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# MEASURE RIGIDITY BEYOND UNIFORM HYPERBOLICITY: INVARIANT MEASURES FOR CARTAN ACTIONS ON TORI

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ABSTRACT. We prove that every smooth action $\alpha$ of $\mathbb{Z}^k$, $k \geq 2$, on the $(k+1)$-dimensional torus whose elements are homotopic to corresponding elements of an action $\alpha_0$ by hyperbolic linear maps preserves an absolutely continuous measure. This is the first known result concerning abelian groups of diffeomorphisms where existence of an invariant geometric structure is obtained from homotopy data.

We also show that both ergodic and geometric properties of such a measure are very close to the corresponding properties of the Lebesgue measure with respect to the linear action $\alpha_0$.

## 1. INTRODUCTION

1.1. **Measure rigidity and hyperbolicity.** It is well-known that in classical dynamical systems, i.e. smooth actions of $\mathbb{Z}$ or $\mathbb{R}$, nontrivial recurrence combined with some kind of hyperbolic behavior produces a rich variety of invariant measures (see for example, [KH] and [KM] for the uniformly and nonuniformly hyperbolic situations correspondingly). On the other hand, invariant measures for actions of *higher rank* abelian groups tend to be scarce. This was first noticed by Furstenberg [F] who posed the still open problem of describing all ergodic measures on the circle invariant with respect to multiplications by 2 and by 3. Great progress has been made in characterizing invariant measures with positive entropy for algebraic actions of higher rank abelian groups; for the measure rigidity results for actions by automorphisms or endomorphisms of a torus see [R, KS1, KS2, KaK1, KaK2, KaSp, EL].

For background on algebraic, arithmetic, and ergodic properties of $\mathbb{Z}^k$ actions by automorphisms of the torus we refer to [KKS]. Recall that an action of $\mathbb{Z}^k$ on $\mathbb{T}^{k+1}$, $k \geq 2$, by automorphisms which are ergodic with respect to the Lebesgue measure is called a (linear) *Cartan action*. Every element of a Cartan action other

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than the identity is hyperbolic and has distinct real eigenvalues, and the centralizer of a Cartan action in the groups of automorphisms of the torus is a finite extension of the action itself ([KKS, Section 4.1]).

A geometric approach to measure rigidity was introduced in [KS1]. It is based on the study of conditional measures on various invariant foliations for the action. Broadly speaking, there are three essential tools or methods within this approach which we list in order of their chronological appearance:

1. The geometry of Lyapunov exponents and derivative objects, in particular Weyl chambers [KS1, KS2, KaK1, KaSp].[1]
2. The noncommutativity and specific commutation relations between various invariant foliations [EK1, EK2, EKL]. [2]
3. Diophantine properties of global recurrence [EL].

In this paper we make the first step in extending measure rigidity from algebraic actions to the general nonuniformly hyperbolic case, i.e. to positive entropy ergodic invariant measures for actions of higher rank abelian groups all of whose Lyapunov characteristic exponents do not vanish. Such measures are usually called *hyperbolic measures*. The theory of hyperbolic measures for smooth actions of higher rank abelian groups is described in Part II of [KaK1]. In Sections 2.1 and 2.2 we briefly mention key elements of that theory relevant for the specific situation considered in this paper.

In this paper we use a counterpart of the method (1) above. We will discuss the scope of this method, difficulties which appear for its extensions, and applications of properly modified versions of other methods to various nonuniformly hyperbolic situations in a subsequent paper.

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### 1.2. **Formulation of results.**

**THEOREM 1.1.** *Any action $\alpha$ of $\mathbb{Z}^k$, $k \geq 2$, by $C^{1+\epsilon}$, $\epsilon > 0$, diffeomorphisms of $\mathbb{T}^{k+1}$, whose elements are homotopic to those of a linear Cartan action $\alpha_0$, has an ergodic absolutely continuous invariant measure.*

The connection between invariant measures of $\alpha$ and those of $\alpha_0$ is established using the following well-known result whose proof we include for the sake of completeness.

**LEMMA 1.2.** *There is a unique surjective continuous map $h\colon \mathbb{T}^{k+1} \to \mathbb{T}^{k+1}$ homotopic to the identity such that $h \circ \alpha = \alpha_0 \circ h$.*

[1] See in particular [KaK1, Section 2.2] for a down-to-earth proof of rigidity of positive-entropy invariant measures for linear Cartan actions. A reader unfamiliar with measure rigidity may look at that section first to get an idea of the basic arguments we generalize in the present paper.

[2] This method was first outlined at the end of [KS1]; note that it is not relevant for the actions on the torus since in this case all foliations commute.

*Proof.* Consider an element $\mathbf{m} \in \mathbb{Z}^k \setminus \{0\}$. By a theorem of Franks ([KH, Theorem 2.6.1]), there exists a unique continuous map $h\colon \mathbb{T}^{k+1} \to \mathbb{T}^{k+1}$ that is homotopic to the identity and satisfies

$$h \circ \alpha(\mathbf{m}) = \alpha_0(\mathbf{m}) \circ h. \tag{1.1}$$

For any other element $\mathbf{m}' \in \mathbb{Z}^k$ consider the map

$$h' = \alpha_0(-\mathbf{m}') \circ h \circ \alpha(\mathbf{m}') \tag{1.2}$$

Using commutativity of both actions $\alpha$ and $\alpha_0$ as well as (1.1) we obtain

$$\begin{aligned} h' \circ \alpha(\mathbf{m}) &= \alpha_0(-\mathbf{m}') \circ h \circ \alpha(\mathbf{m}') \circ \alpha(\mathbf{m}) = \alpha_0(-\mathbf{m}') \circ h \circ \alpha(\mathbf{m}) \circ \alpha(\mathbf{m}') \\ &= \alpha_0(-\mathbf{m}') \circ \alpha_0(\mathbf{m}) \circ h \circ \alpha(\mathbf{m}') = \alpha_0(\mathbf{m}) \circ \alpha_0(-\mathbf{m}') \circ h \circ \alpha(\mathbf{m}') = \alpha_0(\mathbf{m}) \circ h', \end{aligned}$$

i.e. $h'$ satisfies (1.1). Since it is also homotopic to identity, the uniqueness of $h$ forces $h = h'$. Then (1.2) implies $h \circ \alpha(\mathbf{m}') = \alpha_0(\mathbf{m}') \circ h$, so $h$ intertwines the actions $\alpha$ and $\alpha_0$. □

Another way of stating Lemma 1.2 is that the algebraic action $\alpha_0$ is a topological factor of the action $\alpha$ or, equivalently, $\alpha$ is an extension of $\alpha_0$.

**REMARK 1.** If the action $\alpha$ is Anosov, i.e. if $\alpha(\mathbf{m})$ is an Anosov diffeomorphism for some $\mathbf{m}$, then the map $h$ is invertible, and both $h$ and $h^{-1}$ are Hölder [KH, Theorems 18.6.1 and 19.1.2]. This implies various rigidity results for $\mathbb{Z}^k$ Anosov actions on the torus.

- For example, if $\alpha_0$ is a linear $\mathbb{Z}^k$ action on a torus which contains a $\mathbb{Z}^2$ subaction all of whose elements other than identity are ergodic, then any Anosov action $\alpha$ homotopic to $\alpha_0$ preserves a smooth measure. This follows from rigidity of Hölder cocycles over $\alpha_0$ and hence over $\alpha$ applied to the logarithm of the Jacobian for $\alpha$.
- For those cases when the positive-entropy ergodic invariant measure for $\alpha_0$ is unique [EL] the same is true for $\alpha$.

Consider the set of all Borel probability measures $\nu$ on $\mathbb{T}^{k+1}$ such that $(h)_*\nu = \lambda$, where $\lambda$ is Lebesgue measure on $\mathbb{T}^{k+1}$. This set is convex, weak* compact, and $\alpha$-invariant. Hence by Tychonoff theorem it contains a nonempty subset $\mathscr{M}$ of measures invariant under $\alpha$. Since $\alpha_0$ is ergodic with respect to $\lambda$, almost every ergodic component of an $\alpha$-invariant measure $\nu \in \mathscr{M}$ also belongs to $\mathscr{M}$. Let $\mu$ be such an ergodic measure.

Theorem 1.1 follows immediately from Lemma 1.2 and the following theorem, which is the first principal technical result of the present paper.

**THEOREM 1.3.** *Any ergodic $\alpha$-invariant measure $\mu$ such that $(h)_*\mu = \lambda$, where $h$ is the semiconjugacy from Lemma 1.2, is absolutely continuous.*

Since any $\alpha$-invariant measure whose ergodic components are absolutely continuous is itself absolutely continuous we obtain the following

**COROLLARY 1.4.** *Every measure $\nu \in \mathcal{M}$ is absolutely continuous and has no more than countably many ergodic components. Hence $\mathcal{M}$ contains at most countably many ergodic measures.*

In fact a much stronger statement is true. It is proved in [K-RH] using results from the present paper.

**THEOREM.**

1. *The set $\mathcal{M}$ consists of a single measure.*
2. *The semiconjugacy $h$ is a measurable isomorphism between the actions $\alpha$ and $\alpha_0$.*

It is even possible that uniqueness follows from absolute continuity for a single diffeomorphism.

**CONJECTURE 1.5.** *Let $f$ be a $C^2$ diffeomorphism homotopic to a linear hyperbolic automorphism $f_0$ of a torus and let $h$ be the semiconjugacy. Then there is at most one absolutely continuous $f$-invariant measure $\mu$ such that $h_*(\mu) = \lambda$.*

Here we prove a slightly weaker version of part (2) of the above theorem for ergodic measures.

**THEOREM 1.6.** *For any ergodic measure $\mu \in \mathcal{M}$ the semiconjugacy $h$ is finite-to-one in the following sense. There is an $\alpha$-invariant set $A$ of full measure $\mu$ such that for $\lambda$-almost every $x \in \mathbb{T}^{k+1}$, $A \cap h^{-1}(\{x\})$ consists of equal number $s$ of points and the conditional measure induced by $\mu$ assigns every point in $A \cap h^{-1}(\{x\})$ equal measure $1/s$.*

Recall that the Lyapunov characteristic exponents of the linear action $\alpha_0$ are independent of an invariant measure and are equal to the logarithms of the absolute values of the eigenvalues. They all have multiplicity one and no two of them are proportional.

**THEOREM 1.7.** *The Lyapunov characteristic exponents of the action $\alpha$ with respect to any ergodic measure $\mu \in \mathcal{M}$ are equal to the Lyapunov characteristic exponents of the action $\alpha_0$.*

Either of the last two theorems immediately implies the following.

**COROLLARY 1.8.** *The entropy function of $\alpha$ with respect to any measure $\nu \in \mathcal{M}$ is the same as the entropy function of $\alpha_0$ with respect to Lebesgue measure, i.e. for any measure $\nu \in \mathcal{M}$ and any $\mathbf{m} \in \mathbb{Z}^k$ we have $\mathbf{h}_\nu(\alpha(\mathbf{m})) = \mathbf{h}_\lambda(\alpha_0(\mathbf{m}))$.*

Since every element of $\alpha_0$ other than the identity is Bernoulli with respect to the Lebesgue measure, Theorem 1.6 also implies that every element of $\alpha$ is Bernoulli up to a finite permutation.

**COROLLARY 1.9.** *There exist a partition of a set $A$ of full measure $\mu$ into finitely many sets $A_1, \dots, A_m$ of equal measure such that every element of $\alpha$ permutes these sets. Furthermore, there is a subgroup of finite index $\Gamma \subset \mathbb{Z}^k$ such that for any $\gamma \in \Gamma$*

*other than the identity* $\alpha(\gamma)A_i = A_i$, $i = 1,\dots,m$, *and the restriction of* $\alpha(\gamma)$ *to each set* $A_i$ *is Bernoulli.*

*In particular, if all non-identity elements of* $\alpha$ *are ergodic then they are Bernoulli.*

**REMARK 2.** Since by Lemma 2.3 the measure $\mu$ is hyperbolic, Corollary 1.9 follows directly from Theorem 1.3 and the classical result of Pesin [P] that an ergodic hyperbolic absolutely continuous invariant measure for a diffeomorphism is Bernoulli up to a finite permutation.

**REMARK 3.** Theorem 1.1 for $k = 2$ was announced in [KaK1] as Theorem 8.2. The proof in the present paper follows a path different from the one outlined in [KaK1]. There it is derived from Theorem 8.1 about hyperbolic invariant measures for $\mathbb{Z}^2$ actions on three-dimensional manifolds. A proof of the latter theorem (and its $n$-dimensional version) following the outline presented in [KaK1, Section 8.3] and some essential new ingredients will appear in [KKRH].

## 2. LYAPUNOV EXPONENTS, WEYL CHAMBERS, AND INVARIANT "FOLIATIONS" FOR $\alpha$

### 2.1. **Preliminaries.**

2.1.1. *Entropy.* Since $h_*\mu = \lambda$ the measure-theoretic entropy $\mathbf{h}_\mu$ satisfies

$$\mathbf{h}_\mu(\alpha(\mathbf{m})) \ge \mathbf{h}_\lambda(\alpha_0(\mathbf{m})) \ge \max_{1\le i\le k+1} |\log|\rho_i(\mathbf{m})||,$$

where $\rho_i(\mathbf{m})$, $i = 1,\dots,k+1$ are the eigenvalues of the matrix $\alpha_0(\mathbf{m})$.

Since every element of $\alpha_0$ other than identity is hyperbolic this implies in particular that

($\mathscr{E}$) *The entropies* $\mathbf{h}_\mu(\alpha(\mathbf{m}))$ *for all* $\mathbf{m} \in \mathbb{Z}^k \setminus \{0\}$ *are uniformly bounded away from zero.*

2.1.2. *Lyapunov exponents.* The linear functionals $\chi_i = \log|\rho_i|$, $i = 1,\dots,k+1$ on $\mathbb{Z}^k$ are the *Lyapunov characteristic exponents* of the linear action $\alpha_0$ which are independent of an invariant measure. See [KaK1, Section 1.2] for definitions and discussion of Lyapunov characteristic exponents, related notions (Lyapunov hyperplanes, Weyl chambers, *etc.*) and suspensions in this setting. We will use this material without further references.

The following property of linear Cartan actions will play an important role in our considerations, in particular in Section 3.3.

($\mathscr{C}$) *For every* $i \in \{1,\dots,k+1\}$ *there exists an element* $\mathbf{m} \in \mathbb{Z}^k$ *such that* $\chi_i(\mathbf{m}) < 0$ *and* $\chi_j(\mathbf{m}) > 0$ *for all* $j \ne i$. (The same inequalities hold for any other element $\mathbf{m}'$ in the Weyl chamber of $\mathbf{m}$.)

Corresponding notions in a general setting, which includes that of $\mathbb{Z}^k$ actions by measure preserving diffeomorphisms of smooth manifolds, are defined and discussed in Sections 5.1 and 5.2 of the same paper [KaK1]. We will also use those notions without special references.

Let $\tilde\chi_i$, $i = 1,\dots,k+1$, be the Lyapunov characteristic exponents of the action $\alpha$, listed with their multiplicities if necessary. We will eventually show that in our setting the exponents can be properly numbered so that $\tilde{\chi}_i = \chi_i$, $i = 1,\dots,k+1$ (see Section 4.3).

As the first step in this direction we will show in Section 2.3 that exponents for $\alpha$ can be numbered in such a way that they become *proportional* to $\chi_i$ with positive scalar coefficients.

2.1.3. *Suspensions.* Although the Lyapunov characteristic exponents for a $\mathbb{Z}^k$ action are defined as linear functionals on $\mathbb{Z}^k$, it seems natural to extend them to $\mathbb{R}^k$. For example, Lyapunov hyperplanes (the kernels of the functionals) may be irrational and hence "invisible" within $\mathbb{Z}^k$. It is natural to try to construct an $\mathbb{R}^k$ action for which the extensions of the exponents from $\mathbb{Z}^k$ will provide the nontrivial exponents.

This is given by the suspension construction which associates to a given $\mathbb{Z}^k$ action on a space $N$ an $\mathbb{R}^k$ action on a bundle over $\mathbb{T}^k$ with fiber $N$. The topological type of the suspension space depends only on the homotopy type of the $\mathbb{Z}^k$ action. In particular, the suspension spaces for $\alpha_0$ and $\alpha$ are homeomorphic. There is a natural correspondence between the invariant measures, Lyapunov exponents, Lyapunov distributions, stable and unstable manifolds, etc. for the original $\mathbb{Z}^k$ action and its suspension. Naturally, the suspension has $k$ additional Lyapunov exponents corresponding to the orbit directions which are identically equal to zero. In our setting, the semiconjugacy between $\alpha$ and $\alpha_0$ naturally extends to the suspension. The extended semiconjugacy is smooth along the suspension orbits and reduces to the original semiconjugacy in the fiber over the origin in $\mathbb{T}^k$.

At various stages of the subsequent arguments it will be more convenient to deal either with the original actions $\alpha$ and $\alpha_0$ on $\mathbb{T}^{k+1}$ or with their suspensions. So we will take a certain liberty with the notations and *will use the same notations for the corresponding objects*, i.e. $\alpha$ and $\alpha_0$ for the suspension actions, $\tilde{\chi}_i$ and $\chi_i$ for the Lyapunov exponents etc, modifying the notations when necessary, as in $\alpha(\mathbf{m})$ for $\mathbf{m} \in \mathbb{Z}^k$ and $\alpha(\mathbf{t})$ for $\mathbf{t} \in \mathbb{R}^k$.

2.2. **Pesin sets and invariant manifolds.** We will use the standard material on invariant manifolds corresponding to the negative and positive Lyapunov exponents (stable and unstable manifolds) for $C^{1+\epsilon}$ measure preserving diffeomorphisms of compact manifolds. See for example [BP, Chapter 4]. It is customary to use the words "distributions" and "foliations" in this setting although in fact we are dealing with measurable families of tangent spaces defined almost everywhere with respect to an invariant measure, and with measurable families of smooth manifolds, which coincide if they intersect and which fill a set of full measure.

We will denote by $\tilde{W}^-_{\alpha(\mathbf{m})}(x)$ and $\tilde{\mathcal{W}}^-_{\alpha(\mathbf{m})}(x)$ correspondingly the local and global stable manifolds for the diffeomorphism $\alpha(\mathbf{m})$ at a point $x$ that is regular with respect to this diffeomorphism. The global manifold is an immersed Euclidean space and is defined uniquely. Any local manifold is a $C^{1+\epsilon}$ embedded open disc in a Euclidean space. Its germ at $x$ is uniquely defined and for any two choices their intersection is an open neighborhood of the point $x$ in each of them. On a compact set of arbitrarily large measure, called a *Pesin set*, the local stable manifolds can be chosen of a uniform size and depending continuously in the $C^{1+\epsilon}$ topology.

The local and global unstable manifolds $\tilde{W}^{+}_{\alpha(\mathbf{m})}(x)$ and $\tilde{\mathscr{W}}^{+}_{\alpha(\mathbf{m})}(x)$ are defined as the stable manifolds for the inverse map $\alpha(-\mathbf{m})$.

Recall that the stable and unstable manifolds $\tilde{\mathscr{W}}^{-}_{\alpha(\mathbf{m})}$ and $\tilde{\mathscr{W}}^{+}_{\alpha(\mathbf{m})}$ are tangent to the (almost everywhere defined) stable and unstable distributions $E^{-}_{\alpha(\mathbf{m})}$ and $E^{+}_{\alpha(\mathbf{m})}$ accordingly. These distributions are the sums of the distributions corresponding to the negative and positive Lyapunov exponents for $\alpha(\mathbf{m})$ respectively.

At the moment, we do not know the dimensions of those distributions. However, the following lemma shows that $\tilde{\mathscr{W}}^{-}_{\alpha(\mathbf{m})}$ and $\tilde{\mathscr{W}}^{+}_{\alpha(\mathbf{m})}$ are transverse for any regular element $\mathbf{m} \in \mathbb{Z}^k$. An element of $\mathbb{Z}^k$ or $\mathbb{R}^k$ is called *singular* if a nonzero Lyapunov exponent vanishes on it, i.e. the element belongs to a Lyapunov hyperplane. All other elements are called *regular*.

**LEMMA 2.1.** *All Lyapunov exponents of $\mu$ are nonzero, i.e. $\mu$ is a hyperbolic measure for $\alpha$. In particular, $\alpha(\mathbf{m})$ is nonunifomly hyperbolic on $\mathbb{T}^{k+1}$ for any regular element $\mathbf{m} \in \mathbb{Z}^k$.*

*Proof.* Suppose there is an identically zero Lyapunov exponent for $\alpha$. Then $\alpha$ has at most $k$ nonzero Lyapunov exponents. Intersecting the Lyapunov hyperplanes inductively one can easily see that there exists a line in $\mathbb{R}^k$ on which at least $k$ Lyapunov exponents vanish and thus at most one is nonzero. By the Ruelle inequality [KM] this implies that the entropy of the elements of the suspension along that line vanishes. Since there are elements of $\mathbb{Z}^k$ either on the line (if the line is rational) or arbitrary close to it (if it is irrational), there are nonzero elements $\mathbf{m} \in \mathbb{Z}^k$ such that the entropy $\mathbf{h}_{\mu}(\alpha(\mathbf{m}))$ is arbitrarily small. This however contradicts $(\mathscr{E})$. ☐

The corresponding stable and unstable manifolds for the linear action $\alpha_0$ will be denoted by the same symbols without the tilde. Of course those manifolds are affine, and they are defined everywhere, not just on large sets as for the nonlinear action $\alpha$.

### 2.3. Preservation of Weyl chambers under the semiconjugacy.

**LEMMA 2.2.** *For any element $\mathbf{m} \in \mathbb{Z}^k \setminus \{0\}$ the following inclusions hold*

$$h(\tilde{\mathscr{W}}^{-}_{\alpha(\mathbf{m})}(x)) \subset \mathscr{W}^{-}_{\alpha_0(\mathbf{m})}(h(x)) \quad \textit{and} \quad h(\tilde{\mathscr{W}}^{+}_{\alpha(\mathbf{m})}(x)) \subset \mathscr{W}^{+}_{\alpha_0(\mathbf{m})}(h(x)),$$
$$h(\tilde{W}^{-}_{\alpha(\mathbf{m})}(x)) \subset W^{-}_{\alpha_0(\mathbf{m})}(h(x)) \quad \textit{and} \quad h(\tilde{W}^{+}_{\alpha(\mathbf{m})}(x)) \subset W^{+}_{\alpha_0(\mathbf{m})}(h(x)).$$

*on the set of full measure $\mu$ where $\tilde{\mathscr{W}}^{-}_{\alpha(\mathbf{m})}(x)$ and $\tilde{\mathscr{W}}^{+}_{\alpha(\mathbf{m})}(x)$ exist.*

*Proof.* The global stable manifold $\tilde{\mathscr{W}}^{-}_{\alpha(\mathbf{m})}(x)$ is the set of all points $y \in \mathbb{T}^{k+1}$ for which $\mathrm{dist}(\alpha(n\mathbf{m})x, \alpha(n\mathbf{m})y) \to 0$ as $n \to \infty$. Since $h$ is continuous, this implies in turn that $\mathrm{dist}(\alpha_0(n\mathbf{m})h(x), \alpha_0(n\mathbf{m})h(y)) \to 0$ as $n \to \infty$, and hence $h(y)$ belongs to the stable manifold $\mathscr{W}^{-}_{\alpha_0(\mathbf{m})}(h(x))$.

Thus, if $y$ is in the local stable manifold $\tilde{W}^-_{\alpha(\mathbf{m})}(x)$ then $\operatorname{dist}(\alpha(n\mathbf{m})x, \alpha(n\mathbf{m})y)$ remains small for all $n > 0$. Since $h$ is continuous, $\operatorname{dist}(\alpha_0(n\mathbf{m})h(x), \alpha_0(n\mathbf{m})h(y))$ also remains small for all $n > 0$. Since $\alpha_0(\mathbf{m})$ is uniformly hyperbolic, this implies that $h(y)$ belongs to the local stable manifold $W^-_{\alpha_0(\mathbf{m})}(h(x))$.

The corresponding statements for unstable manifolds follow by taking inverses. □

**LEMMA 2.3.** *The Lyapunov half-spaces and Weyl chambers for $\alpha$ with respect to the measure $\mu$ are the same as the Lyapunov half-spaces and Weyl chambers for $\alpha_0$. Hence the Lyapunov exponents for $\alpha$ can be numbered $\tilde{\chi}_i$, $i = 1, \dots, k+1$ in such a way that $\tilde{\chi}_i = c_i \chi_i$, where $c_i$ is a positive scalar.*

*Proof.* Suppose that a Lyapunov hyperplane $L$ of $\alpha_0$ is not a Lyapunov hyperplane of $\alpha$. Then there exist $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^k$ which lie on the opposite sides of $L$ so that $\tilde{W}^-_{\alpha(\mathbf{m})} = \tilde{W}^-_{\alpha(\mathbf{n})}$ but $W^-_{\alpha_0(\mathbf{m})} \neq W^-_{\alpha_0(\mathbf{n})}$.

Let $\Lambda$ be the intersection of a Pesin set for $\alpha(\mathbf{m})$ with a Pesin set for $\alpha(\mathbf{n})$. Consider a point $x \in \Lambda$ such that any open neighborhood of $x$ intersects $\Lambda$ by a set of positive measure $\mu$. By the previous lemma we have

$$h(\tilde{W}^-_{\alpha(\mathbf{m})}(x)) = h(\tilde{W}^-_{\alpha(\mathbf{n})}(x)) \subset (W^-_{\alpha_0(\mathbf{m})}(h(x)) \cap W^-_{\alpha_0(\mathbf{n})}(h(x)))$$

and

$$h(\tilde{W}^+_{\alpha(\mathbf{m})}(x)) \subset W^+_{\alpha_0(\mathbf{m})}(h(x)).$$

Let $R$ be the intersection of $\Lambda$ with a neighborhood of $x$ sufficiently small compared to the size of the local manifolds at points of $\Lambda$. Then since $\mu$ is a hyperbolic measure by Lemma 2.1, for any point $y \in R$ the intersection $\tilde{W}^+_{\alpha(\mathbf{m})}(x) \cap \tilde{W}^-_{\alpha(\mathbf{m})}(y)$ consists of a single point $z_1$. Similarly, $\tilde{W}^-_{\alpha(\mathbf{m})}(x) \cap \tilde{W}^+_{\alpha(\mathbf{m})}(y) = \{z_2\}$ and hence $\tilde{W}^-_{\alpha(\mathbf{m})}(z_1) \cap \tilde{W}^+_{\alpha(\mathbf{m})}(z_2) = \{y\}$. By the previous lemma the latter implies that $W^-_{\alpha_0(\mathbf{m})}(h(z_1)) \cap W^+_{\alpha_0(\mathbf{m})}(h(z_2)) = \{h(y)\}$. Using the inclusions above we see that the image $h(R)$ is in the direct product $V = (W^-_{\alpha_0(\mathbf{m})}(h(x)) \cap W^-_{\alpha_0(\mathbf{n})}(h(x))) \times W^+_{\alpha_0(\mathbf{m})}(h(x))$. Since $W^-_{\alpha_0(\mathbf{m})} \neq W^-_{\alpha_0(\mathbf{n})}$, we conclude that $V$ is contained in a subspace of dimension at most $k$. Hence $\lambda(V) = 0$ which contradicts the fact that $\lambda(h(R)) \geq \mu(R) > 0$. We conclude that any Lyapunov hyperplane of $\alpha_0$ is also a Lyapunov hyperplane of $\alpha$. Recall that $\alpha_0$ is Cartan and thus has the maximal possible number, $k+1$, of Lyapunov hyperplanes. Hence $\alpha$ also has exactly $k+1$ distinct Lyapunov hyperplanes, which coincide with the Lyapunov hyperplanes of $\alpha_0$. In particular, all Lyapunov exponents of $\alpha$ do not vanish.

It follows that for either action there is exactly one Lyapunov exponent that corresponds to a given Lyapunov hyperplane. It remains to check that for every Lyapunov hyperplane $L$ the corresponding Lyapunov exponents of $\alpha$ and $\alpha_0$ are *positively* proportional. Suppose that for some $L$ the corresponding Lyapunov exponents are negatively proportional. Let $W$ be the corresponding Lyapunov foliation for $\alpha_0$. We can take $\mathbf{m}$ close to $L$ in the negative half-space of the corresponding Lyapunov exponent for $\alpha$ and $\mathbf{n}$ sufficiently close to $\mathbf{m}$ across $L$ in the negative half-space for $\alpha_0$, so that $\mathbf{m}$ and $\mathbf{n}$ are not separated from $L$ by any other Lyapunov hyperplane. Then we observe that $\tilde{W}^+_{\alpha(\mathbf{m})} \subset \tilde{W}^+_{\alpha(\mathbf{n})}$ and that $W$ is contained neither in $W^-_{\alpha_0(\mathbf{m})}$ nor in $W^+_{\alpha_0(\mathbf{n})}$.

We choose $\Lambda$, $x$, and $R$ as above. Using Lemma 2.2 we obtain

$$h(\tilde{W}^-_{\alpha(\mathbf{m})}(x)) \subset W^-_{\alpha_0(\mathbf{m})}(h(x)) \quad \text{and} \quad h(\tilde{W}^+_{\alpha(\mathbf{m})}(x)) \subset h(\tilde{W}^+_{\alpha(\mathbf{n})}(x)) \subset W^+_{\alpha_0(\mathbf{n})}(h(x)).$$

As above, these inclusions imply that the image $h(R)$ is contained in the product $V = W^-_{\alpha_0(\mathbf{m})}(h(x)) \times W^+_{\alpha_0(\mathbf{n})}(h(x))$. We observe that $V$ lies in a subspace that does not contain $W(h(x))$ and thus has dimension at most $k$. This again contradicts that $\lambda(h(R)) \geq \mu(R) > 0$. □

Let us summarize the conclusions for the case of Cartan actions.

**COROLLARY 2.4.** *If $\alpha_0$ is Cartan all Lyapunov characteristic exponents for the action $\alpha$ with respect to measure $\mu$ are simple, no two of them are proportional and the counterpart of property ($\mathscr{C}$) holds.*

*For every Lyapunov exponent $\tilde{\chi}_i$ its Lyapunov distribution integrates to an invariant family of one-dimensional manifolds defined $\mu$-almost everywhere. This family will be referred to as the Lyapunov foliation corresponding to $\tilde{\chi}_i$. The semiconjugacy $h$ maps these local (corr. global) manifolds to the local (corr. global) affine integral manifolds for the exponents $\chi_i$.*

## 3. PROOF OF THEOREM 1.3

Throughout this section we fix one of the Lyapunov exponents of $\alpha$. We denote by $L$ the corresponding Lyapunov hyperplane in $\mathbb{R}^k$, by $E$ the corresponding one-dimensional Lyapunov distribution, and by $\tilde{\mathcal{W}}$ the corresponding Lyapunov foliation. Then $\tilde{\mathcal{W}}$ is the one-dimensional stable foliation for some element $\alpha(\mathbf{m})$, $\mathbf{m} \in \mathbb{Z}^k$. The notions of regularity and Pesin sets will refer to the corresponding notions for such an element.

In this section we study properties of the action $\alpha$ related to $\tilde{\mathcal{W}}$. We will show that the conditional measure $\mu_x^{\tilde{\mathcal{W}}}$ on the leaf $\tilde{\mathcal{W}}(x)$ is absolutely continuous for $\mu$-almost every $x$. We then conclude the proof of Theorem 1.3 by showing that the absolute continuity of $\mu$ follows from the absolute continuity of conditional measures for every Lyapunov foliation.

### 3.1. Invariant affine structures on leaves of Lyapunov foliations.

The following proposition gives a family of $\alpha$-invariant affine parameters on the leaves of the Lyapunov foliation $\tilde{\mathcal{W}}$. By an affine parameter we mean an atlas with affine transition maps.

**PROPOSITION 3.1.** *There exists a unique measurable family of $C^{1+\epsilon}$ smooth $\alpha$-invariant affine parameters on the leaves $\tilde{\mathcal{W}}(x)$. Moreover, within a given Pesin set they depend uniformly continuously on $x$ in the $C^{1+\epsilon}$ topology.*

**REMARK 4.** Note that those transition maps may not always preserve orientation. In fact in some situations a measurable choice of orientation is not possible. This however is completely irrelevant for our uses of affine structures.

**REMARK 5.** In the proof below we use the counterpart of the property ($\mathscr{C}$) for $\alpha$ and do not use existence of a semiconjugacy with $\alpha_0$. In fact, the assertion is true under a more general condition. Namely, let $\chi$ be a simple (multiplicity one) Lyapunov exponent for an ergodic hyperbolic measure $\mu$ for a $C^{1+\epsilon}$ diffeomorphism with the extra condition that there are no other exponents proportional to $\chi$ with the coefficient of proportionality greater than one. Then the Lyapunov distribution for $\chi$ is integrable $\mu$-almost everywhere to an invariant family of one-dimensional manifolds and invariant affine parameters still exist.

In the $C^2$ case one-dimensionality of the Lyapunov foliation may be replaced by the following *bunching condition*: Lyapunov exponents may be positively proportional with coefficients of proportionality between 1/2 and 2. The coarse Lyapunov distribution is always integrable and in this case the integral manifolds admit a unique invariant family of smooth affine structures.

The proofs of these statements can be obtained using nonuniform versions of the methods from [G].

*Proof.* The proposition is established using the three lemmas below. We take an element $\mathbf{m} \in \mathbb{Z}^k$ such that $\tilde{\mathscr{W}}$ is the stable foliation of $\alpha(\mathbf{m})$. Then we apply Lemma 3.2 with $f = \alpha(\mathbf{m})$ to obtain the family $H$ of nonstationary linearizations. Lemma 3.3 then shows that these nonstationary linearizations give an affine atlas. Since the linearization $H$ is unique by Lemma 3.4, the family $H$ linearizes any diffeomorphism which commutes with $f$. Indeed, if $g \circ f = f \circ g$, then it is easy to see that $dg^{-1} \circ H_{g(\cdot)} \circ g$ also gives a nonstationary linearization for $f$, and hence $H \circ g = dg \circ H$. Therefore, $H$ provides a nonstationary linearization for every element of the action $\alpha$, i.e. the action is affine with respect to the parameter. □

**LEMMA 3.2.** *Let $\tilde{\mathscr{W}}$ be the one-dimensional stable foliation of a $C^{1+\epsilon}$ nonuniformly hyperbolic diffeomorphism $f$. Then for $\mu$-almost every point $x \in M$ there exists a $C^{1+\epsilon}$ diffeomorphism $H_x \colon \tilde{\mathscr{W}}(x) \to E(x) = T_x\tilde{\mathscr{W}}$ such that*

(i) $H_{f(x)} \circ f = Df \circ H_x$,
(ii) $H_x(x) = 0$ *and* $D_x H_x$ *is the identity map,*
(iii) $H_x$ *depends continuously on* $x$ *in the* $C^{1+\epsilon}$ *topology on a Pesin set.*

*Proof.* We denote by $E$ the one-dimensional stable distribution for $f$. We fix some background Riemannian metric $g$ on $M$ and denote

$$Jf(x) = \|Df(v)\|_{fx} \cdot \|v\|_x^{-1}$$

where $v \in E(x)$ and $\|.\|_x$ is the norm given by $g$ at $x$.

We first construct the diffeomorphism $H_x$ on the local manifold $\tilde{W}(x)$ as follows. Since $E(x)$ is one-dimensional, $H_x(y)$ for $y \in \tilde{W}(x)$ can be specified by its distance to 0 with respect to the Euclidean metric on $E(x)$ induced by $g$. We define this distance by integrating a Hölder continuous density

$$|H_x(y)| = \int_x^y \rho_x(z)\, dz \tag{3.1}$$

where

$$\rho_x(z) = \lim_{n\to\infty} \frac{Jf^n(z)}{Jf^n(x)} = \prod_{k=0}^{\infty} \frac{Jf(f^k(z))}{Jf(f^k(x))}.$$

If $z$ is in the local manifold $\tilde{W}(x)$ then $\mathrm{dist}(f^k(z), f^k(x)) \le C(x)e^{-k\lambda}\mathrm{dist}(z,x)$ for all $k > 0$. In particular, $f^k(z)$ remains in the local manifold $\tilde{W}(f^k(x))$ even though the size of $\tilde{W}(f^k(x))$ may decrease with $k$ at a slow exponential rate. The tangent space $E(s) = T_s\tilde{W}(f^k(x))$ depends Hölder continuously on $s \in \tilde{W}(f^k(x))$, with Hölder exponent $\epsilon$ and a constant which may increase with $k$ at a slow exponential rate. Since $f$ is $C^{1+\epsilon}$, the same holds for $Jf(z)$. We conclude that

$$\left| \frac{Jf(f^k(z))}{Jf(f^k(x))} - 1 \right| \le C(x)\mathrm{dist}(z,x)e^{-k(\lambda+\delta)}.$$

This implies that the infinite product which defines $\rho_x(z)$ converges, and that $\rho_x$ is Hölder continuous on $\tilde{W}(x)$. Moreover, the convergence is uniform when $x$ is in a given Pesin set. Hence $\rho_x$ depends continuously in $C^\varepsilon$ topology on $x$ within a given Pesin set. Since $\rho_x(x) = 1$, we conclude that (3.1) defines a $C^{1+\epsilon}$ diffeomorphism satisfying conditions (ii) and (iii). To verify condition (i) we differentiate $H_{f(x)}(f(y)) = D_x f(H_x(y))$ with respect to $y$ and obtain $\rho_{f(x)}(f(y)) \cdot Jf(y) = Jf(x) \cdot \rho_x(y)$. Since the latter is satisfied by the definition of $\rho$, the condition (i) follows by integration.

Since $f$ contracts $\tilde{\mathcal{W}}$, we can extend $H$ to the global stable manifolds $\tilde{\mathcal{W}}(x)$ as follows. For $y \in \tilde{\mathcal{W}}(x)$ there exists $n$ such that $f^n(y) \in \tilde{W}(f^n x)$ and we can set

$$H_x(y) = Df^{-n} \circ H_{f(x)} \circ f^n(y).$$

This defines $H_x$ on an increasing sequence of balls exhausting $\tilde{\mathcal{W}}(x)$ with conditions (i) and (ii) satisfied by construction. Condition (iii) is satisfied in the following sense. $H_x$ is a $C^{1+\epsilon}$ diffeomorphism with locally Hölder derivative. Its restriction to a ball of fixed radius in $\tilde{\mathcal{W}}(x)$ centered at $x$ depends continuously in $C^{1+\epsilon}$ topology on $x$ within a given Pesin set. □

**REMARK 6.** In general, the regularity of the density $\rho_x$ on $\tilde{\mathcal{W}}(x)$ is the same as the regularity of the differential $Df$, and hence the function $H_x$ is as regular as $f$.

**LEMMA 3.3.** *Under the assumptions of Lemma 3.2, the map*

$$H_y \circ H_x^{-1} \colon\ E(x) \to E(y)$$

*is affine for any x and y on the same leaf of $\tilde{\mathcal{W}}$. Hence the nonstationary linearization H defines affine parameters on the leaves of $\tilde{\mathcal{W}}$.*

*Proof.* By invariance under $f$, it suffices to consider $x$ and $y$ close, and show that the map is affine in a neighborhood of zero. We will show that the differential $D(H_y \circ H_x^{-1})$ is constant on $E(x)$. Consider $z \in \tilde{\mathcal{W}}(x)$ close to $x$ and $y$ and let $\bar{z} = H_x(z)$. From the definition of $H$ we have $D_z(H_x) = \rho_x(z)$ and $D_z(H_y)(z) = \rho_y(z)$.

Hence, using the definition of $\rho$, we obtain

$$D_{\bar z}\left(H_y \circ H_x^{-1}\right) = D_z(H_y)\cdot D_{\bar z}(H_x^{-1}) = D_z(H_y)\cdot (D_z(H_x))^{-1} =$$
$$= \frac{\rho_y(z)}{\rho_x(z)} = \prod_{k=0}^{\infty} \frac{Jf(f^k(z))}{Jf(f^k y)} \cdot \left(\prod_{k=0}^{\infty} \frac{Jf(f^k(z))}{Jf(f^k(x))}\right)^{-1} = \prod_{k=0}^{\infty} \frac{Jf(f^k y)}{Jf(f^k(x))}.$$

We conclude that the differential $D_{\bar z}\left(H_y \circ H_x^{-1}\right)$ is independent of $\bar z$ and thus the map $H_y \circ H_x^{-1}$ is affine. □

**LEMMA 3.4.** *The family of diffeomorphisms $\{H_x\}$ satisfying conditions (i)-(iii) of Lemma 3.2 is unique.*

*Proof.* We note that it is sufficient for the proof to have $H_x$ defined only locally, in a neighborhood of $x$ in $\tilde W(x)$.

Suppose that $H_1$ and $H_2$ are two families of maps satisfying (i)-(iii). Then the family of maps $G = H_1 \circ H_2^{-1}\colon E \to E$ satisfies $G_{f(x)} \circ D_x f = D_x f \circ G_x$, and hence

$$G_x = (D_x f)^{-1} \circ G_{f(x)} \circ D_x f = \cdots = (D_x f^n)^{-1} \circ G_{f^n(x)} \circ D_x f^n.$$

or, since $E$ is one–dimensional,

$$G_x(t) = (Jf^n(x))^{-1} G_{f^n(x)}(Jf^n(x)\cdot t).$$

Since $Jf^n(x) \to 0$ and since $G_x$ depends continuously in the $C^1$-topology on $x$ in a Pesin set, we obtain using returns to such a set that

$$\frac{G_x(Jf^n(x)\cdot t)}{Jf^n(x)\cdot t} \to G'_x(0) = 1$$

and hence

$$G_x(t) = \lim_{n\to\infty} t \cdot \frac{G_{f^n(x)}(Jf^n(x)\cdot t)}{Jf^n(x)\cdot t} = t.$$

Thus $G_x$ is the identity, and $H_1 = H_2$ □

3.2. **Uniform growth estimates along the walls of Weyl chambers.** In the rest of this section we consider suspensions of the actions $\alpha_0$ and $\alpha$. According to our convention we will use the same notations for the suspension actions and associated objects.

We fix a Pesin set $\Lambda$ and a small $r > 0$. For $x \in \Lambda$ we denote by $\tilde B_r(x)$ the ball (interval) in the inner metric of $\tilde{\mathcal W}(x)$ of radius $r$ centered at $x$. An important corollary of the existence of affine parameters is the following estimate of derivatives along $\tilde{\mathcal W}$.

**LEMMA 3.5.** *For a given Pesin set $\Lambda$ and $r > 0$ there exists a constant $C = C(\Lambda, r)$ such that for any $x \in \Lambda$ and $\mathbf t \in \mathbb R^k$ satisfying $\alpha(\mathbf t)x \in \Lambda$*

$$C^{-1}\|D(\alpha(\mathbf t))|_{E(x)}\| \le \|D(\alpha(\mathbf t))|_{E(y)}\| \le C\|D(\alpha(\mathbf t))|_{E(x)}\|$$

*for any $y \in \tilde B_r(x)$ satisfying $\alpha(\mathbf t)y \in \tilde B_r(\alpha(\mathbf t)x)$.*

*Proof.* We use the affine parameter on $\tilde{\mathscr{W}}(x)$ given by Proposition 3.1 with respect to which $\alpha(\mathbf{t})$ has constant derivative. More precisely, using the linearization $H$ along the leaves of $\tilde{\mathscr{W}}$ we can write

$$\alpha(\mathbf{t})|_{\tilde{\mathscr{W}}(x)} = (H_{\alpha(\mathbf{t})x})^{-1} \circ D\alpha(\mathbf{t})|_{E(x)} \circ H_x,$$

and hence

$$D\alpha(\mathbf{t})|_{E(y)} = (D_{\alpha(\mathbf{t})y} H_{\alpha(\mathbf{t})x})^{-1} \circ D\alpha(\mathbf{t})|_{E(x)} \circ D_y H_x.$$

Since $H_z|_{\tilde{B}_r(z)}$ depends continuously in the $C^{1+\epsilon}$ topology on $z$ in the Pesin set $\Lambda$, both $\|D_t H_z\|$ and $\|(D_t H_z)^{-1}\|$ are uniformly bounded above and away from 0 for all $z \in \Lambda$ and $t \in \tilde{B}_r(z)$. Hence the norms of the first and last term in the right hand side are uniformly bounded and the lemma follows. □

We consider $x \in \Lambda$ and the ball $\tilde{B}_r(x) \subset \tilde{W}(x)$. The image $h(\tilde{B}_r(x))$ is contained in $W(x)$. We denote by $m_r(x)$ the radius of the largest ball (interval) in $W(h(x))$ that is centered at $h(x)$ and contained in $h(\tilde{B}_r(x))$. Then $m_r$ is a measurable function on $\Lambda$.

**LEMMA 3.6.** *For any Pesin set $\Lambda$ and $r > 0$ the function $m_r$ is positive almost everywhere on $\Lambda$. Hence for any $\varepsilon > 0$ there exists $m > 0$ and a set $\Lambda_{r,m} \subset \Lambda$ with $\mu(\Lambda \setminus \Lambda_{r,m}) < \varepsilon$ such that $m_r(x) \geq m$ for all $x \in \Lambda_{r,m}$.*

*Proof.* Let $x \in \Lambda$ be such a point that the intersection of $\Lambda$ with any neighborhood of $x$ has positive measure. Furthermore, assume that $x$ is not an endpoint of a complementary interval to the intersection $\tilde{W}(x) \cap \Lambda$. Let $\mathbf{m} \in \mathbb{Z}^k$ be an element such that $\tilde{\mathscr{W}} = \tilde{\mathscr{W}}^-_{\alpha(\mathbf{m})}$. Let $R$ be the intersection of $\Lambda$ with a sufficiently small neighborhood of $x$. If $m_r(x) = 0$ then $h(\tilde{W}^-_{\alpha(\mathbf{m})}(x)) = \{h(x)\}$. This implies, as in Lemma 2.3, that the image $h(R)$ is contained in $W^+_{\alpha_0(\mathbf{m})}(h(x))$. But this implies that $\lambda(h(R)) = 0$, which is impossible since $\lambda(h(R)) \geq \mu(R) > 0$. □

Using the derivative estimate in Lemma 3.5 and the topological semiconjugacy $h$ we obtain in the next lemma the crucial estimate for the derivatives of the elements in the Lyapunov hyperplanes.

We fix a Pesin set $\Lambda$, $r > 0$, and a set $\Lambda_{r,m}$ as in Lemma 3.6.

**LEMMA 3.7.** *For a given set $\Lambda_{r,m}$ there exists a constant $K$ such that for any $\mathbf{t}$ in the Lyapunov hyperplane $L$*

$$K^{-1} \leq \|D(\alpha(\mathbf{t}))|_{E(x)}\| \leq K$$

*if both $x \in \Lambda_{r,m}$ and $\alpha(\mathbf{t})x \in \Lambda_{r,m}$.*

*Proof.* First we note that it suffices to establish the lower estimate, then the upper estimate follows by applying it to $\alpha(-\mathbf{t})$.

By uniform continuity of the semiconjugacy $h$ there exists $\delta > 0$ such that for any $x$ the image $h(\tilde{B}_\delta(x))$ is contained in the ball $B_{m/2}(h(x))$ in $W(x)$. By the choice of $\Lambda_{r,m}$ we also have $B_m(h(x)) \subset h(\tilde{B}_r(x))$. Since $\mathbf{t} \in L$, $\alpha_0(\mathbf{t})$ is an isometry on $\mathscr{W}$, and hence $\alpha_0(\mathbf{t})(B_m(h(x))) = B_m(\alpha_0(\mathbf{t})(h(x)))$. Then since $h$ is a semiconjugacy we obtain

$$B_m(\alpha_0(\mathbf{t})(h(x))) \subset (\alpha_0(\mathbf{t}) \circ h)(\tilde{B}_r(x)) = (h \circ \alpha(\mathbf{t}))(\tilde{B}_r(x)). \tag{3.2}$$

Together with the uniform continuity of $h$ this implies that $\alpha(\mathbf{t})(\tilde{B}_r(x))$ cannot be contained in $\tilde{B}_\delta(\alpha(\mathbf{t})x)$. Indeed, otherwise we would have $B_m(\alpha_0(\mathbf{t})(h(x))) \subset h(\tilde{B}_\delta(\alpha(\mathbf{t})x)) \subset \alpha(\mathbf{t})(\tilde{B}_r(x)) \subset B_{m/2}(\alpha_0(\mathbf{t})(h(x)))$.

Hence there exists $z \in \tilde{B}_r(x)$ with $\operatorname{dist}(\alpha(\mathbf{t})x, \alpha(\mathbf{t})z) = \delta$. We may assume that $\delta < r$ and $z$ is chosen so that $\operatorname{dist}(\alpha(\mathbf{t})x, \alpha(\mathbf{t})y) < \delta$ for all $y \in \tilde{B}_r(x)$ between $x$ and $z$. Using Lemma 3.5 we obtain

$$\delta = \operatorname{dist}(\alpha(\mathbf{t})x, \alpha(\mathbf{t})z) \leq \operatorname{dist}(x,z) \cdot \sup \|D(\alpha(\mathbf{t}))|_{E(y)}\| < r \cdot C \|D(\alpha(\mathbf{t}))|_{E(x)}\|.$$

This implies that $\|D(\alpha(\mathbf{t}))|_{E(x)}\| > \frac{\delta}{Cr}$. □

3.3. **Ergodicity along the walls of Weyl chambers.** We will call an element $\mathbf{t} \in \mathbb{R}^k$ a *generic singular element* if it belongs to exactly one Lyapunov hyperplane. The following lemma presents a variation of an argument from [KS1] for the present setting.

**LEMMA 3.8.** *Let $L$ be one of the Lyapunov hyperplanes in $\mathbb{R}^k$. Let $E$ and $\tilde{\mathcal{W}}$ be the corresponding Lyapunov distribution and foliation of $\alpha$. Then for any generic singular element $\mathbf{t} \in \mathbb{R}^k$ the corresponding partition $\xi_{\alpha(\mathbf{t})}$ into the ergodic components of $\mu$ with respect to $\alpha(\mathbf{t})$ is coarser than the measurable hull $\xi(\tilde{\mathcal{W}})$ of the foliation $\tilde{\mathcal{W}}$.*

*Proof.* Consider a generic singular element $\mathbf{t}$ in $L$. Then the only nontrivial Lyapunov exponent that vanishes on $\mathbf{t}$ is the one with kernel $L$ and the corresponding Lyapunov distribution is $E$. Take a regular element $\mathbf{s}$ close to $\mathbf{t}$ for which this Lyapunov exponent is positive and all other nontrivial exponents have the same signs as for $\mathbf{t}$. Thus $E^+_{\alpha(\mathbf{s})} = E^+_{\alpha(\mathbf{t})} \oplus E$ and $E^-_{\alpha(\mathbf{s})} = E^-_{\alpha(\mathbf{t})}$. The Birkhoff averages with respect to $\alpha(\mathbf{t})$ of any continuous function are constant on the leaves of $\tilde{\mathcal{W}}^-_{\alpha(\mathbf{t})}$. Since such averages generate the algebra of $\alpha(\mathbf{t})$–invariant functions, we conclude that the partition $\xi_{\alpha(\mathbf{t})}$ into the ergodic components of $\alpha(\mathbf{t})$ is coarser than $\xi(\tilde{\mathcal{W}}^-_{\alpha(\mathbf{t})})$, the measurable hull of the foliation $\tilde{\mathcal{W}}^-_{\alpha(\mathbf{t})}$. On the other hand, the measurable hulls $\xi(\tilde{\mathcal{W}}^-_{\alpha(\mathbf{s})})$ and $\xi(\tilde{\mathcal{W}}^+_{\alpha(\mathbf{s})})$ of both $\tilde{\mathcal{W}}^-_{\alpha(\mathbf{s})}$ and $\tilde{\mathcal{W}}^+_{\alpha(\mathbf{s})}$ coincide with the Pinsker algebra $\pi(\alpha(\mathbf{s}))$. Since $\xi(\tilde{\mathcal{W}}^+_{\alpha(\mathbf{s})})$ is coarser than $\xi(\tilde{\mathcal{W}})$, we conclude that

$$\xi_{\alpha(\mathbf{t})} \leq \xi(\tilde{\mathcal{W}}^-_{\alpha(\mathbf{t})}) = \xi(\tilde{\mathcal{W}}^-_{\alpha(\mathbf{s})}) = \pi(\alpha(\mathbf{s})) = \xi(\tilde{\mathcal{W}}^+_{\alpha(\mathbf{s})}) \leq \xi(\tilde{\mathcal{W}}).$$

□

3.4. **Invariance and absolute continuity of conditional measures.** Let $\tilde{\mathcal{W}}$ be one of the Lyapunov foliations of $\alpha$ (recall that it is one–dimensional), and let $L \subset \mathbb{R}^k$ be the corresponding Lyapunov hyperplane. We fix a Pesin set $\Lambda$, $r > 0$, and a set $\Lambda_{r,m}$ as in Lemma 3.6.

**LEMMA 3.9.** *For $\mu$- a.e. $x \in \Lambda_{r,m}$ and for $\mu_x^{\tilde{\mathcal{W}}}$ - a.e. $y \in \Lambda_{r,m} \cap \tilde{B}_r(x)$ there exists an affine map $g \colon \tilde{\mathcal{W}}(x) \to \tilde{\mathcal{W}}(x)$ with $g(x) = y$ which preserves the conditional measure $\mu_x^{\tilde{\mathcal{W}}}$ up to a positive scalar multiple. Furthermore, the absolute value of the derivative of this affine map is bounded away from zero and infinity uniformly in $x$ and $y$. The bounds depend on $r$ and $m$.*

*Proof.* We fix a generic singular element $\mathbf{t} \in L \subset \mathbb{R}^k$. By Lemma 3.8 the partition $\xi_{\alpha(\mathbf{t})}$ into the ergodic components of $\mu$ for $\alpha(\mathbf{t})$ is coarser than the measurable hull $\xi(\tilde{\mathscr{W}})$ of the foliation $\tilde{\mathscr{W}}$. Then there is a set $X_1$ of full $\mu$-measure such that for any $x \in X_1$ the ergodic component $E_x$ of $\alpha(\mathbf{t})$ passing through $x$ is well-defined and contains $\tilde{\mathscr{W}}(x)$ up to a set of $\mu_x^{\tilde{\mathscr{W}}}$-measure 0. Let $\mu_x$ be the measure induced by $\mu$ on $E_x$.

For $n > 0$ we denote by $B^n(x)$ the image under $H_x^{-1}$ of the ball in $T_x\tilde{\mathscr{W}}$ of radius $n$ centered at 0, where $H_x$ comes from Lemma 3.2. We note that the sets $B^n(x)$ exhaust $\tilde{\mathscr{W}}(x)$, i.e. $\tilde{\mathscr{W}}(x) = \bigcup_{n>0} B^n(x)$. For almost every $x$ we can normalize $\mu_x^{\tilde{\mathscr{W}}}$ so that $\mu_x^{\tilde{\mathscr{W}}}(B^n(x)) = 1$ and denote its restriction to $B^n(x)$ by $\mu_x^n$.

We use a fixed Riemannian metric to identify $T_x\tilde{\mathscr{W}}$ with $\mathbb{R}$ and then use $H_x$ to identify $B^n(x)$ with the interval $[-n, n]$. Thus we can consider the system of normalized conditional measures $\mu_x^n$ as a measurable function from the suspension manifold $M$ to the weak* compact set of Borel probability measures on the interval $[-n, n]$. By Luzin's theorem, we can take an increasing sequence of closed sets $K_i$ contained in the support of $\mu$ such that

1. $\mu(K) = 1$, where $K = \bigcup\limits_{i=1}^{\infty} K_i$
2. $\mu_x^n$ depends continuously on $x \in K_i$ with respect to the weak* topology.

Set $X_2 = X_1 \cap K$. Since by definition the transformation $\alpha(\mathbf{t})$ restricted to the ergodic component $E_x$ is ergodic, the transformation induced by $\alpha(\mathbf{t})$ on $X_1 \cap E_x \cap K_i \cap \Lambda_{r,m}$ is also ergodic for any $i$. Hence the set $X_3$, which consists of points $x \in X_2$ whose orbit $\{\alpha(m\mathbf{t})\,x\}_{m\in\mathbb{Z}}$ is dense in a subset of full $\mu_x$ measure of $X_1 \cap E_x \cap K_i \cap \Lambda_{r,m}$ for all $i$, has full measure $\mu$.

Let $x \in X_3 \cap \Lambda_{r,m}$ and $y \in X_3 \cap \Lambda_{r,m} \cap \tilde{B}_r(x)$. Then $x, y \in X_1 \cap E_x \cap K_i \cap \Lambda_{r,m}$ for some $i$. Hence there exists a sequence $m_k \to \infty$ such that the points $y_k = \alpha(m_k\mathbf{t})\,x \in X_1 \cap E_x \cap K_i \cap \Lambda_{r,m}$ converge to $y$. Let us consider the map

$$\phi_k = \alpha(m_k\mathbf{t})|_{\tilde{\mathscr{W}}(x)} \colon \tilde{\mathscr{W}}(x) \to \tilde{\mathscr{W}}(y_k).$$

Since $x$ and $y_k = \alpha(m_k\mathbf{t})\,x$ are both in $\Lambda_{r,m}$, Lemma 3.7 yields $K^{-1} \le \|D_x\phi_k\| \le K$ for all $k$. The map $\phi_k$ is affine with respect to the affine parameters on $\tilde{\mathscr{W}}(x)$ and $\tilde{\mathscr{W}}(y_k)$. By Proposition 3.1, the affine parameters depend continuously in the $C^{1+\epsilon}$ topology on a point in the Pesin set $\Lambda$. Thus the affine parameters at $y_k$ converge to the affine parameter at $y$ uniformly on compact sets in the leaves. Hence, by taking a subsequence if necessary, we may assume that the $\phi_k$ converge uniformly on compact sets to an affine map $g_n \colon \tilde{\mathscr{W}}(x) \to \tilde{\mathscr{W}}(x)$ with $g_n(x) = y$.

Since both $(\phi_k)_*\mu_x^n$ and $\mu_{y_k}^n$ are conditional measures on the same leaf $\tilde{\mathscr{W}}(y_k)$, there exists a constant $c(k) > 0$ such that

$$\mu_{y_k}^n(\phi_k(A)) = c(k)\mu_x^n(A) \quad \text{for any} \quad A \subset B^n(x) \cap \phi_k^{-1}(B_{y_k}^n).$$

Similarly, there exists a constant $c > 0$ such that

$$\mu_y^n(A) = c\mu_x^n(A) \quad \text{for any} \quad A \subset B^n(x) \cap (B_y^n).$$

Since $\mu_x^n$ depends continuously on $x \in K_i$ with respect to the weak* topology, the measures $\mu_{y_k}^n$ weak* converge to the measure $\mu_y^n$. Assuming that the boundary of $A$ relative to the leaf has zero conditional measure, we obtain that

$$c(k)\mu_x^n(A) = \mu_{y_k}^n(\phi_k(A)) \to \mu_y^n(g_n A) = c\mu_x^n(g_n A)$$

and hence

$$\mu_x^n(g_n A) = \frac{\lim c(k)}{c}\mu_x^n(A) \quad \text{for any} \quad A \subset B^n(x) \cap g_n^{-1}(B_y^n).$$

We obtain that $g_n$ preserves the conditional measure $\mu_x^{\tilde{\mathcal{W}}}$ up to a scalar on the set $C^n(x) = B^n(x) \cap g_n^{-1}(B_y^n)$. We note that $C^n(x)$ contains $B^{n/K}(x)$ and also $\tilde{B}_r(x)$, provided $n$ is large enough. Since $\mu_x^{\tilde{\mathcal{W}}}(\tilde{B}_r(x)) > 0$, taking $A = C^n(x)$ we see that $\lim c(k)$ must be positive.

We conclude that for any $n > 0$ there exists a set $X_4$ of full $\mu$-measure such that for any $x \in X_4 \cap \Lambda_{r,m}$ and $y \in X_4 \cap \tilde{B}_r(x) \cap \Lambda_{r,m}$ there exists an affine map $g_n$ of $\tilde{\mathcal{W}}(x)$ such that $g_n(x) = y$ and $g_n$ preserves $\mu_x^{\tilde{\mathcal{W}}}$ up to a positive scalar on $C^n(x)$.

Repeating this construction for every $n > 0$ we can choose a set $X$ of full measure $\mu$ such that for any $x \in X \cap \Lambda_{r,m}$, $y \in X \cap \tilde{B}_r(x) \cap \Lambda_{r,m}$, and any $n$ there exists an affine map $g_n \colon \tilde{\mathcal{W}}(x) \to \tilde{\mathcal{W}}(x)$ satisfying $g_n(x) = y$ and preserving $\mu_x^{\tilde{\mathcal{W}}}$ up to a positive scalar on $C^n(x)$. We note that $\tilde{\mathcal{W}}(x) = \bigcup_{n>0} C^n(x)$. Hence taking a convergent subsequence we obtain that for any $x \in X \cap \Lambda_{r,m}$ and $y \in X \cap \tilde{B}_r(x) \cap \Lambda_{r,m}$ there exists an affine map $g$ of $\tilde{\mathcal{W}}(x)$ with $g(x) = y$ which preserves $\mu_x^{\tilde{\mathcal{W}}}$ up to a positive scalar. This completes the proof of the lemma since we may assume that the set $X$ of full measure is chosen so that for $x \in X \cap \Lambda_{r,m}$ the set $X \cap \tilde{B}_r(x)$ has full $\mu_x^{\tilde{\mathcal{W}}}$-measure. □

**LEMMA 3.10.** *The conditional measures $\mu_x^{\tilde{\mathcal{W}}}$ are absolutely continuous for $\mu$ - a.e. $x$.*

*Proof.* Let $A_x$ be the group of affine transformations of $\tilde{\mathcal{W}}(x)$, and let $G_x$ be the subgroup of $A_x$ consisting of elements which preserve $\mu_x^{\tilde{\mathcal{W}}}$ up to a positive scalar multiple.

Let us first observe that $G_x$ is a closed subgroup. Indeed, if $g_n \to g$ in $A_x$ then $g_n(Z) \to g(Z)$ in the Hausdorff metric whenever $Z \subset \tilde{\mathcal{W}}(x)$ is bounded, so $\mu_x^{\tilde{\mathcal{W}}}(g_n(Z)) \to \mu_x^{\tilde{\mathcal{W}}}(g(Z))$ if the relative boundary of $g(Z)$ has zero conditional measure. This implies that $(g_n)_*\mu_x^{\tilde{\mathcal{W}}} \to g_*\mu_x^{\tilde{\mathcal{W}}}$. We also have $(g_n)_*\mu_x^{\tilde{\mathcal{W}}} = c_n\mu_x^{\tilde{\mathcal{W}}}$, where $c_n = \mu_x^{\tilde{\mathcal{W}}}(Z)/\mu_x^{\tilde{\mathcal{W}}}(g_n(Z))$ for any $Z$. Since $g$ is an invertible affine map we can choose $Z$ such that $\mu_x^{\tilde{\mathcal{W}}}(Z) > 0$, $\mu_x^{\tilde{\mathcal{W}}}(g(Z)) > 0$, and $\mu_x^{\tilde{\mathcal{W}}}(\partial(g(Z))) = 0$. It follows that $c_n \to c = \mu_x^{\tilde{\mathcal{W}}}(Z)/\mu_x^{\tilde{\mathcal{W}}}(g(Z)) > 0$ and $g_*\mu_x^{\tilde{\mathcal{W}}} = c\mu_x^{\tilde{\mathcal{W}}}$.

Since any element $\alpha(\mathbf{t})$ preserves the affine parameters on the leaves of $\tilde{\mathcal{W}}$, it maps the group $A_x$ isomorphically onto $A_{\alpha(\mathbf{t})x}$. Since $\alpha(\mathbf{t})$ also preserves the conditional measures on the leaves of $\tilde{\mathcal{W}}$, it maps the subgroup $G_x$ isomorphically onto $G_{\alpha(\mathbf{t})x}$ on the set of full measure $\mu$ where the conditional measures and affine parameters on the leaves of $\tilde{\mathcal{W}}$ are well defined. Since isomorphism classes of closed subgroups of the group of affine transformations on the line form a separable space, ergodicity of $\alpha(\mathbf{t})$ implies that the groups $G_x$ are isomorphic $\mu$-almost everywhere.

By Lemma 3.9, for a given Pesin set $\Lambda$ and for $\mu_x^{\tilde{\mathcal{W}}}$-almost any $y, z \in \Lambda_{r,m} \cap \tilde{B}_r(x)$ there exists an affine map $g\colon \tilde{\mathcal{W}}(x) \to \tilde{\mathcal{W}}(x)$ preserving $\mu_x^{\tilde{\mathcal{W}}}$ up to a scalar multiple with $g(y) = z$. Thus $G_x$ has an orbit of positive $\mu_x^{\tilde{\mathcal{W}}}$ measure. We note that the measures $\mu_x^{\tilde{\mathcal{W}}}$ are nonatomic for $\mu$-almost every $x$, otherwise the entropy would be zero for any element whose full unstable foliation is $\tilde{\mathcal{W}}$. Then it follows that $G_x$ can not be a discrete subgroup of $A_x$ for $\mu$ almost every $x$. Hence either $G_x = A_x$ or the connected component of the identity in $G_x$ is a one-parameter subgroup of the same type on the set of full $\mu$ measure. Thus either $G_x$ contains the subgroup of translations or it is conjugate to the subgroup of dilations.
(i) First consider the case when $G_x$ contains the subgroup of translations. For any $x$ and $y \in \tilde{\mathcal{W}}(x)$ we define $c_x(y)$ by the equality $g\mu^{\tilde{\mathcal{W}}} = c_x(y)\mu^{\tilde{\mathcal{W}}}$, where $\mu^{\tilde{\mathcal{W}}}$ is the conditional measure on $\tilde{\mathcal{W}}(x)$ and $g$ is a translation such that $g(x) = y$. Note that $c_x(y)$ is well defined. Indeed, such $g$ is unique, and the definition does not depend on a particular choice of $\mu^{\tilde{\mathcal{W}}}$ since the conditional measures are defined up to a scalar multiple. We need to show that $c_x(y) = 1$ for all $y \in \tilde{\mathcal{W}}(x)$.

We note that $c_x(y)$ can be calculated as

$$c_x(y) = \frac{g_*\mu^{\tilde{\mathcal{W}}}(A)}{\mu^{\tilde{\mathcal{W}}}(A)} = \frac{\mu^{\tilde{\mathcal{W}}}(g^{-1}A)}{\mu^{\tilde{\mathcal{W}}}(A)} = \frac{\mu^{\tilde{\mathcal{W}}}(A)}{\mu^{\tilde{\mathcal{W}}}(g(A))}$$

for any set $A$ of positive conditional measure. Since we can take the test set $A$ such that the boundary of $g(A)$ relative to the leaf has zero conditional measure, we conclude that for a fixed $x$ the coefficient $c_x(y)$ depends continuously on $y$.

We see that either for $\mu$- a.e. $x$ $c_x(y) = 1$ for all $y \in \tilde{\mathcal{W}}(x)$, or there exists a set $X$ of positive measure such that $c_x(y)$ is not identically equal to 1 for $x \in X$. In the latter case for some $\epsilon > 0$ we can define a finite positive measurable function

$$\varphi_\epsilon(x) = \inf\{r : \exists\, y \in \tilde{\mathcal{W}}(x)\ s.t.\ d(x, y) < r\ and\ |c_x(y) - 1| > \epsilon\}$$

on some subset $Y \subset X$ of positive $\mu$-measure. By measurability there exists $N$ and a set $Z$ of positive measure on which $\varphi_\epsilon$ takes values in the interval $(1/N, N)$. We will show that

$$\varphi_\epsilon(\alpha(n\mathbf{t})x) \to 0 \text{ as } n \to \infty \tag{3.3}$$

uniformly on $Z$ for an element $\mathbf{t}$ such that $\alpha(\mathbf{t})$ contracts the foliation $\tilde{\mathcal{W}}$. Since this contradicts the recurrence of the set $Z$ we conclude that $c_x(y)$ must be identically equal to 1.

We will prove now that

$$c_x(y) = c_{\alpha(\mathbf{t})x}(\alpha(\mathbf{t})y) \tag{3.4}$$

for $\mu$-a.e. $x$ and $y \in \tilde{\mathcal{W}}(x)$. Since the iterates of $\alpha(\mathbf{t})$ exponentially contract the leaves of $\tilde{\mathcal{W}}$, this invariance property implies that $\varphi_\epsilon(\alpha(n\mathbf{t})x) \le C\lambda^n \varphi_\epsilon(x)$, for some $C, \lambda > 0$, hence (3.4) implies (3.3).

To prove (3.4) we consider the translation

$$f = \alpha(\mathbf{t}) \circ g \circ \alpha(-\mathbf{t}) \in G_{\alpha(\mathbf{t})x}$$

We observe that $f(\alpha(\mathbf{t})x) = \alpha(\mathbf{t})y$ since $gx = y$. Hence we obtain

$$c_{\alpha(\mathbf{t})x}(\alpha(\mathbf{t})y) = \frac{\mu^{\tilde{\mathscr{W}}}(B)}{\mu^{\tilde{\mathscr{W}}}(fB)}$$

for any set $B \subset \tilde{\mathscr{W}}(\alpha(\mathbf{t})x)$ of positive conditional measure. Since $\alpha(\mathbf{t})(g(A)) = f(\alpha(\mathbf{t})A)$ and since $\alpha(\mathbf{t})_*\mu_x^{\tilde{\mathscr{W}}}$ is a conditional measure on the leaf $\tilde{\mathscr{W}}(\alpha(\mathbf{t})x)$ we obtain using $B = \alpha(\mathbf{t})A$ as the test set that

$$c_x(y) = \frac{\mu_x^{\tilde{\mathscr{W}}}(A)}{\mu_x^{\tilde{\mathscr{W}}}(g(A))} = \frac{(\alpha(\mathbf{t})_*\mu_x^{\tilde{\mathscr{W}}})(\alpha(\mathbf{t})A)}{(\alpha(\mathbf{t})_*\mu_x^{\tilde{\mathscr{W}}})(\alpha(\mathbf{t})(g(A)))} = c_{\alpha(\mathbf{t})x}(\alpha(\mathbf{t})y).$$

(ii) Now suppose that $G_x$ is conjugate to the subgroup of dilations. In this case $G_x$ has a fixed point $0_x$ and acts simply transitively on each connected component of $\tilde{\mathscr{W}}(x) \setminus \{0_x\}$. For any $x$ and $y$ in the same component we consider

$$c_x(y) = \frac{Jg \cdot \mu^{\tilde{\mathscr{W}}}(A)}{\mu^{\tilde{\mathscr{W}}}(g(A))}$$

where $g \in G_x$ is such that $g(x) = y$ and $Jg$ is the absolute value of the Jacobian with respect to the affine parameter. To show that measure $\mu_x^{\tilde{\mathscr{W}}}$ is Haar it is sufficient to prove that for any $g \in G_x$

$$g_*\mu_x^{\tilde{\mathscr{W}}} = Jg \cdot \mu_x^{\tilde{\mathscr{W}}} \tag{3.5}$$

For that it suffices to show that $c_x(y) = 1$ identically on $\tilde{\mathscr{W}}(x)$ for $\mu$-almost every $x$. This can be established by repeating the argument of the previous case. The only difference is that to prove (3.4) we need to note that for the map

$$f = \alpha(\mathbf{t}) \circ g \circ \alpha(-\mathbf{t}) \in G_{\alpha(\mathbf{t})x}$$

we have $Jf = Jg$. □

Notice that at the end we proved that $G_x = A_x$ for almost every $x$.

3.5. **Conclusion of the proof.** In order to prove that $\mu$ is an absolutely continuous measure it is sufficient to show that for a certain element $\alpha(\mathbf{m})$

($\mathscr{P}$) *The entropy* $\mathbf{h}_\mu(\alpha(\mathbf{m}))$ *is equal both to the sum of the positive Lyapunov exponents and to the absolute value of the sum of the negative Lyapunov exponents.* (See [L, LY]).

First recall that there are $2^{k+1} - 2$ Weyl chambers for $\alpha_0$ and any combination of positive and negative signs for the Lyapunov exponents, except for all positive or all negative, appears in one of the Weyl chambers. The same is true for $\alpha$ by Lemma 2.3. Denote the Lyapunov exponents for $\alpha$ by $\chi_1, \dots, \chi_{k+1}$. Let $\mathscr{C}_i$, $i = 1, \dots, k+1$, be the Weyl chamber on which the $\chi_i > 0$ and $\chi_j < 0$ for all $j \neq i$. Notice that we use notations different from those of Section 2.1.

Consider $\mathbf{m} \in \mathscr{C}_i$. Since the conditional measure on $\tilde{\mathscr{W}}^+_{\alpha(\mathbf{m})}$ is absolutely continuous by Lemma 3.10, we obtain that

$$\mathbf{h}_\mu(\alpha(\mathbf{m})) = \chi_i(\mathbf{m})$$

for any $\mathbf{m} \in \mathscr{C}_i$. By the Ruelle entropy inequality $\mathbf{h}_\mu(\alpha(\mathbf{m})) \le -\sum_{j\ne i} \chi_j(\mathbf{m})$ and hence

$$\sum_{j=1}^{k+1} \chi_j(\mathbf{m}) \le 0.$$

If $\sum_{j=1}^{k+1} \chi_j(\mathbf{m}) = 0$ then ($\mathscr{P}$) holds and the proof is finished.

Thus we have to consider the case when $\sum_{j=1}^{k+1} \chi_j(\mathbf{m}) < 0$ for all $\mathbf{m}$ in all Weyl chambers $\mathscr{C}_i,\ i = 1,\dots k+1$. This implies that $\bigcup_{i=1}^{k+1} \mathscr{C}_i$ lies in a negative half-space of the linear functional $\sum_{j=1}^{k+1} \chi_j$. But this is impossible since there exist elements $\mathbf{t}_i \in \mathscr{C}_i,\ i = 1,\dots k+1$ such that $\sum_{i=1}^{k+1} \mathbf{t}_i = 0$. □

## 4. PROOF OF THEOREMS 1.6 AND 1.7

4.1. **Rigidity of the expansion coefficients.** We consider the suspension action of $\alpha$. Let $\chi$ be one of the Lyapunov exponents of $\alpha$. Let $E$ be the corresponding Lyapunov distribution and $L = \ker \chi \subset \mathbb{R}^k$ be the corresponding Lyapunov hyperplane.

**LEMMA 4.1.** *The restriction of $\alpha$ to $L$ is ergodic.*

*Proof.* By Lemma 3.8, the partition $\xi_L$ by ergodic components of $\mu$ is coarser than the measurable hull $\xi(\tilde{\mathscr{W}})$ of the foliation $\tilde{\mathscr{W}}$, which in turn coincides with the Pinsker algebra of a regular element in $\mathbb{R}^k$. Since we have established that $\mu$ is absolutely continuous, the Pinsker algebra on $\mathbb{T}^{k+1}$ of a regular element in $\mathbb{Z}^k$ is at most finite [P].

Then on the suspension manifold $M$ the Pinsker algebra is given by the corresponding finite partitions of the fibers of the suspension. Since $L$ is an irrational hyperplane in $\mathbb{R}^k$, its action on $\mathbb{T}^k$ in the base of the suspension is uniquely ergodic, and hence $\xi_L$ is at most finite. Since the action $\alpha$ is ergodic, $\xi_L$ is trivial since the stationary subgroup in $\mathbb{R}^k$ of any $L$-invariant set has to have finite index and hence must coincide with $\mathbb{R}^k$. □

**LEMMA 4.2.** *There is a measurable metric on $E$ with respect to which*

$$\|D\alpha(\mathbf{t})\,v\| = e^{\chi(\mathbf{t})}\|v\| \tag{4.1}$$

*for any $\mathbf{t} \in \mathbb{R}^k$, $\mu$-a.e. $x$, and any $v \in E(x)$. Such a measurable metric is unique up to a scalar multiple.*

*Proof.* First we construct a measurable metric $g$ on $E$ which is preserved by an ergodic element $\mathbf{t} \in L$. In other words, (4.1) is satisfied with respect to $g$ for this element $\mathbf{t}$. Then we will show that such a metric is unique up to a scalar multiple. The uniqueness easily implies that (4.1) is satisfied for all $\mathbf{t} \in \mathbb{R}^k$.

Let $\Lambda' = \Lambda_{r,m}$ and constant $K$ be as in Lemma 3.7. Ergodicity of $\alpha|_L$ implies that there exists an ergodic element $\mathbf{t} \in L$. We fix such an element $\mathbf{t}$, and let $X$

be an invariant set of full measure consisting of points whose $\alpha(\mathbf{t})$ orbits visit $\Lambda'$ with frequency $\mu(\Lambda')$.

We fix some background Riemannian metric $g_0$ on $M$. We use the notations $D_x^E\alpha(\mathbf{t}) = D(\alpha(\mathbf{t}))|_{E(x)}$ and

$$\|D_x^E\alpha(\mathbf{t})\| = \|D_x^E\alpha(\mathbf{t})(v)\|_{\alpha(\mathbf{t})x} \cdot \|v\|_x^{-1}$$

where $v \in E(x)$ and $\|.\|_x$ is the norm given by $g_0$ at $x$.

We define a measurable renormalization function $\phi$ as follows.

$$\phi(x) = \sup\{\|D_x^E\alpha(n\mathbf{t})\| : n \in \mathbb{N},\ \alpha(n\mathbf{t})x \in \Lambda'\} \tag{4.2}$$

We note that by Lemma 3.7 the supremum is bounded by $K$ for any $x \in \Lambda'$. More generally, the supremum is finite for any point whose $\alpha(\mathbf{t})$ orbit visits $\Lambda'$. Thus the function is well defined and finite on $X$. Using (4.2) we obtain

$$\begin{aligned}\frac{\phi(\alpha(\mathbf{t})x)}{\phi(x)} &= \frac{\sup\{\|D_{\alpha(\mathbf{t})x}^E\alpha(n\mathbf{t})\| : n \in \mathbb{N},\ \alpha((n+1)\mathbf{t})x \in \Lambda'\}}{\sup\{\|D_x^E\alpha(n\mathbf{t})\| : n \in \mathbb{N},\ \alpha(n\mathbf{t})x \in \Lambda'\}}\\ &= \frac{\sup\{\|D_{\alpha(\mathbf{t})x}^E\alpha(n\mathbf{t})\| : n \in \mathbb{N},\ \alpha((n+1)\mathbf{t})x \in \Lambda'\}}{\sup\{\|D_x^E\alpha(\mathbf{t})\| \cdot \|D_x^E\alpha(n\mathbf{t})\| : n \in \mathbb{N},\ \alpha((n+1)\mathbf{t})x \in \Lambda'\}} = \|D_x^E\alpha(\mathbf{t})\|^{-1}\end{aligned}$$

This means that with respect to the renormalized Riemannian metric $g = \phi g_0$ we have

$$\|D_x^E\alpha(\mathbf{t})\|_g = \|D_x^E\alpha(\mathbf{t})\| \cdot \frac{\phi(\alpha(\mathbf{t})x)}{\phi(x)} = 1.$$

Suppose that (4.1) is satisfied for the fixed $\mathbf{t}$ with respect to another Riemannian metric $\psi g_0$ on $E$. Then equation (4.1) implies that

$$\|D_x^E\alpha(\mathbf{t})\| \cdot \frac{\psi(\alpha(\mathbf{t})x)}{\psi(x)} = \|D_x^E\alpha(\mathbf{t})\|_{\psi g_0} = 1 = \|D_x^E\alpha(\mathbf{t})\|_{\phi g_0} = \|D_x^E\alpha(\mathbf{t})\| \cdot \frac{\phi(\alpha(\mathbf{t})x)}{\phi(x)}$$

and hence

$$\frac{\psi(\alpha(\mathbf{t})x)}{\phi(\alpha(\mathbf{t})x)} = \frac{\psi(x)}{\phi(x)}$$

By ergodicity of $\alpha(\mathbf{t})$ we conclude that $\psi = \kappa\phi$, where $\kappa$ is a constant.

For any other element $\mathbf{s} \in \mathbb{R}^k$ consider the metric $\alpha(\mathbf{s})_* g$. By commutativity, this metric is again preserved by $\alpha(\mathbf{t})$. From the uniqueness we obtain $\alpha(\mathbf{s})_* g = \kappa(\mathbf{s}) \cdot g$ where $\kappa(\mathbf{s})$ is a positive constant. Let us show that

$$\log\kappa(\mathbf{s}) = \chi(\mathbf{s}).$$

Indeed, let $\Lambda$ be a set of positive measure on which $C^{-1} < \phi < C$ for some constant $C$. Since

$$\kappa(\mathbf{s}) = \|D_x^E\alpha(n\mathbf{s})\|_g = \|D_x^E\alpha(\mathbf{s})\| \cdot \frac{\phi(\alpha(n\mathbf{s})x)}{\phi(x)}$$

we obtain

$$C^{-2}\kappa^n(\mathbf{s}) < \|D_x^E\alpha(n\mathbf{s})\| < C^2\kappa^n(\mathbf{s}) \tag{4.3}$$

if both $x$ and $\alpha(n\mathbf{s})x$ are in $\Lambda$. By recurrence, for almost every $x \in \Lambda$ there exists a sequence of natural numbers $n_i \to \infty$ such that $\alpha(n_i\mathbf{s}) \in \Lambda$. Since for almost every $x$

$$\chi(\mathbf{s}) = \lim_{i\to\infty} n_i^{-1} \log \| D_x^E \alpha(n_k \mathbf{s}) \|$$

we conclude using (4.3) that $\chi(\mathbf{s}) = \log \kappa(\mathbf{s})$. □

### 4.2. **Smoothness of the semiconjugacy along the Lyapunov foliations.**

**LEMMA 4.3.** *For almost every $x$ the semiconjugacy $h$ intertwines the actions of the groups of translations of $\tilde{\mathcal{W}}(x)$ and $\mathcal{W}(h(x))$. More precisely, for any translation $\tilde{\tau}$ with respect to the affine structure on $\tilde{\mathcal{W}}(x)$ there is a translation $\tau$ of $\mathcal{W}(h(x))$ with $h \circ \tilde{\tau} = \tau \circ h$.*

*Proof.* The proof of this lemma closely follows the proof of Lemma 3.9. Let $\Lambda$ be a Pesin set. By Lemma 3.1, the affine parameters depend continuously in $C^{1+\epsilon}$ on a point in $\Lambda$.

By Luzin's theorem, the measurable metric from Lemma 4.2 is uniformly continuous on sets of large measure. Hence we can take an increasing sequence of closed sets $K_i$ such that

1. $\mu(K) = 1$, where $K = \bigcup_{i=1}^{\infty} K_i$
2. the measurable metric depends continuously on $x \in K_i$.

As in the previous lemma we fix an ergodic element $\mathbf{t} \in L$. Then the transformation induced by $\alpha(\mathbf{t})$ on $K_i \cap \Lambda$ is also ergodic for any $i$. Hence, there is an invariant full measure $\mu$ set $X \subset K$ of points $x$ whose orbit $\{\alpha(m\mathbf{t})\,x\}_{m\in\mathbb{Z}}$ is dense in $K_i \cap \Lambda$ for all $i$.

Let $x \in X$ and $y \in \tilde{\mathcal{W}}(x) \cap X \cap \Lambda$. Then $y \in K_i \cap \Lambda$ for some $i$. Hence there exists a sequence $m_k \to \infty$ such that the points $y_k = \alpha(m_k\mathbf{t})\,x \in K_i \cap \Lambda$ converge to $y$. Let us consider the affine map

$$\phi_k = \alpha(m_k\mathbf{t})|_{\tilde{\mathcal{W}}(x)} : \tilde{\mathcal{W}}(x) \to \tilde{\mathcal{W}}(y_k).$$

We normalize the affine parameters using the measurable metric. Then $\phi_k$ is an isometry with respect to the normalized parameters at $x$ and $y_k$. The normalized parameters vary continuously on $K_i \cap \Lambda$. Since $y$ and $y_k$ are both in $K_i \cap \Lambda$, the normalized affine parameters at $y_k$ converge to the normalized affine parameter at $y$ uniformly on compact sets. Hence, by taking a subsequence if necessary, we may assume that the $\phi_k$ converge to an isometry $g : \tilde{\mathcal{W}}(x) \to \tilde{\mathcal{W}}(x)$ with $g(x) = y$. We also note that $y_k \to y$ implies that $h(y_k) \to h(y)$, and the maps

$$\psi_k = \alpha_0(m_k\mathbf{t})|_{\mathcal{W}(h(x))} : \mathcal{W}(h(x)) \to \mathcal{W}(h(y_k)).$$

are isometries. By taking a subsequence if necessary, we may assume that $\psi_k$ converge to an isometry $f : \mathcal{W}(h(x)) \to \mathcal{W}(h(x))$ with $f(h(x)) = h(y)$. Since $h$ is a semiconjugacy we obtain $h \circ g = f \circ h$.

Let $G_x$ be the set of all isometries $g$ of $\tilde{\mathcal{W}}(x)$ for which there exists an isometry $f$ of $\mathcal{W}(x)$ with $h \circ g = f \circ h$. It is easy to see that $G_x$ is a closed subgroup of the group of affine transformations of $\tilde{\mathcal{W}}(x)$.

Since a set of full measure can be exhausted by Pesin sets we obtain that for almost every point $x$ and for $\mu_x^{\tilde{\mathcal{W}}}$-almost every $y \in \tilde{\mathcal{W}}(x)$ there exists an isometry $g_{xy} \in G_x$ with $g(x) = y$. We note that by Lemma 3.10, for almost every point $x$ the conditional measure $\mu_x$ is Haar with respect to the affine parameter. Hence we conclude that for almost every point $x$ there is a dense set of points $y \in \tilde{\mathcal{W}}(x)$ for which there exists an isometry $g_{xy} \in G_x$. Since $G_x$ is closed this implies that $G_x$ acts transitively on $\tilde{\mathcal{W}}(x)$ and thus contains the subgroup $\mathcal{T}_x$ of translations of $\tilde{\mathcal{W}}(x)$. The corresponding isometries of $\tilde{\mathcal{W}}(x)$ also have to be translations and the lemma follows. □

**LEMMA 4.4.** *For almost every point $x$ and every Lyapunov foliation $\tilde{\mathcal{W}}$ the semiconjugacy $h$ is a $C^{1+\epsilon}$ diffeomorphism from $\tilde{\mathcal{W}}(x)$ into $\mathcal{W}(h(x))$.*

*Proof.* This follows immediately from Lemma 4.3. Indeed, the correspondence $\tilde{\tau} \to \tau$ is a continuous isomorphism between the groups of translations $\tilde{\mathcal{T}}$ and $\mathcal{T}$ of $\tilde{\mathcal{W}}(x)$ and $\mathcal{W}(x)$ respectively. Hence there exists $a \in \mathbb{R}$ such that if $\tilde{\tau}(y) = y + t$ for $y \in \tilde{\mathcal{W}}(x)$ then $\tau(z) = z + at$ for $z \in \mathcal{W}(h(x))$. Then $h \circ \tilde{\tau} = \tau \circ h$ implies that $h|_{\tilde{\mathcal{W}}x}$ is a linear map with respect to the affine parameter on $\tilde{\mathcal{W}}(x)$ and the standard affine parameter on $\mathcal{W}(x)$. Since the affine parameter on $\tilde{\mathcal{W}}(x)$ is given by a $C^{1+\epsilon}$ diffeomorphism, then so is $h$. □

4.3. **Conclusion of the proof of Theorem 1.7.** Fix a Lyapunov foliation $\tilde{\mathcal{W}}$. Let $\Lambda$ be a set of positive measure such that the semiconjugacy $h$ is differentiable at every $x \in \Lambda$ and the derivative $L(x)$ of the $h$ along $\tilde{\mathcal{W}}$ and its inverse are both bounded by a constant $C$. Such a set exists by Lemma 4.4. Suppose that both $x$ and $\alpha(\mathbf{t})(x)$ are in $\Lambda$. Let $v$ be a tangent vector at $x$ to $\tilde{\mathcal{W}}$. We have

$$\|D(\alpha(\mathbf{t}))v\| = L(x)\|D(\alpha_0(\mathbf{t}))|_{\mathcal{W}(h(x))}\|L^{-1}(\alpha(\mathbf{t})(x)\|v\|. \tag{4.4}$$

Since $\alpha_0$ is a linear action, $\|D(\alpha_0(\mathbf{t}))|_{\mathcal{W}(h(x))}\| = \exp\chi(\mathbf{t})$, where $\chi$ is the Lyapunov exponent of $\alpha_0$ corresponding to the foliation $\mathcal{W}$. Since by assumption

$$C^{-1} < \min\{L(x), L^{-1}(\alpha(\mathbf{t})(x))\} < \max\{L(x), L^{-1}(\alpha(\mathbf{t})(x))\} < C$$

we obtain from (4.4)

$$C^{-2}\exp\chi(\mathbf{t})\|v\| < \|D(\alpha(\mathbf{t}))v\| < C^2\exp\chi(\mathbf{t})\|v\|. \tag{4.5}$$

Now let $\tilde{\chi}$ be the Lyapunov exponent of the action $\alpha$ corresponding to the foliation $\tilde{\mathcal{W}}$. Take $\mathbf{s} \in \mathbb{R}^k$ such that $\alpha_0(\mathbf{s})$ is ergodic (the set of such $\mathbf{s}$ is dense in $\mathbb{R}^k$). Then for almost every $x \in \Lambda$ one can find a sequence of natural numbers $n_k \to \infty$ such that $\alpha(n_k\mathbf{s}) \in \Lambda$. Since for almost every $x$ and for $v \in T_x\tilde{\mathcal{W}}$

$$\lim_{k\to\infty} \frac{\log\|D\alpha(n_k\mathbf{s})(v)\|}{n_k} = \tilde{\chi}(\mathbf{s})$$

we conclude from (4.5) that $\tilde{\chi}(\mathbf{s}) = \chi(\mathbf{s})$. Since this is true for a dense set of $\mathbf{s}$, this implies that $\tilde{\chi} = \chi$ □

4.4. **Conclusion of the proof of Theorem 1.6.** For every Weyl chamber $\mathscr{C}_i$ we choose an element $\mathbf{m}_i \in \mathbb{Z}^k \cap \mathscr{C}_i$. For every $\mathbf{m}_i$ we choose a Pesin set $\Lambda_i$ and let $\Lambda = \bigcap_i \Lambda_i$. For a point $x$ in $\Lambda$ we denote by $B_r(x)$ the ball in $\mathbb{T}^{k+1}$ of radius $r$ centered at $x$. We fix $r$ sufficiently small compared to the size of the local manifolds at points of $\Lambda$.

**LEMMA 4.5.** *The semiconjugacy $h$ is injective on $B_r(x) \cap \Lambda$ for any $x \in \Lambda$.*

*Proof.* Let $y$ be a point in $B_r(x) \cap \Lambda$ different from $x$. Then there exists an element $\mathbf{m}_i$ such that $\tilde{W}^+_{\alpha(\mathbf{m}_i)}(x)$ is $k$-dimensional and does not contain $y$. Indeed, the intersection of all $k$-dimensional local unstable manifolds through $x$ contains only $x$ itself. We will denote in this proof $\tilde{W}^+_{\alpha(\mathbf{m}_i)}$ by $\tilde{F}$ and the complementary one-dimensional local Lyapunov foliation $\tilde{W}^-_{\alpha(\mathbf{m}_i)}$ by $\tilde{W}$. By Lemma 2.2, $h(\tilde{F}(z)) \subset F(h(z))$ and $h(\tilde{W}(z) \subset W(h(z)))$ for any $z \in B_r(x) \cap \Lambda$, where $F$ and $W$ are the corresponding local foliations for $\alpha_0$. Since both $x$ and $y$ are in $B_r(x) \cap \Lambda$, the intersection $\tilde{W}(x) \cap \tilde{F}(y)$ consists of exactly one point $z \in B_r(x)$. Then $h(z)$ is the unique point in the intersection $W(h(x)) \cap F(h(y))$. Suppose now that $h(x) = h(y)$. Then $h(z) = W(h(x)) \cap F(h(x))$, which means that $h(z) = h(x)$ and thus $h$ is not injective on $W(x)$. The latter, however, contradicts the fact that, according to Lemma 4.4, $h$ is a diffeomorphism on $W(x)$. □

Now we can complete the proof of Theorem 1.6 as follows. By compactness, the set $\Lambda$ can be covered by finitely many balls $B_r(x)$. Then the previous lemma implies that $h^{-1}(h(x)) \cap \Lambda$ is finite for any $x \in \Lambda$. Since we can choose the set $\Lambda$ to have arbitrarily large measure $\mu$, by taking an increasing sequence of such sets we can obtain an invariant set $A$ with $\mu(A) = 1$ such that $h^{-1}(h(x)) \cap A$ is at most countable for all $x \in A$. Now we consider the measurable partition of $A$ into the preimages of points under $h$. Since the elements of this partition are at most countable, the conditional measures are discrete. We see that $h|_A$ gives a countable extension of $(\alpha_0, \lambda)$. Since $(\alpha, \mu)$ is ergodic it follows that the the conditional measures of all points in the fiber are the same and hence the extension is finite. □

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