that for any admissible permutation the whole simplex of interval exchange transformations with this permutation and with Lebesgue measure satisfies collective property P. Theorem 5.25 then follows from Proposition 5.27 and Theorem 5.26.

An earlier and more elementary example of use of direct methods for showing prevalence of certain properties concerns spectral properties of exchanges of three intervals.

Many interesting phenomena in ergodic theory can be realized within the class of interval exchange transformations. In particular, this is connected with a possibility to realize certain kinds of symmetry within this class. Notice in particular that any piecewise constant finite extension of a rotation (or, more generally of an interval exchange transformation) can be represented as an interval exchange transformation. See Section 5.8.4.

THEOREM 5.28 [81]. *Almost every exchange of three intervals has simple singular continuous spectrum*.

This result follows form existence of both good cyclic approximation and good approximation of type $(n, n+1)$ (Section 5.4.2), which are constructed using properties of approximation of parameters by rationals, and Propositions 5.39 and 5.40.

Renormalization dynamics and advanced results. A powerful indirect approach to the study of interval exchange transformations is based on renormalization type dynamics introduced by Rauzy [133]. It was first developed by Veech [153] for his proof of Theorem 5.25. Let us mention a couple of relevant results.

Veech [154] has proved:

THEOREM 5.29. *Almost every interval exchange transformation is of rank one*.

Recently Avila and Forni [20] solved the long-standing open problem.

THEOREM 5.30. *Almost every interval exchange transformation is weakly mixing.*

The following related question by Veech remains open.

PROBLEM 5.31. Is it true that almost every interval exchange of $m \geqslant 3$ intervals is simple?

5.3.2. *Substitution dynamical systems.* Another interesting class of examples related to symbolic dynamics comes from what is called substitutions. Literature on substitution dynamical systems is quite extensive, maybe a bit out of proportion of the place of the subject within the general context of ergodic theory and symbolic dynamics. In particular, a detailed albeit not fully up-to-date account of the spectral properties for this class of systems exists in book form [128]. We restrict ourselves to the definition and a couple of interesting examples.

We consider a finite set $A = \{0, 1, 2, \dots, n-1\}$. We let $A^* = \bigcup_{k \geqslant 1} A^k$ be the set of all finite words in the alphabet of A.

DEFINITION 5.32. A *substitution* ζ on A is a map from A to A^*. It defines a map from A^* to A^* in the following way: if $x = x_0 x_1 \dots x_n \in A^*$, then

$$\zeta(x) = \zeta(x_0)\zeta(x_1)\dots\zeta(x_n).$$

This obviously extends to a map from A^N to A^N.

We consider substitutions such that

(a) the length of $\zeta^n(i)$ goes to infinity when $n \to \infty$ for every $i \in A$,

(b) there exists a symbol 0 in A such that $\zeta(0)$ starts with (0),

(c) there exists an integer k such that for every two $i, j \in A$, $\zeta^k(i)$ contains j.

A substitution satisfying (a), (b) and (c) is called *primitive*. The most famous transformation which can be described by a substitution is the *Morse sequence*, which is defined on the alphabet 0, 1 by

$$\zeta(0) = 01, \qquad \zeta(1) = 10.$$

THEOREM 5.33. *Given a primitive substitution ζ any fixed point $x = \zeta(x)$, $x \in A^N$ (which is easily shown to exist) has on orbit closure X on which the shift T is a uniquely ergodic transformation (independent of the fixed point).*

THEOREM 5.34. *All the transformations (X, T) as described in the previous theorem are finite rank transformations.*

Another important example is the *Rudin–Shapiro sequence* which is generated by the following primitive substitution on 4 symbols:

$$\zeta(0) = 02, \qquad \zeta(1) = 32, \qquad \zeta(2) = 01, \qquad \zeta(3) = 31.$$

The remarkable spectral property of the transformation T associated to the Rudin–Shapiro sequence is the following, first proved by T. Kamae [74]

THEOREM 5.35. *T has a Lebesgue component in its spectral measure.*

Since T is of finite rank, this implies U_T is a unitary operator with finite spectral multiplicity and a Lebesgue component in its spectrum. We note however that U_T also has a discrete component in its spectrum. This is one of very few known examples with Lebesgue component of finite multiplicity in the spectrum. Other examples are discussed in Section 5.8.5; these constructions are somewhat more flexible than Rudin–Shapiro; in particular, they can be made weakly mixing (Theorem 5.75).

Notice that no examples are known with a *simple* Lebesgue component in the spectrum as well as with Lebesgue (or absolutely continuous) spectrum of finite multiplicity.

5.4. *Periodic approximation*

The method of periodic approximations is in a number of respects parallel and complementary to the cutting and stacking method. It is based on the ideas of fast approximation of a measure preserving transformations in weak as opposed to uniform topology. This allows to define approximating transformations everywhere if need arises. The method has been introduced in [81]; see also [29]. For the most up-to-date albeit not comprehensive presentation of the methods and some of its applications see [78]. We mostly follow the last source in this section.

5.4.1. *Periodic processes.* Let (X, μ) be a Lebesgue space. A *periodic tower* t is an ordered sequence of disjoint subsets $t = \{c_1, \dots, c_h\}$ of X having equal measure which we will usually denote $m(t)$. The number $h = h(t)$ will be called *the height* of the tower t. Associated with a tower, there will be a cyclic measure-preserving permutation σ sending c_1 to c_2, c_2 to c_3, etc., and c_n to c_1. The set c_1 will be called *the base* of the tower.

DEFINITION 5.36. A *periodic process* is a collection of disjoint towers covering X, together with an equivalence relation among these towers which identifies their bases. A periodic process which consists of a single tower is called a *cyclic process*.

The notion of periodic tower is a counterpart of the notion of tower in the construction of uniform approximation while the notion of periodic process corresponds that of cityscape from Section 5.1.2.

The partition into all elements of all towers will normally be denoted by ξ, sometimes with indices. The permutation σ sends every element of ξ into the next element of its tower in cyclic order. Another partition naturally associated with a periodic process consists of the unions of bases of towers in each equivalence class and their images under the iterates of σ, where when we go beyond the height of a certain tower in the class we drop this tower and continue until the highest tower in the equivalence class has been exhausted. We will denote this partition by η, with appropriate indices. Obviously $\eta \leqslant \xi$.

DEFINITION 5.37. The sequence $(\xi_n, \eta_n, \sigma_n)$ of periodic processes is called *exhaustive* if $\eta_n \to \varepsilon$ as $n \to \infty$, i.e. for every measurable set $A \subset X$ there exists a sequence of sets $A_n \in \mathfrak{B}(\eta_n)$ such that $\mu(A \Delta A_n) \to 0$ as $n \to \infty$. An exhaustive sequence of periodic processes $(\xi_n, \eta_n, \sigma_n)$ is called *consistent* if for every measurable set $A \subseteq X$, the sequence $\sigma_n A$ converges to a set B, i.e. $\mu(\sigma_n A \Delta B) \to 0$ as $n \to \infty$.

Since $\xi_n \geqslant \eta_n$, then for an exhaustive sequence of periodic processes, $\xi_n \to \varepsilon$ as $n \to \infty$. For a consistent exhaustive sequence of periodic processes, independently of particular realizations of σ_n as measure-preserving transformations, the sequence $\{\sigma_n\}$ converges in the weak topology. For a given transformation T and an exhaustive sequence of periodic processes $(\xi_n, \eta_n, \sigma_n)$, a sufficient condition for the weak convergence of $\sigma_n \to T$ is $d(\xi_n, T, \sigma_n) = \sum_{c \in \xi_n} \mu(Tc \Delta \sigma_n c) \to 0$ as $n \to \infty$.

DEFINITION 5.38. If the last condition is satisfied we will say that the exhaustive sequence of periodic processes $(\xi_n, \eta_n, \sigma_n)$ forms a *periodic approximation* of T. In particular, if the periodic processes are cyclic the periodic approximation is called *cyclic*.

5.4.2. *Speed of approximation*. The *type* of approximation is defined in [78, Definition 1.9]. It involves a somewhat technical equivalence relation between sequences of periodic processes. However there is going to be no ambiguity for natural types of approximation discussed below, such as cyclic, type $(n, n+1)$ and so on. Given a type $\mathcal{T} = \{\tau_n\}$ in that sense defined above and a sequence $g(n)$ of positive numbers, we will say that a measure preserving transformation T *admits a periodic approximation of type* $\{\tau_n\}$ *with speed* $g(n)$ if for a certain subsequence $\{n_k\}$ there exists an exhaustive sequence of periodic processes $(\xi_k, \eta_k, \sigma_k)$ of type τ_{n_k} such that

$$d(\xi_k, T, \sigma_k) < g(n_k).$$

The speed of approximation will usually be measured against a certain *characteristic parameter* q depending on the type. There is a natural notion of a good speed of approximation, which generally means that a typical orbit of the limit transformation reproduces the behavior of one of the orbits of the approximation for sufficiently many periods. Usually the characteristic parameter q is chosen in such a way that *good approximation means approximation with any speed of the form* $g(q) = o(1/q)$. In the particular case of cyclic approximation the only parameter for a cyclic process is the height q of its single tower, which naturally serves as the characteristic parameter. Cyclic approximation with speed $o(1/q)$ is usually called *good cyclic approximation*. Good cyclic approximation is characteristic for the elliptic paradigm in smooth ergodic theory (see Section 2.2.4). Principal properties of transformations allowing good cyclic approximation which are thus typical for the elliptic paradigm are summarized in the following proposition. When it is possible we also describe weaker conditions.

PROPOSITION 5.39. *If T admits a good cyclic approximation then*:

(1) *T is ergodic. This remains true for a cyclic approximation with speed $(4-\theta)/q$ for any fixed $\theta > 0$* [81,29].

(2) T *is not mixing*. *This remains true for the speed* $(2-\theta)/q$, $\theta > 0$ [81,29].
(3) *The maximal spectral type of* T *is singular* [81]. *This remains true for speed* $(1-\theta)/n$ *for any* $\theta > 0$ [78, *Section* 3].
(4) T *is rigid* [81]. *This property implies* (3).
(5) T *is standard*; *this remains true for the speed* $(2-\theta)/q$, $\theta > 0$.
(6) T *is rank one*.
(7) T *has simple spectrum*. *This follows from* (6). *This property remains true for speed* $(1-\theta)/q$ *for any* $\theta > 0$ [81,29].

Good cyclic approximation does not allow to distinguish between transformations with pure point, mixed or continuous spectrum. In fact, every ergodic transformation with pure point spectrum admits good cyclic approximation [75, Section 8]. Here we give an example of another approximation property which guarantees weak mixing.

The type of periodic approximation is generated by periodic processes equivalent to processes consisting of two substantial towers t_1, t_2 whose heights differ by 1. Equivalently the heights of the two towers are equal to n and $n+1$ and for some $r > 0$,

$$m(t_1) > r/n \quad \text{and} \quad m(t_2) > r/n. \tag{5.3}$$

This type of approximation is said to be of *type* $(n, n+1)$. This type of approximation is related with the rank two property (see Section 5.2.2) and it implies rank two if the speed is sufficiently high; however the extra property that the roof of each tower returns mostly to the base of the same tower makes it stronger. For approximation of type $(n, n+1)$ the choice of the characteristic parameter is ambiguous. There are two natural ways to define it according to what properties of the limit transformation T we want to study. Namely, we can either take the characteristic parameters q as the length of one of the cycles (n or $n+1$), or as the period $n(n+1)$ of the permutation σ. We will call the approximation of type $(n, n+1)$ with speed $o(1/n)$ *good* and the approximation with speed $o(1/n(n+1))$ *excellent*. One some occasions it will be necessary to assume that the two towers involved in the approximation are equivalent. This simply insures that the partitions generated by the union of the bases of the towers and the iterates of this set is fine. The corresponding approximation will be called *linked approximation of type* $(n, n+1)$.

PROPOSITION 5.40 [82, Theorem 5.1]. *If a transformation* T *admits a good linked approximation of type* $(n, n+1)$ *or if* T *is ergodic and admits a good approximation of type* $(n, n+1)$ *then* T *has continuous spectrum*.

SKETCH OF PROOF. The proof is very similar to the proof of weak mixing for the Chacon transformation (Theorem 5.12). Namely an eigenfunction with eigenvalue λ would have to be almost constant on a typical level of the linked towers and hence on the base. But since return to the base happens mostly in two successive moments n and $n+1$ which implies that both λ^n and λ^{n+1} are close to one and hence in the limit $\lambda = 1$ which contradicts ergodicity. □

The property of approximation of type $(n, n+1)$ (linked or not) is compatible with cyclic approximation with arbitrary high speed. This allows to demonstrate in very simple concrete examples how transformations admitting good periodic approximation may have mixed or continuous spectrum, see, for example, Theorem 5.28. Another historically important two-parametric family of examples is the two point extension of the rotation R_α with the switch of levels on the interval $[0, \beta]$. For almost every (α, β) the spectrum in the space of "odd" functions is simple, singular and continuous.

5.4.3. *Further properties and applications.* Some more elaborate versions of periodic approximation either compatible with fast cyclic approximation or not produce interesting properties.

PROPOSITION 5.41. *If T admits an excellent linked approximation of type $(n, n+1)$ then the maximal spectral multiplicity $M_{T\times T}$ (cf. 4.1) of $T \times T$ is finite and is less than or equal to $2[1/2r(1-r)]$, where r is the constant from (3.6). In particular, if $r > 1/2 - \sqrt{3}/6$ then $M_{T\times T} \leqslant 4$.*

SKETCH OF PROOF. The Cartesian square of the periodic process approximating T is a periodic process approximating $T \times T$ which, in this case, includes two substantial towers of height $n(n+1)$ and each of these towers has measure at least $r(1-r)$. For this periodic process, the length of the maximal cycle is equal to the period of the permutation $\sigma \times \sigma$. Furthermore, if the original approximation is excellent then the approximation of $T \times T$ is good when measured against this parameter. We consider the invariant subspace generated by characteristic functions of the bases of two towers of height $n(n+1)$ and apply Theorem 1.21 to this subspace. □

There is a natural generalization of an approximation of type $(n, n+1)$ which is useful for dealing with higher Cartesian powers. It involves several substantial towers whose heights are consecutive integers. A version of this property is also crucial in the proof of the genericity of the following useful property due to Stepin and Oseledec [148]; see also [149].

DEFINITION 5.42. Given $0 \leqslant \alpha \leqslant 1$, a measure-preserving transformation T is called *α-weak mixing* if for some sequence $n_k \to \infty$ and for every set A,

$$\lim_{k\to\infty} \mu\big(T^{n_k} A \cap A\big) = \alpha\mu(A)^2 + (1-\alpha)\mu(A).$$

An equivalent formulation of α-weak mixing is that the operators $U_T^{n_k}$ converge in the weak operator topology to $(1-\alpha)\,\mathrm{Id} + \alpha P_c$ where P_c is the orthogonal projection to the one-dimensional space of constants. 0-weak mixing corresponds to rigidity, whereas 1-weak mixing corresponds to the usual notion of weak mixing. Although the terminology may suggest it, α-weak mixing does not imply β-weak mixing for $\beta < \alpha$. On the contrary, α-weak mixing for any $\alpha > 0$ implies 1-weak mixing.

PROPOSITION 5.43. *If T is α-weak mixing for some $0 < \alpha < 1$ and ρ is the maximal spectral type for $U_T|_{L_2^0(X,\mu)}$, then all of the convolutions $\rho^{(m)}$ for $m = 1, 2, \ldots$ are pairwise singular.*

Now let us show how to derive α-weak mixing from an approximation. We consider a process with s linked towers $t_1, \ldots, t_s$ of consecutive heights $q, q+1, q+2, \ldots, q+s-1$, where q will serve as the parameter for good approximation. If $m(t_i) = \mu_i$, $i = 1, \ldots, s$, we will call such an approximation a *linked approximation of type* $(q, q+1, \ldots, q+s-1; \mu_1, \ldots, \mu_s)$.

PROPOSITION 5.44. *Given α, $0 \leqslant \alpha \leqslant 1$, if T admits a good linked approximation of type*

$$\left(q, q+1, \ldots, q+s-1;\ \frac{1-\alpha}{q}, \frac{\alpha}{(q+1)(s-1)}, \ldots, \frac{\alpha}{(q+s-1)(s-1)}\right)$$

for an arbitrary large s, then T is α-weak mixing.

An application of α-weak mixing, given by del Junco and Lemańczyk [32], is that it implies a kind of "rigidity of joinings" property.

THEOREM 5.45. *Let $(X, \mathcal{A}, m, T)$ be α-weakly mixing with $0 < \alpha < 1$. Consider $S = \prod_{i \in N}(X_i, \mathcal{A}_i, m_i, T_i)$, where $(X_i, \mathcal{A}_i, m_i, T_i)$ is a copy of $(X, \mathcal{A}, m, T)$ for each $i \in N$. If $\mathcal{B}$ is an S invariant subalgebra of $\prod_{i \in M} \mathcal{A}_i$ restricted to which S acts isomorphically to a factor of T, then $\mathcal{B}$ is a factor of some $\mathcal{A}_i$.*

The proof uses Proposition 5.43 and the property is already sufficient to produce, with the help of the same techniques, some of the examples which can be obtained using transformations which have minimal self-joinings. The authors have given an extension of the notion of α-weak mixing, $(\alpha_1, \alpha_2, \ldots, \alpha_s)$-weak mixing, such that transformations which satisfy it can be used as building blocks to exhibit most of the examples of the "counterexample machine" of D. Rudolph. It is interesting that this can be reached out of purely spectral properties. However $(\alpha_1, \alpha_2, \ldots, \alpha_s)$-weak mixing transformations are only produced through constructions involving some grafting of "mixing rank one type" objects, which hinders any simple presentation.

B. Fayad [46] developed a novel concept of periodic approximation where at each given moment only a small part of the space returns close to itself but over the time most points experience this return infinitely many times. The goal was to find a criterion of singular spectrum which is compatible with mixing. Abstract description of the property in purely measurable terms in somewhat cumbersome and in [46] a structure of metric space is assumed. Then the property of *slowly coalescent periodic approximation* involves systems of balls of decreasing size returning to themselves at exponentially growing moments of time with exponentially small relative error in such a way that almost every point belongs to infinitely many such balls.

PROPOSITION 5.46. *Any transformation which admits slowly coalescent periodic approximation has singular spectrum* (*not necessarily continuous*).

5.4.4. *Genericity of periodic approximation* [78, Section 2]. Many important properties generic for measure preserving transformations in weak topology can be deduced for the following result (see [78, Theorem 2.1]).

THEOREM 5.47. *Given a type $T = \{\tau_n\}$ and a speed $g(n)$, the set of all measure-preserving transformations of a Lebesgue space which admit a periodic approximation of type T with speed $g(n)$ is a residual set* (*i.e. it contains a dense G_δ set*) *in the weak topology.*

In particular, all properties discussed earlier in this section which follow form a certain type of periodic approximation belong to this category. For convenience we formulate this as a separate statement.

COROLLARY 5.48. *A generic measure preserving transformation in the weak topology is weakly mixing* (*hence ergodic*), *rigid* (*hence is not mildly mixing*), *has simple singular spectrum such that the maximal spectral type in L_0^2 together with all its convolutions are mutually singular and supported by a thin set on any given scale*.

We will see later that one can add to this list homogeneous spectrum of multiplicity two for the Cartesian square, see Section 5.8.2, and other properties.

5.5. *Approximation by conjugation*

5.5.1. *General scheme*. Approximation by conjugation is a method of producing transformations admitting fast periodic approximation as well as some other transformations with interesting properties by conjugating elements (usually periodic) of actions of compact groups (usually S^1, but sometimes $\mathbb{T}^k$ and others) and taking limits in various topologies. This method is particularly suitable for smooth realizations of measure preserving transformations with various properties. It was first introduced in [18]; this is still the basic source on the subject. For an account of some recent development as well as an up-to-date perspective on the topic see [47]. A purely measurable version of the method and some of its applications are described in [78, Section 8]. Since most applications of the method still deal with the smooth situation, we will present the set-up and results for that case. We present a general overview of the method following [47].

Let M be a differentiable manifold with a non-trivial smooth circle action $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, $S_{t+1} = S_t$, preserving a smooth volume. Every smooth S^1 action preserves a smooth volume ν which can be obtained by taking any volume μ and averaging it with respect to the action: $\nu = \int_0^1 (S_t)_*\mu\, dt$. Similarly $\mathcal{S}$ preserves a smooth Riemannian metric on M obtained by averaging of any smooth Riemannian metric.

Volume preserving maps with various interesting, often surprising, topological and ergodic properties are obtained as limits of volume preserving periodic transformations

$$f = \lim_{n\to\infty} f_n, \quad \text{where } f_n = H_n S_{\alpha_{n+1}} H_n^{-1} \tag{5.4}$$

with $\alpha_n = \frac{p_n}{q_n} \in \mathbb{Q}$ and

$$H_n = h_1 \circ \cdots \circ h_n, \tag{5.5}$$

where every h_n is a volume preserving diffeomorphism of M that satisfies

$$h_n \circ S_{\alpha_n} = S_{\alpha_n} \circ h_n. \tag{5.6}$$

In certain versions of the method the diffeomorphisms h_n are chosen not preserving the volume but distorting it in a controllable way; this, for example, is the only interesting situation when M is the circle (see, e.g., [79, Section 12.6]).

Usually at step n, the diffeomorphism h_n is constructed first, and α_{n+1} is chosen afterwards close enough to α_n to guarantee convergence of the construction. For example, it is easy to see that for the limit in (5.4) to exist in the C^∞ topology it is largely sufficient to ask that

$$|\alpha_{n+1} - \alpha_n| \leqslant \frac{1}{2^n q_n \|H_n\|_{C^n}}. \tag{5.7}$$

The power and fruitfulness of the method depend on the fact that the sequence of diffeomorphisms f_n is made to converge while the conjugates H_n diverge often "wildly" albeit in a controlled (or prescribed) way. Dynamics of the circle actions and of their individual elements is simple and well-understood. In particular, no element of such an action is ergodic or topologically transitive, unless the circle action itself is transitive, i.e. $M = S^1$. To provide interesting asymptotic properties of the limit typically the successive conjugates spread the orbits of the circle action $\mathcal{S}$ (and hence also those of its restriction to the subgroup C_q of order q for any sufficiently large q) across the phase space M making them almost dense, or almost uniformly distributed, or approximate another type of interesting asymptotic behavior. Due to the high speed of convergence this remains true for sufficiently long orbit segments of the limit diffeomorphism. To guarantee an appropriate speed of approximation extra conditions on convergence of approximations in addition to (5.7) may be required.

There are many variations of the construction within this general scheme. In different versions of the approximation by conjugation method one may control the asymptotic behavior of almost all orbits with respect to the invariant volume, or of all orbits. Somewhat imprecisely we will call those versions ergodic and topological.

Ergodic constructions deal with measure-theoretic (ergodic) properties with respect to a given invariant volume, such as the number of ergodic components (in particular ergodicity), rigidity, weak mixing, mixing, further spectral properties. Topological constructions deal with minimality, number of ergodic invariant measures (e.g., unique ergodicity) and their supports, presence of particular invariant sets, and so on.

Control over behavior of the orbits of approximating periodic diffeomorphisms f_n in (5.4) on the nth step of the construction is typically provided by taking an invariant under S_{α_n} (and hence under $S_{\frac{1}{q_n}}$) collection of "kernels", usually smooth balls, and redistributing them in the phase space in a prescribed fashion (also $S_{\frac{1}{q_n}}$ invariant). In ergodic constructions one requires the complement to the union of the kernels to have small volume and hence most orbits of $\mathcal{S}$ (and consequently of any finite subgroup C_q for a sufficiently large q) to spend most of the time inside the kernels. In the topological versions the kernels need to be chosen in such a way that *every* orbit of $\mathcal{S}$ spends most of the time inside the kernels. This requires more care and certain attention to the geometry of orbits.

A natural way of selecting the kernels, their intended images, and constructing a map h_n satisfying (5.6) is by taking a fundamental domain Δ for S_{α_n} (or, equivalently, for $S_{\frac{1}{q_n}}$) choosing kernels and images inside Δ, constructing a diffeomorphism of Δ to itself identical near its boundary which sends kernels into their intended images, and extending the map to the images $S_{\frac{k}{q_n}}$, $k = 1, \dots, q_n - 1$, by commutativity. This method in particular is used in the construction of ergodic diffeomorphisms conjugate to a rotation on manifolds other than the circle as well as in a number of constructions where topological properties are involved. However in order to achieve other ergodic properties, for example weak mixing, it is necessary to use more general constructions.

5.5.2. *Generic constructions*. The first group of results obtained by the approximation by conjugation method deals with realization of certain ergodic properties in the category of C^∞ diffeomorphisms of a compact manifold preserving a smooth volume, i.e. a volume given by a positive C^∞ function in every local C^∞ coordinate system. First recall that all volumes with fixed total volume on a given manifold are conjugate by a C^∞ diffeomorphism [113]. Before we start listing properties which can be produced in the framework of the method it is useful to mention that the constructions come in two different varieties which will be called generic and non-generic; justification for this terminology will become apparent soon.

In the constructions of the first kind (generic) it is sufficient *to control the behavior of approximating and hence resulting diffeomorphisms on a series of growing but unrelated time scales*. To carry out those construction the commutativity condition (5.6) is not necessary. In fact the conjugating maps H_n while formally can be written as products as in (5.5) are not constructed as such. Instead an approximate version of the desired property is achieved by conjugation and care is taken that the sequence f_n converges. The approximate pictures may look quite whimsical (see, e.g., the original weak mixing construction in [18, Section 5] and a modern version in [67]), but as long as a diffeomorphism is close enough to conjugates of rotations appearing in such pictures the property is guaranteed. A natural setting for those constructions is categorical. One considers the space $\mathcal{A}$, the closure of diffeomorphisms of the form gS_tg^{-1} in C^∞ topology. Here we fix a volume ν invariant by the action $\mathcal{S}$ and consider all C^∞ diffeomorphisms g preserving ν. Notice that $\mathcal{A}$ is a complete metrizable space and hence Baire category theorem can be used.

This was first noticed in [18, Section 7] in connection with ergodic properties with respect to the invariant volume and was used in [41] to control topological properties. In fact, for a proof of genericity in $\mathcal{A}$ of a property exhibited by a construction of this sort no actual inductive construction is needed. One just needs to show that an approximate picture at each scale appears for an open dense subset of conjugates of rotations. If appearance in an approximate picture at infinitely many growing scales guarantees the property then by the Baire category theorem the property holds for a dense G_δ subset on $\mathcal{A}$.

THEOREM 5.49. *For any positive function $g(n)$ the space $\mathcal{A}$ contains a dense G_δ subset of weakly mixing diffeomorphisms which admit cyclic approximation with the speed g* [18]. *Furthermore, transformations in that set are α-weak mixing for every α, $0 \leqslant \alpha \leqslant 1$.*

If the action $\mathcal{S}$ is fixed point free then $\mathcal{A}$ contains a dense G_δ subset of uniquely ergodic diffeomorphisms [41].

Even if the action $\mathcal{S}$ has fixed points or if the manifold M has a boundary the number of invariant measures can be controlled and is generically the minimal possible. Here is a nice low-dimensional example.

Let M be one of three manifolds: the disc $\mathbb{D}^2$, the annulus $[0, 1] \times S^1$ or the sphere S^2, λ Lebesgue measure and $\mathcal{S}$ action by rotations (uniquely defined on the disc and the annulus and defined by a choice of axis on the sphere). Let us call Lebesgue measure on the manifold, the δ-measures at the fixed points of the rotations and Lebesgue measures on the boundary components the *natural measures*.

THEOREM 5.50 [47, Theorem 3.3]. *Let M be $\mathbb{D}^2$, $[0, 1] \times S^1$ or S^2, and S_t be the standard action by rotations. Diffeomorphisms that have exactly three ergodic invariant measures, namely the natural measures on M, form a residual set in the space $\mathcal{A}'$: the closure in the C^∞ topology of the conjugates of rotations with conjugates fixing the fixed points of $\mathcal{S}$ and every point of the boundary.*

5.5.3. *Non-generic constructions.* In the constructions of the second kind approximations at different steps of the construction are linked and hence in principle *the asymptotic behavior of the resulting diffeomorphism is controlled for all times*. Constructions of this kind appear most naturally when the resulting diffeomorphism is constructed to be measure-theoretically conjugate to a map of a particular kind, but they also appear when one constructs transformations with more than one ergodic component [157]. This category also includes mixing constructions which were first introduced for time changes for flows on higher-dimensional tori [43,44] and were developed in [47, Section 6] in the context of the approximation by conjugation method. In the latter case one needs to start from a smooth action of a torus rather than of a circle.

Non-standard realizations of Liouvillean rotations. Recall that a number α is called *Liouvillean* if it allows approximation by rationals better than any negative power of denominators.

THEOREM 5.51. *Let α be an arbitrary Liouvillean number. Then arbitrary close to S_α in C^∞ topology there exists a diffeomorphism preserving the volume ν, ergodic and measurably conjugate to the rotation R_α.*

This result was proved in [18, Section 4] for a dense set of α; the proof for arbitrary Liouvillean α is forthcoming [49].

Let us explain why this result may be considered definitive.

In the case of the disc or the annulus with the standard action by rotations the diffeomorphisms in question act as rotations R_α on boundary component(s).

Numbers other than Liouvillean are called *Diophantine*. For Diophantine rotation numbers such a realization on the disc or annulus (with rotation on the boundary) is impossible since due to M. Herman's "last geometric theorem" (to be published posthumously) any such diffeomorphism has uncountably many invariant circles and hence cannot be ergodic.

Other realization results. Possibilities of realizing of particular transformations or members of particular families within the framework of the approximation by conjugation method has not been explored systematically; see [47, Section 7] for a sample of open questions as well as a discussion of prospects and difficulties. As is the case or rotations it looks that realization is often possible for certain subsets of transformations from finite- or infinite-parameter families for the sets of parameters which are residual but very "thin" in the metric sense. However, unlike the rotation situation it is hard to expect definitive results. We restrict ourselves to a sample of results for that kind.

THEOREM 5.52 [18, Section 6]. *For any natural number n there is a dense set of vectors $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ whose coordinates satisfy no rational relation such that there exist a diffeomorphism $f \in \mathcal{A}$ arbitrary close to S_β for some β and measurably conjugate to the translation T_α on the torus $\mathbb{T}^n$.*

There exists a dense in the product topology set of vectors $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{R}^\infty$ whose coordinates satisfy no rational relation such that there exist a diffeomorphism $f \in \mathcal{A}$ arbitrary close to S_β for some β and measurably conjugate to the translation

$$T_\alpha : x \to x + \alpha \pmod 1$$

on the torus $\mathbb{T}^\infty$.

THEOREM 5.53. *Arbitrary close to any transformation S_β for any β there exists a non-standard ergodic diffeomorphism.*

The proof is based on a smooth realization of a version of Feldman's construction described in [78, Section 8].

5.5.4. *Toral actions and mixing transformations*. The use of approximation type techniques to produce mixing transformations and flows was pioneered by B. Fayad [43]. He used reparametrizations of linear flows on the tori of dimension $\geqslant 3$ to produce mixing by carefully controlling behavior of the sequence of overlapping time scales. See Section 5.6.3 for a brief outline of the method. In [47] the techniques of reparametrization of linear flows on $\mathbb{T}^3$ were combined with the explicit approximation by conjugation methods. The basic setting is a compact smooth manifold M with non-trivial smooth $\mathbb{T}^3$ action $\mathcal{S} = \{S_v\}_{v \in \mathbb{R}^3}$, $S_{v+k} = S_v$ if $k \in \mathbb{Z}^3$ and a smooth volume μ preserved by $\mathcal{S}$.

THEOREM 5.54 [47, Theorem 6.2]. *There exists a sequence $\gamma_n \in \mathbb{Q}^3$ and a sequence H_n of diffeomorphisms preserving μ such that the sequence $H_n S_{t\gamma_n} H_n^{-1}$ converges in the C^∞ topology to a flow preserving μ and mixing for this measure.*

5.6. *Time change*

5.6.1. *General results*. Given a flow, T_t, the operation of *time change* produces a flow with the same orbits as T_t but evolving at a different speed and with an invariant measure accordingly changed with a suitable density.

Time changes can be described in terms of $\mathbb{R}$-valued untwisted one-cocycles over the flow, see [8, Section 1.3m] for a discussion in very general context and [78, Section 9.3] for basic definitions in the specific setting of flows. This operation is important in the topological and differentiable dynamics where the time change is assumed correspondingly continuous and differentiable. We already discussed a specific case of time change in Section 2.2.4.

Since every flow by Ambrose–Kakutani theorem can be represented as a special flow over a measure preserving transformation, the time change produces a special flow over the same transformation with a different roof function. In particular, if the roof functions φ and ψ for special flows over the same measure preserving transformation T are *cohomologous*, i.e.

$$\phi = \varphi + h \circ T - h$$

for a measurable function h then the special flows are isomorphic. The function h which in the case of ergodic T is uniquely defined mod 0 up to a constant is sometimes called *transfer function*.

The basic properties which are preserved by any time change are ergodicity (more generally, the structure of the decomposition into ergodic components) and the property of entropy to be zero, a positive number, or infinity. Other spectral and non-spectral invariants are in general not preserved.

Still it is a meaningful question to ask how the spectral properties of a flow may be modified by a time change.

Two basic general results in this direction show that stochastic properties may be improved by a proper time change. They are due to Kochergin [90] and Ornstein and Smorodinsky [120] correspondingly.

THEOREM 5.55.
(1) *For any ergodic flow there exists a time change which is mixing* [90].
(2) *For any ergodic flow with positive entropy there exists a time change which is a K-flow* [120].

In both cases the time change can be chosen arbitrary close to identity in a variety of senses; for example, if a flow is represented as a special flow over a transformation the roof function can be changed arbitrary little in the uniform norm.

5.6.2. *Continuous and almost differentiable time changes.* It is interesting and in fact remarkable that in the continuous category the time changes described in the previous theorem can be made continuous and in differentiable category "almost" differentiable (with derivative discontinuous only at one point). This follows from the analysis of cohomology classes of cocycles which produce time changes. If two cocycles are cohomologous then corresponding time changes are metrically isomorphic by a conjugacy which moves each point along its orbit according to the solution of the cocycle equation. We follow the presentation of [78, Section 10.2].

THEOREM 5.56. *Let $\mathcal{L} \subset L^1(X, \mu)$ be a linear subspace of L^1 dense in the L^1 topology and closed in the L^∞ topology (uniform convergence almost everywhere). Then for every*

$f \in L^1(X, \mu)$ the set $\mathcal{L}_f = \{h \in \mathcal{L}$: h is cohomologous to $f\}$ is dense in the L^∞ topology in the set $\{h \in \mathcal{L},\ \int h\,d\mu = \int f\,d\mu\}$.

If we put $\mathcal{L} = C(X)$, the space of all continuous functions, we immediately obtain the following statement which was originally proved in [120].

COROLLARY 5.57. *Let X be a compact metric space, μ be a Borel probability nonatomic measure on X, $T : X \to X$ be a measure-preserving transformation (not necessarily continuous). Then every real-valued cocycle $f \in L^1(X, \mu)$ is cohomologous to a continuous cocycle. Moreover the set of continuous cocycles cohomologous to f is dense in uniform topology in the space of all continuous functions with the same integral as f.*

Corollary 5.57 can be strengthened by specifying the values of a continuous function cohomologous to f on any closed set F so that $\mu(X \backslash F) > 0$. Pushing the method described above a bit further one obtains the result advertized above which looks quite striking at first glance.

THEOREM 5.58. *Let M be a compact differentiable manifold, μ be a Borel probability measure on M, $T : M \to M$ be a measure-preserving transformation. Then every real-valued cocycle $f \in L^1(M, \mu)$ is cohomologous to a continuous cocycle $\bar{f}$ which is continuously differentiable except at a single point.*

SKETCH OF PROOF. First, one finds a continuous cocycle f_1 cohomologous to f which is continuously differentiable outside a ball B_1 of radius, say, $1/2$ and can be extended to a continuously differentiable function. This is possible by a stronger version of Corollary 5.57 mentioned above. Then one approximates f_1 in uniform topology by a continuously differentiable cocycle g_1 which coincides with f outside B_1. If the L^1 norm of $f_1 - g_1$ is small enough one can find a cocycle f_2 cohomologous to f_1 (and hence to f) which coincides with f_1 outside a smaller ball $B_2 \subset B_1$ of radius $1/4$ and extends to a continuously differentiable function and such that the support of the transfer function ψ_1 has measure less than $1/2$. Continuing by induction one constructs on the nth step the cocycle f_n continuously differentiable outside of a ball $B_n \subset B_{n-1}$ of radius 2^{n+1} which coincides with f_{n-1} outside of the ball B_{n-1} and extends to a continuously differentiable function and such that a transfer function ψ_n connecting f_n with f_{n-1} is supported on a set of measure less than 2^n. In the limit the function $\bar{f} = \lim_{n\to\infty} f_n$ is continuous everywhere and continuously differentiable outside of the single point $\bigcap_{n=1}^{\infty} B_n$. By the Borel–Cantelli lemma the series $\sum_{n=1}^{\infty} \psi_n$ converges and hence gives a transfer function between f_1 and $\bar{f}$. Since f_1 is cohomologous to f this finishes the proof. □

5.6.3. *Regular time change in various classes of systems.* Notice that the property of almost differentiability in Theorem 5.58 cannot be replaced by any reasonable *uniform* property stronger than continuity.

Hyperbolic and parabolic systems. For example, Hölder time changes behave quite differently for many classes of dynamical systems such as Anosov flows or special flows over subshifts of finite type [79, Section 19.2], [78, Sections 11.3–4]. In those cases on the one hand, there are infinitely many moduli for existence of a *measurable* solution of the cohomological equation, and, on the other, robustness of spectral properties. The spectrum is either countable Lebesgue or that plus pure point component with single frequency. The latter is impossible for example for contact Anosov flows. Thus, in the hyperbolic (and to a certain extent partially hyperbolic situation [78, Section 11.5]) spectral properties exhibit robustness under reasonably regular time changes with the countable Lebesgue spectrum prevailing.

A somewhat similar albeit more subtle and less understood situation exists for parabolic systems. Since these effects are in essence different from those produced by combinatorial constructions which dominate this part of the survey we will discuss the topic later in Section 6.3.

Elliptic systems: codimension one. Now we will consider specific situations where interesting effects can be achieved by producing a nice (smooth, analytic Hölder, etc.) time change with interesting properties by means of a construction which successfully controls behavior at various time scales. In this respect this class of constructions fits with the general theme of this part of the survey. We already discussed time changes in a linear flow on the two-dimensional torus in Section 2.2.4. We will discuss the situation in more detail and comment on methods used. First, notice that for any irrational slope and for a sufficiently smooth time change (or, equivalently, the roof function for the special flow) the resulting flow (or the time one transformations) allows sufficiently good cyclic approximation to guarantee simple singular spectrum and the absence of mixing, see Proposition 5.39; the latter property also follows under much weaker assumptions from Theorem 5.61 below. Weak mixing of course does not follow from cyclic approximation. It can be produced by several different methods. To produce genericity one can use perturbation with small sinusoidal waves similar to those described below for producing mixing in higher dimension. A more interesting method deals with the study of special flows with a fixed roof function and varying translation in the base. This method leads to a conclusion that, while other types of behavior are possible, under certain assumptions, weak mixing is the only alternative to at least measurable conjugacy to the linear flow.

First, if the roof function is a trigonometric polynomial of if the translation is Diophantine and the function is C^∞ then the roof function is cohomologous (with the transfer function which is correspondingly itself a trigonometric polynomial of a C^∞ function) to its average and hence the flow is smoothly conjugate to a constant time suspension or, equivalently to a linear flow.

Shklover [146] proved the following converse to the statement about trigonometric polynomials.

THEOREM 5.59. *For any real-analytic function f other than trigonometric polynomials* (*in other words those with infinitely many non-zero Fourier coefficients*) *there is always an α such that the special flow over R_α with the roof function f is weakly mixing*.

A more quantitative statement connecting the approximation in the base with the decay of Fourier coefficients for the roof function is in [78, Theorem 13.7].

THEOREM 5.60. *Let $h(x)=\sum_{n\neq 0} h_n \exp 2\pi i n x$ be a C^2 real valued function on S^1 with zero average. Suppose for a certain sequence of rational numbers p_n/q_n,*

$$\frac{q_n|\alpha - p_n/q_n|}{\sum_{k=1}^{\infty}|h_{kq_n}|}\to 0 \quad \textit{and} \quad \frac{|h_{q_n}|}{\sum_{k=1}^{\infty}|h_{kq_n}|}>c>0.$$

Then for any h_0 and r the cocycle $\exp ir(h_0+h(z))$ is not a coboundary and consequently the special flow over R_α with the roof function $h_0+h(z)$ is weakly mixing.

Developing this method and using new ideas involving a central limit theorem to treat the case of intermediate approximation Fayad and Windsor proved in [50] that under stronger conditions on regularity of decay of Fourier coefficients than in Theorem 5.60 (satisfied, for example, when they are close enough to a geometric progression) there is a dichotomy between solvability of the cohomological equation in L^2 (and hence the pure point spectrum with two "right" frequencies) and weak mixing.

The following general criterion for absence of mixing was found in [76].

THEOREM 5.61. *Any special flow with the roof function of bounded variation over an interval exchange transformation is not mixing.*

The proof is based on using the return properties of the base transformation (Theorem 5.22) and the bounded variation of the roof function to show that returns in the base produce returns for the flow within a bounded time. Thus since a bounded from below proportion of measure returns close to itself in the base at a certain sequence of time moments growing to ∞ the same can be said about a fixed proportion of measure for the flow for a sequence of fixed length time segments. This contradicts mixing.

This result has been recently strengthened by Fraczek and Lemańczyk.

THEOREM 5.62 [57]. *Any special flow with the roof function of bounded variation over an ergodic interval exchange transformation is disjoint from any mixing flow.*

Mixing can be produced with a minimal loss of regularity. For example, any Lipschitz time change in a linear flow on $\mathbb{T}^2$ is not mixing by Theorem 5.61. One the other hand Kochergin proved the following converse to that statement.

THEOREM 5.63 [93]. *For any modulus of continuity ω weaker than Lipschitz, i.e. such that $\lim_{t\to 0}\frac{t}{\omega(t)}=0$ one can find a linear flow on T^2 and a time change with modulus of continuity ω which is mixing.*

Equivalently, one can find a rotation R_α and a function f with modulus of continuity ω such that the special flow over R_α with the roof function f is mixing.

The construction is of inductive character producing approximate mixing on growing but overlapping time scale and is somewhat similar to a more subtle and specialized version of the general construction from [90].

Elliptic systems: higher codimension. An essential new phenomenon for time changes in linear flows on $\mathbb{T}^k$, $k \geqslant 3$, or, equivalently, in special flows over translations on $\mathbb{T}^k$, $k \geqslant 2$, is a possibility of mixing in very regular situations including real analytic [43]. This is a very special situation impossible in the Diophantine context and non-generic in Liouvillean. It is produced by an inductive combinatorial construction which we briefly outline for the case of a special flow over a translation of $\mathbb{T}^2$ with coordinates (x, y).

If we assume that the rotation in the x direction is periodic with period n consider an addition to a given roof function of the form $a \sin 2\pi nx$ then the successive returns will develop sinusoidal waves which at the time scale grater than n will produce approximate mixing for sets transversal to the x direction. Now we add a very small translation in the x direction to keep this effect for the perturbed system for a long enough time until the effect of a similar perturbation in the y direction of much greater frequency but much smaller magnitude takes over. This relies on a proper very special choice of periodic approximations in the x and y direction. The scales when mixing is produced by the stretching in the two directions overlap but because of the independence of the perturbations they do not interfere and cancel each other. Thus genuine albeit fairly slow mixing is achieved for the limit transformation whose base translation has the form (α, β) with $\alpha = \sum_{k=1}^{\infty} \frac{1}{n_k}$, $\beta = \sum_{k=1}^{\infty} \frac{1}{m_k}$ with $n_k \ll m_k \ll n_{k+1}$ and the roof function is of the form

$$\sum_{k=1}^{\infty} a_k \sin n_k x + b_k \sin m_k y \tag{5.8}$$

with $a_k \gg b_k \gg a_{k+1}$. The construction can be carried out in such a way that the function (5.8) is real analytic.

There are variations of this method where instead of sinusoidal waves different more elaborate shapes are used. For example, using some version of Dirichlet kernels one can combine mixing with Fayad's criterion of slowly coalescent periodic approximation for singularity of the spectrum which is compatible with mixing, see Proposition 5.46.

THEOREM 5.64 [46]. *There exists a C^{∞} time change of a linear flow on $\mathbb{T}^3$ which is mixing and has singular spectrum.*

5.7. *Inducing*

The operation analogous to time change in the discrete case is the operation of inducing. The natural topology in the space of measurable subsets of a given space (X, μ) is given by the metric

$$d(A, B) = \mu(A \Delta B).$$

Denote the collection of all classes mod 0 of measurable sets provided with this metric by $\mathcal{X}$.

The following result is a counterpart of Theorem 5.55.

THEOREM 5.65.

(1) *Any ergodic transformation induces mixing on a dense in $\mathcal{X}$ class of sets* [58].

(2) *Any ergodic transformation with positive entropy induces K-automorphisms on a dense in $\mathcal{X}$ class of sets* [120].

The method of proving Theorems 5.55 and 5.65 is similar in spirit to cutting and stacking constructions albeit limited to introduction of spacers since the return maps for the towers are fixed.

De la Rue improved the first statement of the previous theorem:

THEOREM 5.66 [35]. *An ergodic transformation induces a transformation with Lebesgue spectrum on a dense in $\mathcal{X}$ class of sets*.

Multiplicity of Lebesgue spectrum in this construction is not known. Thus the following problem is open:

PROBLEM 5.67. Does any ergodic transformation with zero entropy induce a transformation with countable Lebesgue spectrum?

De la Rue in [36] has produced a spectral type which cannot be obtained in a standard transformation, i.e. on any induced of an irrational rotation. We will discuss this result based on the theory of Gaussian dynamical systems in Section 6.4.3.

Positive answer to the following problem would require an essentially new construction.

PROBLEM 5.68. Does any ergodic transformation with zero entropy induce a transformation with simple spectrum?

Conze [27] has proved that it is in fact generic that an induced of an ergodic transformation is weakly mixing. Notice that mixing is not generic.

In [78, Section 7] transformations induced by a standard transformation on various sets are considered. The following result is parallel to Theorem 5.47.

THEOREM 5.69. *Let T be a standard measure-preserving transformation. Given a type $\mathcal{T}$ and a speed $g(n)$, the set of all $A \in \mathcal{X}$ such that the induced transformation T_A admits a periodic approximation of type $\mathcal{T}$ with speed $g(n)$ is a residual set in $\mathcal{X}$.*

All the standard corollaries follow such as simple continuous singular spectrum which is mutually singular with all its convolutions. Since inducing (and the inverse operation of taking a *special transformation* which is the discrete time equivalent of the special flow) involves cohomological equations with integer values, interesting questions related with behavior of regular (analytic, smooth, etc.) real-valued cocycles which played the central role in Section 5.6.3 do not have direct equivalents in this setting.

5.8. *Spectral multiplicity, symmetry and group extensions*

5.8.1. *Introduction*. Most of this section deals with the realization problem for various sets of essential values of spectral multiplicity; see the preview in Section 3.6.2. Success in treating of this problem via appropriate constructions is based on the combination of two principal elements:

(i) *Symmetry* which allows to produce for certain classes of transformations various intertwining operators in L^2 (often but not always coming from commuting measure preserving transformations) which interchanges various subspaces and hence guarantees that certain parts of the spectrum come with multiplicity, and

(ii) *Approximation* which shows that "minimal" multiplicities compatible with the symmetry are actually realized. Approximation properties come from combinatorial constructions. Sometimes it is sufficient to consider generic data within a given class of transformations; in other cases more careful inductive process might be needed.

The subtlety of using approximation techniques is in that it is not always sufficient to produce approximation which allows to obtain an above estimate for the multiplicity using Theorem 1.21 or something similar but (in the case of non-homogeneous spectrum) one needs separate estimates in various subspaces responsible for parts of the spectrum with different values of the multiplicity function.

5.8.2. *Homogeneous spectrum of multiplicity two and Cartesian products*. Ergodic measure preserving transformations with homogeneous spectrum of multiplicity two were found simultaneously and independently by Ryzhikov [144] and Ageev [16]. They used approach of [78] and improved the estimate given by Proposition 5.41.

THEOREM 5.70. *For a generic in the weak topology measure preserving transformation T the Cartesian square $T \times T$ has homogeneous spectrum with multiplicity two*.

PROOF. The symmetry here is the involution $J : (x, y) \mapsto (y, x)$ which guarantees that essential values of the spectral multiplicity are even (see Proposition 4.2) and the approximation is, first, good cyclic approximation for T which insure simple spectrum and hence, multiplicity two for the part of the spectrum coming from functions depending only on one coordinate and, second, a slightly generalized version of good approximation of type $(n, n + 1)$ (see Section 5.4.2). Namely, for a given natural number m we will consider a good linked approximation of type $(n, n + m)$ by periodic processes with two towers whose size is bounded away from zero and heights differing by m. Existence of this kind of approximation guarantees that weak limit of powers of T contains a linear combination $\alpha \operatorname{Id} + (1 - \alpha)T$. This of course means that the limit of U_{T^n} in the strong operator topology contains $\alpha \operatorname{Id} + (1 - \alpha)U_T$.

It is sufficient to prove that the maximal spectral multiplicity of $U_{T \times T}$ is at most two. Thus the theorem will follow from the following lemma

LEMMA 5.71. *If T admits a good cyclic approximation and a good approximation of type $(n, n + m)$ for any natural m and f is a cyclic vector for U_T then the functions $f(x)f(y)$ and $f(x)f(Ty)$ generate L^2*.

PROOF. Since f is a cyclic vector for U_T the functions of the form $f(T^k x)f(T^m y)$ generate L^2 for the Cartesian product. Thus it is sufficient to show that any function of the form $f(x)f(T^m y)$ belongs to the space generated by $f(x)f(y)$ and $f(x)f(Ty)$ which we will denote by H. To simplify notations let us denote $f(T^m x)f(T^k y)$ by $m \times k$ and use similar notation for linear combinations of such functions. From the invariance one gets for every $m \in \mathbb{Z}$,

$$m \times m \in H \quad \text{and} \quad m \times (m+1) \in H.$$

Thus from the approximation criterion $(\alpha 0 + (1-\alpha)m) \times (\alpha 0 + (1-\alpha)m) \in H$ hence by invariance $0 \times m + m \times 0 \in H$. Similarly by taking limits of some iterates of 0×1 we obtain $0 \times m + (m-1) \times 1 \in H$ and hence

$$m \times 0 + (m-1) \times 1 \in H. \tag{5.9}$$

Using these inclusions inductively for $m = 2, 3, \ldots$ we obtain that $m \times 0 \in H$. For $m = 2$ one obtains $0 \times 2 + 1 \times 1 \in H$ and hence $0 \times 2 \in H$. Assuming that $k \times 0 \in H$ for $k \leqslant m$, in particular, $(m-1) \times 0 \in H$ and hence $m \times 1 \in H$ we get from (5.9) that $(m+1) \times 0 \in H$. □

This finishes the proof of the theorem. □

Looking back at the structure of the spectrum for the Cartesian square described in Proposition 4.2 we deduce interesting arithmetic structure of the maximal spectral type for a transformation T whose Cartesian square has spectrum of multiplicity two. First, any measure μ of the maximal spectral type is singular with respect to its convolution $\mu * \mu$ and, second for almost every with respect to $\mu * \mu$ $\lambda \in S^1$ the conditional of $\mu \times \mu$ on the circle $\lambda_1 \lambda_2 = \lambda$ is concentrated in two symmetric points $(\lambda_1^0, \lambda_2^0)$ and $(\lambda_2^0, \lambda_1^0)$.

A more sophisticated analysis allows to describe essential values of spectral multiplicity for the mth Cartesian power of a generic measure preserving transformation where the symmetry is given by the symmetric group S_m of permutations of components and where the maximal spectral multiplicity is at least $m!$.

THEOREM 5.72 [144,16]. *For a generic measure preserving transformation T the mth, $m \geqslant 3$, Cartesian power $T^{(m)}$ has $m-1$ different values of the spectral multiplicity*: $m, m(m-1), m(m-1)(m-2), \ldots, m!$.

5.8.3. *Homogeneous spectrum of arbitrary multiplicity and group actions*. Measure preserving transformations with homogeneous spectrum of arbitrary multiplicity (including new examples with multiplicity two) were recently found by Ageev [17] using a different type of symmetry. His main idea is quite brilliant although in retrospect it looks natural.

Ageev considers the following group G_m. It is a finite extension of $\mathbb{Z}^m$ and has generators $T_1, \ldots, T_m, S$ where $T_1, \ldots, T_m$ commute, $T_1 \cdot T_2 \cdot \cdots \cdot T_m = \mathrm{Id}$ and $T_{i+1} = S \cdot T_i \cdot S^{-1}$ for $i = 1, \ldots, m-1$. Notice that S^m commutes with $T_1, \ldots, T_m$ and thus the group G_m is an m-fold extension of the Abelian group with generators $T_1, \ldots, T_{m-1}, S^m$.

THEOREM 5.73. *For a generic action α of the group G_m by measure preserving transformations of Lebesgue space the transformation $\alpha(S^m)$ has homogeneous spectrum of multiplicity m.*

The upper bound on the spectral multiplicity is provided by simplicity of the spectrum for S; this can be achieved using a proper version of periodic approximation theory for actions of G_m. It is a standard corollary of the Spectral Theorem 1.8 that then the spectrum of the mth power has multiplicity at most m. Spectral theory for this group provides for symmetry. In particular if S is weakly mixing (which can also be guaranteed by approximation arguments) there are m mutually orthogonal S^m invariant subspaces where the restriction of the Koopman operator are unitarily equivalent so by Corollary 1.20 the values of spectral multiplicity are multiples of m.

5.8.4. *Non-homogeneous spectrum, group extension and factors.* These examples which produced successively more general sets of values of spectral multiplicities from $\{1, m\}$ [134], to finite [135] and infinite [65] sets containing 1 and invariant under taking the least common multiple, to arbitrary sets containing 1 [100], are all based on finite and, more generally, compact group extensions of transformations admitting good cyclic approximation with cocycles possessing certain symmetry. The idea actually goes back to the work of Oseledets [123] who was the first to construct an example of a measure preserving transformation with non-simple spectrum of bounded multiplicity. However, his upper estimate based on Theorem 5.24 was very crude. Oseledets' example was the starting point for Robinson who introduced finer methods of estimating the multiplicity from above. Here we will describe Robinson's first construction since it shows both the symmetry and approximation elements in a clear and suggestive way. We follow [78].

We will consider T, the double group extension of a transformation T_0. $T : X \times \mathbb{Z}/m\mathbb{Z} \times \mathbb{F}_p \to X \times \mathbb{Z}/m\mathbb{Z} \times \mathbb{F}_p$ where p is a prime number specified below and $\mathbb{F}_p$ is the finite field with p elements, of the following special form

$$T(x, y, z) = \big(T_0 x, \gamma(x) + y, \phi(y) + z\big). \tag{5.10}$$

Here $\gamma : X \to \mathbb{Z}/m\mathbb{Z}$ is a measurable function which will be specified to provide approximation properties needed to the above estimate of the spectral multiplicity. For any m there exists a prime number p and an isomorphism $\phi : \mathbb{Z}_m \to G \subseteq \mathbb{F}_p$, where G is a subgroup of the multiplicative group $\mathbb{F}_p^*$ of the finite field $\mathbb{F}_p$ with p elements. These are the data which go to the second extension.

THEOREM 5.74. *For a generic in weak topology T_0 and a generic in L_1 set of cocycles γ the transformation T defined by* (5.10) *is weakly mixing and has $\{1, m\}$ as the set of essential values of the spectral multiplicity.*

REMARK. In fact, genericity arguments are not necessary as the proof below shows. The required conditions are certain approximation properties which can be guaranteed by choosing, for example, a certain exchange of three intervals as T_0 and a certain piecewise constant function as γ.

PROOF. Associated with a finite group extension there is a natural orthogonal decomposition of L_2 into U_T-invariant subspaces corresponding to the characters of the group. The additive characters of $\mathbb{F}_p$ are given by $\chi_w(z) = \exp 2\pi i z w/p$ where $w \in \mathbb{F}_p$, so that if T is given by (5.10) we obtain an invariant orthogonal decomposition

$$L_2(X \times \mathbb{Z}/m\mathbb{Z} \times \mathbb{F}_p) = \bigoplus_{w \in \mathbb{F}_p} H_w,$$

where

$$H_w = \{\chi_w(z) f(x, y) \colon f \in L_2(X \times \mathbb{Z}/m\mathbb{Z})\}.$$

Let us define a permutation $\sigma : \mathbb{F}_p \to \mathbb{F}_p$ by $\sigma(w) = \phi(1)w$. For $w \neq 0$ we also define the operator

$$S_w : H_w \to H_{\sigma(w)} \quad \text{by} \quad S_w(\chi_w(z) f(x, y)) = \chi_{\sigma(w)} f(x, y+1).$$

Since $\chi_{\sigma(w)}(\phi(y)) = \chi_w(\phi(y+1))$, one has $U_T|_{H_{\sigma(w)}} \cdot S_w = S_w \cdot U_T|_{H_w}$. Now let us examine the permutation σ. It fixes 0 and has $m' = \frac{p-1}{m}$ cycles of length m. This explains how the operators S_w permute the subspaces H_w. We will choose an arbitrary element θ_k, $k = 1, \ldots, m'$, from the kth cycle of σ, and for $j = 0, \ldots, m-1$ we will define the subspace

$$H^j = H_{\sigma^j(\theta_1)} \oplus H_{\sigma^j(\theta_2)} \oplus \cdots \oplus H_{\sigma^j(\theta_{m'})}.$$

We will also define

$$H^* = H_0.$$

It is clear that $L_2(X \times \mathbb{Z}/m\mathbb{Z} \times \mathbb{F}_p) = H^* \oplus H^0 \oplus \cdots \oplus H^{m-1}$. The linear operator

$$S^j : H^j \to H^{j+1}, \quad j \neq *,$$

is defined in the natural way so that

$$S^j|_{H_w} H_w = H_{\sigma(w)} \subseteq H^{j+1}.$$

It follows that

$$S^j \cdot U_T|_{H^j} = U_T|_{H^{j+1}} \cdot S^j$$

and thus since the spectra in all of the spaces H^j are identical, the maximal spectral multiplicity of T is at least m. To obtain the estimate of the maximal spectral multiplicity for T from above we will need two types of approximation for the first extension T_1. In particular, these will guarantee that U_{T_1}, or equivalently $U_T|_{H^*}$, has simple continuous spectrum.

They are, (i) a good linked approximation of type $(n, n+1)$ (see Proposition 5.40) and (ii) a certain good approximation with m towers of equal height, which are related to each other by shifts $(x, y) \to (x, y+k)$. By extending the approximation for T_1 to the second extension, we obtain from (ii): (iii) a good approximation for T with m towers. Since at least one of the towers has size close to $1/m$, Theorem 5.10 implies that maximal spectral multiplicity for T is no greater than m. This in particular implies ergodicity of T since otherwise there would be invariant functions in every H^j in addition to constants contradicting the above estimate for the spectral multiplicity. This in turn implies weak mixing since otherwise there would be eigenfunctions with the same eigenvalue in every H^j and their ratios would produce non-constant invariant functions. By a combinatorial analysis of the approximating cocycles γ_n, measurable with respect to the partitions involved in the approximation of T_0, one can show that (i), (ii) and (iii) hold for a generic set of cocycles γ in the L_1 topology. Since the maximal spectral types in all H^j are identical the above estimate of the maximal spectral multiplicity by m implies that the spectra in those subspaces are simple and with maximal spectral type singular with respect to that in H^*. This implies that set of essential values of spectral multiplicity is $\{1, m\}$. □

For constructions with many values of spectral multiplicity the algebraic or "symmetry" part is more complicated but similar in principle. For infinite sets of values finite extensions are not sufficient and other compact group extensions are used. The most general case is represented by [100, Algebraic Lemma]. Approximation part has to be done differently though. The above estimate is not sufficient to conclude that all components in the spectrum which come from the algebraic construction are mutually singular and have maximal possible multiplicity. The solution is to consider approximation constructions directly for operators in invariant subspaces, to produce simple spectrum for those operators and guarantee mutual singularities of spectra.

5.8.5. *Finite extensions and spectral properties*. In [69] a construction was found which produced finite extensions of simple systems with certain functions with Lebesgue spectral measure. Based on this work Matthew and Nadkarni [112] have constructed a two points extension of an adding machine which they showed has a Lebesgue component of multiplicity 2. The Matthew–Nadkarni example involves a construction of a cocycle over the adding machine which takes values in $\mathbb{Z}/2\mathbb{Z}$ in such a way that the corresponding two point extension possesses a natural partition in two sets of equal measures whose iterates are pairwise independent. By replacing the adding machine in the base and modifying the construction appropriately Ageev [15] proved

THEOREM 5.75. *For any $n \geqslant 1$ there exists a weakly mixing transformation with essential values of spectral multiplicity $\{1, 2n\}$ where the component of multiplicity $2n$ is Lebesgue.*

The construction is also a finite extension, but this time, of a weakly mixing rank one transformation. See also [103] for examples with Lebesgue component of any given even multiplicity in the spectrum.

6. Key examples outside combinatorial constructions

6.1. *Introduction*

Of the four principal classes of systems which appear in smooth dynamics, two, hyperbolic and (typical) partially hyperbolic, are well understood from the point of view of ergodic theory. Modulo some sufficiently trivial modifications ergodic behavior of such systems with respect to an absolutely continuous invariant measure (as well as some other good invariant measures, such as maximal entropy or more general Gibbs measures) is described by the Bernoulli model which has countable Lebesgue spectrum and is classified up to a measurable isomorphism by the single invariant, entropy [3, Sections 2.3 and 3]. On the other hand, it is worth noticing that certain partially hyperbolic systems exhibit complicated and non-standard ergodic behavior. For example there are partially hyperbolic volume preserving diffeomorphisms which are K but not Bernoulli [77].

Elliptic systems admit in addition to the basic model of the toral translation a variety of behaviors which are well modeled by several kinds of combinatorial constructions discussed above.

The remaining class, parabolic systems, characterized by moderate and more or less uniform growth of orbit complexity do not naturally appear in the context of combinatorial constructions. In fact, it would be fair to say that many of the examples of the greatest intrinsic interest produced by combinatorial construction display phenomena which are difficult to render in the smooth situation.

In the next two sections we briefly review ergodic properties of two classes of parabolic systems which appear most naturally and are best understood. Key results concerning those systems are among the deepest in the field of ergodic theory and they yield remarkable applications outside the field, see [10]. In the last section we discuss another class of examples which came from probability theory and which provide a remarkably flexible and powerful tool for the spectral realization problem; in particular, the first example of a measure preserving transformation with simple continuous spectrum was found among Gaussian systems by Girsanov in 1958 [61] almost a decade earlier than direct methods based on rank and periodic approximation were developed.

6.2. *Unipotent homogeneous systems*

6.2.1. *Definitions and simple examples.* A homogeneous system has naturally defined *linear part* namely the adjoint action on the Lie algebra of G.

If all eigenvalues of the linear part of a homogeneous map are equal to one the map is called *unipotent*. A one-parameter group of unipotent maps is called a *unipotent flow*. If the linear part is semisimple, i.e. linearizable over complex numbers the flow acts by isometries with respect to a Riemannian metric and hence the spectrum is always pure point. Linear flows on the torus are examples; more generally this can happen on Euclidean manifolds (see Section 1.4b and Theorem 2.3.3 in [10]).

More interesting behavior appears when the linear part has non-trivial Jordan blocks. For example, mixture of pure point and countable Lebesgue spectrum appears in homogeneous flows on nilpotent groups which in many respects are similar to unipotent affine maps on the torus like those in Examples 3.17 and 3.18.

6.2.2. *Horocycle flows and property R.* Horocycle flows which appeared in Section 2.1.3 are the simplest and best understood non-trivial examples among unipotent flows on homogeneous spaces of semisimple Lie groups.

We showed that they have countable Lebesgue spectrum which appears quite often in ergodic theory. However, beyond that horocycle flows possess very striking ergodic properties which imply strong rigidity statements. These properties are summarized in the following theorems due to M. Ratner [130]:

THEOREM 6.1. *If λ is an ergodic self-joining of a horocycle flow which is not the product measure, then it is a finite extension of its two marginals.*

REMARK. This statement is very close to simplicity. Simplicity is saying that $\mathcal{V} = \mathcal{H}$, here we have that $\mathcal{V}$ and $\mathcal{H}$ both have finite fibers in $\mathcal{V} \vee \mathcal{H}$.

THEOREM 6.2. *Horocycle flows have the pairwise independently determined property* (*see Definition* 4.15).

This implies mixing of all orders for the horocycle flows. As a consequence of these theorems, Ratner has obtained the following rigidity results:

THEOREM 6.3. *If two horocycle flows are measure theoretically isomorphic they are algebraically isomorphic.*

THEOREM 6.4. *Every factor of a horocycle flow is algebraic.*

THEOREM 6.5. *The time one transformation of every horocycle flow is a factor of a simple transformation. In case the subgroup γ is maximal and not arithmetic* [4, *Section* 1.5c], *the horocycle flow has minimal self-joinings as an $\mathbb{R}$ action.*

A key property for the understanding of the horocycle flow is the R property of Ratner which can be formulated in a general context.

Let T_t be a flow on a metric space with σ-compact metric d preserving a Borel measure.

DEFINITION 6.6. The flow T_t has the property R_p, $p \neq 0$ if the following is true:

For every $\varepsilon > 0$ and $N > 0$ there exist $\alpha(\varepsilon), \delta(\varepsilon, N) > 0$ and a subset $A(\varepsilon, N) \subset X$ such that $m(A) > 1 - \varepsilon$ with the property that if $x, y \in A$ and $d(x, y) < \delta(\varepsilon, N)$ and y is not on the T_t orbit of x, then there are $L = L(x, y)$ and $M = M(x, y) \geqslant N$ with $M/L \geqslant \alpha$ such that if

$$K^{+}_{-}(x, y) = \bigl\{n \in Z \cap [L, L + M]\colon\, d\bigl(T_{np}(x), T_{(n\pm1)p}(y)\bigr) < \varepsilon\bigr\},$$

then

$$|K^+|/M > 1 - \varepsilon \quad \text{or} \quad |K^-|/M > 1 - \varepsilon.$$

It is remarkable that this property of "slow relative drift of nearby points" is also satisfied by the Chacon transformation.

It is not known how far the R-property is from simplicity.

PROBLEM 6.7. Does there exist a flow satisfying the R-property such that its time one map is disjoint from all simple transformations?

6.2.3. *Ratner theory.* Recall that spectral properties of unipotent homogeneous systems are fairly standard: as for all homogeneous systems in general the mixture of pure point and countable Lebesgue spectrum. In the most interesting case of unipotent maps and flows on homogeneous spaces of semisimple Lie groups the spectrum is countable Lebesgue.

On the other hand, these systems exhibit very interesting ergodic properties beyond spectrum. For example, they provide examples of infinitely many systems with countable Lebesgue spectrum and zero entropy which are pairwise not Kakutani equivalent, namely different Cartesian powers of any horocycle flow [129]. The distinguishing invariant is of "slow entropy" type but adapted to the Kakutani rather than Hamming metrics in the spaces of sequences coding orbit segments; see [75] for the discussion of metrics and [83] for a general discussion of these invariants.

Isomorphisms, factors and joining between unipotent systems can be systematically studied with the powerful tool, the Ratner Measure Rigidity Theorem [132] which basically states that any invariant Borel probability ergodic measure is of algebraic nature. For a detailed exposition of Ratner theory and its applications see [10, Section 3].

It is worth noticing that while great attention has been paid to the number theoretical applications of Ratner's rigidity for unipotent systems there has been no systematic study of its implications to the ergodic theory of such actions, as has been done for the horocycle flows. Certainly it deserves to be looked at.

6.3. *Effects of time change in parabolic systems*

We will now complete the discussion of Section 5.6.3 of known spectral and other ergodic properties which appear under sufficiently nice time change in principal classes of systems.

6.3.1. *Time change in horocycle flows.* Let v be the vector field generating a horocycle flow. In [98] it is proved that if C^1 time change is not too large, namely if $f - \mathcal{L}f > 0$ where $\mathcal{L}$ is the derivative with respect to the geodesic flow then the flow generated by fv is mixing. The idea of the proof is of course to show that there is enough uniform twist across the orbits so that a small piece gets spread sufficiently uniformly across the space. However, the rate of mixing is not controlled well enough to guarantee absolutely continuous or Lebesgue spectrum. Still this looks plausible.

CONJECTURE 6.8. *Any flow obtained by a sufficiently smooth time change from a horocycle flow has countable Lebesgue spectrum.*

Cohomological equations over the horocycle flows has been throughly studied by Flaminio and Forni in [53]; see [78, Section 11.6.2] for a summary. While the results (growing number of invariant distributions of increasing orders) indicate complex structure of measurable isomorphism classes they do not shed direct light on spectral or other ergodic properties of time changes.

In an earlier work Ratner [131] shows that rigidity of isomorphisms between horocycle flows is partly inherited by time changes with very moderate degree of regularity in the sense that isomorphic time changes appear only for isomorphic horocycle flows. A key ingredient in the proof is showing property R for this class of time changes.

6.3.2. *Flows on surfaces of higher genus*. Another class of parabolic systems after unipotent homogeneous systems is represented by area preserving flows on surfaces with finitely many fixed points. In this case the section maps on transversals are one-dimensional, in fact they are interval exchange transformations. On the other hand, the slowdown near a fixed point leads to strong stretching which albeit not uniform in space is somewhat similar in effects with the uniform transverse stretching in unipotent systems.

A model example. The simplest example where it is evident that the slowdown and not transverse dynamics plays the main role in determining the asymptotic behavior is a flow on $\mathbb{T}^2$ obtained from an irrational linear flow by slowing down near a single point. In order to have a absolutely continuous measure preserved the inverse of the velocity change function must be integrable and the measure will still have a singularity. An alternative way is to change the flow in a neighborhood of a point so that in a local linear coordinate system (x, y) in which the linear flow is generated by the vector field $\partial/\partial x$ and hence is Hamiltonian with Hamiltonian function y to have the new flow with the Hamiltonian which locally has the form $y(x^2 + y^2)^k$ and gradually changes to y. One can make the change carefully so that the section map on a circle which still be a rotation and the flow will be isomorphic to the special flow with the roof function smooth except of one point near which it has an integrable singularity of a power type. In contrast with the case when the roof function has bounded variation such a flow is mixing [91]. The method is similar to that of [98] albeit the estimates are more subtle. Notice that unlike the latter case flows here the direction of stretching is different on two sides of the singularity.

Degenerate and non-degenerate saddles. A natural class of systems of this kind consists of area preserving flows on surfaces of genus $\geqslant 2$ with singularities of the saddle type. To include the previous example one may also allow a finite number of stopping points. The section map on a transversal is an interval exchange transformation and return time function has singularities at the endpoints of the intervals. There is an interesting difference between non-degenerate saddles (zeroes of the first order for the vector field) and other degenerate saddles which include stopping points (the latter can be considered as saddles with two separatrices). Non-degenerate saddles produce milder symmetric *logarithmic* singularities of the return time functions whereas others produce power singularities; in the latter case if the flow is ergodic it is mixing [91]. This in particular implies the following existence result:

THEOREM 6.9. *There is an area preserving mixing flow of class C^∞ on any close surface other than the sphere, projective plane and Klein bottle.*

On the other hand, if the section map happens to be a rotation then any flow with only non-degenerate saddles is not mixing [92].

An interesting phenomenon appears when the singularities of the return time function are logarithmic but asymmetric; this still may produce mixing [87]. This situation appears, for example, on the torus for a flow with a separatrix loop.

Thus sufficiently strong stretching due to power or asymmetric logarithmic singularities of the return time function produces mixing while slightly weaker symmetric logarithmic singularities do not if the base transformation is a rotation (this can be explained from the point of view of Fourier analysis, see [105]). However mixing properties of typical flows on higher genus surfaces, namely flows with zeroes of order one, remain unknown.

PROBLEM 6.10. Does there exist a mixing special flow over an interval exchange transformation with the roof function smooth except for symmetric logarithmic singularities at the interval endpoints?

Also little is known about the spectral properties of mixing flows. Some estimate of correlation decay have been obtained but they are too weak to conclude that the spectrum is absolutely continuous. Nothing is also known about multiplicity of the spectrum.

Cohomological equations. Cohomological equations over interval exchange transformations and related classification of flows on surfaces have been studied by Forni in two very powerful papers [55,56]. Those results contain some of the deepest insights into interplay between ergodic theory and harmonic analysis. There are important applications to the speed of convergence of ergodic averages for various classes of functions. See [5] for an exposition of Forni's work.

However, as is the case with horocycle flows, there are no direct implications for spectral and other invariant under metric isomorphism ergodic properties of the flows.

6.4. *Gaussian and related systems*

6.4.1. *Spectral analysis of Gaussian systems*. For a detailed introduction to the subject see [29, Chapter 14].

Recall that from the "classical" ergodic point of view, given a measure preserving transformation T on a measure space (X, μ) and a measurable function f on X, the sequence $Y_n = f \circ T^n$, $n \in \mathbb{Z}$, defines a stationary stochastic process. A stochastic process can then be considered as a measure preserving transformation together with a measurable function f.

DEFINITION 6.11. A stationary process $X_n, n \in \mathbb{Z}$, with zero mean defined on a probability space $(\Omega, \mathcal{A}, P)$ is called *Gaussian* if for all $n \in \mathbb{Z}$, $m \in \mathbb{N}$ the law of the m-tuple $(X_n, X_{n+1}, \dots, X_{n+m-1})$ is Gaussian (and independent of n). The shift transformation T_σ defined by $T(X_n)_{n\in\mathbb{Z}} = (X_{n+1})_{n\in\mathbb{Z}}$ is obviously measure preserving. It is often called the *Gaussian dynamical system* generated by the process X_n.

The spectral measure of the Koopman operator associated to T restricted to the closure of the space of linear combinations of X_n is called the *spectral measure of the Gaussian process*.

We will soon see how this measure determines the maximal spectral type of the corresponding Gaussian dynamical system. The covariance matrix of the stationary Gaussian process $X_n, n \in \mathbb{Z}$, $E(X_n X_{n+m})$ is entirely determined by the spectral measure σ:

$$E(X_n X_{n+m}) = \int_{S^1} e^{ixm}\, d\sigma.$$

Conversely, given a positive symmetric measure σ on the circle, there exists a stationary Gaussian process with zero mean $X_n^\sigma, n \in \mathbb{Z}$, with associated shift transformation T_σ such that

$$E(X_n X_{n+m}) = \int_{S^1} e^{ixm}\, d\sigma.$$

A way to construct X_n^σ is to first consider a probability space $(\Omega, \mathcal{A}, P)$ on which a family $Z_n, n \in \mathbb{Z}$, is defined, consisting of independent Gaussian random variables with law $N(0,1)$ and with H being the L^2-closure of the linear span of the $Z_n, n \in \mathbb{Z}$. The Z_n are thus an orthonormal basis for H and every element in H is a random variable with zero mean and a Gaussian distribution law. Consider the operator U_σ on H which is isometric to the unitary operator M on $L^2(S^1, d\sigma)$ defined by $g \to e^{ix} g$ (as in Theorem 1.1), by means of an isometry V between H and $L^2(S^1, d\sigma)$. ($U_\sigma = V^{-1} M V$.) Then

$$X_n^\sigma = U_\sigma^n\big(V^{-1} 1\big), \quad n \in \mathbb{Z},$$

is a Gaussian process which obviously satisfies

$$E\big(X_n^\sigma X_{n+m}^\sigma\big) = \int_{S^1} e^{ixm}\, d\sigma.$$

Let $\mathcal{B}(H)$ be the smallest σ-algebra which makes all elements in H measurable. Then $L^2(\mathcal{B}(H))$ is the direct sum of orthogonal spaces $H^{(n)}, n \in \mathbb{N}$ (the Wiener chaos) where $H^{(n)}$ is the orthocomplement of the direct sum of the $H^{(k)}$, $1 \leqslant k \leqslant n-1$, in the closure of the linear space generated by the polynomials of degree n in variables which are in H. These spaces are invariant under U_{T_σ} and the spectral measure of U_{T_σ} restricted to $H^{(n)}$ is the n-fold convolution $\sigma^{(n)}$. Thus we can calculate the maximal spectral type of the Gaussian system.

PROPOSITION 6.12. *The maximal spectral type of the Gaussian transformation T_σ is the sum of the spectral measure σ and all it convolutions $\sigma^{(n)}$.*

COROLLARY 6.13. *T_σ is ergodic only when σ is non-atomic, and in that case it is weakly mixing.*

If a symmetric measure σ on S^1 is the sum of two symmetric measures σ_1 and σ_2 which are mutually singular, T_σ is isomorphic to the direct product $T_{\sigma_1} \times T_{\sigma_2}$. Therefore, decomposing σ as the sum of its singular part σ_s and of its absolutely continuous part σ_a, we see that, since a Gaussian process with singular spectral measure has 0 entropy, T_σ is isomorphic to a factor of the product of a zero entropy transformation by an infinite entropy Bernoulli shift, and is itself of this form, as an application of general theorems. If in the preceding construction, we consider the more general situation where the operator U on H has no longer simple spectrum, we still obtain a transformation, which is no longer described by a single Gaussian process, which we call *generalized Gaussian*. Generalized Gaussian processes share many properties with ordinary Gaussian processes.

A version of the generalized Gaussian construction for more general groups provides a general way to construct many spectrally (and hence metrically) non-isomorphic actions by measure preserving transformations [4, Section 4.4]. For such groups as semisimple Lie groups of rank $\geqslant 2$ and lattices in such groups whose actions possess strong rigidity properties which render many standard constructions trivial this is the only known way to produce many non-isomorphic actions.

6.4.2. *Spectral multiplicity for Gaussian systems.* In order for U_{T_σ} to have simple spectrum it is necessary for all $\sigma^{(n)}$ to be pairwise singular.

On the other hand, the spectrum is simple if there is a set K such that (i) $K \cup -K$ has full σ-measure, and (ii) all its elements are independent over the rationals, that is if

$$\lambda_1, \dots, \lambda_n \in K, \quad \text{and} \quad (m_1, \dots, m_n) \in \mathbb{Z}^n \setminus \{0\}, \quad \text{then} \quad m_1\lambda_1 + \dots + m_n\lambda_n \neq 0. \tag{6.1}$$

We use here additive coordinate on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. The first proof of existence of a measure preserving transformation with a simple but not pure point spectrum was given by Girsanov in [61] using the Gaussian system of this kind. A stronger condition which implies (6.1) is the following:

($\mathfrak{K}$) *Every continuous function on the set K of modulus* 1 *is a uniform limit of characters*. A closed set satisfying condition ($\mathfrak{K}$) is called a *Kronecker set*. D. Newton [116] first used Kronecker sets to construct Gaussian systems with simple spectrum. His examples were Gaussian systems with spectral measures supported by the union of a Kronecker set K and $-K$. Let us call such a measure *Kronecker*. Also using a construction of a mixing measure suggested by Rudin [139] Newton found a mixing Gaussian transformation with simple spectrum.

PROPOSITION 6.14. *For Gaussian systems the multiplicity function is multiplicative almost everywhere with respect to a measure of the maximal spectral type.*

COROLLARY 6.15. *Either the spectrum of a Gaussian transformation is simple or the maximal spectral multiplicity is unbounded.*

PROPOSITION 6.16. *There exists σ such that T_σ has non-simple spectrum and for which the multiplicity function is finite almost everywhere.*

Corollary 6.15 and Proposition 6.16 explain why finding systems with non-simple spectrum of bounded multiplicity was considered an interesting problem when Gaussian systems and their modifications provided the only models with interesting spectral properties. After the initial success in the study of Gaussian systems there was a hope to organize a good part of ergodic theory around a generalized version of the Gaussian model reflected in [147]. One of the original impulses which led to the development of the theory or periodic approximations and similar geometric methods came from attempts to understand how restrictive were the assumptions on which this approach was based. The answer on the occasion was that they almost never held in natural geometric situations.

6.4.3. *Spectrally defined isomorphism in Gaussian and similar systems.* Foias and Stratila in [54] showed that Newton's examples have a remarkable property which makes them similar to transformations with pure point spectrum, in fact like translations on continuum-dimensional tori.

THEOREM 6.17. *Let σ be a Kronecker measure. Then if $(X, \mathcal{A}, m, T)$ is ergodic and if $f \in L^2(X)$ satisfies $\nu_f = \sigma$, the process $T^n f$, $n \in \mathbb{Z}$, is Gaussian.*

One important consequence of this theorem is the following [150].

THEOREM 6.18. *Let σ be a Kronecker measure and T_σ the associated Gaussian transformation. All ergodic self-joinings of T_σ remain generalized Gaussian.*

One can prove that the conclusion of this theorem holds for measures σ such that the associated Gaussian T_σ has simple spectrum. Those processes such that all their ergodic joinings remain generalized Gaussian are called GAG and are the subject of a comprehensive study in [107]. They can be thought of as a limit of a product of pairwise disjoint simple transformations. Let us say that σ for which the conclusion of Theorem 6.17 holds has the *F.S.-property*. There are examples in [107] where measures satisfying the F.S.-property have as support S the union of two disjoint Kronecker sets without S itself being Kronecker. An interesting question is the following:

PROBLEM 6.19. Does there exist a mixing measure which possesses the F.S.-property?

F. Parreau (unpublished) has produced a mildly mixing measure with the F.S.-property. Notice that Kronecker systems are rigid. This is a direct consequence of the property of Kronecker sets that every continuous function is a uniform limit of characters. The rigidity is just this statement applied to the constant function 1. We are now going to show that, with the use of Gaussian processes, it is easy to produce two transformations which are weakly isomorphic but not isomorphic. Given a Gaussian process X_n^σ to which we associate the shift transformation T_σ (acting on $(X, \mathcal{A}, m)$), if we let H be the L^2-closure of the linear span of X_n^σ, $n \in \mathbb{Z}$, we have seen that every unitary operator U on H gives rise to a measure preserving transformation τ_U (the one coming from the Gaussian processes associated to U when H is decomposed into an orthogonal sum of U-cyclic subspaces). If we take

$$UX = -X,$$

the map τ_U is an involution which commutes with T_σ; the σ-algebra $\mathcal{B}$ of τ_U-invariant sets defines a factor (which we call $\widehat{T_\sigma}$) of T_σ and

$$L^2(\mathcal{B}) = \sum_{n \geqslant 0} H^{(2n)}.$$

Such a factor was first defined by Newton and Parry. We take σ such that σ is continuous and $\sigma^{(n)} \perp \sigma^{(m)}$, $n \neq m$. We define

$$T_1 = \prod_{k \in \mathbb{N}} T_{k,\sigma},$$

where every $T_{k,\sigma}$, $k \in \mathbb{N}$, is a copy of T_σ and

$$T_2 = \widehat{T_\sigma} \times T_1.$$

It is therefore obvious that T_1 and T_2 are weakly isomorphic.

THEOREM 6.20. *T_1 and T_2 are weakly isomorphic but not isomorphic.*

PROOF. In L^2 of the space on which T_1 lives, U_{T_1} (the unitary operator associated to T_1) has spectral measure σ on

$$\sum_{k \in \mathbb{N}} \oplus H_k = \overline{H}.$$

The spectral measure of U_{T_1} on $\overline{H}^\perp$ is singular with respect to σ (because of the hypothesis on σ). Again, because

$$\sigma^{(2n)} \perp \sigma,$$

the spectral measure of U_{T_2} on $\overline{H}_2^\perp$ is singular with respect to σ. ($\overline{H}_2$ is $\sum_{k \in \mathbb{N}} \oplus H_k$ in the space on which T_2 lives.) Therefore, if T_1 and T_2 are isomorphic then the associated isometry must send $\overline{H}$ onto $\overline{H}_2$, which is impossible. □

Note that it is also very easy to construct weakly isomorphic but not isomorphic transformations from a transformation which has MSJ (and therefore from Chacon transformations). This was done by D. Rudolph before the Gaussian example described above. The first example of two weakly isomorphic but not isomorphic transformations is due to S. Polit. Kwiatkowski, Lemańczyk and Rudolph [101] have constructed an example of two smooth dynamical systems which are weakly isomorphic but not isomorphic. The following result by De la Rue shows that Girsanov examples and more general transformations with simple spectrum coming from the Gaussian construction are quite different from transformations with simple spectrum constructed by more geometric methods in earlier parts of this survey.

THEOREM 6.21 [37]. *A Gaussian transformation cannot be locally rank one.*

Another result in a similar vein was proved by del Junco and Lemańczyk [31] who extended an earlier result by Thouvenot [151].

THEOREM 6.22. *Gaussian transformations are disjoint from simple transformations.*

One more striking property of Kronecker Gaussian systems is the De la Rue result [36] mentioned before that there are maximal spectral types which appear for zero entropy ergodic transformations but not for standard ones (Kakutani equivalent to adding machines and irrational rotations). This follows form the Foias–Stratila theorem and the following fact proved by De la Rue.

THEOREM 6.23. *There exists a Kronecker measure such that the corresponding Gaussian transformation is not standard.*

An interesting open problem tying together the themes of this section and Section 5.5 is the following:

PROBLEM 6.24. Does there exist a volume preserving diffeomorphism of a compact differentiable manifold which is measurably conjugate to a Gaussian system?

More specifically, given a non-trivial volume preserving smooth action $\mathcal{S}$ of S^1 on a compact differentiable manifold M, does there exist a diffeomorphism measurably conjugate to a Kronecker Gaussian system in the space $\mathcal{A}$, the closure of conjugates of the elements of $\mathcal{S}$ (see Section 5.5.2)?

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