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Measurable rigidity and disjointness for $\mathbb{Z}^k$ actions by toral automorphisms

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Abstract. We investigate joinings of strongly irreducible totally non-symplectic Anosov $\mathbb{Z}^k$, $k \geq 2$ actions by toral automorphisms. We show that the existence of a non-trivial joining has strong implications for these actions, in particular, that the restrictions of the actions to a finite index subgroup of $\mathbb{Z}^k$ are conjugate over $\mathbb{Q}$. We also obtain a description of the joining measures modulo the classification of zero entropy measures for the actions.

1. *Introduction*

In this paper we consider the measurable structure of certain irreducible Anosov actions of $\mathbb{Z}^k$, $k \geq 2$ by automorphisms of the torus $\mathbb{T}^m$ with respect to the Lebesgue measure. The individual elements of such actions are Bernoulli automorphisms [**9**]; hence they are metrically isomorphic if they have the same entropies [**12**]. In addition, the set of conjugacies and the set of joinings of two Bernoulli automorphisms are large and do not admit a reasonable description.

In contrast to the properties of the individual elements many natural actions of higher rank Abelian groups have remarkable rigidity properties. In [**6, 7**] Katok and Spatzier studied invariant measures with a non-vanishing entropy function for a class of algebraic actions of $\mathbb{Z}^k_+$, $\mathbb{Z}^k$ and $\mathbb{R}^k$, $k \geq 2$ which, in particular, includes irreducible $\mathbb{Z}^k$ actions by ergodic automorphisms of the torus. Based on the results from [**6, 7**], it has been shown in [**4**] that measurable conjugacies, centralizers and factors for these actions on the torus are essentially algebraic. See [**3**] for a complete and detailed exposition of the properly modified principal result from [**6, 7**] for the case of actions by toral automorphisms. In particular, this exposition corrects various inaccuracies from the original papers.

Conjugacies, centralizers and factors give rise to special kinds of joinings between actions, see Section 2.1 and [**4**]. Thus the natural next step in the rigidity programme would be to describe all joinings between actions in a given class.

Note that *any* invariant measure of an action by toral automorphisms gives rise to a self-joining (see Section 2.1). Since the description of invariant measures with a vanishing entropy function is an open and notoriously difficult problem at the moment we can only hope to describe joinings in terms of invariant measures. A more accessible question which is addressed in the present paper is: What are the implications of the existence of a non-trivial joining between two actions? We treat the leading case of totally non-symplectic actions by toral automorphisms which contain, in particular, the maximal rank (Cartan) actions [**4**]. In fact we also show that, in our case, all joinings up to a finite factor are associated with invariant measures.

Our solution is based on further development of the main technique from [**6**], the consideration of conditional measures on various invariant foliations of the product action, complemented by a new argument, a sort of relative version of the Ruelle inequality between entropy and Lyapunov exponents (Lemma 4.9).

In the last section we discuss possible generalizations to other classes of actions.

2. *Preliminaries*

2.1. *Ergodic toral automorphisms and joinings.* We denote by $GL(m,\mathbb{Z})$ the group of integral $m \times m$ matrices with determinant 1 or -1. Any matrix $A \in GL(m,\mathbb{Z})$ defines an automorphism of the torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ which we denote by F_A; it is ergodic with respect to the Lebesgue measure on $\mathbb{T}^m$ if and only if no eigenvalue of A is a root of unity. Furthermore, in this case A has eigenvalues of an absolute value greater than one and F_A is a Bernoulli automorphism [**9**].

Any $\mathbb{Z}^k$ action α by automorphisms of $\mathbb{T}^m$ is given by an embedding $\rho_\alpha : \mathbb{Z}^k \to GL(m,\mathbb{Z})$ such that $\alpha(n) = F_{\rho_\alpha(n)}$, where $n = (n_1, \dots, n_k) \in \mathbb{Z}^k$.

Let α_1 and α_2 be two $\mathbb{Z}^k$ actions by automorphisms of $\mathbb{T}^{m_1}$ and $\mathbb{T}^{m_2}$ correspondingly. The action α_2 is called *an algebraic factor* of α_1 if there exists an epimorphism h: $\mathbb{T}^{m_1} \to \mathbb{T}^{m_2}$ such that $h \circ \alpha_1 = \alpha_2 \circ h$.

A Borel probability measure μ on $\mathbb{T}^{m_1} \times \mathbb{T}^{m_2}$ is called a *joining* of α_1 and α_2 if it is invariant under the diagonal (product) action $\alpha_1 \times \alpha_2$ of $\mathbb{Z}^k$ on $\mathbb{T}^{m_1} \times \mathbb{T}^{m_2}$ and projects onto the Lebesgue measures on $\mathbb{T}^{m_1}$ and $\mathbb{T}^{m_2}$. A joining of an action with itself is called a *self-joining*. The Lebesgue measure on $\mathbb{T}^{m_1} \times \mathbb{T}^{m_2}$ is called a *product* or *trivial joining*. Two actions are called *disjoint* if the only joining is the product measure.

Any isomorphism H between two action-preserving measures μ_1 and μ_2 correspondingly generates a joining measure via the pull-back: $(\mathrm{Id} \times H)^*\mu_1 = (H^{-1} \times \mathrm{Id})^*\mu_2$; the same is true for a factor map, i.e. a non-invertible measure-preserving semiconjugacy. Furthermore, any measurable factor of an action generates a self-joining via the relative product construction.

For an action α by automorphisms of the torus $\mathbb{T}^m$ any α-invariant Borel probability measure μ and any non-singular integer 2×2 matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with non-zero elements generate the self-joining measure

$$(\pi^{a,b})^*\lambda \times (\pi^{c,d})^*\mu,$$

where λ is the Lebesgue measure and $\pi^{k,l}(x, y) = kx + ly$, $x, y \in \mathbb{T}^m$.

2.2. *Irreducible $\mathbb{Z}^k$ actions on $\mathbb{T}^m$.* We give a brief description of the algebraic structure of irreducible $\mathbb{Z}^k$ actions on $\mathbb{T}^m$, see [**1, 2**] for more details.

The action α is called *irreducible* if any non-trivial algebraic factor has finite fibres. In [**6, 7**] irreducible actions are called completely irreducible. The following proposition gives an alternative description of irreducible $\mathbb{Z}^k$ actions.

PROPOSITION 2.1. [**1**] *Let α be a $\mathbb{Z}^k$ action by automorphisms of $\mathbb{T}^m$. The following properties are equivalent:*

(1) *α is irreducible;*

(2) *$\rho_\alpha(\mathbb{Z}^k)$ contains a matrix with a characteristic polynomial irreducible over $\mathbb{Q}$;*

(3) *$\rho_\alpha(\mathbb{Z}^k)$ does not have a non-trivial invariant rational subspace; and*

(4) *any α-invariant closed proper subgroup of $\mathbb{T}^m$ is finite.*

The action α is called *strongly irreducible* if the action of any finite index subgroup of $\mathbb{Z}^k$ is irreducible. Equivalently, α is strongly irreducible if and only if the orbit of any non-trivial rational subspace is infinite.

Any matrix $A \in GL(m, \mathbb{Z})$ with a characteristic polynomial irreducible over $\mathbb{Q}$ has simple eigenvalues and hence is diagonalizable over $\mathbb{C}$. Moreover, the centralizer of A in $M(m, \mathbb{Q})$, all $m \times m$ matrices with rational entries, coincides with the polynomials of A with rational coefficients.

Let α be an irreducible $\mathbb{Z}^k$ action on $\mathbb{T}^m$ and let $G_\alpha = \rho_\alpha(\mathbb{Z}^k) \subset GL(m, \mathbb{Z})$. By Proposition 2.1 there exists an $A \in G_\alpha$ with an irreducible characteristic polynomial. Since G_α is commutative we see that all matrices in G_α are simultaneously diagonalizable over $\mathbb{C}$. If, in addition, all eigenvalues of A are real then the eigenvalues of any matrix in G_α are also real. In this case all matrices in G_α are simultaneously diagonalizable over $\mathbb{R}$; in other words G_α is conjugate to a subgroup of the group of all diagonal matrices in $GL(m, \mathbb{R})$.

It follows that the tangent bundle $T\mathbb{T}^m = \mathbb{T}^m \times \mathbb{R}^m$ splits into the direct sum of invariant subbundles corresponding to one-dimensional eigenspaces of real eigenvalues and two-dimensional planes of pairs of complex eigenvalues. Each of these subbundles is tangent to a totally irrational homogeneous foliation of $\mathbb{T}^m$.

2.3. *Lyapunov exponents.* If v lies in one of the subbundles then the Lyapunov exponent $\lambda(\alpha(n), v)$ is equal to the logarithm of the absolute value of the corresponding eigenvalue of element $\alpha(n)$. Moreover, $\lambda(\alpha(\cdot), v)$ is an additive functional on $\mathbb{Z}^k$. We see that the eigenspace splitting of $T\mathbb{T}^m$ is a refinement of the Lyapunov decomposition of $T\mathbb{T}^m$ into Lyapunov subspaces for the action α. For a detailed discussion of Lyapunov exponents for $\mathbb{Z}^k$ and $\mathbb{R}^k$ actions on manifolds see [**6**].

It is convenient for us to operate with $\mathbb{R}^k$ actions so we would like to pass from an action of $\mathbb{Z}^k$ to the corresponding action of $\mathbb{R}^k$. This is the so-called suspension construction.

Suppose $\mathbb{Z}^k$ acts on $\mathbb{T}^m$. Embed $\mathbb{Z}^k$ as a lattice in $\mathbb{R}^k$. Let $\mathbb{Z}^k$ act on $\mathbb{R}^k \times \mathbb{T}^m$ by $z(x, m) = (x - z, z \cdot m)$ and form the quotient

$$M = \mathbb{R}^k \times \mathbb{T}^m/\mathbb{Z}^k.$$

Note that the action of $\mathbb{R}^k$ on $\mathbb{R}^k \times \mathbb{T}^m$ by $x \cdot (y, n) = (x + y, n)$ commutes with the $\mathbb{Z}^k$-action and therefore descends to M. This $\mathbb{R}^k$-action is called the *suspension* of the $\mathbb{Z}^k$-action.

Note that any $\mathbb{Z}^k$-invariant measure on $\mathbb{T}^m$ can be lifted to a unique $\mathbb{R}^k$-invariant measure on M and conversely any invariant measure for the suspension induces a unique invariant measure for the original action.

The manifold M is a fibration over the 'time' torus $\mathbb{T}^k$ with the fibre $\mathbb{T}^m$. We note that TM splits into the direct sum $TM = T_f M \oplus T_o M$ where $T_f M$ is the subbundle tangent to the $\mathbb{T}^m$ fibres and $T_o M$ is the subbundle tangent to the orbit foliation. The Lyapunov exponent corresponding to $T_o M$ is always identically zero. To exclude this trivial case, when we speak of Lyapunov exponents we will always mean the Lyapunov exponents corresponding to $T_f M$. These Lyapunov exponents of the $\mathbb{R}^k$ action are the extensions of the Lyapunov exponents of the $\mathbb{Z}^k$ action to the linear functionals on $\mathbb{R}^k$.

The kernels of the non-zero Lyapunov exponents are called *Lyapunov hyperplanes*. The pre-images of the positive half-line are called *positive Lyapunov half-spaces*. If there are no non-trivial identically zero Lyapunov exponents then the action is called an *Anosov* action.

An element $a \in \mathbb{R}^k$ is called *regular* if it does not belong to any Lyapunov hyperplane. All other elements are called *singular*. We call a singular element *generic* if it belongs to only one Lyapunov hyperplane. A regular element for an Anosov action is called an *Anosov element*.

Lyapunov exponents may be proportional to each other with positive or negative coefficients. In this case, they define the same Lyapunov hyperplane. An Anosov action α is called *totally non-symplectic* or *TNS* (see [**5**]) if there are no Lyapunov exponents proportional to each other with negative coefficients; or, equivalently, any two positive Lyapunov half-spaces have a non-empty intersection. For any TNS $\mathbb{Z}^k$ action $k \geq 2$ automatically since for $k = 1$ all Lyapunov exponents are proportional. A typical example of an irreducible TNS action is the Cartan action of $\mathbb{Z}^{m-1}$ on $\mathbb{T}^m$, $m \geq 3$ (see [**4**]).

For an element $a \in \mathbb{R}^k$ the *stable*, *unstable* and *centre* distributions E_a^+, E_a^- and E_a^0 are defined as the sum of the Lyapunov spaces for which the value of the corresponding Lyapunov exponent on a is negative, positive and zero respectively. Since all elements of any irreducible action are diagonalizable, the derivative of any singular element a acts isometrically on its centre distribution. So in this case E_a^0 coincides with the isometric distribution E_a^I. For an element a we will denote the integral foliations of the stable, unstable, centre and isometric distributions E_a^-, E_a^+, E_a^0 and E_a^I by W_a^-, W_a^+, W_a^0 and W_a^I.

For any α-invariant measure μ and invariant subfoliation F of the unstable foliation W_a^+ we define the *entropy with respect to* μ *along the foliation* F as the supremum of the conditional entropies $H_\mu(\xi/\alpha(a)\xi)$ over all increasing measurable partitions ξ subordinated to the foliation F. We denote this entropy by $h_\mu^F(a)$. We have

$$h_\mu(\alpha(a)) = h_\mu^{W_a^+}(a),$$

where $h_\mu(\alpha(a))$ is the entropy of the transformation $\alpha(a)$ with respect to the measure μ.

3. *The Main Theorem*

THEOREM 3.1. *Let (α_1, λ_1) and (α_2, λ_2) be strongly irreducible Anosov TNS actions of $\mathbb{Z}^k$ by automorphisms of $\mathbb{T}^{m_1}$ and $\mathbb{T}^{m_2}$, where λ_1 and λ_2 are Lebesgue measures on $\mathbb{T}^{m_1}$ and $\mathbb{T}^{m_2}$ respectively. If a non-trivial joining measure $\mu_{\mathbb{T}^m}$ on $\mathbb{T}^m = \mathbb{T}^{m_1} \times \mathbb{T}^{m_2}$ exists then the following statements are true.*

(1) $m_1 = m_2$.

(2) *The Lyapunov exponents of the actions α_1 and α_2 are identical.*

(3) *The entropy of every element in $\mathbb{Z}^k$ with respect to the joining measure $\mu_{\mathbb{T}^m}$ is equal to its entropy with respect to the Lebesgue measure in each factor.*

(4) *There exists a subgroup $\Gamma \subset \mathbb{Z}^k$ of finite index such that the actions α_1 and α_2 restricted to Γ are conjugate over $\mathbb{Q}$.*

(5) *If the joining measure $\mu_{\mathbb{T}^m}$ is ergodic it decomposes as $(1/N)(\mu_1 + \cdots + \mu_N)$, where each μ_i, $i = 1, \ldots, N$, is an invariant measure for the restriction of $\alpha_1 \times \alpha_2$ to Γ and is an extension of a zero-entropy measure in an algebraic factor of dimension $m_1 = m_2$ with Haar measures in the fibres. This algebraic factor is isomorphic to a finite factor of the actions α_1 and α_2 restricted to Γ.*

4. *Proof of Theorem 3.1*

4.1. *Scheme of proof.* We denote by $\alpha_{\mathbb{T}^m}$ the product action $\alpha_1 \times \alpha_2$ of $\mathbb{Z}^k$ on $\mathbb{T}^m = \mathbb{T}^{m_1} \times \mathbb{T}^{m_2}$. Let $\mu_{\mathbb{T}^m}$ be a non-trivial joining measure on $\mathbb{T}^m$. We use the suspension construction to pass to the corresponding $\mathbb{R}^k$ action (α, μ) on a manifold M. Using ergodic decomposition we may assume without loss of generality that the $\mu_{\mathbb{T}^m}$ and hence μ are ergodic.

Step 1. (Section 4.2) We start by considering a Lyapunov foliation of the action α on M. If the corresponding Lyapunov exponent is also a Lyapunov exponent in both factors then the Lyapunov foliation splits as the sum of two invariant subfoliations which correspond to the Lyapunov foliations in the factors. In this case, instead of considering the whole Lyapunov foliation we consider each of these invariant subfoliations separately.

Let F be a foliation as described earlier. For any such foliation we establish the following fact.

LEMMA 4.1. *The conditional measures of μ on F are atomic.*

We first prove that the conditional measures are Haar measures on certain affine spaces (Lemmas 4.2–4.4, and checking the ergodicity condition for Lemma 4.2). Next we show that if those affine spaces have a positive dimension then both actions cannot be strongly irreducible (Lemma 4.5).

Step 2. (Section 4.3) We compare the Lyapunov exponents of the actions α_1 and α_2. We show that the conditional measures of μ are atomic on some sufficiently large subfoliation G of the unstable foliation of some Anosov element $b \in \mathbb{R}^k$. Specifically this subfoliation G contains all Lyapunov foliations corresponding to Lyapunov exponents for the actions α_1 and α_2 positive at b which do not have matching exponents for the other action (taking multiplicities into account), as well as at least one Lyapunov foliation for each matching pair. The atomicity is proved by induction by first adding foliations corresponding to exponents which are positively proportional to a given one in increasing order (Lemma 4.6) and then foliations corresponding to the non-proportional exponents (Lemma 4.7).

Step 3. (Section 4.4) Lemma 4.9 implies that the entropy of $\alpha(b)$ with respect to the joining measure is less than the sum of the remaining positive Lyapunov exponents, i.e. exponents not coming from the foliation G. Thus if exponents of α_1 and α_2 are not identical (with multiplicities), then the entropy of $\alpha(b)$ is less than the entropy with respect to the Lebesgue measure of $\alpha_1(b)$ or $\alpha_2(b)$ which is impossible since these measures are the factors of the joining measure. This shows that the Lyapunov exponents of α_1 and α_2 must be identical, in particular $m_1 = m_2$. In this case this argument also shows that for any element b the conditional measures of μ are atomic on 'half' of its unstable foliation so that the entropy of b with respect to the joining measure cannot be more than the entropy in each factor.

Step 4. (Section 4.5) Having thus proven that the Lyapunov exponents of actions α_1 and α_2 and their multiplicities coincide we now consider a Lyapunov foliation for the product action and the conditional measures of the joining measure on that foliation. As in the arguments in Step 1 these conditionals are Haar measures on affine subspaces. However, now the atomic option (zero-dimensional spaces) is excluded either by an entropy argument as in Step 3, or, more straightforwardly, by considering projections. Thus Lemma 4.5 can be applied to deduce that up to a subgroup of finite index the joining measure decomposes into Haar measures on parallel rational subtori. By irreducibility the dimensions of these subtori cannot be either greater or less than the dimension of the factors. Thus it is equal and the invariance of such a subtorus implies the existence of a rational conjugacy between the restrictions of α_1 and α_2 to a finite index subgroup. Finally, the decomposition of the joining measure according to the partition into parallel invariant subtori provides the structure described in the theorem.

Now we proceed to the details.

4.2. *Proof of Lemma 4.1.* We use three lemmas which appeared as [**6**, Lemmas 5.4–5.6] (see also [**3**, Lemmas 3.2–3.4]) to describe the conditional measures on the foliation F.

Step 1.1: Algebraicity of conditional measures. Since F is either a Lyapunov foliation or an invariant subfoliation of a Lyapunov foliation we can take a generic singular element a in the corresponding Lyapunov hyperplane so that $F \subset W_a^I$. We will verify the 'ergodicity' assumption of Lemma 4.2 ($\xi_a \leq \xi(F)$) later.

Suppose that $a \in \mathbb{R}^k$ is a generic singular element and $F \subset W_a^I$ is some α-invariant subfoliation of W_a^I. Denote by $B_1^F(x)$ the unit ball in $F(x)$ about x with respect to the flat metric. Let μ_x^F denote the system of conditional measures on F normalized by the requirement $\mu_x(B_1^F(x)) = 1$ for all x in the support of μ. For a detailed discussion of conditional measures on foliations see [**6**]. We denote by ξ_a the partition into ergodic components of a and by $\xi(F)$ the measurable hull of F.

LEMMA 4.2. *Suppose that $\xi_a \leq \xi(F)$. Then for μ-a.e. x, the support of μ_x^F is the orbit of the closed subgroup G_x of isometries of $F(x)$ which preserve μ_x^F up to a scalar multiple. Furthermore, for μ_x^F-a.e. $y \in F(x)$, μ_y^F is the image of μ_x^F under an isometry in G_x.*

LEMMA 4.3. *In addition to the assumptions in Lemma 4.2, let F be contained in the intersection of W_a^I with a Lyapunov subspace for a non-zero Lyapunov exponent λ. Then for μ-a.e. x, the support S_x of μ_x^F is an affine subspace of $F(x)$.*

LEMMA 4.4. *Under the assumptions of Lemma 4.3, μ_x^F is a Haar measure on S_x.*

These three lemmas show (modulo the assumption of Lemma 4.2) that, for μ-a.e. x, μ_x^F is a Haar measure on an affine subspace S_x of the leaf $F(x)$.

Step 1.2: Checking the ergodicity assumption. We now verify the assumption

$$\xi_a \leq \xi(F)$$

of Lemma 4.2. This is the only place in the proof where the TNS condition is used. Let $W_{a,1}^-$ be the subfoliation of W_a^- that corresponds to the Lyapunov exponents of α_1, in other words it is the maximal invariant subfoliation of W_a^- which is subordinate to the partition of M into horizontal tori. The Hopf argument shows that $\xi_a \leq \xi(W_a^-)$. Indeed, if f is a continuous function on M then the forward ergodic averages

$$f^+(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(a^n x)$$

are constant along stable manifolds of a as they contract exponentially under a. Since the continuous functions are dense in $L^2_\mu(M)$, it follows that any invariant L^2_μ-function is constant a.e. on F with respect to the conditional measure induced by μ.

Let $\mathcal{P}$ be the Lyapunov hyperplane that contains a. Since α_1 is a TNS action we see that all Lyapunov exponents of α_1 with kernel $\mathcal{P}$ are positively proportional. This implies that we can take a regular element b in the positive half-space $\mathcal{P}^+$ sufficiently close to a so that $\alpha_1(b)$ is Anosov and $W_{a,1}^- = W_{b,1}^-$, where $W_{b,1}^-$ is again the part of W_b^- that corresponds to the Lyapunov exponents of α_1.

We note that $\xi(W_{b,1}^-) = \xi(W_{b,1}^+)$ since they are both equal to the conditional Pinsker algebra of b. This can be seen in [**11**].

Now using the decomposition into Lyapunov spaces we see that $W_{a,1}^- = W_{b,1}^-$ implies that $W_{a,1}^0 \subset W_{a,1}^0 \oplus W_{a,1}^+ = W_{b,1}^+$ and hence that $\xi(W_{b,1}^+) \leq \xi(W_{a,1}^0)$.

Combining all the inequalities between σ-algebras established earlier we obtain

$$\xi_a \leq \xi(W_{a,1}^-) = \xi(W_{b,1}^-) = \xi(W_{b,1}^+) \leq \xi(W_{a,1}^0) \leq \xi(F).$$

Step 1.3: Excluding the positive dimension case. To complete the proof of Lemma 4.1 it remains to consider the case when conditional measures are Haar measures on affine subspaces of positive dimension.

LEMMA 4.5. *Let F be an invariant foliation. Suppose that for μ-a.e. x, the conditional measure μ_x^F is a Haar measure on an affine subspace S_x with* $\dim S_x = l \geq 1$. *Then there exist a finite index subgroup Γ of $\mathbb{Z}^k$, rational subtori $T_i \subset \mathbb{T}^m$, $i = 1, \dots, N$, of the same dimension, and Borel probability measures μ_i, $i = 1, \dots, N$, on $\mathbb{T}^m$ such that*

(1) *for μ-a.e. x, the closure of S_x is a translation of T_i for some i;*

(2) *the joining measure decomposes as $\mu_{\mathbb{T}^m} = (1/N)(\mu_1 + \cdots + \mu_N)$;*

(3) *μ_i, $i = 1, \ldots, N$ is invariant under the group of translations in the direction of T_i; and*

(4) *T_i and μ_i, $i = 1, \ldots, N$, are invariant under $\alpha(\Gamma)$.*

Proof. Let us denote the closure of S_x by $T(x)$. $T(x)$ is a rational subtorus which corresponds to the minimal rational subspace that contains S_x. Hence $\dim T(x)$ is an invariant function. Since the joining is ergodic we conclude that $\dim T(x)$ is constant μ-a.e. Let us call two points x and y equivalent if $T(x) = T(y)$. This equivalence relation gives rise to a measurable partition of $\mathbb{T}^m$ into rational subtori.

We first show that the conditional measures on these tori are Lebesgue. Indeed, let us consider a typical torus T with conditional measure μ_T. T is foliated by affine subspaces S_x in such a way that the conditional measures of μ_T on S_x are Haar measures for μ_T-a.e. x. One can assign an orthonormal basis to each subspace S_x in a measurable way. This produces a measurable $\mathbb{R}^l$ action on T which preserves μ_T. Let us take the ergodic decomposition of μ_T with respect to this $\mathbb{R}^l$ action and consider one ergodic component. The direction of S_x is invariant under the action since the subspace S_x is the same for all points on one trajectory. Hence almost all subspaces S_x within one ergodic component have the same direction, i.e. they are parallel. It follows that the measure on this ergodic component is invariant under translations in this direction. By the construction of T the closure of S_x equals T for all x. This implies that the measure on any ergodic component is a Haar measure on T. Hence μ_T is the Lebesgue measure on T.

Since there can be, at most, countably many classes of parallel tori we conclude that there exists a torus T_1 such that the set E_1 consisting of all points x for which $T(x)$ is parallel to T_1 has a positive measure. By recurrence, for any generator A_j of the action α there exists $n > 0$ such that $\mu_{\mathbb{T}^m}(E_1 \cap A_j^n(E_1)) > 0$. This means that for any point x in this intersection the element A_j^n maps $T(x)$ to a parallel torus. Since A_j^n is an affine map it follows that A_j^n preserves this class of parallel tori. Hence $A_j^n(E_1) = E_1$. In the same way it follows that the set E_1 is invariant with respect to the action of a finite index subgroup $\Gamma \subset \mathbb{Z}^k$. The orbit of the set E_1 consists of finitely many sets $E_1, \ldots, E_N$ of equal measure which correspond to the elements of $\mathbb{Z}^k/\Gamma$. By the ergodicity of the joining the union of these sets has a full measure. Hence $\mu_{\mathbb{T}^m} = (1/N)(\mu_1 + \cdots + \mu_N)$, where $\mu_i = \mu_{\mathbb{T}^m}|_{E_i}$. We note that $T(x)$ is parallel to the torus $T_i = a(T_1)$ for μ_i-a.e. x, where a is an element of the action α which maps E_1 to E_i. We conclude that T_i and μ_i are invariant under $\alpha(\Gamma)$ and that μ_i is invariant under the group of translations in the direction of T_i. □

We have proved so far that for μ-a.e. x, μ_x^F is a Haar measure on an affine subspace S_x of the leaf $F(x)$. Since $\dim S_x$ is an invariant function and the joining measure is ergodic we conclude that either for μ-a.e. x, μ_x^F is atomic or for μ-a.e. x, μ_x^F is a Haar measure on an affine subspace S_x with $\dim S_x = l \geq 1$. In the latter case we apply Lemma 4.5. The foliation F is subordinate either to the first or second factor. Assume the former. Then the rational subtori T_i given by Lemma 4.5 are contained in $\mathbb{T}^{m_1}$. Since the action α_1 is strongly irreducible and the tori T_i are invariant under the action of some finite index subgroup we conclude that $T_i = \mathbb{T}^{m_1}$ for all i. This implies that the conditional measures corresponding to the partition of $\mathbb{T}^m$ into 'horizontal' tori $\mathbb{T}^{m_1} \times \{x\}$, $x \in \mathbb{T}^{m_2}$ are Lebesgue. Since the measure μ_2 in the factor is also Lebesgue we conclude that $\mu_{\mathbb{T}^m}$ is Lebesgue on $\mathbb{T}^m$, which contradicts the assumption that the joining is not trivial. The other case is completely symmetric.

Thus the proof of Lemma 4.1 is complete. □

4.3. *Atomicity of the joining measure on a large foliation.* Lemma 4.1 implies that the conditional measures of μ are atomic on any invariant foliation of M which corresponds to some Lyapunov exponent of α_1 or α_2. We need to prove that the Lyapunov exponents of α_1 and α_2 are identical including multiplicities.

Step 2.1: Construction of a sufficiently large foliation. Suppose the opposite. Then there exists an Anosov element b such that different Lyapunov exponents take different values on $\alpha(b)$ and the positive Lyapunov exponents of $\alpha_1(b)$ and $\alpha_2(b)$ are not the same. We would like to construct an invariant splitting $W_b^+ = F \oplus G$ such that the conditional measures on G are atomic and G is sufficiently large, i.e. it contains all Lyapunov foliations corresponding to the Lyapunov exponents for the actions α_1 and α_2 positive at b which do not have matching exponents for the other action, taking multiplicities into account, as well at least one of the Lyapunov foliations for each matching pair.

The foliation W_b^+ splits as a sum of Lyapunov foliations corresponding to the Lyapunov exponents of the factors α_1 or α_2 which are positive on b. Let W be one of these foliations with the corresponding Lyapunov exponent λ. If λ is not a Lyapunov exponent in the other factor then we assign W to G. If λ is a Lyapunov exponent in the other factor then we compare the multiplicities of λ in the two factors. We assign the higher-dimensional foliation to G and the lower-dimensional foliation to F. Thus the foliation G is sufficiently large in the previous sense.

It is clear that the invariant splitting $W_b^+ = F \oplus G$ thus obtained has the property that the intersection of any Lyapunov foliation of α with G is subordinate to one of the factors. If the Lyapunov exponents of $\alpha_1(b)$ and $\alpha_2(b)$ are not the same then the sum of Lyapunov exponents of b corresponding to F is strictly less than the sum of all positive Lyapunov exponents of $\alpha_1(b)$ or $\alpha_2(b)$. Let us assume that it is less than the sum of all positive Lyapunov exponents of $\alpha_1(b)$. This sum is equal to $h_{\lambda_1}(\alpha_1(b))$, where λ_1 is the Lebesgue measure on $\mathbb{T}^{m_1}$.

Step 2.2: Atomicity of conditional measures. We will show that the conditional measures for the foliation G are atomic.

Restrict the action α to a generic 2-plane in $\mathbb{R}^k$ which contains b and intersects Lyapunov hyperplanes by generic lines. Let us consider the lines $L_1, \dots, L_l$ that correspond to the Lyapunov exponents positive on b. Then foliation $G \subset W_b^+$ splits into the sum of invariant subfoliations $G = \bigoplus W_i$ where $W_i \subset (W_{a_i}^I \cap W_b^+)$ and a_i is a generic singular element on L_i. We can reorder the lines $L_1, \dots, L_l$ and choose elements a_i in such a way that $(W_1 \oplus \dots \oplus W_{i-1}) \subset (W_{a_i}^- \cap W_b^+)$ for $i = 2, \dots, l$.

Consider the decomposition $W_i = \sum W_i^\lambda$ into its intersections $W_i^\lambda = W_i \cap W^\lambda$ with the Lyapunov foliations. We note that, by the construction of G, each W_i^λ is subordinate to one of the factors hence the conditional measures on each W_i^λ are atomic by Lemma 4.1.

If W_i is subordinate to one of the factors then the ergodicity assumption of Lemma 4.2 is fulfilled and this lemma shows that the conditional measures on W_i are supported on smooth submanifolds of the leaves. If W_i contains Lyapunov foliations from both factors then we see that there are no Lyapunov exponents of the whole action α which are negatively proportional to the exponents corresponding to W_i. In this case the ergodicity assumption of Lemma 4.2 follows from (the simpler version of) the argument in the proof of Lemma 4.1 and we also obtain that the conditional measures on W_i are supported on smooth submanifolds of the leaves. Then Lemma 4.6 shows that the conditional measures on the foliation W_i are atomic. Once we know that the conditional measures on W_i, $i = 1, \dots, l$, are atomic we apply Lemma 4.7 inductively to conclude that the conditional measures on $G = W_1 \oplus \cdots \oplus W_l$ are atomic.

Thus to complete the proof of the atomicity of the conditional measures for the foliation G it remains to prove Lemma 4.6 and Lemma 4.7. Note that we can use the inverse of element b from Step 2.1 as the element b required in Lemmas 4.6 and 4.7.

LEMMA 4.6. *Let F be an invariant subfoliation of foliation $W_a^I \cap W_b^-$ for some element b and let $F = \sum_\lambda (F \cap W^\lambda)$ be the splitting into intersections with Lyapunov foliations. Assume that the conditional measures on all foliations $F \cap W^\lambda$ are atomic and that the support S_x of measure μ_x^F is a smooth submanifold of $F(x)$. Then the conditional measures on F are also atomic.*

The proof of this lemma follows the proof of Lemma 5.8 in [**6**] and we include it for the sake of completeness.

Proof. The support S_x of measure μ_x^F is a smooth submanifold which intersects every $F \cap W^\lambda$ in, at most, one point. Let λ be the smallest Lyapunov exponent on b. Let D be the distribution of the tangent spaces of S_x. It is measurable, b-invariant and C^∞ on $F(x)$. Since D cannot intersect the component in the W_λ-direction in a subspace of positive dimension, D must be tangent to the sum $\sum_{\mu \neq \lambda} W_\mu$ by b-invariance. By taking the Lyapunov exponents inductively in increasing order, we see that D is trivial and μ_x^F are atomic. □

LEMMA 4.7. *Let W be an invariant subfoliation of W_a^I and F be an invariant subfoliation of W_a^- for some element a. Suppose that $F \oplus W \subset W_b^-$ for some element b and that the conditional measures of μ on both foliations F and W are atomic. Then the conditional measures on the foliation $F \oplus W$ are also atomic.*

This lemma is very similar to Lemma 5.10 in [**6**] and the proof employs the idea of the proof of Lemma 5.10. While the latter lemma is correct its proof in [**6**] lacks some essential details. We give a new argument to complete the proof.

Proof. We prove the lemma inductively by adding one-dimensional subfoliations of W to the foliation F one by one until we exhaust the whole $F \oplus W$. Then the next lemma which contains the main technical argument shows that on each step of this process we obtain a foliation with atomic conditional measures. We note that W splits into one-dimensional invariant foliations corresponding to the real eigenvalues of $\alpha(a)$ and $\alpha(b)$ and two-dimensional invariant foliations corresponding to pairs of complex eigenvalues of $\alpha(a)$ and $\alpha(b)$. In order to add a two-dimensional foliation we consider the proper multiples $ta, sb \in \mathbb{R}^k$ of elements a and b for which the corresponding eigenvalues of

$\alpha(ta)$ and $\alpha(sb)$ are real. Then the two-dimensional foliation splits into a sum of one-dimensional subfoliations invariant with respect to $\alpha(ta)$ and $\alpha(sb)$. Then these one-dimensional foliations can be added using the next lemma with elements ta and sb in place of a and b.

LEMMA 4.8. *Let F_1 and F_2 be invariant subfoliations of W_a^I and F be an invariant subfoliation of W_a^- for some element a. Suppose that F_2 is one-dimensional, $(F \oplus F_1 \oplus F_2) \subset W_b^-$ for some element b and that the conditional measures of μ on both foliations $F_1 \oplus F_2$ and $F \oplus F_1$ are atomic. Then the conditional measures on the foliation $F \oplus F_1 \oplus F_2$ are also atomic.*

Proof of Lemma 4.8. Since the conditional measures on the foliations $F_1 \oplus F_2$ and $F \oplus F_1$ are atomic we can find a set of full measure and its compact subset K of measure at least 0.99 which intersect any leaf of the foliations $F_1 \oplus F_2$ and $F \oplus F_1$ in at most one point.

Consider a measurable partition, subordinate to $F \oplus F_1 \oplus F_2$, which consists 'mainly' of small 'rectangles' of the same size with sides parallel to the foliations F, F_1 and F_2 and has the following property:

The measure of the set Int_γ *is at least* 0.99 *for some $\gamma > 0$, where* Int_γ *consists of points inside rectangles on the distance at least γ from the relative (to the leaf of $F \oplus F_1 \oplus F_2$) boundary of the rectangle that contains the point.*

We would like to consider only the 'good' part of the measure μ, so we introduce a new measure μ_X by $\mu_X(\cdot) = \mu(\cdot \cap X)$, where $X = K \cap \mathrm{Int}_\gamma$ with $\mu(X) \geq 0.98$. Let us consider a system of conditional measures of μ_X with respect to the measurable partition into the rectangles (the remaining elements have μ_X measure 0). These measures will be referred to as the conditional measures of the rectangles. We regard each rectangle as a direct product of its F, F_1 and F_2 directions.

We observe the following dichotomy:

(1) For every rectangle, in a set of positive measure, at least one-third of its conditional measure is concentrated on a single $F \oplus F_1$ leaf, hence at one point; or

(2) there exists a set Y with $\mu_X(Y) > 0.97$ which consists of whole rectangles and a number $d > 0$ such that for any rectangle in Y any subrectangle that carries at least one-third of the conditional measure has width at least d along the F_2 direction.

In the first case the existence of atoms for the conditional measures of the rectangles implies the existence of atoms for the conditional measures on the foliation $F \oplus F_1 \oplus F_2$. Since this foliation is contracted by b we can see that the existence of atoms forces the conditional measures on $F \oplus F_1 \oplus F_2$ to be atomic.

In the latter case each rectangle in Y can be split into three subrectangles of width at least d along the one-dimensional F_2 direction so that both the left and the right ones have a conditional measure of at least one-third (we may assume that the conditional measures do not have atoms since otherwise we could argue as in the first case).

Now we regard the intersection of the compact set K with any rectangle as a graph of a continuous (not necessarily everywhere defined) function from the F direction to the $F_1 \oplus F_2$ direction. The family of these functions is equi-continuous. We would like to show that this equi-continuity is in contradiction to recurrence under the action of a properly chosen element.

Since b contracts $F \oplus F_1 \oplus F_2$ we can find a sufficiently large number n such that the size of the image of any rectangle under the action of b^n is γ-small, i.e. has a diameter less than γ. The distance along the F_2 direction between the images of the right and left subrectangles, which was at least d, becomes at least d'. Let us fix some $\epsilon < d'$ and consider $\delta > 0$ given by the equi-continuity. Since a acts isometrically on $F_1 \oplus F_2$ and contracts F we can choose k so large that the image of any rectangle under $a^k b^n$ is δ-small in the F direction. We see that the element $a^k b^n$ satisfies the following conditions:

(1) the image of any rectangle is δ-narrow in the F direction;
(2) the distance along the F_2 direction between the images of the right and the left subrectangles is at least ϵ;
(3) the diameter of the image of any rectangle is less than γ, hence it cannot intersect the γ-interiors of two different rectangles simultaneously.

Under these conditions the images of the right and left subrectangles cannot intersect $Z = X \cap Y$ simultaneously. But this implies that $\mu_X(a^k b^n(Z) \cap Z) \leq \frac{2}{3}\mu_X(Z)$ which is impossible since $\mu(Z) = \mu_X(Z) \geq 0.95$. This shows that the second alternative of the dichotomy is impossible and proves that the conditional measures on $F \oplus F_1 \oplus F_2$ are atomic. □

Thus the proof of Lemma 4.7 is complete. □

4.4. *Equality of exponents.* We will now complete the proof of equality of Lyapunov exponents for actions α_1 and α_2.

Step 3.1: The entropy lemma.

LEMMA 4.9. *Let F, G and $H = F \oplus G$ be invariant subfoliations of W_b^+ for some element b. Suppose that the conditional measures of μ on G are atomic. Then $h_\mu^H(b) \leq \chi_F$, where χ_F is the sum of the Lyapunov exponents (counted with multiplicities) that correspond to the foliation F.*

Proof. Since the conditional measures on the foliation G are atomic there exist a set of full measure and its compact subset K of measure at least $1 - \epsilon$ which intersects any leaf of the foliation G in at most one point.

We would like to construct a partition of M convenient for calculating the entropy along the foliation H. We start with a coarse measurable partition of M, subordinate to H, which consists mainly of identical parallelepipeds with edges of length ρ and sides parallel to the foliations F and G. These parallelepipeds will be referred to as cubes. The remaining elements of the partition can be chosen to be polytopes with diameters not more than $C_1\rho$ and containing balls of radius at least $c_2\rho$. For ρ sufficiently small this partition can be chosen in such a way that the total measure of the remaining elements and σ-neighbourhoods of the boundaries of the cubes is small. More precisely we assume that $\mu(\mathrm{Int}_\sigma) > 1 - \epsilon$ for some $0 < \sigma < \rho$, where Int_σ is the union of all relative σ-interiors of the cubes. Then $\mu(K_\sigma) > 1 - 2\epsilon$, where $K_\sigma = K \cap \mathrm{Int}_\sigma$.

Let us fix a large m and take $d > 0$ satisfying $2e^{m\lambda}d < \sigma$, where λ denotes the maximal Lyapunov exponent of b corresponding to H. The intersection of K with any cube is a graph of a continuous (not necessarily everywhere defined) function from the F direction to the G direction; moreover, the family of these functions is equi-continuous. So for the given $d > 0$, we can take $\delta > 0$ such that if $x, y \in K$ lie in one cube with a distance along the F direction which is less than δ then the distance along the G direction is less than d. The F direction of each cube Q can be partitioned into identical cubes Q_i of diameter δ. Then for each Q_i there exists a cube $Q_i^{j_0}$ along the G direction of diameter at most $C_3 d$ containing a ball of radius d such that the set $Q \cap K_\sigma$ is covered by rectangles $R_i = Q_i \oplus Q_i^{j_0}$. The remaining part of each Q can be covered by rectangles $R_i^j = Q_i \oplus Q_i^j$, where Q_i^j has a diameter of at most $C_3 d$ and contains a ball of radius $d/2$. We see that all these rectangles satisfy the following properties: $\operatorname{diam}(R_i^j) \le C_3 d$ and $\operatorname{Vol}(R_i^j) \ge c_4 \delta^{\dim F} d^{\dim G}$.

If d and c_4 are chosen small enough all remaining elements of the initial coarse partition can be partitioned into sets satisfying the same properties. We denote the constructed partition by $\xi = \{C_\alpha\}$. Since d and δ can be chosen to be as small as we wish we can assume that $h_\mu(b^m, \xi) > h_\mu^H(b^m) - \epsilon$. Let us introduce the notation $\phi(x) = -x \log x$, $b^m \xi = \{D_\beta\}$ and let μ_β be the conditional measure on D_β. Then we obtain

$$\begin{aligned} m h_\mu^H(b) - \epsilon &= h_\mu^H(b^m) - \epsilon \le h(\xi, b^m \xi) \\ &= \int h(\xi, D_\beta)\, d\mu(\beta) \\ &= \int \sum_{\alpha : C_\alpha \cap K_\sigma \ne \emptyset} \phi(\mu_\beta(C_\alpha))\, d\mu(\beta) + \int \sum_{\alpha : C_\alpha \cap K_\sigma = \emptyset} \phi(\mu_\beta(C_\alpha))\, d\mu(\beta). \end{aligned}$$

Since $\operatorname{diam}(D_\beta) \le e^{m\lambda} 2d < \sigma$ the intersection $D_\beta \cap C_\alpha \cap K_\sigma$ can be non-empty only for C_α inside one cube of the coarse partition. The number of these C_αs can be estimated using the volume in the F direction.

The F cross sections of D_β have volume $\operatorname{Vol}_F(D_\beta) \le C_5 e^{m\chi_F} \delta^{\dim F}$ and the F cross sections of C_α have volume $\operatorname{Vol}_F(C_\alpha) \ge c_6 \delta^{\dim F}$. So for the given D_β there are at most $C_7 e^{m\chi_F}$ elements C_α that intersect $D_\beta \cap K_\sigma$. This implies that

$$\begin{aligned} \int \sum_{\alpha : C_\alpha \cap K_\sigma \ne \emptyset} \phi(\mu_\beta(C_\alpha))\, d\mu(\beta) &\le \int \log \operatorname{card}\{\alpha : C_\alpha \cap D_\beta \cap K_\sigma \ne \emptyset\}\, d\mu(\beta) \\ &\le \log C_7 + m\chi_F. \end{aligned}$$

Since $\operatorname{Vol}(D_\beta) \le C_8 e^{m\lambda \dim H} \delta^{\dim F} d^{\dim G}$ and $\operatorname{Vol}(C_\alpha) \ge c_4 \delta^{\dim F} d^{\dim G}$ for any α, the number of elements C_α that intersect D_β is bounded above by $C_9 e^{m\lambda \dim H}$.

Let $r(\beta) = \mu_\beta(D_\beta \setminus K_\sigma)$. Then since

$$-\left(\frac{1}{N}\sum_{i=1}^N x_i\right) \log\left(\frac{1}{N}\sum_{i=1}^N x_i\right) = \phi\left(\frac{1}{N}\sum_{i=1}^N x_i\right) \ge \frac{1}{N}\sum_{i=1}^N \phi(x_i)$$

by taking $N = \log \operatorname{card}\{\alpha : C_\alpha \cap D_\beta \cap K_\sigma = \emptyset\}$ we obtain

$$\begin{aligned}\int \sum_{\alpha: C_\alpha \cap K_\sigma = \emptyset} & \phi(\mu_\beta(C_\alpha))\, d\mu(\beta) \\ & \leq \int r(\beta)(\log \operatorname{card}\{\alpha : C_\alpha \cap D_\beta \cap K_\sigma = \emptyset\} - \log r(\beta))\, d\mu(\beta) \\ & \leq \mu(M \setminus K_\sigma)(\log C_9 + m\lambda \dim H) - \int r(\beta) \log r(\beta)\, d\mu(\beta) \\ & \leq 2\epsilon(\log C_9 + m\lambda \dim H) + 1.\end{aligned}$$

Finally we obtain

$$mh_\mu^H(b) - \epsilon \leq \log C_7 + m\chi_F + 2\epsilon(\log C_9 + m\lambda \dim H) + 1.$$

Since C_7 and C_9 do not depend on m, taking the limit as $m \to \infty$ we obtain

$$h_\mu^H(b) \leq \chi_F + 2\epsilon\lambda \dim H.$$

Since ϵ is arbitrary the lemma follows. □

Step 3.2: Excluding non-equal exponents. We proved in §4.3 that the conditional measures on G are atomic. If the Lyapunov exponents are not identical to the multiplicities then Lemma 4.9 shows that the sum of the Lyapunov exponents of b corresponding to F gives the upper estimate for $h_\mu(\alpha(b))$ so that we obtain $h_\mu(\alpha(b)) < h_{\lambda_1}(\alpha_1(b))$. Since α_1 is a factor of α this is impossible and we conclude that the Lyapunov exponents of α_1 and α_2 must be identical.

4.5. *Conjugacy over $\mathbb{Q}$ and the structure of the joining measure.* In this section we complete the proof of the Main Theorem. We show that $\mu_{\mathbb{T}^m}$ has a specific structure and the restrictions of α_1 and α_2 to some finite index subgroup $\Gamma \subset \mathbb{Z}^k$ are conjugate over $\mathbb{Q}$.

Step 4.1: Lyapunov conditionals are non-atomic. We consider a Lyapunov foliation W for α. Since the Lyapunov exponents of α_1 and α_2 are the same, W splits into the direct sum of two invariant subfoliations of equal dimensions corresponding to the factors. We note that the conditional measures on W cannot be atomic. The easiest way to prove this is to assume the opposite. Then the partition into complete fibres of W is measurable with respect to the joining measure and hence so are its projections onto the factors. This is a contradiction since, for the factors, the conditional measures are Lebesgue measures and the partitions are not measurable.

Alternatively, one can take an Anosov element b for which W is unstable. Then we can split the whole unstable foliation of b as $W_b^+ = F \oplus G$, where G contains W and the invariant subfoliations corresponding to the first factor of all other Lyapunov foliations which are unstable for b. If the conditional measures on W were atomic we could show, as in Section 4.3, that the conditional measures on G are also atomic. Then we could apply Lemma 4.9 to conclude that the entropy of b with respect to the joining measure is less than the entropy of b in the second factor, which is impossible.

Step 4.2: Algebraicity of Lyapunov conditionals. Since we already know that there are no negatively proportional Lyapunov exponents for the product action we can use (a simpler version of) the argument in the proof of Lemma 4.1 to verify the assumption of Lemma 4.2. Then we can apply Lemmas 4.2, 4.3 and 4.4 to describe the conditional measures μ_x^W for the foliation W. We obtain that for μ-a.e. x, μ_x^W is a Haar measure on an affine subspace S_x of the leaf $W(x)$. We have just proved that the conditional measures on W are not atomic. Since $\dim S_x$ is an invariant function and the joining measure is ergodic we conclude that for μ-a.e. x, μ_x^W is Haar on an affine subspace S_x with $\dim S_x = l \geq 1$. We can now apply Lemma 4.5. We conclude that the joining measure decomposes as $\mu_{\mathbb{T}^m} = (1/N)(\mu_1 + \cdots + \mu_N)$. Each μ_i, $i = 1, \ldots, N$, is invariant under the restriction of α to a certain finite index subgroup $\Gamma \subset \mathbb{Z}^k$. Each μ_i is also invariant under the group of translations in the direction of a rational torus $T_i \subset \mathbb{T}^m$. The tori T_i, $i = 1, \ldots, N$, have the same dimension and are invariant under $\alpha(\Gamma)$. Let us consider T_1.

Step 4.3: Equality with the dimension of factors and rational conjugacy. We claim that $\dim T_1 = m_1 = m_2$. If $\dim T_1 > m_1$ then the intersection of T with each factor has a positive dimension. Since the action in the factors is strongly irreducible we see that T_1 must contain each factor, i.e. $T_1 = \mathbb{T}^m$. Thus the invariance with respect to translations in the direction of T_1 implies that μ_1 is Lebesgue on $\mathbb{T}^m$. Since the dimensions of the tori T_i are the same we conclude that all measures μ_i, $i = 1, \ldots, N$ are Lebesgue. This implies that $\mu_{\mathbb{T}^m}$ is Lebesgue on $\mathbb{T}^m$ which contradicts the assumption that the joining is not trivial. Thus we have proved that $\dim T_1 \leq m_1$. On the other hand, since the projections of T_1 onto the factors are rational subtori invariant under $\alpha(\Gamma)$ their dimensions must be either zero or m_1 by strong irreducibility. Since $\dim T_1$ is positive we conclude that $\dim T_1 = m_1 = m_2$.

The rational subtorus T_1 lifts to a rational subspace in $\mathbb{R}^m$ of dimension $m_1 = m_2$ which is invariant for the restriction of the action $\alpha_1 \times \alpha_2$ to Γ. Hence this subspace is a graph of a conjugacy over $\mathbb{Q}$ between actions α_1 and α_2 restricted to Γ.

Step 4.4: The structure of joinings. The existence of T_1 also produces an algebraic factor on a torus of dimension $m_1 = m_2$ for the restriction of $(\alpha_1 \times \alpha_2, \mu_{\mathbb{T}^m})$ to Γ. We note that the conditional measures of μ_1 along the fibres are Haar. Let us denote this factor by α_3. It is easy to see that α_3 is algebraically isomorphic to finite factors for the restrictions of α_1 and α_2 to Γ. Thus α_3 is also strongly irreducible and a TNS action of $\mathbb{Z}^k$. Hence the invariant measure for α_3 either is Lebesgue or has zero entropy with respect to all elements of the action α_3. We note that the former case implies that the joining was trivial. We conclude that μ_1 is an extension of a zero-entropy measure in this algebraic factor with Haar measures in fibres. The other measures μ_i have the same structure since the actions $(\alpha_1 \times \alpha_2|_\Gamma, \mu_i)$, $i = 1, \ldots, N$ are algebraically isomorphic.

This competes the proof of Theorem 3.1. □

5. *Open problems and comments*

5.1. *Actions by toral automorphisms.* As we mentioned earlier the natural problem of describing all joinings, even self-joinings, is very difficult since it includes the description of all invariant measures for the actions. However, there are several natural extensions of the results of the present paper which look more feasible.

The rigidity of isomorphisms, centralizers and factors holds for a rather broad class of actions by toral automorphisms satisfying the following 'genuine higher-rank' assumption [**4**].

($\mathcal{R}$)*: The action* α *contains a group, isomorphic to* $\mathbb{Z}^2$*, which consists of ergodic automorphisms.*

Naturally, $\mathbb{Z}^k$ actions satisfying condition ($\mathcal{R}$) may be reducible and in fact may have factors which reduce to actions of rank lower than k, but still greater than one. The most general 'rigidity of joinings' conjecture which avoids description of all joinings may be formulated as follows.

CONJECTURE 5.1. *If two* $\mathbb{Z}^k$ *actions by toral automorphisms satisfying condition* ($\mathcal{R}$) *have a non-trivial joining then there are subgroups of the finite index in* $\mathbb{Z}^k$ *whose actions have algebraically isomorphic factors which reduce to* $\mathbb{Z}^l$ *actions satisfying condition* ($\mathcal{R}$) *for some* l*,* $2 \le l \le k$*.*

In the irreducible case the only algebraic factors are those with finite fibres. Furthermore, it is possible that no reduction to a subgroup of finite index would be necessary. Thus for irreducible actions the rigidity of the joining conjecture takes the following form.

CONJECTURE 5.2. *If two irreducible* $\mathbb{Z}^k$ *actions by toral automorphisms satisfying condition* ($\mathcal{R}$) *have a non-trivial joining then they are isomorphic over* $\mathbb{Q}$*, or, equivalently, each is algebraically isomorphic to a finite factor of the other.*

There are three difficulties on the way from our Theorem 3.1 to the proof of Conjecture 5.2:
(1) the presence of zero exponents;
(2) the presence of negatively proportional exponents; and
(3) the possibility of irreducible but not strongly irreducible actions.

Allowing zero exponents and keeping the rest of the assumptions of Theorem 3.1 allows the part of the proof that gives the equality of multiplicities of the non-zero Lyapunov exponents to be carried out. The equality of multiplicities of the zero exponent does not follow.

The TNS condition, which does not hold if some non-zero exponents are negatively proportional, is used to check the ergodicity assumption in Lemma 4.2. While this condition looks technical it is central for initiating the whole machinery of conditional measures. The hope here is that for special invariant measures of products which correspond to joinings this assumption or some substitute could be derived.

5.2. *Other algebraic actions of higher-rank Abelian groups.* Actions by toral automorphisms form only one of several classes of actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$, $k \ge 2$, which exhibit various rigidity phenomena. These phenomena extend, on the one hand, to certain classes of Anosov and partially hyperbolic homogeneous and affine actions on coset and double coset spaces of Lie groups (see e.g. [**8**]); and, on the other hand, to actions by automorphisms of compact Abelian groups which are more general than the torus [**10**].

A natural question is to ask about the existence of non-trivial joinings among actions in these classes as well as between actions from different classes. The techniques of this paper can be adapted and extended to obtain various results in these directions although our current understanding of the situation is far from definitive. These questions will be treated in a subsequent paper.

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