Continuation of the preprint

"MORE ABOUT BIRKHOFF PERIODIC ORBITS AND MATHER SETS FOR TWIST MAPS"

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In general, there may be more than one critical point for $L_{p,q}$ inside $\Phi^{\psi}_{p,q}$. However, there is always at least one such point ψ' which allows the description as a "mountain pass." Namely, let us consider various smooth paths $\alpha : [0,1] \to \Phi^{\psi}_{p,q}$ such that

$$\alpha(0) = \psi \quad \text{and} \quad \alpha(1) = \tilde{\psi}.$$

Let us denote

$$L^{\min}_{p,q} = \min_{\Phi_{p,q}} L_{p,q} = L_{p,q}(\psi)$$

and

$$L^{\min\max}_{p,q} = \inf_{\alpha} \max_{0\le t\le 1} L_{p,q}(\alpha(t)).$$

The infimum in the last formula is always achieved for some α and some value of t such that $\alpha(t)$ is a critical point of $L_{p,q}$ which we shall denote by ψ'.

Obviously we have for the function $M(x)$ defined by (17)

$$(27) \qquad L^{\min\max}_{p,q} \ge L^{\max\min}_{p,q} \overset{\text{def}}{=} \max_{\psi(0)\le x\le\psi(1)} M(x)$$

$$= \max_{\psi(0)\le x\le\psi(1)} \min_{\varphi\in\Phi^{x,\psi}_{p,q}} L_{p,q}(\varphi)$$

We shall call the Birkhoff orbit Γ determined by the map ψ i.e. any orbit which minimizes the functional $L_{p,q}$ a minimal Birkhoff orbit of type (p,q) and the orbit Γ' determined by any map ψ' satisfying the above description a minimax orbit associated with Γ.

The quantity

$$\Delta L_{p,q} \stackrel{\text{def}}{=} L_{p,q}^{\min\max} - L_{p,q}^{\min}$$

will play an important role in the subsequent discussion.

5. Orbits homoclinic to a Mather set

We start from some general estimates for the functionals $L_{p,q}$. First, let us notice that $L_{p,q}$ may be extended in a natural way from the space $\Phi_{p,q}$ to the space $\prod_{p,q}$ of all maps $\varphi : \mathbb{Z} \to \mathbb{R}$ such that

$$\varphi(n+q) = \varphi(n) + 1 \quad \text{and}$$

$$g_0(\varphi(n)) \le \varphi(n+p) \le g_1(\varphi(n))$$

with usual identifications.

Every map $\varphi \in \prod_{p,q}$ determines two functions $h_1, h_2 : \mathbb{Z} \to [0,1]$ (see [2], Section 4 before formula (6)). Since we are going to deal with various φ's at the same time we shall denote those functions h_1^φ and h_2^φ. Furthermore we shall say that $\varphi \in \prod_{p,q}$ is precise at n if $h_1^\varphi(n) = h_2^\varphi(n)$. In this case we shall denote the common value of h_1^φ and $h_2^\varphi(n)$ by $h^\varphi(n)$

Let for $\varphi, \varphi' \in \prod_{p,q}$

$$P_{\varphi,\varphi'} = \begin{cases} n \in \mathbb{Z}; \ \varphi \text{ and } \varphi' \text{ are precise at } n-p,\ n,\ n+p \\ \text{and all three differences } \varphi'(n-p) - \varphi(n-p), \\ \varphi'(n) - \varphi(n),\ \varphi'(n+p) - \varphi(n+p) \text{ have the same sign.} \end{cases}$$

Using Lipschitz twist condition in the same way as in the proof of Proposition 1 in [2] we see that for $n \in P_{\varphi,\varphi'}$

(28) $$|h^{\varphi}(n) - h^{\varphi'}(n)| < c_1 |\varphi'(n) - \varphi(n)|$$

where c_1 is a constant depending only on f. Consequently since f and f^{-1} are Lipschitz maps

$$|h^{\varphi}(n-p) - h^{\varphi'}(n-p)| < c_2 |\varphi'(n) - \varphi(n)|$$

and

$$|h^{\varphi}(n+p) - h^{\varphi'}(n+p)| < c_2 |\varphi'(n) - \varphi(n)|$$

for another constant c_2.

<u>Lemma 5.1</u>: There exists a constant c depending only on f but not on p and q such that for $\varphi, \varphi' \in \prod_{p,q}$ and for every set $P \subset P_{\varphi,\varphi'}$

$$|L_{p,q}(\varphi) - L_{p,q}(\varphi')| \le c\Big(\sum_{n\in[0,q-1]\cap P} (\varphi(n)-\varphi'(n))^2 + \sum_{n\in[0,q-1]\setminus P} |\varphi(n)-\varphi'(n)|\Big).$$

<u>Proof</u>: By the definition of $L_{p,q}$ (see [2], Section 4, Figure 1 and below) we have

$$L_{p,q}(\varphi) - L_{p,q}(\varphi') = \sum_{i=0}^{q-1} \mu(T(\varphi(n),\varphi(n+p)))$$

$$- \mu(T(\varphi'(n),\varphi'(n+p))) = \sum_{i=0}^{q-1} \mu_n$$

where μ_n's are defined by the last equality.

Let us denote the segments $\{\varphi(n)\}\times[0,1]$ and $\{\varphi'(n)\}\times[0,1]$ by I_n and I'_n correspondingly. Let, moreover $\mathcal{D}(n)$ be the domain bounded by the bottom boundary component B of the strip S, FI_{n-p}, I_n and FI'_{n-p} and $\mathcal{D}'(n)$ be the domain bounded by B, I_n, FI_{n-p} and I'_n. We have $\mu_n = \mu(\mathcal{D}(n+p)) - \mu(\mathcal{D}'(n+p))$.

Since the measure μ is preserved by F and

$$F^{-1}\mathcal{D}(n+p) \subset [\varphi(n),\varphi'(n)] \times [0,1], \tag{29}$$

we have

$$\begin{aligned} \mu_n &\le \mu([\varphi(n),\varphi'(n)] \times [0,1]) \\ &+ \mu([\varphi(n+p),\varphi'(n+p)] \times (0,1]) < c_3((\varphi'(n)-\varphi(n) \\ &+ \varphi'(n+p) - \varphi(n+p)). \end{aligned} \tag{30}$$

On the other hand, if $n \in P_{\varphi,\varphi'}$, we can obtain a better estimate for $|\mu(\mathcal{D}(n+p)) - \mu(\mathcal{D}'(n))|$.

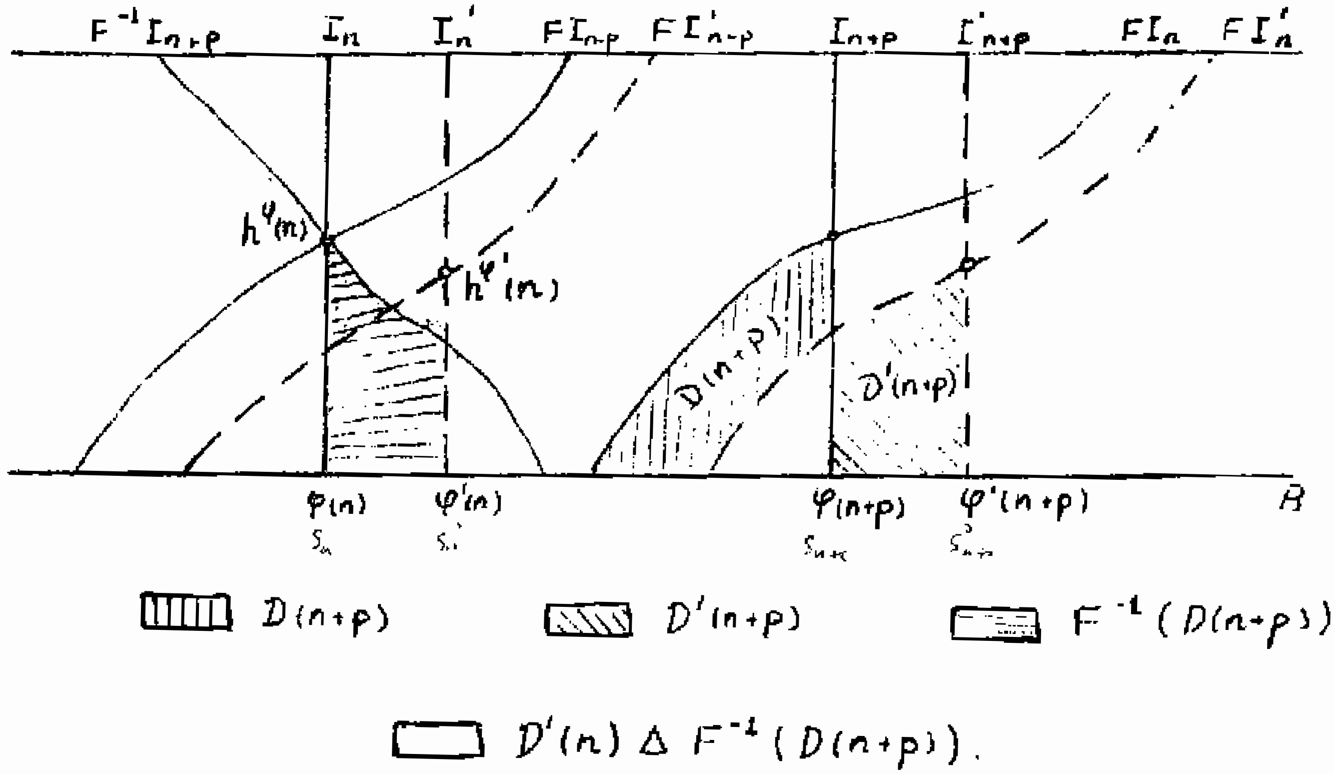

Fig. 5

The top component of the boundary of $\mathcal{D}'(n)$ passes through the point $(\varphi'(n), h^{\varphi'}(n))$ and has a bounded slope. Similarly, the top component of the boundary of $F^{-1}(\mathcal{D}(n+p))$ passes through $(\varphi(n), h^{\varphi}(n))$ and also has a bounded slope. Hence, it follows from (28) that

$$(31) \qquad |\mu(\mathcal{D}(n+p)) - \mu(\mathcal{D}'(n))| < c_4(\varphi'(n)-\varphi(n))^2.$$

Now we can finish the proof the lemma. Obviously

$$\sum_{n=0}^{q-1} \mu_n = \sum_{n=0}^{q-1} \mu_{pn}.$$

Let us divide the sequence $0, p, 2p, \ldots (q-1)p$ into successive intervals belonging to the set P and to its complement.

Let $\{np,\dots,(n+k-1)p\}$ be an interval belonging to P. We have

$$
\begin{aligned}
&\mu_{np} + \dots + \mu(n+k-1)p \\
&= \sum_{i=1}^{k} \mu(\mathcal{D}(n+i)p) - \mu(\mathcal{D}'(n+i)p) \\
&\le \Big|\sum_{i=1}^{k} \mu(\mathcal{D}'(n+i-1)p) - \mu(\mathcal{D}'(n+i)p)\Big| \\
&\quad + \sum_{i=1}^{k} |\mu(\mathcal{D}(n+i)p) - \mu(\mathcal{D}'(n+i-1)p)| \\
&= |\mu(\mathcal{D}'(n+k)p) - \mu(\mathcal{D}'(np)| \\
&\quad + \sum_{i=1}^{k} |\mu(\mathcal{D}(n+i)p) - \mu(\mathcal{D}'(n+i-1)p)|.
\end{aligned}
$$

The first term does not exceed

$$
\begin{aligned}
&c_5(|\varphi'((n+k)p) - \varphi((n+k)p| + |\varphi'(np) - \varphi'(np)|) \\
&< c_6(|\varphi'((n+k)p) - \varphi((n+k)p)| + |\varphi'((n-1)p) - \varphi'((n-1)p)|).
\end{aligned}
$$

The last inequality is simply a corollary of Lipschitz condition. The second term is estimated through (31).

Finally we get

$$
\begin{aligned}
(32)\quad &\mu_{np}+\dots+\mu(n+k-1)p < c_7(|\varphi'((n-1)p)-\varphi((n-1)p)| \\
&+ |\varphi'((n+k)p)-\varphi((n+k)p)| + \sum_{i=0}^{k-1} (\varphi'((n+i)p)-\varphi((n+i)p))^2.)
\end{aligned}
$$

Adding these inequalities for intervals comprising P and using estimate (30) for the other terms we obtain the statement of the lemma. □

Remark: In subsequent arguments we shall use not only Lemma 5.1 itself but also the "local" estimates (30) and (32) which give the upper bounds for "general" and "good" terms correspondingly in the increment of the functional.

Let us consider the situation similar to that discussed in Section 3. Namely, let α be an irrational number from the twist interval for f (see [2], Section 1), $\frac{p_m}{q_m}$ be a sequence of rational numbers converging to α, Γ_m and Γ'_m be correspondingly a minimal Birkhoff periodic orbits of type (p_m, q_m) and a minimax orbit associated with Γ_m. Without loss of generality we can assume that Γ_m converge in Hausdorff topology to a Mather set Γ and $\Gamma_m \cup \Gamma'_m$ converge to a Mather set $\tilde{\Gamma}$. Obviously, $\tilde{\Gamma} \supset \Gamma$ and by Proposition 3 from [2] the rotation numbers for both sets Γ and $\tilde{\Gamma}$ are equal to α. If the set $\tilde{\Gamma}$ is minimal it coincides with Γ. Otherwise $\tilde{\Gamma} = \Gamma_0 \cup \mathcal{D}$ where Γ_0 is a minimal Cantor set and $\mathcal{D}$ is non-empty and consists of orbits homoclinic (asymptotic in both directions) for the set Γ_0 and moving through consecutive holes in Γ_0.

Proposition 5: If $\tilde{\Gamma}$ is a minimal set then

$$\lim_{m\to\infty} \Delta L_{p_m, q_m} = 0.$$

Proof: Let $\psi^{(m)} \in \Phi_{p_m, q_m}$ and $\varphi^{(m)} \in \Phi^{\psi^{(m)}}_{p_m, q_m}$ be the maps associated with the orbits Γ_m and Γ'_m correspondingly. Under the assumption of the proposition $\tilde{\Gamma}$ is either a circle with a dense orbit or a Cantor set. In the first case

$$\max_n (\psi^{(m)}(n+1) - \psi^{(m)}(n)) \to 0 \quad \text{as} \quad m \to \infty$$

and since for every n, $\psi^{(m)}(n) < \varphi^{(m)}(n) < \psi^{(m)}(n+1)$ we have

$$\max_n(\varphi^{(m)}(n) - \psi^{(m)}(n)) \to 0 \quad \text{as} \quad m \to \infty. \tag{33}$$

Let us apply Lemma 5.1 with $\varphi = \psi^{(m)}$ and $\varphi' = \varphi^{(m)}$. In this case $P_{\varphi,\varphi'} = \mathbb{Z}$ so that by that lemma

$$\begin{aligned} \Delta L_{p_m,q_m} &= L_{p_m,q_m}(\varphi^{(m)}) - L_{p_m,q_m}(\psi^{(m)}) \\ &\leq c \sum_{m=0}^{q-1} (\varphi^{(m)}(n) - \psi^{(m)}(n))^2. \end{aligned}$$

This inequality together with (33) and

$$\sum_{m=0}^{q-1} \varphi^{(m)}(n) - \psi^{(m)}(n) < 1$$

imply that $\Delta L_{p_m,q_m} \to 0$ or $m \to \infty$.

Now we proceed to the remaining case where $\tilde{\Gamma} = \Gamma$ is a minimal Cantor set. This case is certainly much more delicate than the previous one. Let H be a hole in Γ defined by its two endpoints. In terms of the universal covering we can assume that

$$\psi^{(m)}(0) \to \psi_0 \quad \psi^{(m)}(1) \to \psi_1 > \psi_0.$$

If the sequence $\varphi^{(m)}(0)$ has a limit point different from both ψ_0 and ψ_1 then the set $\tilde{\Gamma}$ would contain a point inside the hole H. But that contradicts the minimality of $\tilde{\Gamma}$. Thus, we can assume without loss of generality that either $\varphi^{(m)}(0) \to \psi_0$ or $\varphi^{(m)}(0) \to \psi_1$.

Since for every $\varepsilon > 0$ there are only finitely many, say $m(\varepsilon)$, holes in $\tilde{\Gamma}$ of length $\geq \frac{\varepsilon}{2}$ then for any sufficiently large n there are at most $m(\varepsilon)$ numbers $n : 0 \leq n \leq q_m - 1$ such that

$$\psi^{(m)}(n+1) - \psi^{(m)}(n) \geq \varepsilon.$$

Let us denote $A_{m,\varepsilon} = \{n \in \mathbb{Z},\ \psi^{(m)}(n+1) - \psi^{(m)}(n) \geq \varepsilon\}$. Since $\psi^{(m)}(n) < \varphi^{(m)}(n) < \psi^{(m)}(n+1)$ it is obvious that for $n \notin A_{m,\varepsilon}$

(34) $$\varphi^{(m)}(n) - \psi^{(m)}(n) < \varepsilon \quad \text{and}$$
$$\psi^{(m)}(n+1) - \varphi^{(m)}(n) < \varepsilon.$$

<u>Lemma 5.2</u>: $\min\{|n_1 - n_2| : n_1 - 1 \in A_{m,\varepsilon}, n_2 \in A_{m,\varepsilon}\} \to \infty$ as $m \to \infty$.

<u>Proof</u>: If the statement is false then there exists a hole H in Γ, a sequence of points $z_n \in \Gamma_m$ converging to the left end of H and a uniformly bounded sequence of integers k_m such that $f^{k_m} z_m$ converge to the right end of another hole H'. But one derives from the convergence that $k_m \equiv$ const., say k, and thus $H' = f^k H$. Then by the continuity of f $f^{k_m} z_m$ converge to the left end of H', a contradiction. □

Let us call <u>a segment of length s</u> any finite sequence of integers of the form

$$S = \{k, k+p_m, \ldots, k + (s-1)p_m\}.$$

Let us pick an arc Δ on the circle which does not intersect any hole in Γ of length $\frac{\ell(\Delta)}{2}$, where $\ell(\Delta)$ is the length of Δ, and denote

$$B_{m,\Delta} = \{n \in \mathbb{Z}, \psi^{(m)}(n) \in \Delta \bmod 1\}.$$

It follows immediately from the strict ergodicity of f on Γ and from the uniform convergence of Γ_m to Γ that there exists an $N = N(\Delta)$ such that for any sufficiently large m any segment of length N intersects the set $B_{m,\Delta}$.

We shall call a segment S an A-segment if

$$S \cap A_{m,\varepsilon} \neq \phi, \quad S \cap B_{m,\Delta} = \{k\} \quad \text{and} \quad k + sp_m \in B_{m,\Delta}.$$

It follows from the previous remark that any A-segment has length less than N. Moreover, Lemma 5.2 implies that any A-segment S is disjoint with any segment of the form $S' + 1$ where S' is any A-segment. Let $\mathbf{A}$ be the set of all A-segments which begin between 0 and $q - 1$.

Lemma 5.3: There exists a constant c such that for every $\varepsilon > 0$

$$\limsup_{m\to\infty} \sum_{S\in\mathbf{A}} \Big| \sum_{n\in S} \mu(T(\psi^{(m)}(n),\psi^{(m)}(n+p_m)) - \mu(T(\psi^{(m)}(n+1),\psi^{(m)}(n+p_m+1)))\Big| < c\ell(\Delta).$$

Proof: Let us denote the expression inside the absolute value sign by $\mathcal{D}_S$. We shall call an A-segment S a left A-segment if $\mathcal{D}_S < 0$ and a right A-segment if $\mathcal{D}_S > 0$.

Now we are going to change the map $\psi^{(m)}$ along A-segments, trying to minimize the value of the functional L_{p_m,q_m}. Namely, let

$$(35)\quad \chi^{(m)}(n) = \begin{cases} \psi^{(m)}(n-1) & \text{if } n-1 \text{ belongs to a left A-segment} \\ \psi^{(m)}(n+1) & \text{if } n \text{ belongs to a right A-segment} \\ \psi^{(m)}(n) & \text{otherwise.} \end{cases}$$

Obviously $\chi^{(m)}(n+q_m) = \chi^{(m)}(n)$. The definition also implies that for any n $\chi^{(m)}(n+1) \geq \chi^{(m)}(n)$ and by extending f to a larger annulus, if necessary, we can assume that

$$g_0(\chi^{(m)}) \leq \chi^{(m)}(n+p_m) \leq g_1(\chi^{(m)}).$$

Thus, $\chi^{(m)} \in \Phi_{p_m,q_m}$ and since $\psi^{(m)}$ minimizes the functional L_{p_m,q_m} on Φ_{p_m,q_m} we have

$$(36)\qquad L_{p_m,q_m}(\chi^{(m)}) \geq L_{p_m,q_m}(\psi^{(m)}).$$

The following notations will be useful in the computation of the difference between the values of the functional at $\chi^{(m)}$ and $\psi^{(m)}$. Let

A_L be the set of all left A-segments which begin in $[0,q-1]$.

A_R be the corresponding set of all right A-segments

B_L and B_L be the sets of beginnings of segments from A_L and A_R correspondingly

E_L and E_L be the set of ends of segments from A_L and A_R correspondingly.

We shall suppress the dependence on m in the subsequent computation

We have from (35) and (30)

$$
\begin{aligned}
&\mathcal{L}_{p,q}(\chi) - \mathcal{L}_{p,q}(\psi) \\
&= \sum_{S\in\mathbf{A}_L} \mathcal{D}_S - \sum_{S\in\mathbf{A}_R} \mathcal{D}_S \\
&\quad + \sum_{n-1\in B_L} \mu(T(\psi(n-p),\psi(n-1))) - \mu(T(\psi(n-p),\psi(n))) \\
&\quad + \sum_{n-1\in E_L} \mu(T(\psi(n-1),\psi(n+p))) - \mu(T(\psi(n),\psi(n+p))) \\
&\quad + \sum_{n\in B_R} \mu(T(\psi(n-p),\psi(n+1))) - \mu(T(\psi(n-p),\psi(n))) \\
&\quad + \sum_{n\in E_R} \mu(T(\psi(n+1),\psi(n+p+1))) - \mu(T(\psi(n),\psi(n+p))) \\
&\le -\sum_{S\in\mathbf{A}} |\mathcal{D}_S| + c\Big(\sum_{n-1\in B_L\cup E_L} \psi(n)-\psi(n-1) + \sum_{n\in B_R\cup E_R} \psi(n+1)-\psi(n)\Big).
\end{aligned}
\tag{37}
$$

Since all A-intervals are disjoint,

$$\sum_{n\in B_L} \psi(n) - \psi(n-1) < 2\ell(\Delta) \quad \text{and} \quad \sum_{n\in B_R} \psi(n+1) - \psi(n) < 2\ell(\Delta)$$

for sufficiently large m. Similarly, the endpoints of all A-intervals belong to an interval of length comparable with Δ. Thus by (36) and (37) for any sufficiently large m

$$\sum_{S\in\mathbf{A}} |\mathcal{D}_S| < c\ell(\Delta). \qquad \square$$

Since the orbits Γ'_n converge in Hausdorff topology to the same set as Γ_n we have

$$\limsup_{m\to\infty} \min\Big(\sup_{n\in\mathbf{Z}}(\varphi^{(m)}(n) - \psi^{(m)}(n)), \sup_{n\in\mathbf{Z}}(\psi^{(m)}(n+1) - \varphi^{(m)}(n))\Big) = 0$$

Thus, if we fix δ; $\varepsilon > \delta > 0$ and a natural number N then for any sufficiently large m we have for all $a \in A_{m,\varepsilon}$ and $k : |k| < N$ either

$$(38) \qquad \varphi^{(m)}(n+p_m k) - \psi^{(m)}(n+p_m k) < \delta \qquad \text{or}$$

$$(39) \qquad \psi^{(m)}(n+1+p_m k) - \varphi^{(m)}(n+p_m k) < \delta .$$

Let us denote

$$L_{m,\varepsilon} = \{n \in A_{m,\varepsilon} \text{ and (38) holds}\}$$

$$R_{m,\varepsilon} = \{n \in A_{m,\varepsilon} \text{ and (39) holds}\}.$$

If $\delta < \frac{\varepsilon}{2}$ then the sets $L_{m,\varepsilon}$ and $R_{m,\varepsilon}$ are disjoint.

<u>Lemma 5.4</u>. $\min\{|n_1-n_2|, n_1 \in L_{m,\varepsilon}, n_2 \in R_{m,\varepsilon}\} \to \infty$ as $m \to \infty$.

<u>Proof</u>. The argument is completely parallel to the one from the proof of Lemma 5.2. Namely, if the assertion is not true then there exists a sequence of points $z_m \in \Gamma'_m$ converging to the left end of a hole H (these points correspond to $\varphi(n_1)$'s) and a bounded sequence of integers k_m such that $f^{k_m} z_m$ converge to the right end of another hole. The rest of the argument is the same as in Lemma 5.2. □

This lemma implies, in particular, that if the interval Δ is chosen sufficiently small then for every sufficiently large m every A-segment intersects only one of two sets $L_{m,\varepsilon}$ and $R_{m,\varepsilon}$. Let us call corrresponding A-segments L-segments and R-segments. Let m be sufficiently large and S be an R-segment. Since its length does not exceed $N(\Delta)$ we have from (39)

(40) $$\sum_{n\in S} \psi^{(m)}(n+1) - \varphi^{(m)}(n) < N(\Delta)\delta.$$

The number $N(\Delta)\delta$ can be arbitrarily small since δ can be chosen after Δ has been fixed.

Now we can finish the proof of the proposition. Let us denote as before

$$\mu_n = \mu(T(\varphi^{(m)}(n), \varphi^{(m)}(n+p_m))) - \mu(T(\psi^{(m)}(n), \psi^{(m)}(n+p_m))).$$

The difference

$$\Delta L_{p_m,q_m} = L_{p_m,q_m}(\varphi^{(m)}) - L_{p_m,q_m}(\psi^{(m)}) = \sum_{n=0}^{q_m-1} \mu_n = \sum_1 + \sum_2$$

where $\sum_1$ is the sum of μ_n's over all n belonging to R-segments and $\sum_2$ is the sum of μ_n's over all other n. Obviously

$$\sum_1 = \sum_{S-\text{ an R-segment}} \sum_{n\in S} \mu_n .$$

Let S be an R-segment. Then

$$\sum_{n\in S} \mu_n = -\mathcal{D}_S + \sum_{n\in S} \mu(T(\varphi^{(m)}(n), \varphi^{(m)}(n+p_m)))$$
$$- \mu(T(\psi^{(m)}(n+1), \psi^{(m)}(n+p_m+1)))$$

so that by (30) and (40)

$$\sum_{n\in S} \mu_n < |\mathcal{D}_S| + c \sum_{n\in S} \psi^{(m)}(n+1) - \varphi^{(m)}(n) < |\mathcal{D}_S| + cN(\Delta)\delta.$$

Since the total number of R-segments does not exceed $m(\varepsilon)$ one has using Lemma 5.3

$$(41) \qquad \sum_1 < \sum |D_s| + cm(\varepsilon)N(\Delta)\delta \;\le\; c(\ell(\Delta) + m(\varepsilon)N(\Delta)\delta).$$

By fixing Δ small enough after ε has been fixed and choosing δ small enough afterwards we can make $\sum_1$ arbitrarily small. It remains to estimate $\sum_2$. This sum splits into pieces of the form

$$(42) \qquad \mu_{kp_m} + \dots + \mu_{(k+s-1)p_m}.$$

(Remember that indices can always be changed by multiples of q_m) so that

$$(43) \qquad \begin{aligned} &\psi^{(m)}(kp_m) \in \Delta \mod 1 \text{ and} \\ &\psi^{(m)}((k+s)p_m) \in \Delta \mod 1. \end{aligned}$$

This follows from the construction of an A-segment of which R-segment is a special case.

For each sum (42) we can apply the estimate (32) assuming $\varphi = \psi^{(m)}$ $\varphi' = \varphi^{(m)}$ since in this case $P_{\varphi,\varphi'} = \mathbb{Z}$. This result is the following inequality

$$(44) \qquad \begin{aligned} &\mu_{kp_m} + \dots + \mu_{(n+s-1)p_m} \\ &\le c((\varphi^{(m)}(kp_m) - \psi^{(m)}(kp_m) + \varphi^{(m)}((k+s)p_m) - \psi^{(m)}((k+s)p_m)) \\ &\quad + \sum_{i=0}^{j-1}(\varphi^{(m)}((k+i)p_m) - \psi^{(m)}((k+i)p_m))^2). \end{aligned}$$

Adding inequalities (44) for each sum (42) comprising $\sum_2$ and taking into account (43) we see that the total sum of linear terms does not exceed a constant multiple of the length of the interval

Δ. Each of the differences which appears in the square is less than ε by (31) and (38) and the total sum of those differences does not exceed 1. That means that the sum of the quadratic terms does not exceed ε. Since ε can be taken arbitrarily small (41) and (44) imply the statement of the Proposition. □

Now we are ready to prove our second and last main result.

Theorem 2. Let Γ be a Mather set which is the limit of minimal Birkhoff periodic orbits of types (p_m, q_m). Then either Γ is a circle or Γ is not a minimal set or there exists a non-minimal Mather set $\Gamma' \supset \Gamma$.

Proof. By Proposition 5 we can assume that $\lim_{n\to\infty} \Delta L_{p_m, q_m} = 0$. Let us fix a point x belonging to a hole H for Γ and consider a map, φ_x^m minimizing L_{p_m, q_m} on the space $\Phi^{x,\psi^{(m)}}_{p_m, q_n}$ (cf. Lemma 2.2). By Lemma 2.3 and by (27)

$$|h^{\varphi_x^m}(0) - h^{\varphi_x^m}(q_m)| \to 0 \quad \text{as} \quad m \to \infty$$

so that two semi-orbits for f (positive and negative) determined by φ_x^m as $m \to \infty$ have the same initial condition and form together an orbit which is asymptotic to Γ. Theorem 1 implies that by adding this orbit to Γ we obtain a non-minimal Mather set $\Gamma' \supset \Gamma$. □

Remark: Actually if $\Delta L_{p_m, q_m} \to 0$ and Γ is not a circle we constructed this way a lot of orbits asymptotic to Γ. Those orbits form a closed set which intersects every vertical interval $\{\varphi\} \times [0,1]$.

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