MORE ABOUT BIRKHOFF PERIODIC ORBITS AND

MATHER SETS FOR TWIST MAPS

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This is a preliminary version of results obtained mostly in collaboration with John Mather. The final version will appear as a joint paper. The reasons for this separated presentation are mostly stylistical. Although the main ideas involved both in the formulation and in the proofs appeared in collaboration, each of us has his own favourite way to present these ideas. Mather prefers a rather analytical language which he first developed in his landmark paper [1], whereas I follow a more geometric style of my paper [2] where I gave simpler proofs and slightly generalized Mather's results.

This text is a continuation of [2] so that we shall use all definitions and notations from this paper without any special notice. We shall always assume that f is a twist homeomorphism of the annulus preserving a measure positive on open sets. Additional assumptions will be mentioned in appropriated places.

1. Homoclinic and heteroclinic orbits for Birkhoff orbits

Let $\frac{p}{q}$ be a rational number belonging to the twist interval for f. Let us consider a sequence of irrational numbers α_n converging to $\frac{p}{q}$ and minimal Mather sets E_n with rotation numbers α_n. By replacing α_n by a subsequence, if necessary, we can always assume that E_n converge in Hausdorff topology to a set E which by Proposition 3 [2] is a Mather set for f with rotation number $\frac{p}{q}$. All non-wandering points of the restriction $f|_E$ are, of course, Birkhoff periodic points of type (p,q). Let E_o be the subset of E consisting of all those points. We claim that $E \setminus E_o$ is a non-empty set, unless E_o is a circle. Moreover, let us consider the projection of E_o to the circle. The complement to this projection consists of finite or countable number of intervals.

Proposition 1 : Every such interval contains a projection of a non-periodic point from E .

Proof : If among the sets E_n there are infinitely many circles then E is also a circle and the statement is obviously correct. To prove it in the case where all but finitely many of E_n's are Cantor sets, we shall consider the behaviour of holes in those sets as n goes to infinity. A hole H in a Mather set is determined by two points of the set, whose projections to the circle bound an interval which is complemetary to the projection of the set. The length of this interval will be called the length of the hole H and denoted $\ell(H)$. The image $f(H)$ of the hole H has an obvious meaning.

Let H be a hole in E determined by points x and y . In order to prove the proposition it is enough to show that either x or y is not a periodic point. To prove that, let us consider a sequence H_n of holes in E_n converging to H . Such a sequence obviously exists. Since the rotation numbers of the sets E_n are irrational, for every hole H_n the projections of all its images are disjoint so that

$$(1) \qquad \sum_{m=0}^{\infty} \ell(f^m H_n) \leq 1 \quad .$$

Let us assume now that the hole H is periodic, i.e. both x and y are periodic points. Then for some $c > 0$ and every $N \geq 1$

$$(2) \qquad \sum_{m=0}^{N} \ell(f^m H) > cN \quad .$$

Since H_n converge to H we have for every m

$$(3) \qquad \ell(f^m H_n) \longrightarrow \ell(f^m H) \quad .$$

Obviously (1), (2) and (3) are incompatible. □

For every non-periodic point $z \in E$, its α-limit set is the orbit of the periodic point which is projected into an endpoint of the complementary interval to E_o, containing the projection of z the set $\omega(z)$ is the orbit of the point which projects into the other end of the same interval. Thus, z is either a homo- or a heteroclinic point to some periodic points. Let us summarize the information we have obtained for the typical case when E_o is a finite set.

Corollary 1 : Let E_o consist of finitely many Birkhoff periodic orbits, say $\gamma_1,\dots,\gamma_s$, ordered accordingly to a cyclic order on the circle. Then there exist heteroclinic (homoclinic if $s = 1$) orbits, $\sigma_1,\dots,\sigma_s$, such that either

$$\begin{aligned}
\gamma_1 &= \omega(\sigma_s) = \alpha(\sigma_1) \\
\gamma_2 &= \omega(\sigma_1) = \alpha(\sigma_2) \\
&\vdots \\
\gamma_s &= \omega(\sigma_{s-1}) = \alpha(\sigma_s) \quad .
\end{aligned}$$

or

$$\begin{aligned}
\gamma_1 &= \alpha(\sigma_s) = \omega(\sigma_1) \\
\gamma_2 &= \alpha(\sigma_1) = \omega(\sigma_2) \\
&\vdots \\
\gamma_s &= \alpha(\sigma_{s-1}) = \omega(\sigma_s) \quad .
\end{aligned}$$

In both cases we can say that $\sigma_1,\dots,\sigma_s$ form a homoclinic link for $\gamma_1,\dots,\gamma_s$.

2. Conditional minima for the Lagrangian

Let us consider a Birkhoff periodic orbit for f. By lifting this orbit to the universal cover and projecting the lift to the line we obtain a map

$$\psi : \mathbb{Z} \longrightarrow \mathbb{R}$$

(cf. details in [2], section 4).

Let us consider the following subspace $\Phi^{\psi}_{p,q}$ of the space $\Phi_{p,q}$:

$$\Phi^{\psi}_{p,q} = \{\varphi \in \Phi_{p,q} \ , \ \psi(n) \leq \varphi(n) \leq \psi(n+1) \ , \ n \in \mathbb{Z}\} \quad .$$

This space is naturally fibered by smaller spaces $\Phi^{x,\psi}_{p,q}$ where $\psi(0) \leq x \leq \psi(1)$

$$\Phi^{x,\psi}_{p,q} = \{\varphi \in \Phi^{\psi}_{p,q} \ , \ \varphi(0) = x\} \quad .$$

Notice that in those cases the identification of integral shifts is not necessary.

Obviously $\Phi^{\psi}_{p,q}$ and all $\Phi^{x,\psi}_{p,q}$ are compact. Let us show that they are non-empty. For the map ψ can be extended to a homeomorphism $\tilde{\psi} : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\tilde{\psi}(t+q) = \tilde{\psi}(t)+1$$

and

$$g_o(\tilde{\psi}(t)) \leq \tilde{\psi}(t+p) \leq g_1(\tilde{\psi}(t)) \quad .$$

Such an extension can be constructed separately on each interval $[k,k+1]$, $k = 0,\ldots,q-1$ and then continued by periodicity. Since for some $t \in \mathbb{R}$ we have $\tilde{\psi}(t) = x$ we obtain an element of $\Phi^{x,\psi}_{p,q}$ by putting $\varphi(n) = \tilde{\psi}(t+n)$.

The following lemma will be used later in this section for the study of the Lagrangian $L_{p,q}$ restricted to the spaces $\Phi^{x,\psi}_{p,q}$ as well as in section 4 for the construction of a second "minimax" Birkhoff periodic orbit associated with an orbit which gives minimum to $L_{p,q}$.

Lemma 2.1. If $\varphi \in \Phi^{\psi}_{p,q}$ and $\varphi(k) = \psi(k+1)$ then for that φ , $h_1(k) \geq h_2(k)$ and the inequality is strict iff $\varphi(k-p) < \psi(k-p+1)$.

Similarly, if $\varphi(k) = \psi(k)$ then $h_1(k) \leq h_2(k)$ and this inequality is strict iff $\varphi(k+p) > \psi(k+p)$.

Proof : Let us consider the first case (cf. Figure 1).

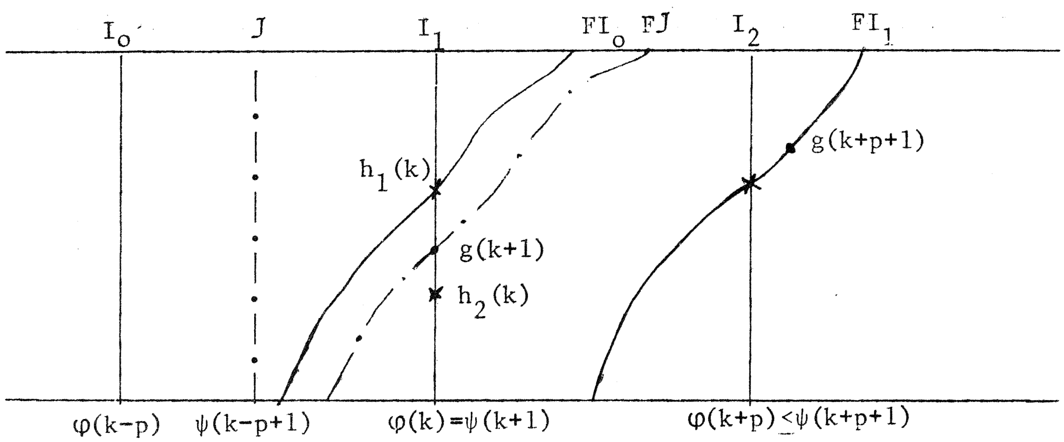

Figure 1

Let $(\psi(n),g(n))$ be the coordinate form of the Birkhoff orbit determined by ψ . From the assumption of the lemma and the twist condition we have

$$h_1(k) \geq g(k+1)$$

and the inequality is strict iff $\varphi(k-p) < \psi(k-p+1)$. However,

$$F(\psi(k+1),g(k+1)) = (\psi(k+p+1),g(k+p+1))$$

and since $\psi(k+p+1) \geq \varphi(k+p)$ the point $(\psi(k+p+1),g(k+p+1))$ lies further to the right or coincides with $F(\varphi(k),h_2(k))$. This implies that $h_2(k) \leq g(k+1)$. This proves the first statement of the lemma.

The second case is considered similarly ; namely we prove that $h_2(k) \geq g(k)$ and the inequality is strict iff $\varphi(k+p) \geq \psi(k+p)$ and then show that $g(k) \geq h_1(k)$. □

Lemma 2.2. : If the functional $L_{p,q}$ restricted to the space $\Phi^{x,\psi}_{p,q}$ reaches its local minimum at φ then for that φ and for $n \not\equiv 0 \pmod q$

$$h_1(n) = h_2(n) .$$

Proof : Let us first put together the complete definition of the space $\Phi^{x,\psi}_{p,q}$. Namely, this space consists of all maps $\varphi : \mathbb{Z} \longrightarrow \mathbb{R}$ such that

(4) $$\varphi(0) = x \quad ,$$
$$\varphi(n+q) = \varphi(n)+1 \quad ,$$

(5) $$g_o(\varphi(n)) \leq \varphi(n+p) \leq g_1(\varphi(n)) \quad ,$$

(6) $$\psi(n) \leq \varphi(n) \leq \psi(n+1) \quad .$$

and the topology in this space is defined by the embedding into $\mathbb{R}^q$ given by $\varphi \longrightarrow (\varphi(0),\ldots,\varphi(q-1))$.

Not only the formulation but also the proof of our lemma is but a minor modification of those for the lemma from section 4 of [2] .

The argument from that proof applies directly to the case when inequalities (6) are strict for all n . That argument shows that if $h_1(n) \neq h_2(n)$ for some n which is not an integer multiple of q , then φ can be modified by an arbitrary small amount within $\Phi^{x,}_{p,q}$ so that the value of the functional

$L_{p,q}$ decreases. For, it is obvious that the perturbation keeps all conditions except for, probably condition (4). However, by (6) $\varphi(1) > \varphi(0) > \varphi(-1)$ so that $\varphi(0)$ remains unchanged by the perturbation.

Thus, it remains to prove that if for $\varphi \in \Phi^{x,\psi}_{p,q}$ one of the inequalities (6) is not strict and for some n, $n \not\equiv 0 \pmod q$, $h_1(n) \neq h_2(n)$ then the value of $L_{p,q}$ can be decreased by an arbitrary small perturbation within $\Phi^{x,\psi}_{p,q}$. In this case there exists k, $0 \leq k < q$ such that either

(7) $$\varphi(k-p) < \psi(k-p+1) \;,\; \varphi(k) = \psi(k+1)$$

and then by lemma 2.1,

(8) $$h_1(k) > h_2(k)$$

or

(9) $$\varphi(k) = \psi(k) \;,\; \varphi(k+p) > \psi(k+p)$$

and then again by lemma 2.1,

$$h_2(k) > h_2(k) \quad .$$

In the first case (i.e. where (7) and (8) are satisfied) we have

(10) $$\varphi(k) = \psi(k+1) > g_o(\psi(k-p+1)) > g_o(\varphi(k-p))$$

and

(11) $$\varphi(k+p) \leq \psi(k+p+1) < g_1(\psi(k+p)) = g_1(\varphi(k)) \;.$$

The inequalities (8), (10) and (11) assure that we remain in $\Phi^{x,\psi}_{p,q}$ if we replace $\tilde{\varphi}$ by $\tilde{\varphi}_\varepsilon$ where

$$\tilde{\varphi}_\varepsilon(n) = \begin{cases} \varphi(n) & , \quad n \neq k, \bmod q \\ \varphi(n)-\varepsilon & , \quad n = k, \bmod q \end{cases}$$

for sufficiently small $\varepsilon > 0$.

Now the argument from the proof of the above mentioned lemma applies and shows that $L_{p,q}(\tilde{\varphi}_\varepsilon) < L_{p,q}(\varphi)$. (cf. figure 2, [2])

The case when conditions (8) are satisfied is considered similarly. In this case we apply the argument in the situation reflected by Figure 3 from [2].□

Since the space $\Phi^{x,\psi}_{p,q}$ is compact and non-empty and the functional $L_{p,q}$ is continuous, this functional always reaches its minimum on $\Phi^{x,\psi}_{p,q}$. By lemma 2.2 the map φ where this minimum is reached determines a map $\mathbb{Z} \longrightarrow S$, $n \longrightarrow (\varphi(n),h(n))$ such that for $n \neq 0 \mod q$

$$F(\varphi(n),h(n)) = (\varphi(n+p),h(n+p))$$

and since $\varphi(0) = x$ and p and q are relatively prime

$$F^q(x,h(0)) = (x,h(q)) \qquad .$$

The projection to the annulus gives an orbit of a point y which moves around the circle respecting the order defined by the iterates of the original Birkhoff periodic point z and after q iterates returns to the same vertical segment.

Let us denote

(12) $$f^n z = (\varphi_n, r_n) \qquad \text{as in [2] and}$$

(13) $$f^n y = (\varphi'_n, r'_n) \quad , \quad n = 0,1,\dots,q \qquad .$$

Condition (6) allows us to prove the following extension of Proposition 1 from [2] .

<u>Proposition 2</u> : If $|\varphi_n - \varphi'_m| < r$ then $|r_n - r'_m| < \gamma(r)$ and if $|\varphi'_n - \varphi'_m| < r$

then $|r'_n - r'_m| < \gamma(r)$ for $m,n = 1,\dots,q-1$ and for the same function γ as in Proposition 1 [2] .

The proof of the mentioned proposition directly applies to this case. The exclusion of $m,n = 0$ or q allows to move points both forward and backward which is crucial for the application of the twist condition.

Simply by reformulating the statements of two propositions we obtain the following result.

Corollary 2 : The orbit of z and a part of the orbit of g namely $f^n y$, $n = 1,\dots,q-1$ form together a graph of a function defined on a finite subset of the circle with the values at $[0,1]$. This function has module of continuity γ ; for a Lipschitz-twist map this function satisfies the Lipschitz condition with the constant depending only on f .

The points y and f^q_y are excluded from this picture. Apparently, it is unavoidable if z is an arbitrary Birkhoff periodic point. Very likely candidate for a counterexample is the case when both eigenvalues at z are -1. However, if ψ is a point of minimum for the functional $L_{p,q}$ and the invariant measure is given a density bounded away from 0 and ∞ then the two remaining points can be included into the picture. We shall discuss this case assuming in addition in order to avoid some unpleasant, but purely technical complications that f is a Lipschitz twist map. Throughout the rest of this section we assume that all these extra conditions are satisfied.

Let k be the only number $0 \le k \le q-1$ such that the coordinate $\varphi'_o = \varphi'_q$ lies on the circle between φ_o and φ_k .

Proposition 3 : There exists a constant C such that $\max(|r'_o - r_o|, |r'_q - r_o|) < C \min(|\varphi'_o - \varphi_o|, |\varphi'_o - \varphi_k|)$.

The proof of this proposition is based on several lemmas which will also be used in subsequent sections. We shall work on the universal covering again. Let us remind that $\varphi(0) = x$ and $\psi(0) \leq x \leq \psi(1)$.

Lemma 2.3 : There exists a constant C' such that $L_{p,q}(\varphi)-L_{p,q}(\psi) > C'|h(q)-h(0)| \times \min(|h(q)-h(0)|, x-\psi(0), \psi(1)-x)$.

Proof : Without loss of generality we can assume that x is not too close to $\psi(1)$. This case can be reduced to the one we are going to treat by considering F^{-1} instead of F and changing notation appropriately.

Cases $h(q) > h(0)$ and $h(q) < h(0)$ are presented on figures 2 and 3 which reproduce in our notation corresponding figures from [2] . Both cases are treated similarly but we shall give more details for the latter one because it is the only case which is needed for the proof of Proposition 3.

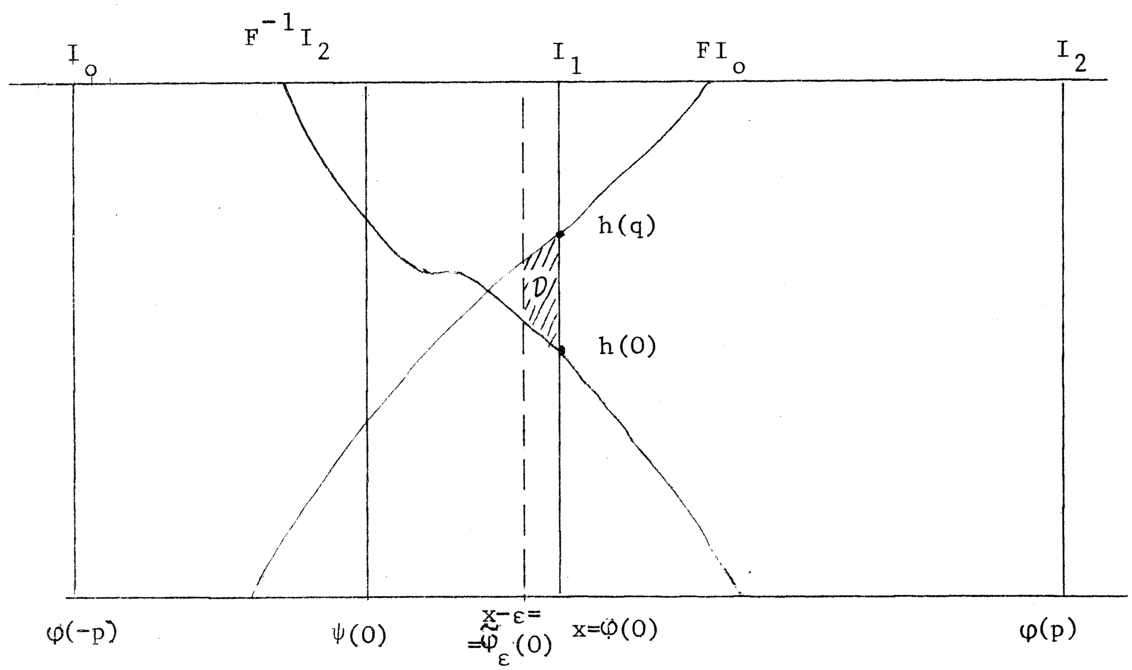

Figure 2

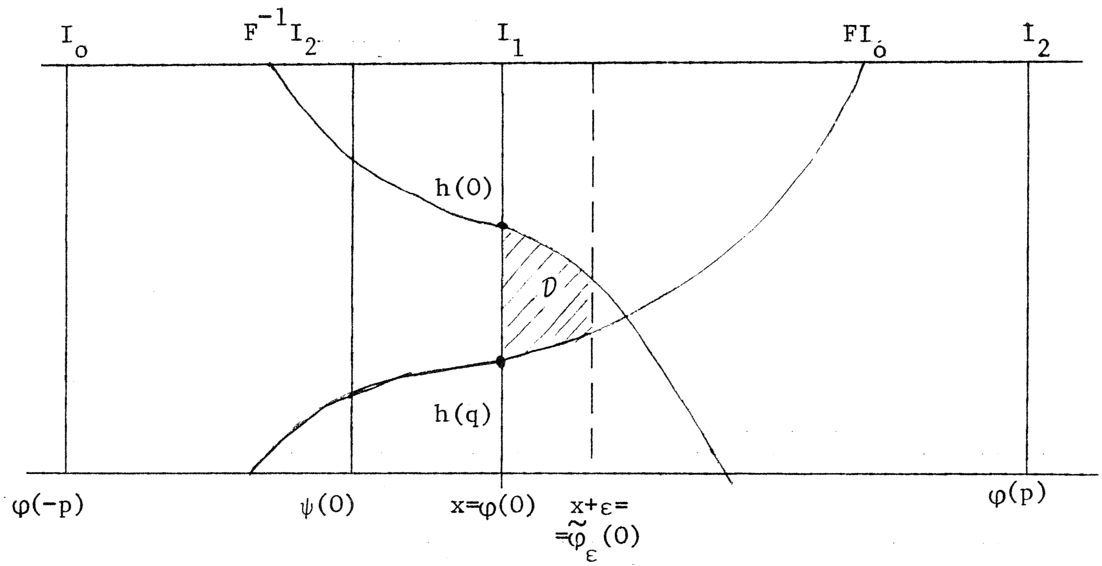

Figure 3

In this case (cf. Figure 3) we replace φ by $\tilde{\varphi}_\varepsilon$ where

$$\tilde{\varphi}_\varepsilon(n) = \begin{cases} \varphi(n) & \text{if } n \not\equiv 0 \mod q \\ \varphi(n)+\varepsilon & \text{if } n \equiv 0 \mod q \end{cases}$$

and thus diminish the value of the functional $L_{p,q}$ by the measure of the shaded domain $\mathcal{D}$ (see [2], p. 13).

Because of the assumption about the invariant measure we have

$$L_{p,q}(\tilde{\varphi}_\varepsilon) < L_{p,q}(\varphi) - C_1 \text{ area } \mathcal{D} \tag{14}$$

where C_1 is independent of ψ . By the Lipschitz twist condition the curves $F^{-1}I_2$ and FI_o bounding $\mathcal{D}$ have bounded slope so that

$$\text{area } \mathcal{D} > C_2(h(0)-h(q))(\min(\varepsilon,h(0)-h(q))) \tag{15}$$

for another constant C_2 .

If ε is sufficiently small then $\tilde{\varphi}_\varepsilon \in \Phi_{p,q}$. On the other hand, we can assume that $x-\psi(0)$ is sufficiently small because otherwise the desired inequality is trivial. Thus, in particular, $\tilde{\varphi}_{x-\psi(0)} \in \Phi_{p,q}$ and since $L_{p,q}$ reaches its minimum at ψ we have from (14) and (15)

$$L_{p,q}(\psi) \leq L_{p,q}(\tilde{\varphi}_{x-\psi(0)}) < L_{p,q}(\varphi) - C_1C_2(h(q)-h(0)) \times$$

$$\times (\min(h(0)-h(q), x-\psi(0))).$$

If $h(q) > h(0)$ we move $\varphi(0)$ leftward (figure 2) and proceed with similar argument. □

<u>Lemma 2.4</u>. There exists a constant C'' such that

$$L_{p,q}(\varphi) < L_{p,q}(\psi) + C''(\min(x-\psi(0),\psi(1)-x))^2 \quad .$$

<u>Proof</u> : As before we can assume that x is not too close to $\psi(1)$. First we shall show that $L_{p,q}(\varphi)-L_{p,q}(\psi)$ is bounded from above by a constant which depends only on f but not on a particular Birkhoff periodic orbit. It is enough to show that the function $M(x)$ defined by

$$M(x) = \min_{\varphi\in\Phi_{p,q}^{x,\psi}} L_{p,q}(\varphi) \tag{17}$$

is a Lipschitz function of x with a uniform Lipschitz constant. But that easily follows from the construction of the previous lemma. Namely, since for small ε $M(x+\varepsilon) \leq L_{p,q}(\tilde{\varphi}_\varepsilon)$ we have

$$|M(x+\varepsilon)-M(x)| < C_3 \ \text{area}\ \mathcal{D} < C_3\varepsilon \quad , \tag{18}$$

where C_3 depends only on the density of the invariant measure. Now we can assume that x is close to $\psi(0)$.

Let $\widetilde{\psi}_\varepsilon$ be defined by

$$\widetilde{\psi}_\varepsilon(n) = \begin{cases} \psi(n) & \text{if } n \not\equiv 0 \mod q \\ \\ \psi(n)+\varepsilon & \text{if } n \equiv 0 \mod q \end{cases} .$$

If ε is small enough then $\widetilde{\psi}_\varepsilon \in \Phi_{p,q}^{\psi(0)+\varepsilon,\psi}$; in particular we can assume that $\widetilde{\psi}_{x-\psi(0)} \in \Phi_{p,q}^{x,\psi}$ and since φ is a point of minimum for $L_{p,q}$ on $\Phi_{p,q}^{x,\psi}$ we have

$$L_{p,q}(\widetilde{\psi}_{x-\psi(0)}) \geq L_{p,q}(\varphi) \tag{1.9}$$

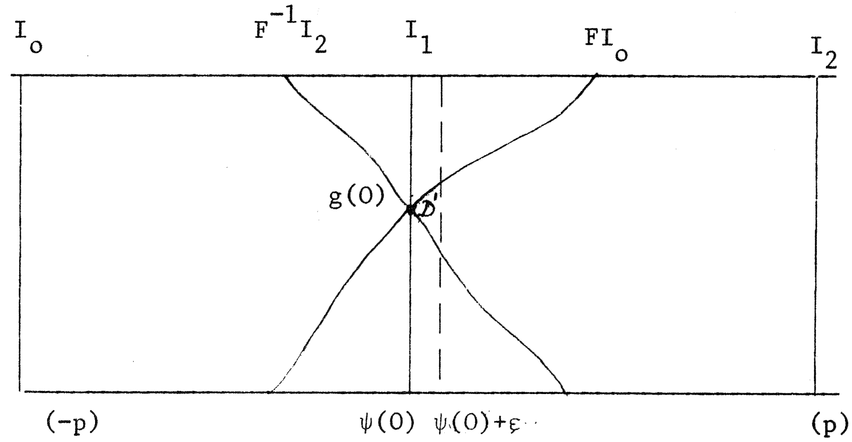

Figure 4

From the same consideration as above we obtain that

$$L_{p,q}(\widetilde{\psi}_\varepsilon) = L_{p,q}(\psi) + \mu(\mathcal{D}') \leq \tag{20}$$

$$\leq L_{p,q}(\psi) + C_3 \text{ area } \mathcal{D}' \leq L_{p,q}(\psi)+C_3C_4\varepsilon^2 .$$

The lemma follows now from (18), (19) and (20). $\square$

Lemma 2.5. $h(0)-h(q) < C'''(x-\psi(0))$.

<u>Proof</u> : From Lemmas 2.4 and 2.3 we obtain

$$L_{p,q}(\varphi) \leq L_{p,q}(\psi) + C''(x-\psi(0))^2 \leq$$

$$\leq L_{p,q}(\varphi) - C'(h\ (0)-h(q)) \times \min(h(0)-h(q), x-\psi(0)) + C''(x-\psi(0))^2 \quad .$$

This implies that

$$h(0)-h(q) < C'''(x-\psi(0)) \quad . \qquad \square$$

<u>Proof of Proposition 3</u> : The subsequent argument is essentially the repetition of the argument from the proof of Proposition 1 from [2].

Let us first assume that $h(0) < g(0)$ and consider the images of three points $(\psi(0), g(0)), (x, h(0))$ and $(\psi(0), h(0))$. The first two images are respectively $(\psi(p), g(p))$ and $(\varphi(p), h(p))$. Let us denote $F(\psi(0), h(0))$ by $(\tilde{\psi}, \tilde{h})$. By the Lipschitz twist condition

$$\tilde{\psi} < \psi(p) - C_5(g(0)-h(0))$$

and since F is a Lipschitz map

$$\varphi(p) > \tilde{\psi} - C_6(x-\psi(0)) \quad .$$

But since $\varphi(p) > \psi(p)$ we have

(21) $$g(0)-h(0) < C_7(x-\psi(0)) \quad .$$

Similarly, assuming that $h(q) > g(0)$ and taking pre-images of appropriate points we obtain

(22) $$h(q)-g(0) < C_8(x-\psi(0)) \quad .$$

If $h(q) < g(0)$ we have from Lemma 2.5 and (21)

$$h(q) > h(0)-C'''(x-\psi(0)) > g(0) - C_9(x-\psi(0)) \quad . \tag{23}$$

Similarly, if $h(0) > g(0)$ we derive from Lemma 2.5 and (22)

$$h(0) < h(q) + C'''(x-\psi(0)) < g(0) + C_9(x-\psi(0)) \; . \tag{24}$$

Inequalities (21) and (24) imply that

$$|h(0)-g(0)| < C(x-\psi(0))$$

and similarly from (22) and (23) we see that

$$|h(q)-g(0)| < C(x-\psi(0)) \quad ,$$

Those two inequalities together with the similar ones for the case when x is close to $\psi(1)$ imply the statement of the proposition. □

Now we can formulate a stronger version of uniform Lipschitz property for the orbit of y and z .

Corollary 3 : The points $f^n y$, $f^n z$, $n = 0,\ldots,q-1$ form a graph of a Lispschitz function with a Lipschitz constant depending only on f defined on a finite subset of the circle. The same is true for the points $f^n(f^q y)$, $f^n z$, $n = 0,-1,\ldots,-q+1$.

3. "Stable and unstable manifolds" for Mather sets

In [2] section 3 we used a simple limit process to obtain Mather sets from Birkhoff periodic orbits. Exactly the same process allows to constuct orbits which are asymptotic to Mather sets from orbits constructed in the previous section. The properties of those orbits depend on the information for the rational rotation number case. Thus, we shall get less in a more general situation which was summarized in Corollary 2 and more for a more special situation described in Corollary 3 .

Let us fix an irrational number α from the twist interval for f and consider a sequence of rational numbers $\frac{p_n}{q_n}$ converging to α and corresponding sequence of Birkhoff periodic orbits of type (p_n, q_n) we shall denote those orbits $\Gamma_n = \{z_n, \dots, f^{q-1} z_n\}$. Without loss of generality we can assume that Γ_n converge in Hausdorff topology to a Mather set E ; by [2] , Proposition 3, $\rho(E) = \alpha$. Let us assume that E is not a circle. Then it consists of a minimal Cantor set E_o and a certain number (finite or infinite) of orbits whose α- and ω-limit sets coincide with E_o .

Let $\varphi \in S^1 \setminus$ projection of E_o . Let us consider for every n the orbit $\Gamma_{n,\varphi} = \{y_n, \dots, f^{q_n} y_n\}$ as described in the previous section. Remind that both y_n and $f^{q_n} y_n$ are projected into φ . Without loss of generality we can assume that $\Gamma_n \cup \Gamma_{n,\varphi}$ converge in Hausdorff topology to a set which we denote by E_φ .

In particular, $y_n \longrightarrow y = (\varphi, r)$ $f^{n} y_n \longrightarrow y' = (\varphi, r')$. It follows from Corollary 2 that $E_\varphi \setminus (\{y\} \cup \{y'\})$ is a graph of a function with module of continuity γ defined on a certain closed subset of the circle. Moreover, if $y = y'$ then $fE_\varphi = E_\varphi$;if $y \neq y'$ then $fE_\varphi \,\Delta\, E_\varphi = y$, $f^{-1} E_\varphi \,\Delta\, E_\varphi = y'$.

Suppose that φ belongs to the hole H_o for E_o in the sense that the complementary interval to the projection of E_o which corresponds to E_o contains φ . Then the projection of f^n_y , $n \geq 0$ belongs to $f^n H_o$.

This follows from the convergence of $\Gamma_{n,\varphi}$ and the preservation of order for $\Gamma_n \cup \Gamma_{n,\varphi}$. In particular, $\alpha(y') = \omega(y) = E_o$. Moreover, every orbit in E_φ different from the orbits of y and y' has E_o as both its α- and ω-limit sets.

If the assumptions of Proposition 3 are satisfied we can add an extra bit of information, namely each of the sets $E_\varphi \setminus \{y\}$ and $E_\varphi \setminus \{y'\}$ is a graph of a Lipschitz function.

If we take for every Γ_n all orbits obtained by minimizing the functional L_{p_n,q_n} on $\Phi^{x,\psi}_{p_n,q_n}$ for every x, we obtain for every n a closed set $\tilde{\Gamma}_n \supset \Gamma_n$ which in general is not a graph of the function. However the Hausdorff limit of this set which we shall denote by $\tilde{E}$ has some nice properties which are summarized in the following theorem :

<u>Theorem 1</u> : (i) $\tilde{E} = E_1 \cup E_2$ and for every point $y \in E_1$, $\omega(y) = E_o$ and for $y \in E_2$, $\alpha(y) = E_o$. Moreover, the projections of the point $f^n y$ for $y \in E_1$, $n > 0$ (corr. $y \in E_2$, $n < 0$) lie in the n-th image of the holes in E_o containing the projection of y.

(ii) E_1 and E_2 intersect every vertical interval $\{\varphi\} \times [0,1]$ at least at one point.

(iii) If $(\varphi, r) \in fE_1$ and $(\varphi_o, r_o) \in E_o$ then $|r-r_o| \leq \gamma(|\varphi-\varphi_o|)$ for $\gamma(r)$ from Proposition 1 of [1] . Similarly for $(\varphi, r) \in f^{-1}E_2$.

(iv) If the conditions of Proposition 3 are satisfied then for $(\varphi, r) \in E_1 \cup E_2$ $(\varphi_o, r_o) \in E_o$,

$$|r-r_o| \leq K|\varphi-\varphi_o|$$

where the constant K depends only on f.

Remark : It is interesting to know whether the set $E_1 \cap E_2 \setminus E_o$ is non-empty. If it is, it consists of orbits homocyclic to E_o . We can prove the existence of homoclinic orbits to E_o using a different method, namely an infinite-dimensional version of the method from the next section.

4. Minimax Birkhoff orbits

For the rest of this paper we shall assume that f is a Lipschitz twist map preserving a measure given by a positive Lipschitz density. A natural local coordinate system on $\Phi_{p,q}$ is given by coordinates $\varphi_i = \varphi(i)$, $i = 0,\dots,q-1$. This coordinate system becomes global on $\Phi^{\psi}_{p,q}$. In this case $h_1(n)$ and $h_2(n)$ for every n are Lipschitz functions of coordinates φ_i . Let $\rho(x,y)dx\,dy$ be the lift of the invariant measure to the universal covering. Then our computations of the variation of $L_{p,q}$ in [2] pp. 13-14 show that for $\varphi = (\varphi_o\cdots\varphi_{q-1}) \in \Phi_{p,q}$

$$\frac{\partial L_{p,q}}{\partial\varphi i}(\varphi) = \int_{h_2(i)}^{h_1(i)} \rho(\varphi_i,t)\,dt \tag{25}$$

and those derivatives are Lipschitz functions too. In particular if φ is a critical point of $L_{p,q}$ then $h_2(n) = h_1(n)$ for all n , and φ determines a Birkhoff periodic orbit. If the map ψ determines a Birkhoff periodic orbit then there are always two critical points for $L_{p,q}$ in the space $\Phi^{\psi}_{p,q}$, namely ψ and $\widetilde{\psi}$ given by $\psi(n) = \psi(n+1)$. Both of them obviously lie on the boundary of $\Phi^{\psi}_{p,q}$ and $L_{p,q}(\psi) = L_{p,q}(\widetilde{\psi})$.

Proposition 4 : The functional $L_{p,q}$ has at least one critical point φ inside $\Phi^{\psi}_{p,q}$.

Proof : First of all, we can by a simple trick omit inequalities (5) from the definition of the space $\Phi^{\psi}_{p,q}$. We simply continue f to a larger annulus by adding sufficiently strong twist and then (6) imply (5).

Thus $\Phi_{p,q}$ is simply a parallelepiped in the space $\mathbb{R}^q$ given by inequalities

$$\psi(0) \leq \varphi_o \leq \psi(1)$$

$$\psi(1) \leq \varphi_1 \leq \psi(2)$$

(26)

$$\psi(q-1) \leq \varphi_{q-1} \leq \psi(q) = \psi(0)+p$$

and ψ and $\tilde{\psi}$ are two vertices of this parallelepiped. To simplify the notations we shall denote $\Phi^{\psi}_{p,q}$ by Π . Let us consider the vector-field $V(\varphi) = -\text{grad } L_{p,q}(\varphi)$ on $\Phi^{\psi}_{p,q}$. By (25) it is a Lipschitz vector field and ψ and $\tilde{\psi}$ are attracting points for it. If we can show that orbits of V which begin at the boundary of $\Phi^{\psi}_{p,q}$ stay within $\Phi^{\psi}_{p,q}$ that would imply that $L_{p,q}$ has at least one more critical point which is not a local minimum. But this statement about the orbits follows easily from Lemma 2.1. For; if φ belongs to a face of the boundary, i.e. if only one of inequalities (26) becomes an equality, say $\varphi_i = \psi(i)$ then by that lemma $h_1(i) < h_2(i)$ so that V is transversal to the face $\varphi_i = \psi(i)$ and looks inside the parallelepiped. A similar argument works if $\varphi_i = \psi(i+1)$. Now imagine that some orbit of V_φ which begins at the boundary eventually leave $\Phi^{\psi}_{p,q}$. Then the same is true for an open set of initial conditions on the boundary and this open set must leave the parallelepiped through a set which contains an open set and subsequently intersects a face, which is impossible. Thus $L_{p,q}$ has a critical point φ in $\Phi^{\psi}_{p,q}$ which is not a local minimum. But Lemma 2.1 shows that on the boundary the vector-field is non-zero except for ψ and $\tilde{\psi}$. Thus, $\varphi \in \text{Int } \Phi^{\psi}_{p,q}$. □

Let us reformulate the result in terms of orbits on the annulus.

<u>Corollary 4</u> : Under the assumptions of Proposition 4 let $\Gamma = \{z,\ldots,f_z^{q-1}z\}$, $f^n z = (\varphi_n, r_n)$, $n = 0,\ldots,q-1$ be the Birkhoff periodic orbit determined by ψ . Then there exists another Birkhoff periodic orbit $\Gamma = \{y,\ldots,f_\varphi^{q-1}y\}$ such that $f^n y = (\varphi_n', r_n')$ and every φ_n' lies on the circle between φ_n and the next of φ_i's counterclockwise. In particular, $\Gamma \cup \Gamma'$ is a Mather set.

REFERENCES

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[2] A. Katok, Some remarks on Birkhoff and Mather twist map theorems, to appear in Ergodic Theory and Dynamical Systems.