# ERGODIC FLOWS GENERATED BY A SYSTEM OF WEAKLY INTERACTING OSCILLATORS

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Over the last 10-15 years metric properties of smooth and, particularly, Hamiltonian dynamical systems were studied in many papers (see e.g. [1]-[6]).

As far as the author is aware of, all Hamiltonian systems with more than one degree of freedom for which ergodicity[1] has been proven, were flows with mixing (and even, with the only exception of the horocycle flow, turned out to be $K$-flows with transversal foliations).

Recently D. V. Anosov and the author [7] suggested an inductive construction allowing to obtain examples of ergodic diffeomorphisms of class $C^\infty$ with a smooth invariant measure on a wide class of manifolds. This construction leads to diffeomorphisms with metric properties quite different from properties of $K$-systems. These diffeomorphisms are not mixing, admit good cyclic approximation by periodic transformations in the sense of [8], and their spectrum may be discrete, continuous or mixed.

In this paper we set out a construction of ergodic Hamiltonian systems of a special kind that have many features in common with the construction from [7]. We consider a flow on $\mathbb{R}^{2m}$ $(m \geq 2)$ defined by a system of Hamiltonian differential equations with Hamiltonian

$$H(p,q) = \sum_{i=1}^{m} \alpha_i (p_i^2 + q_i^2) + H_1(p,q), \tag{1}$$

where $\alpha_1, \ldots, \alpha_m$ are arbitrary positive rational numbers, $H(p,q) > 0$ for $(p,q) \neq (0,0)$, and the perturbation $H_1(p,q)$ is constructed by the appropriate inductive process. Our result is the following

**Theorem.**[2] *Let natural numbers $s, k, 2 \leq k \leq m$ and an arbitrary small $\varepsilon > 0$ be given. There exists an infinitely differentiable function*

Translation from Russian by Svetlana Katok.

[1]We call a flow, defined by a system of Hamiltonian differential equations, ergodic if it is ergodic on surfaces of constant energy $H = c$ for almost all admitted values $c$. A similar meaning is used in the terms "mixing", "$K$-flow", "spectrum" etc. in relation to Hamiltonian systems.

[2]*Translator's note:* It was pointed out in [10] that the formulation of this theorem had an error. In fact, one can only claim that this flow is ergodic on each

*$H_1(p, q)$ which together with its partial derivatives up to order $s$ are bounded by the modulus in the whole space $\mathbb{R}^{2m}$ by the number $\varepsilon$, and such that the flow defined by the system of differential equations with Hamiltonian (1) is ergodic on each manifold $H^{-1}(c)$, $c > 0$ and has discrete spectrum generated by $k$ independent frequencies equal for all $c$.*

For $H_1 \equiv 0$ the function $H$ in (1) defines an integrable system of $m$ independent oscillators. From the point of view of the modern theory of perturbations of integrable Hamiltonian systems allowing to find $m$-dimensional invariant tori of the perturbed system, this case is quite degenerate. Our result shows that the conditions appearing in this theory are essential.

**Remark.** *The additional condition of commensurability of frequencies of oscillators of the unperturbed system, naturally, is not essential if we consider the flow in the bounded part of the space (for example, for bounded values of $H$).*

The function $H$ is constructed as a limit of fast converging, together with all derivatives, sequence of functions $H^{(n)}(p, q)$ such that the system with the Hamiltonian $H^{(n)}(p, q)$ defines a periodic flow $\{S_t^{(n)}\}$, provided that

$$H^{(0)}(p, q) = \sum_{i=1}^{m} \alpha_i (p_i^2 + q_i^2).$$

We will describe the main steps of the construction of the sequence $H^{(n)}$, limiting ourselves, in order not to complicate the notations, to the case $m = 2$ (in this case, of course, $k = 2$).

Let us introduce some notations. We denote by $\{S_t^{\alpha\beta}\}$ a flow defined by the system of equations with the Hamiltonian

$$\begin{aligned} F_{\alpha\beta}(p, q) &= \alpha(p_1^2 + q_1^2) + \beta(p_2^2 + q_2^2); \\ R_{\alpha\beta}^c &= \{(p, q) \,:\, F_{\alpha\beta}(p, q) = c\}; \\ M &= \{(p, q) \,:\, p_1 = q_1 = 0 \text{ or } p_2 = q_2 = 0\}; \\ L_{\alpha\beta}^c &= \{(p, q) \,:\, f_{\alpha\beta}(p, q) < c, (p, q) \notin M\}. \end{aligned}$$

The function $H^{(n)}(p, q)$ is defined by an inductive relation

$$H^{(n+1)}(p, q) = H^{(n)}(p, q) + F_{\delta_n 0}(B_{n+1}^{-1}(p, q)), \tag{2}$$

manifold $H^{-1}(c)$ and for almost all $c$ has discrete spectrum with $k$ equal independent frequences.

i.e.

$$H^{(n)}(p,q) = F_{\alpha_1\alpha_2}(p,q) + \sum_{k=0}^{n-1} F_{\delta_k 0}(B_{k+1}^{-1}(p,q)), \tag{3}$$

where $B_{n+1} = A_1 \circ \cdots \circ A_{n+1}$, and $A_{n+1}$ is the canonical diffeomorphism of the space $\mathbb{R}^{2m}$ onto itself whose properties will be listed below, $\delta_n$ is a (sufficiently small) positive number. $A_{n+1}$ and $\delta_n$ are parameters of the construction, moreover $\delta_n$ is chosen after the diffeomorphism $A_{n+1}$ is already constructed.

First of all, we require the diffeomorphism $A_{n+1}$ to be identity in some neighborhood of the set $M$ and outside of $L^{2^n}_{\alpha_1^{(n)}\alpha_2}$, where $\alpha_1^{(n)} = \alpha_1 + \sum_{k=0}^{n-1} \delta_k$. Secondly, $A_{n+1}$ should commute with the flow $\{S_t^{\alpha_1^{(n)}\alpha_2}\}$, or, which is the same, fix every ellipsoid $R^c_{\alpha_1^{(n)}\alpha_2}$. Since it is assumed that this requirement was satisfied for the construction of $A_k$ with $k \le n$, it follows from (3) that

$$H^{(n)}(p,q) = F_{\alpha_1^{(n)}\alpha_2}(B_n^{-1}(p,q)).$$

Before formulating the third condition on $A_{n+1}$, we have to make several remarks. By uniform continuity of the map $B_n$, there exists $\gamma_n > 0$ such that $\|B_n(x) - B_n(y)\| < \frac{1}{2^n}$ if $\|x - y\| < \gamma_n$.

Orbits of the flow $\{S_t^{\alpha_1^{(n)}\alpha_2}\}$ are closed, moreover, the periods of all orbits lying outside $M$ are the same. Identifying the points lying on one orbit we introduce on $L^{2^n}_{\alpha_1^{(n)}\alpha_2}$ a structure of a circle bundle over some three-dimensional open manifold $N$. Let $\pi : L^{2^n}_{\alpha_1^{(n)}\alpha_2} \to N$ be the projection to the base. Maps and flows acting on $L^{2^n}_{\alpha_1^{(n)}\alpha_2}$ and commuting with $\{S_t^{\alpha_1^{(n)}\alpha_2}\}$ project naturally to $N$. Let $Q_n$ be the period of the factor-flow $\{S_t^{10}|_N\}$. The diffeomorphism $A_{n+1}$ should be the identity outside of some compact $L_0 \subset L^{2^n}_{\alpha_1^{(n)}\alpha_2}$ and such that the factor-flow $\{A_{n+1} \circ S_t^{10} \circ A_{n+1}^{-1}|_N\} = \{S'_t\}$ acts on each set $\pi(R^c_{\alpha_1^{(n)}\alpha_2} \setminus M)$, $\frac{1}{2^{n-1}} \le c \le 2^{n-1}$ "almost ergodically". This means that it is possible to delete from $N$ a subset $N'$ whose intersection with each $\pi(R^c_{\alpha_1^{(n)}\alpha_2} \setminus M)$, $\frac{1}{2^{n-1}} \le c \le 2^{n-1}$ has small (given in advance) conditional measure in such a way that the intersection of some fundamental domain $\Delta$ of the diffeomorphism $S'_{\frac{Q_n}{k_n}}$ ($k_n$ is some natural number) and the sets $S'_{\frac{kQ_n}{k_n}}\Delta$, $k = 0, \ldots, k_n - 1$ with the set $\pi(R^c_{\alpha_1^{(n)}\alpha_2} \setminus M) \setminus N'$ have diameter less than $\gamma_n$. As $\Delta$ we choose

$$\Delta = \bigcup_{0\le t<\frac{Q_n}{k_n}} A_{n+1} S_t^{10} \pi(L^{2^n}_{\alpha_1^{(n)}\alpha_2} \cap L),$$

where $L$ is the hyperplane $p_2 = 0$ which is the fundamental domain for $\{S_t^{10}\}$ and $\mathbb{R}^4 \setminus M$.

The existence of the canonical diffeomorphism $A_{n+1}$ possessing the required properties should certainly be proven. In the author's opinion, there should be a proof of general nature based on the deformation of symplectic structures (some considerations to this effect are contained in [9]). The author's proof is of more direct nature. It is based on introduction of a special system of canonical coordinates, that allows to reduce the problem to a consideration of canonical (i.e. simply area-preserving) diffeomorphisms of the two-dimensional plane.[3] However, this proof can be modified to a more general situation.

The map $A_{n+1}$ extends as the identity to the whole space $\mathbb{R}^4$. After that the choice of sufficiently small $\delta_n$ in the formula (2) allows to achieve the following:

1. Sufficient closeness of $H^{(n)}(p,q)$ and $H^{(n+1)}(p,q)$ with a given number of derivatives, and this closeness in its turn provides convergence of the sequence $H^{(n)}(p,q)$ with all derivatives, sufficient smallness of $H_1(p,q)$ in the formula (1), and closeness of the conditional measures on the surfaces $H_c = \{(p,q) : H(p,q) = c\}$ and $H_c^n = \{(p,q) : H^n(p,q) = c\}$.
2. "Almost ergodicity" of the flow $\{S_t^{(n+1)}\}$ on each manifold $R^c_{\alpha_1^{(n)}\alpha_2} \setminus M$, $\frac{1}{2^{n-1}} \le c \le 2^{n-1}$ in the sense analogous to the one described above.
3. To construct periodic flows $\{\hat{S}_t^{(n)}\}$ preserving $H_c$ and close to the flows $\{S_t^{(n)}\}$ so that for some $Q_n$ the sequence of diffeomorphisms $\hat{S}^{(n)}_{Q_n}$ defines on $H_c$ a cyclic a.p.t. of the flow $\{S_t\}$ defined by the function $H(p,q)$ (with sufficiently high speed) which allows to construct a coordinated system of metric isomorphisms between $\{\hat{S}_t^{(n)}\}$ on $H_c$ and a periodic winding $\{R_t^{(n)}\}$ of the two-dimensional torus: $R_t^{(n)}(\varphi_1,\varphi_2) = (\varphi_1 + \alpha_1^{(n)} t, \varphi_2 + \alpha_2 t)\}$ converging to a metric isomorphisms of $\{S_t\}$ on $H_c$ and an ergodic winding $\{R_t\} = \lim\{R_t^{(n)}\}$.

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