MINIMAL ORBITS FOR SMALL PERTURBATIONS OF COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS

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1. Introduction. In this paper we summarize an attempt to carry out certain aspects of Aubry–Mather theory for twist maps (see e.g. [M1] [AL] [K] [Mos2] [B1]) to Hamiltonian systems with more than two degrees of freedom. In a sense, the paper should be considered as a continuation of [BK]. Although the minimal motions for an arbitrary "admissible" rotation vector, may not exist (see [He2], p. 54) the combination of KAM theory with some of the methods developed in [BK] still yields a considerable information about those motions. Our two main results are:

(i) the orbits in KAM tori are minimal and they are the only minimal motions for corresponding rotation vectors (Section 5, Theorem 1);

(ii) there are infinitely many rotation vectors for which KAM tori do not exist but minimal motions do exist and the closure of these minimal motions contains all the KAM tori (Section 6, Theorem 2).

Both results hold for small perturbations of completely integrable Hamiltonian systems satisfying convexity assumptions (see Section 2 below and [BK], Sections 2 and 7). For (ii) we naturally assume that the set of KAM tori is nowhere dense.

John Mather has a proof of (i) or a close result (personal communication) which looks considerably more involved that the proof presented in this paper.

We will use notations from [BK]. We will only present arguments for the discrete-time case, namely for symplectic maps. The reduction of the more traditional continuous-time case of Hamiltonian vector-fields to the discrete-time one is explained in Section 7 of [BK]. Furthermore, some of the arguments and estimates presented in [BK] for Birkhoff periodic orbits work almost literally for more general types of minimal orbits and orbit segments considered in this paper. In such cases we will give precise formulations and will refer to an appropriate place in [BK] for elaboration.

This paper grew out of the talk given on September 17, 1988 at the Conference on Hamiltonian Dynamics at La Jolla, California dedicated to John Greene's sixtieth birthday and was certainly influenced by Greene's work. I would like to thank Robert McKay for inviting me to speak at the conference. Earlier discussions with Victor Bangert were useful in developing some of the ideas which led to this paper.

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2. Minimal states. Let us briefly recall the notations and assumptions from [BK], Section 2. We will consider the space

$$M = \mathbf{T}^n \times \mathbf{R}^n = \{\varphi_1, \ldots, \varphi_n, r_1, \ldots, r_n); \quad \varphi_i \in \mathbf{R}/\mathbf{T}, r_i \in \mathbf{R}\}$$

provided with the standard symplectic form $\Omega = \sum_{i=1}^{n} d\varphi_i \wedge dr_i$ and a symplectic diffeomorphic embedding

$$f : \mathbf{T}^n \times U \to M$$

where $U \subset \mathbf{R}^n$ is diffeomorphic to an open n-disc. We assume that f is a small perturbation of an integrable map $f_0 : f_0(\varphi, r) = (\varphi + \alpha(r), r)$. Let us denote by F and F_0 the lifts of f and f_0 correspondingly to the universal cover of $\mathbf{T}^n \times U$. Let $F_0(x, r) = (x + a(r), r)$. In terms of generating functions, F_0 is generated by a function $h(x' - x)$ and F by a function

$$H(x, x') = h(x' - x) + P(x, x')$$

where P is periodic, i.e.; $P(x + m, x' + m) = P(x, x')$ for all $m \in \mathbf{T}^n$, and is small with several derivatives. Precise assumptions on the size of P will vary. The strongest assumptions will be those to guarantee that the map f has sufficiently many invariant KAM tori close to the tori $r =$ const. for the unperturbed map f_0.

On the other hand, we will assume that the function h is strictly differentiably convex, i.e. its Hessian is a positive definite quadratic form.

At the end of Section 2 of [BK] an extension of H to the whole space $\mathbf{R}^n \times \mathbf{R}^n$ is described which allows to keep the smallness of the perturbation part P of the generating function. We will use that extension, but unlike [BK] we will still denote the extended function by $H = h + P$.

Fix $x, y \in \mathbf{R}^n$ and a natural number q. Let

$$\Psi_q^{x,y} = \{x_0 = x, x_1 \ldots x_{q-1}, x_q = y;\ x_1, \ldots x_{q-1} \in \mathbf{R}^n\}.$$

Let us introduce the Lagrangian $L_q^{x,y} : \Psi_q^{x,y} \to \mathbf{R}$ by

$$L_q^{x,y}(x_0, x_1, \ldots, x_q) = \sum_{i=0}^{q-1} H(x_i, x_{i+1}).$$

Any critical point $(x_0, x_1, \ldots, x_q)$ of the Lagrangian determines a unique orbit segment of F such that $(x_i, r_i) = F^i(x_0, r_0)$ and vice versa, for any such orbit segment the sequence of first coordinates $(x, x_1, \ldots, x_{q-1}, y)$ is a critical point for $L_q^{x,y}$. Sometimes we will call the elements of the spaces $\Psi_q^{x,y}$ *states* and the critical points of $L_q^{x,y}$ *equilibrium states*.

The convexity of h and the smallness of P imply that $L_q^{x,y}$ is a proper function bounded from below and hence that it reaches its absolute minimum which we will denote by $\ell_q^{x,y}$.

Definition. Any state $(x, x_1, \ldots, x_{q-1}, y) \in \Psi_q^{x,y}$ for which $L_q^{x,y}(x, x_1 \ldots, x_{q-1}, y) = \ell_q^{x,y}$ is called a *minimal state*. Corresponding orbit segments for F and f are called *minimal orbit segments.*

Let δ_k be the C^k norm of the function P. For a state $\overline{x} \in \Psi_q^{x,y}$ let us denote $x_i - x_{i-1} = a_i$. The following statements are direct counterparts of corresponding results from [BK] for minimal periodic orbits. The letter C with various indices denotes constants which depend only on the unperturbed generating function h, i.e. on the map f_0.

Lemma 1. *If $\overline{x} \in \Psi_q^{x,y}$ is an equilibrium state then $|a_{i+1} - a_i| < C_1\delta_1$ for $i = 1, \ldots, q-1$. If $\overline{x}$ is a minimal state then $|a_{i+1} - a_i| < C_2\delta_0^{1/2}$.*

Proof. See [BK], Lemma 1.

Lemma 2. *Let $\overline{x}$ be a minimal state for $L_q^{x,y}$ and $q > C_3\delta_1^{-1/2}$. Then $|a_i - a_j| < C_4\delta_1^{1/2}$.*

Similarly, if $q > C_5\delta_0^{-1/4}$ then $|a_i - a_j| < C_6\delta_0^{1/4}$.

Proof. See the proof of Lemma 2 from [BK]. It works verbatim in our case if either $C\delta_1^{-1/2} < \min(i,j)$ or $\max(i,j) < q - C\delta_1^{-1/2}$ for the first statement and either $C\delta_0^{-1/4} < \min(i,j)$ or $\max(i,j) < q - C\delta_0^{-1/4}$ for the second. If both conditions are violated let $k = \left[\frac{q}{2}\right]$ and then our inequality for q guarantees that one of the conditions holds for the pair (i,k) and the other for the pair (j,k). Since $|a_i - a_j| \le |a_i - a_k| + |a_k - a_j|$, by doubling the constant we obtain the desired inequalities for arbitrary i and j.

□

Lemma 2 implies that if the vector $v = \dfrac{y-x}{q} \in a(U)$ and is not too close to the boundary of $a(U)$ then all minimal orbit segments for f corresponding the minimal states from $\Psi_q^{x,y}$ belong to $\mathbf{T}^n \times U$ and consequently they are orbit segments of the original map. Thus the way we extended the generating function is unimportant.

Lemma 3. *Let $(\varphi_i, r_i) = f^i(\varphi_0, r_0)$, $i = 0, 1, \ldots q$ be a minimal orbit segment, $i, j, k \in \{0, 1, \ldots q\}$ be different. Then*

$$|r_i - r_k| < C_7(\mathrm{dist}(\varphi_i, \varphi_j) + \mathrm{dist}(\varphi_i, \varphi_k))^{1/2}$$

Proof. See [BK], Proposition 5.

3. Minimal orbits

Definition. An orbit of f is *minimal* if every finite segment of it is a minimal orbit segment.

For the geodesic problem on a Riemannian manifold, the corresponding concept of minimal geodesic was considered by Morse [Mo], Hedlund [H] and recently by Bangert [B2].

DEFINITION. An orbit of f has *rotation vector* v if for some (and hence any) lift $(x_m, r_m) = F^m(x_0, r_0)$, $m \in \mathbb{Z}$

$$\lim_{m \to \pm\infty} \frac{x_m - x_0}{m} = v. \tag{1}$$

More generally, the *rotation set* of the orbit, is the set of all limit points of $\frac{x_m - x_0}{m}$. For $(\varphi, r) \in \mathbb{T}^n \times U$ let $\rho(\varphi, r)$ be the vector opposite to the difference between the x coordinates of any lift of (φ, r) and its F-image.

DEFINITION. For any probability Borel f-invariant measure μ the *rotation vector of* μ, $\rho(\mu)$ is

$$\rho(\mu) = \int_{\mathbb{T}^n \times U} \rho(\varphi, r) d\mu$$

By Birkhoff Ergodic Theorem for any finite f-invariant measure μ, μ-almost every orbit has a rotation vector i.e. the limit in the left-hand part of (1) exists. In particular, if μ is ergodic, μ-almost every orbit of f has the same rotation vector equal to $\rho(\mu)$.

PROPOSITION 1. *Let* $(\varphi_m, r_m) = f^m(\varphi_0, r_0)$, $m \in \mathbb{Z}$ *be a minimal orbit of* f.

If $\varphi = \varphi_k$ is a non-isolated point for the set $\Phi = \{\varphi_i\}_i \in \mathbb{Z}$ then for every integer i

$$|r_i - r_k| \le C_8(\text{dist}(\varphi_i, \varphi))^{1/2} \tag{2}$$

If the orbit is non-periodic and φ is an isolated point in Φ, then (2) holds for all the integers i, except maybe for one.

If the orbit is periodic with the minimal period q then (2) holds for all except may be one $i \in \{0, 1, \ldots, q-1\}$.

Proof. If φ is a non-isolated point in Φ then for any i one can choose an integer j such that $\text{dist}(\varphi, \varphi_i) < \text{dist}(\varphi, \varphi_i)$ and (2) immediately follows from Lemma 3.

Otherwise choose j such that dist $(\varphi, \varphi_j) < 2 \inf_{i \in \mathbb{Z}} \text{dist}(\varphi, \varphi_i)$ and again apply Lemma 3.

□

PROPOSITION 2. *If a minimal orbit has rotation vector* v *then for all integers* n, $|r_n - a^{-1}(v)| < C_9 \delta_1^{1/2}$.

Proof. Consider the lift of the orbit and the corresponding minimal state $(\ldots x_{-1}, x_0, x_1 \ldots)$. Take a sufficiently large $N > |n|$ such that

$$\left| \frac{x_N - x_{-N}}{2N+1} - v \right| < \delta_1^{1/2}. \tag{3}$$

Since $\overline{x} = (x_{-N'} \ldots, x_N) \in \Psi_{2N}^{x_{-N'} x_N}$ is a minimal state, by Lemma 2 for any $i, j \in \{-N+1, \ldots, N\}$ $|a_i - a_j| < C_4 \delta_1^{1/2}$ where as before $a_i = x_i - x_{i-1}$.

We have $\sum_{i=N+1}^{N} a_i = x_N - x_{-N}$, hence by (3) for any i

$$|a_i - v| < (C_4+1)\delta_1^{1/2}. \tag{4}$$

But since $\overline{x}$ is an equilibrium state we have

$$r_i = \frac{\partial H}{\partial x}(x_i, x_{i+1}) = dh(a_i) + \frac{\partial P}{\partial x}(x_i, x_{i+1}), \tag{5}$$

where $\left\|\frac{\partial P}{\partial x}\right\| \le \delta_1$ by the definition of δ_1, and $dh(v) = a^{-1}(v)$. Thus by (4) and (5) $|r_n - a^{-1}(v)| < \left\|\frac{\partial P}{\partial x}\right\| + |dh(a_i) - a^{-1}(v)| \le \delta_1 + |dh(a_i) - dh(v)| \le \delta_1 + \|dh\|\,(C_4+1)\delta_1^{1/2}$.

□

The next statement is an immediate corollary of the definitions of a minimal orbit segment and a minimal orbit. We formulate it separately because of its importance for the theory.

PROPOSITION 3. *Let $x \in \mathbf{R}^n, x = \lim_{m\to\infty} x_m$ and suppose there are $k_m \to \infty, l_m \to \infty$ and minimal states $\overline{x}^{(m)} = (x_0^{(m)}, \dots, x_{k_m+l_m}^{(m)}) \in \Psi_{k_m+l_m}^{x_0^{(m)}, x_{k_m+l_m}^{(m)}}$ such that $x_m = x_{k_m}^{(m)}$. Then the F-orbit of x is minimal. In particular, the closure of a minimal f-orbit consists of minimal f-orbits.*

Let us fix a vector $v \in a(U)$ not too close to the boundary of $a(U)$. Consider a sequence of minimal states in $\Psi_{2q_m}^{x_m, x_m+2q_m v}$ where x_m and $q_m \to \infty$ are chosen in such a way that the sequence $x_m + q_m v$ converges. This is always possible because x_m can be moved by any integer vector so we can assume that the vectors $x_m + q_m v$ lie within the unit cell and then take a converging subsequence. Put a uniform normalized δ-measure on the projection of the minimal orbit segment determined by each minimal state and take a weakly converging subsequence. The limit measure is f-invariant and has rotation vector v. It is also supported on the set of minimal orbits since for every point $(\varphi, r) \in \operatorname{supp}\mu$ the x-coordinate of every lift of φ satisfies the assumption of Proposition 3. If μ also happens to be ergodic, then μ-almost every orbit has rotation vector v.

4. Minimal action function

LEMMA 4. *For any $x, y, w \in \mathbf{R}^n$ and a natural number q*

$$|\ell_q^{x,x+w} - \ell_q^{y,y+w}| \le C \tag{6}$$

where C is independent of x, y, w, q.

Proof. First, let us notice that for every $p \in \mathbf{Z}^n$, $\ell_q^{x+p,\, x+w+p} = \ell_q^{x,\, x+w}$. Then let us find $p \in \mathbf{Z}^n$ such that the distance between $x+p$ and y is less than $n^{1/2}$. Then compare the values of the Lagrangian at any minimal state $(y, y_1, \dots, y_{q-1}, y+w) \in \Psi_q^{y,\, y+w}$ and at the state $(x+p, y_1, \dots, y_{q-1}, x+p+w) \in \Psi_q^{x+p,\, x+p+2}$. The first value is equal to $\ell_q^{y,\, y+w}$ by definition, the second is greater or equal than $\ell_q^{x,\, x+w}$. Thus we have $\ell_q^{x,\, x+w} \leq \ell_q^{y,y+w} + |H(x+p, y_1) - H(y, y_1)| + |H(y_{q-1}, x+p+w) - H(y_{q-1}, y+w)| \leq \ell_q^{y,\, y+w} + |h(x+p-y_1) - h(y-y_1)| + |h(x+p+w-y_{q-1}) - h(y+w-y_{q-1})| + |P(x+p, y_1) - P(y, y_1)| + |P(y_{q-1}, x+p+w) - P(y_{q-1}, y+w)| \leq \ell_q^{y,y+w} + 2|h(x+p-y)| + 2\delta_0$. To obtain the last inequality we used the convexity of the (extended) function h. Since $\|x+p-y\| < n^{1/2}$, the lemma is proved.

□

PROPOSITION 4. *The ratio* $\dfrac{\ell_q^{x,\, x+qv}}{q}$ *has a limit as* $q \to \infty$, *which is independent of* x.

Proof. By the definition of the minimal state one has

$$\ell_{q_1+q_2}^{x,x+(q_1+q_2)v} \leq \ell_{q_2}^{x,x+q_1 v} + \ell_{q_2}^{x+q_1 v, x+(q_1+q_2)v} \tag{7}$$

Combining (6) and (7) one derives "almost subadditivity" of $\ell_q^{x,x+qv}$ in q

$$\ell_{q_1+q_2}^{x,x+(q_1+q_2)v} \leq \ell_{q_1}^{x,x+q_1 v} + \ell_{q_2}^{x,x+q_2 v} + C$$

which implies the existence of $\lim_{q\to\infty} \dfrac{\ell_q^{x,x+qv}}{q}$. Lemma 4 implies that the limit does not depend on x.

□

We will denote $\lim_{q\to\infty} \dfrac{\ell_q^{x,x+qv}}{q}$ by $\mathcal{L}(v)$ and will call $\mathcal{L}$ the *minimal action function*.

PROPOSITION 5. $\mathcal{L}$ *is a convex function.*

Proof. Using the definition of the minimal state and Lemma 4 we have

$$\ell_{2q}^{x,x+v+w} \leq \ell_q^{x,x+v} + \ell_q^{x+v,x+v+w} \leq \ell_q^{x,x+v} + \ell_q^{x,x+w} + C$$

which immediately implies that $\mathcal{L}\left(\dfrac{v+w}{2}\right) \leq \dfrac{\mathcal{L}(v) + \mathcal{L}(w)}{2}$.

□

Let us fix a lift of $\varphi \in \mathbf{T}^n$ and denote it as usual by x; let furthermore $x'(x,r)$ be the first coordinate of $F(x,r)$. The periodicity of H implies that the function $H(x, x'(x,r))$ does not depend on the choice of a particular lift of φ and thus can be denoted by $\mathcal{H}(\varphi, r)$.

PROPOSITION 6. $\mathcal{L}(v) = \inf \int \mathcal{H} d\mu$ *where the infinum is taken over all probability* f*-invariant measures* μ *with rotation vector* v.

Proof. (1) In order to establish the inequality

$$\mathcal{L}(v) \leq \int \mathcal{H} d\mu \tag{8}$$

for every μ with rotation vector v it is enough to do that only for ergodic measures μ. For, otherwise take the ergodic decomposition of μ,

$$\mu = \int_Z \mu_z d\nu.$$

We have $v = \rho(\mu) = \int_Z \rho(\mu_z)d\nu$, since the rotation vector is defined as an integral of the shift function $\rho(\varphi, r)$.

On the other hand, also by definition

$$\int \mathcal{H} d\mu = \int_Z \left(\int \mathcal{H} d\mu_z\right) d\nu.$$

Assuming that $\mathcal{L}(\rho(\mu_z)) \leq \int \mathcal{H} d\mu_z$ and using the convexity of $\mathcal{L}: \mathcal{L}(v) \leq \int_Z \mathcal{L}(\rho(\mu_z))d\nu$ we obtain (8).

For an ergodic measure μ with $\rho(\mu) = v$, as we mentioned already, μ-almost every point (φ, r) has rotation vector v. That means that for every lift (x, r) of (φ, r), the x-coordinate of $F^q(x, r)$ has the form

$$x + w_q(x, r) = x + qv + o(q).$$

Thus

$$\ell_q^{x, x+w_q(x,r)} \leq \sum_{i=0}^{q-1} \mathcal{H}(f^i(\varphi, r)) = q \int \mathcal{H} d\mu + o(q).$$

Now, let us take a minimal state $\overline{x} \in \Psi_q^{x, x+w_q(x,r)}$ and use Lemma 2 which allows to replace its last $C\delta_1^{-1/2}$ $o(q)$ terms to obtain a state $\overline{\overline{x}} \in \Psi_q^{x, x+qv}$ such that the norm of the difference of corresponding coordinates for $\overline{x}$ and $\overline{\overline{x}}$ does not exceed $C'\delta^{1/2}$. This implies that $\ell_q^{x, x+qv} \leq q\mathcal{L}(\overline{\overline{x}}) \leq \ell_q^{x, x+w_q(x,r)} + o(q) \leq q \int \mathcal{H} d\mu + o(q)$. Dividing by q and taking limit we obtain (8).

(2) In order to construct a measure μ with rotation vector v (unfortunately, not necessarily ergodic) for which $\mathcal{L}(v) = \int \mathcal{H} d\mu$ we take a sequence of absolute minima in $\Psi_{2q}^{x-qv,\, x+qv}$ and denote by μ_q the normalized δ-measures on corresponding minimal orbit segments for f. Since for any q

$$\frac{1}{2q} \ell_{2q}^{x-qv, x+qv} = \int \mathcal{H}\, d\mu_q,$$

for any weak limit point μ of the sequence μ_q, $\mathcal{L}(v) = \int \mathcal{H}\, d\mu$.

PROPOSITION 7. *If v is an extreme point of the convex set $\mathcal{L}_v = \{w : \mathcal{L}(w) \leq \mathcal{L}(v)\}$ then there exists an ergodic measure μ with rotation vector v for which*

$$\mathcal{L}(v) = \int \mathcal{H} d\mu$$

and whose support consists of minimal orbits.

Proof. Consider the construction from part (2) of the previous proposition. The support of the limit measure μ consists of orbits which are limits of minimal orbit segments of increasing length. In order to see that one should remember that any subsegment of a minimal orbit segment is minimal. If the measure μ is ergodic, it satisfies the assertion of the proposition.

Otherwise consider the ergodic decomposition of $\mu, \mu = \int_Z \mu_z d\nu$, so that $v = \int_Z \rho(\mu_z) d\nu$ and by Proposition 6

$$\mathcal{L}(v) = \int \mathcal{H} d\mu = \int_Z \left(\int \mathcal{H} d\mu_z \right) d\nu \geq \int_Z \ell(\rho(u_z)) d\nu \tag{9}$$

Since $\mathcal{L}$ is convex and v is an extreme point of the set $\mathcal{L}_v$ then either

$$\mathcal{L}(v) < \int_Z \ell(\rho(\mu_z)) d\nu$$

which contradicts (9) or for ν-almost every $z \in Z$, $\rho(\mu_z) = v$ and every such measure μ_z satisfies the assertion of the proposition.

□

5. Minimality of KAM tori. Let us summarize without going into much technical details some of the results concerning the existence of invariant tori which are needed for the subsequent discussion (see e.g. [Bo]; the original proof for the real analytic case is in [A] and useful discussions are in [Mos1] and [He1]):

There exists a closed set $V \subset a(U)$ of a large relative Lebesque measure such that for any map f for which the perturbation part P of the generating function H is small enough with sufficiently many derivatives and for any $v \in V$, there is an f-invariant torus

$$T_{f,v} = \{(\varphi, r) : r = g_{f,v}(\varphi)\}$$

such that the restriction of f to $T_{f,v}$ has rotation vector v. Those tori are usually called KAM tori. The set V is defined by arithmetic conditions which have to do with rational approximation. Let us point out two important uniformity properties of KAM tori which will play an important part in our proofs of minimality and uniqueness.

First, the difference $g_{f,v}(\varphi) - a^{-1}(v)$ is uniformly small with several derivatives.

Secondly, the restriction of the map f to the torus $T_{f,v}$ is smoothly conjugate to the translation $L_v : \ \varphi \to \varphi + v$ via a diffeomorphism whose φ-coordinate $\psi_{f,v} : \mathbf{T}^n \to \mathbf{T}^n$, is uniformly in v and f close to the identity with several derivatives.

Not every invariant torus of the form graph g where $g : \mathbf{T}^n \to U$ which the map f may happen to possess is necessarily smooth and even if it is, it might not satisfy the uniform estimates mentioned above. Such extra or "accidental" tori are not covered by our results concerning minimality and uniqueness although they satisfy local versions of some of those properties.

Let us fix a KAM torus $T_{f,v}$ and consider the following symplectic coordinate change $S^{f,v} = S_1^{f,v} \circ S_2^{f,v}$, where

$$S_1^{f,v}(\varphi, r) = (\varphi, r - g_{f,v}(\varphi)), \qquad S_2^{f,v}(\varphi, r) = (\psi_{f,v}(\varphi), (d\psi^*_{f,v})^{-1}_{\varphi} r).$$

Here $(d\psi^*_{f,v})\varphi$ denotes the matrix transposed to that of the derivative of $\psi_{f,v}$ at φ. The map $S_1^{f,v}$ is symplectic because the torus $T_{f,v}$ is a Lagrangian manifold. From now on let us suppress the dependence on f and v in our notation.

The coordinate change S transforms our KAM torus $T_{f,v} = T$ into the standard torus $r = 0$ and the map f restricted to T, into the linear translation L_v. It is important to remember that the estimates given by Lemmas 1-3 and results based on those lemmas remain true, maybe with different constants, due to the uniformity of the functions $g_{f,v}$ and maps $\psi_{f,v}$.

In general, generating functions change under symplectic coordinate changes in a complicated way. However the map S_2 is "Lagrangian" and it carries out the generating function. The lift of the map g to $\mathbf{R}^n$ has the form dG where $G : \mathbf{R}^n \to \mathbf{R}$. It is easy to see that the map S_1 changes the generating function by adding a coboundary $G(x') - G(x)$. Thus, minimal orbit segments and minimal orbits are preserved under the coordinate change S. S_2 does not change the minimal action function $\mathcal{L}$ either; S_1 may only add a linear term to it and thus does not change properties like strict convexity, etc.

After the coordinate change the generating function takes the form

$$Q(x' - x - v) + P(x, x')$$

where Q is a positive definite quadratic form and both the first and the second derivatives of P vanish at $x' - x = v$. By adding a constant we may assume that P itself vanishes too.

Thus, in a fixed neighborhood of the plane $x' - x = v$

$$P(x, x') < CQ(x' - x - v)^{3/2} \leq \frac{1}{2}\, Q(x' - x - v) \tag{10}$$

Since derivatives of the maps S and S^{-1} are uniformly bounded, that neighborhood contains pairs (x_i, x_{i+1}) for any minimal state from $\psi_q^{x, x+qv}$. We will use our new coordinates to calculate those minimal states.

LEMMA 5. *The only minimal state in* $\psi_q^{x, x+qv}$ *is* $(x, x+v, \dots, x+qv)$.

Proof. Let $(x = x_0, x_1, \dots, x_{q-1},\ x + qv = x_q)$ be a minimal stade. Then, using (10) we have

$$L_1^{x, x+qv}(x_0, x_1, \dots, x_{q-1}, x_q) = \sum_{i=0}^{q-1} Q(x_{i+1} - x_i - v) + P(x_i, x_{i+1}) \geq$$

$$\frac{1}{2} \sum_{i=0}^{q-1} Q(x_{i+1} - x_i - v) > 0 \text{ unless } x_{i+1} = x_i + v \text{ i.e. } x_i = x_0 + iv,\ i = 0, 1, \dots, q.$$

□

Corollary 1. *Any KAM torus consists of minimal orbits.*

Lemma 6. *There exists $\varepsilon > 0$ which depends only on the unperturbed map f_0 and on the C^k size of the perturbation for some k, such that if for a vector v there is a KAM torus with rotation vector v, than for $\|v' - v\| < \varepsilon$*

$$\mathcal{L}(v) + \ell_v(v'-v) + Q_v(v'-v) - C\|v'-v\|^3 \leq \mathcal{L}(v') \leq \mathcal{L}(v) + \ell_v(v'-v) + Q_v(v'-v) + C\|v'-v\|^3,$$

where ℓ_v is a linear function and Q_v is a positive definite quadratic form.

Proof. The coordinate changes described at the beginning of this section may only change $\mathcal{L}$ by a linear term. Thus we will use the new coordinates for our calculations.

Let $\overline{x} \in \psi_q^{x,x+qv'}$ be a minimal state.

We have

$$qQ(v'-v) + Cq\|v'-v\|^3 \geq qQ(v'-v) + \sum_{i=0}^{q-1} P(x+(i+1)v', x+iv') =$$
$$L_q^{x,x+qv'}(x, x+v', \ldots, x+(q-1)v', x+qv') \geq \ell_q^{x,x+qv'} =$$
$$L_q^{x,x+qv'}(x_0, x_1, \ldots, x_{q-1}, x_q) = \sum_{i=0}^{q-1} Q(x_{i+1} - x_i - v) + P(x_{i+1}, x_i).$$

Using (10) and the convexity of Q we can continue

$$\sum_{i=0}^{q-1} Q(x_{i+1} - x_i - v) + P(x_i, x_{i+1}) \geq \sum_{i=0}^{q-1} Q(x_{i+1} - x_i - v) - C\big(Q(x_{i+1} - x_i - v)\big)^{3/2} \geq$$
$$q(Q(v'-v) - CQ(v'-v)^{3/2}) \geq qQ(v'-v) - C'q\|v'-v\|^3.$$

□

Theorem 1. *If for a vector v there exists a KAM torus with that rotation vector then every minimal orbit with rotation vector v (or even those whose rotation set contains v) belongs to the torus.*

Proof. Take the closure of such a minimal orbit. It is an f-invariant closed set. Its intersection with the torus is also f-invariant and closed. Since every orbit of f on the torus is dense, the intersection is either empty or coincides with the torus. By Proposition 1 in the latter case there is no room for elements of a minimal orbit outside the torus; hence the whole orbit lies on the torus.

Consider the former case. Take a lift of the orbit, $F^m(x,r) = (x_m, r_m), m \in \mathbb{Z}$. By our assumption there exists either a sequence $q_m \to \infty$ or $q_m \to -\infty$ such that $x_{q_m} = x + q_m v + o(q_m)$. The two cases are completely symmetric so we assume the first.

Since the orbit closure does not intersect the torus,
$Q(x_{i+1}-x_i-v)\geq\delta>0$ for all i. Thus, using (10) we obtain

$$L_{q_m}^{x,x_{q_m}}(x,x_1,\dots,x_{q_m})=\sum_{i=0}^{q_m-1}Q(x_{i+1}-x_i-v)+P(x_i,x_{i+1})\geq\frac{1}{2}\sum_{i=0}^{q_m-1}Q(x_{i+1}-x_i-v)>\frac{q_m\delta}{2}.$$

On the other hand, let $u_m=\dfrac{x_{q_m}-x}{q_m}$ so that $u_m=v+o(1)$ and let us calculate and estimate using (10) again

$$L_{q_m}^{x,x_{q_m}}(x,x+u_m,x+2u_m,\dots,x+(q_m-1)u_m,x_{q_m})=\sum_{i=0}^{q_m-1}Q(u_m)+P(x+iu_m,x+(i+1)u_m)\leq\frac{3}{2}q_mQ(u_m).$$

Since for large enough m, $Q(u_m)<\dfrac{\delta}{3}$, $(x,x_1,\dots,x_{q_m})$ is not a minimal orbit segment so our orbit is not minimal.

□

6. Other minimal orbits. Now, having established the minimality of KAM tori and their rather strong uniqueness properties we are going to look for minimal orbits which are not associated with these tori. Let W be the set of all vectors for which KAM tori do not exist and let $W_M\subset W$ be the set all $w\in W$ for which a minimal orbit with the rotation vector w exists. Let A_K be the set of rotation vectors for which a KAM torus exists. If follows from Proposition 7 that the convex hull of $A_M=A_K\cup W_M$ contains W. Now we are going to strengthen that statement.

PROPOSITION 8. *The convex hull of W_M contains W.*

Proof. By definition A_M is the set of all vectors v for which minimal orbits with rotation number v exists. By the convexity of the minimal action function $\mathcal{L}$ every w is a linear combination of extreme points of the set $\mathcal{L}_w$ which by Proposition 7 belong to A_M. What remains to prove is that for $w\in W\setminus W_M$ none of those extreme ponts can belong to the set A_K, i.e. correspond to a KAM torus. But that follows from Lemma 6, since after a coordinate change which only changes $\mathcal{L}$ by a linear term any vector $v\in A_K$ becomes an isolated minimum of $\mathcal{L}$.

□

THEOREM 2. *Unless KAM tori exist for every v, there are infinitely many v for which a KAM torus does not exist but minimal orbits with rotation number v exist. Furthermore, the closure of the set of such v contains the boundary of the set $a(U)\setminus W$.*

Proof. Follows immediately from Proposition 8.

The most natural approach to the construction of minimal orbits is the one indicated in part (2), Proposition 6 and further discussed in the proof of Proposition 7. In other words, one should hope that in general minimal orbit segments with correct average behavior converge to a minimal orbit with the desirable rotation vector. The kind of convergence discussed in our proof of Propositions 6 and 7 is the weak convergence of δ-measures on minimal orbit segments. Another type of convergence would be the convergence in Hausdorff topology on the space of compact subsets of $\mathbf{T}^n \times U$. In order to guarantee the convergence one has to choose the initial condition carefully. A natural way to do that is to start from minimal Birkhoff periodic orbits constructed in [BK] (see Corollaries 3 and 4 of that paper). Any segment of such an orbit, whose length does not exceed the period, is a minimal orbit segment in our sense.

Now we will give a simple criterion for the existence of minimal orbits with a given rotation number whose validity is supported by rather convincing numerical experiments performed by Mark Muldoon.

As before, let $v \in a(U)$ be any vector not too close to the boundary of $a(U)$.

PROPOSITION 9. *Suppose there are sequences $x^{(m)}, y^{(m)} \in \mathbf{R}^n, q_m \to \infty$, such that*

$$v_m = \frac{y^{(m)} - x^{(m)}}{q_m} \to v,$$

and minimal states

$$\left(x_0^{(m)} = x^{(m)}, x_1^{(m)}, \ldots, x_{q_n-1}^{(m)}, x_{q_n}^{(m)} = y^{(m)}\right) \in \Psi_{q_n}^{x^{(m)}, y^{(m)}},$$

such that for $i = 1, \ldots, q-1$

$$\text{dist}\left(x_i^{(m)}, x^{(m)} + iv^{(m)}\right) < C$$

where C is independent of m.

Then there exists a minimal orbit with rotation vector v.

Proof. Using the periodicity of the generating function we can assume that all $x^{(m)}_{[\frac{q_m}{2}]}$ lie in the unit cell. Hence, by compactness one can also assume that $x^{(m)}_{[\frac{q_m}{2}]} \to x$ and corresponding r-coordinates also converge to r. By Proposition 3 the orbit $F^k(x, r) = (x_k, r_k)$ is minimal. Fixing k, putting $i = k + \left[\frac{q_m}{2}\right]$ in (11) and letting m go to ∞ we have for sufficiently large m

$$\text{dist}(x_k^{(m)}, x^{(m)} + kv) < 3C$$

and hence

$$\lim_{k \to \pm\infty} \frac{x_k - x}{k} = v.$$

□

Let us discuss some of numerical results supporting the validity of the criterion of the last proposition and possible shapes of minimal orbits for irrational rotation vectors which do not produce KAM tori. The results were obtained by Mark Muldoon in his Ph.D Thesis [Mu]. Related calculations can be found in a preprint by Kook and Meiss [KM].

Muldoon studies Birkhoff minimizing periodic orbits for four-dimensional symplectic maps. The symbol $(p,q)/m$ describes such an orbit with rotation vector $\left(\frac{p}{m}, \frac{q}{m}\right)$. A typical example presented by Figure 3 displays a highly disconnected behavior for a fairly small approximation of the integrable map due to a near resonance. Nevertheless, the picture is quite far from a uniform two-dimensional Cantor set which would appear for a product of two twist maps. This is probably related to very different behavior of two positive Lyapunov exponents corresponding to the minimizing orbits. We refer to [Mu] for further discussion.

Figure 4 shows that the bounded deviation condition of Proposition 9 is highly plausible.

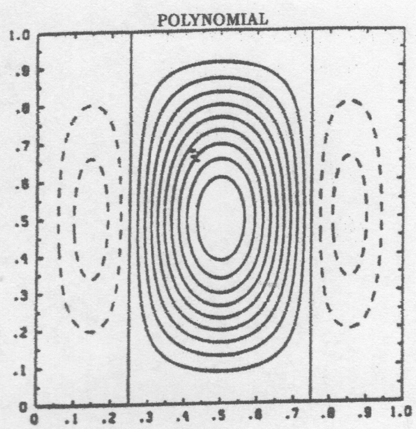

Figure 1

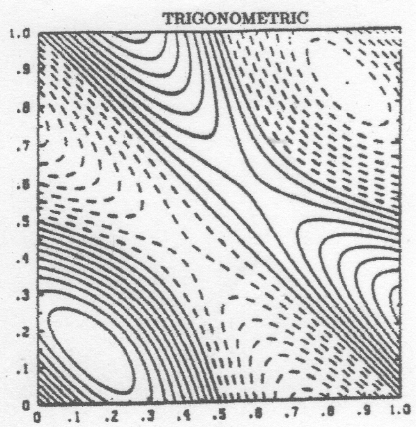

Figure 2

Pictures of the perturbations to the generating function. We study maps generated by functions of the form

$$H_\varepsilon(x,x') = h(x'-x) + P_\varepsilon(x,x'),$$

where

$$h(x'-x) = \frac{1}{2}\|x'-x\|^2\,, \qquad P_\varepsilon(x,x') = \varepsilon P(x),$$

and

$$P(x) = \begin{cases} \textit{either} \\ P_{\text{trig}}(x) = \dfrac{1}{M_{\text{trig}}}\left\{\dfrac{1}{2}(\sin 2\pi x_0 + \sin 2\pi x_1) + \sin 2\pi(x_0+x_1)\right\}, \\ \textit{or} \\ P_{\text{poly}}(x) = \dfrac{1}{M_{\text{poly}}}\left\{\left(\left[x_0^2(1-x_0)^2(x_0-\frac{3}{4})\left(\frac{1}{4}-x_0\right)\right)\right][x_1^2(1-x_1)^2]\right\}. \end{cases}$$

The x_i, $i=0,1$ in the formula above are real numbers, the components of the argument of the function $P(x)$. Call the first perturbation the trigonometric perturbation and the second the polynomial perturbation. The constants M_{trig} and M_{poly} are chosen so that $\max_{x\in T^n} P(x) = 1$. These are standard-like perturbations, they depend on x, but not on its successor, x'. Using the definition of a generating function one finds the map:

$$p'(x,p) = p - \varepsilon\left[\frac{\partial P}{\partial x}(x),\right]$$
$$x'(x,p) = x - p + \varepsilon\left[\frac{\partial P}{\partial x}(x)\right].$$

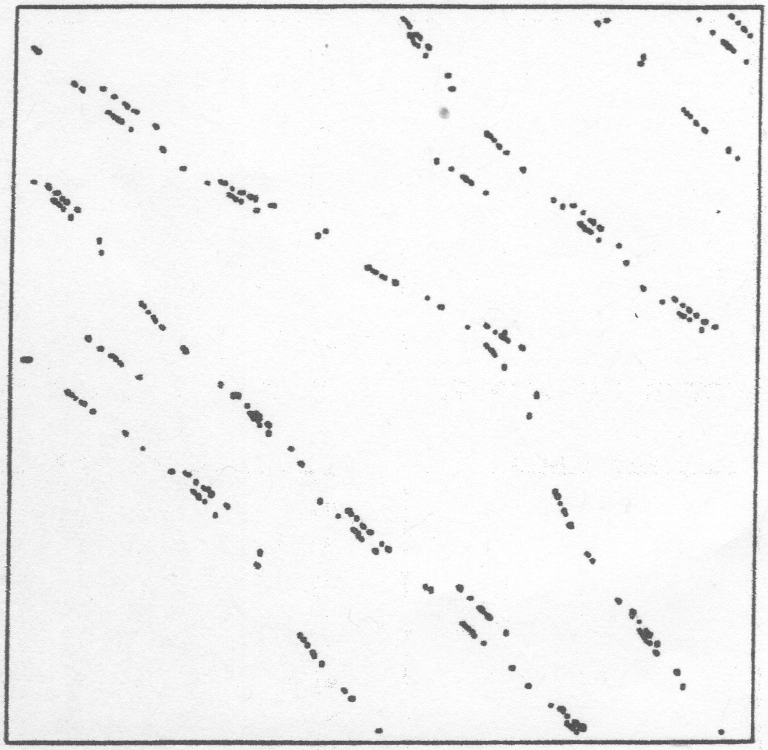

Figure 3

trigonometric perturbation
(1432, 1897)/2513

ε	0.0285
grad size	$6.259.10^{-6}$
shadow	$9.404.10^{-5}$

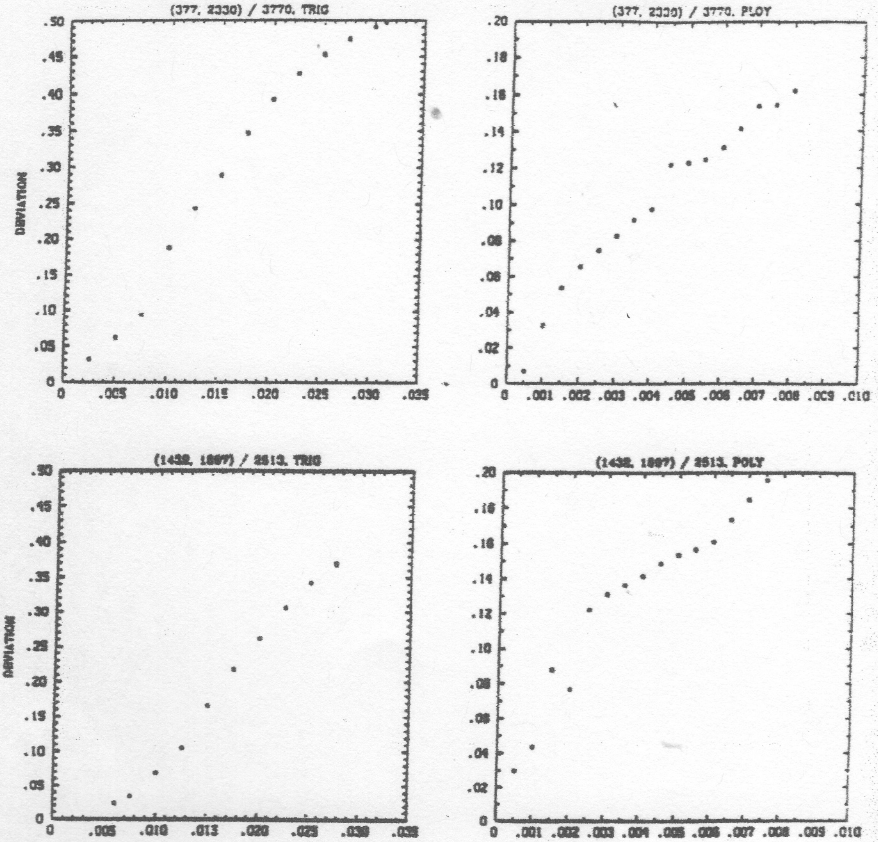

Figure 4

Deviation of points in the minimizing state from the position they would have if the state were unperturbed. The squares in the plots above represent the largest deviation found. The labels at the tops of the frames indicate the perturbation and rotation number.

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