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## Bernoulli Diffeomorphisms on surfaces

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#### Introduction

The smooth ergodic theory deals with the metric (ergodic) properties of classical dynamical systems (i.e., diffeomorphisms and smooth one-parameter flows on smooth manifolds) with respect to a positive smooth invariant measure. By a positive smooth measure we mean a measure on a smooth (usually  $C^{\infty}$ ) manifold (possibly with boundary) which is given by a positive smooth density in every local coordinate system. We shall say that a diffeomorphism  $f: M \to M$  is a Bernoulli diffeomorphism if (1) f preserves some smooth positive probability measure  $\mu$  on M and (2) considered as an automorphism of the Lebesgue space  $(M, \mu)$ , the mapping f is metrically isomorphic to a Bernoulli shift.

Recently Ja. B. Pesin ([1], [2], [3]) established remarkable connections between the Lyapunov characteristic exponents and the ergodic properties of classical dynamical systems on compact manifolds. One of the main results of Pesin can be described as follows. Almost every ergodic component of a diffeomorphism  $f: M \to M$  with respect to a smooth invariant measure, belonging to the set

 $\Lambda = \{x \in M, ext{ upper Lyapunov exponent } \chi^+(x, v) 
eq 0 ext{ for every tangent vector } v \in T_x M \}$  ,

has a positive measure, and the restriction of f on such a component is metrically isomorphic to the direct product of a Bernoulli shift and a cyclic permutation of a finite set. If  $\Lambda = M(\mod 0)$  and some extra conditions concerning unstable manifolds hold, then f is a Bernoulli diffeomorphism ([3], Theorems 7.5, 7.6, 7.7, 7.8, 8.1).

In this paper we construct a diffeomorphism satisfying the strong version of Pesin's conditions on every two-dimensional manifold. Thus, we prove the existence of Bernoulli diffeomorphisms on such manifolds. In the

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main part of the construction we deal with the two-dimensional disc. Reduction of the general case to this particular case is given by a purely differential topology construction and does not depend on the ergodic properties of the example on the disc. We make this reduction for the n-dimensional case because we hope that our result admits an n-dimensional generalisation.

This work has been done in co-operation with V.Z. Grines. Non-mathematical circumstances have forced separate publication.

#### 1. Preliminaries

1. The following standard notations are frequently used in the paper:

$$\mathbf{D}_r^n = \{(x_1, \cdots, x_n) \in \mathbf{R}^n, x_1^2 + \cdots + x_n^2 \leq r^2\}$$
,  
 $\mathbf{D}^n = \mathbf{D}_1^n, S^n = \partial \mathbf{D}^n, \mathbf{T}^n = \mathbf{R}^n / \mathbf{Z}^n$ .

The probability Lebesgue measures on all these manifolds are always denoted by  $\lambda$ .

2. We use two ways to produce the induced map of measures by the given point map. First, if  $(X, \mu)$  is a measure space and  $f: X \to Y$  is a map then the formula  $(f_*\mu)(A) = \mu(f^{-1}(A))$  defines the measure on Y.

On the other hand, let M, N be *n*-dimensional manifolds and  $f: M \to N$ be a map with the following property: the restriction  $f|_A$  is a smooth map for some A such that  $\mu(M \setminus A) = 0$  for every smooth positive measure  $\mu$  on M and  $\nu(f(M \setminus A)) = 0$  for every smooth positive measure  $\nu$  on N. Smooth measures are defined locally by differential *n*-forms so that in this case we can define the induced measure  $f^*\nu$  for every smooth measure  $\nu$  on N. If  $f|_A$  is injective then  $f^*f_*\mu = \mu$ . If, moreover,  $\nu(N \setminus f(M)) = 0$  then  $f_*f^*\nu = \nu$ .

3. We shall widely use definitions, notations and results concerning Lyapunov characteristic exponents, local stable and unstable manifolds and ergodic properties of measure-preserving diffeomorphisms. This material is contained in [3], especially in Sections 3, 4, 7, 8. The multiplicative ergodic theorem which plays the fundamental role in this region is proved in [4] (Theorems 1-4).

4. It is useful for our purposes to define classes of functions and diffeomorphisms on the two-dimensional disc  $\mathbf{D}^2$  which are "sufficiently flat" near  $\partial \mathbf{D}^2$ . We shall say that a sequence  $\rho = (\rho_0, \rho_1, \cdots)$  of real-valued continuous functions on  $\mathbf{D}^2$  is admissible if every function  $\rho_n$ ,  $n = 0, 1, 2, \cdots$ , is non-negative and positive inside the disc. Let

$$C^{\infty}_{
ho}(\mathbf{D}^2)=\Big\{h\in C^{\infty}(\mathbf{D}^2),\ orall n\geqq 0,\ \existsarepsilon_n>0:\ orall(x_1,\ x_2)\in \mathbf{D}^2,\ x_1^2+x_2^2\geqq (1-arepsilon_n)^2\ ,$$

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for all non-negative integers  $i_1$ ,  $i_2$ ,  $i_1 + i_2 = n$ ,

$$\left|rac{\partial^n h(x_{\scriptscriptstyle 1},\,x_{\scriptscriptstyle 2})}{\partial^{i_1} x_{\scriptscriptstyle 1} \partial^{i_2} x_{\scriptscriptstyle 2}}
ight| < 
ho_{\scriptscriptstyle n}(x_{\scriptscriptstyle 1},\,x_{\scriptscriptstyle 2}) 
ight\} \;.$$

Any diffeomorphism  $f: \mathbf{D}^2 \to \mathbf{D}^2$  can be represented in the coordinate form:  $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$ . Let us define

 $\mathrm{Diff}_{
ho}^{\infty}(\mathbf{D}^2)=\{f\in\mathrm{Diff}^{\infty}(\mathbf{D}^2):f_i(x_1,x_2)-x_i\in C_{
ho}^{\infty}(\mathbf{D}^2),\ i=1,\,2\}\;.$ 

These definitions can be extended immediately from  $D^2$  to  $D^n$ .

Similarly we can define the classes  $C_{\rho}^{\infty}(\mathbf{R}^{n}, 0)$  and  $\operatorname{Diff}_{\rho}^{\infty}(\mathbf{R}^{n}, 0)$  of "flat" germs of functions and diffeomorphisms near the origin in  $\mathbf{R}^{n}$  marked by a sequence  $\rho = (\rho_{0}, \rho_{1}, \cdots)$  of germs  $\rho_{m}$  such that  $\rho_{m}(x) \geq 0$  and  $\rho_{m}(x) > 0$  for  $x \neq 0$ . In the one-dimensional case we need the corresponding one-sided definition of classes  $C_{\rho}^{\infty}(\mathbf{R}^{+}, 0)$  for functions defined on the half-line.

5. The following two propositions are used for extending our examples from the disc to an arbitrary manifold.

PROPOSITION 1.1. Let  $M^n$  be an n-dimensional  $C^{\infty}$  manifold and  $F: \mathbf{D}^n \to M^n$ be a continuous mapping such that the restriction  $F|_{\operatorname{Int} \mathbf{D}^n}$  is a diffeomorphic embedding. Then there exists an admissible sequence of functions  $\rho$  such that for every  $h \in C^{\infty}_{\rho}(\mathbf{D}^n)$  and  $f \in \operatorname{Diff}^{\infty}_{\rho}(\mathbf{D}^n)$  the formulae

$$\widehat{h}(y) = egin{cases} h(F^{-1}y) & if \ y \in F(\operatorname{Int} \mathbf{D}^n) \ 0 & otherwise \end{cases}$$

and

(1.1) 
$$\hat{f}(y) = \begin{cases} F(f(F^{-1}y)) & \text{if } y \in F(\operatorname{Int} \mathbf{D}^n) \\ y & \text{otherwise} \end{cases}$$

define a  $C^{\infty}$  function on  $M^n$  and a  $C^{\infty}$  diffeomorphism of  $M^n$ .

PROPOSITION 1.2. Let  $\mu$  be an arbitrary positive smooth measure on a compact connected n-dimensional  $C^{\infty}$  manifold M (possibly with boundary). There exists a continuous mapping  $F: \mathbf{D}^n \to M$  with the following properties:

- 1.1. The restriction  $F|_{Int \mathbf{D}^n}$  is a diffeomorphic embedding;
- 1.2.  $F(\mathbf{D}^n) = M;$
- 1.3.  $\mu(M\setminus F(\mathbf{D}^n)) = 0;$
- 1.4.  $F_*\lambda = \mu$ .

The proof of Proposition 1.1 is routine so we omit it. The proof of the second proposition rests upon the following fact from differential topology.

PROPOSITION 1.3. For every  $C^{\infty}$  n-dimensional compact connected manifold M there exists a  $C^{\infty}$  mapping  $f: \mathbf{D}^n \to M$  such that  $f(\mathbf{D}^n) = M$  and the

restriction  $f|_{Int \mathbf{D}^n}$  is a diffeomorphic embedding.

Undoubtedly, this fact is well-known, although I find it difficult to give the exact reference. It can be proved by induction with the use of a smooth triangulation of M. Let  $\sigma_1, \dots, \sigma_N$  be the *n*-dimensional simplexes of such a triangulation, so ordered that each simplex  $\sigma_k$  has a common (n-1)dimensional face with some simplex  $\sigma_i$ , i < k. The inductive statement is formulated as follows:

For every  $k = 1, \dots, N$  there exists a  $C^{\infty}$  mapping  $f^k: \mathbf{D}^n \to M$  with the following properties:

1.5.  $f^k(\mathbf{D}^n) = \bigcup_{i=1}^k \sigma_i$ .

1.6. The restriction  $f^{k}|_{Int \mathbf{D}^{n}}$  is a diffeomorphic embedding.

1.7. For every  $i, 1 \leq i \leq k$ , and every (n - 1)-dimensional face  $\sigma$  of the simplex  $\sigma_i$  there exists a point  $x \in \partial \mathbf{D}^n$  such that  $f^k(x) \in \sigma$  and  $f^k$  is regular at the point x.

The statement for k = 1 is almost obvious. The inductive transition is reduced to the following local statement.

Let  $\sigma'$ ,  $\sigma''$  be two smooth *n*-dimensional simplexes with a common (n-1)-dimensional face  $\sigma$  and x be an interior point of  $\sigma$ . Then for every neighborhood U of the point x there exists a  $C^{\infty}$  mapping  $g: \sigma' \to \sigma' \cup \sigma''$  such that

1.8. 
$$g(\sigma') = \sigma' \cup \sigma'';$$

1.9. g coincides with identity outside U;

1.10. The restriction  $g|_{Int \sigma'}$  is a diffeomorphic embedding;

1.11. For every (n-1)-dimensional face  $\hat{\sigma}$  of  $\sigma''$ , there exists a regular point x of g such that  $g(x) \in \hat{\sigma}$ .

This statement means that a small part of  $\sigma'$  may be "blown up" onto  $\sigma''$  through a small "hole" in the face  $\sigma$  without perturbation of the remaining part of  $\sigma'$ .

The inductive transition from k to k + 1 follows immediately. For, let  $\sigma'$  be such a simplex  $\sigma_i$ ,  $i \leq k$ , that has a common (n-1)-face  $\sigma$  with  $\sigma_{k+1}$ , and let  $\sigma''$  be  $\sigma_{k+1}$ . Construct the map  $g: \sigma_i \to \bigcup \sigma_{k+1}$  satisfying properties 1.8-1.11 and prolong this map identically to the map  $g: \bigcup_{i=1}^k \sigma_i \to \bigcup_{i=1}^{k+1} \sigma_i$ . Then the map  $f^{k+1} = gf^k$  satisfies conditions 1.5-1.7 for k + 1.

6. Proof of Proposition 1.2. Let f be mapping satisfying the assertions of Proposition 1.3. Then  $f^*\mu = \rho\lambda$  where  $\rho$  is a  $C^{\infty}$  function which is positive inside  $\mathbf{D}^n$ . We are going to construct a homeomorphism  $h: \mathbf{D}^n \to \mathbf{D}^n$ which is  $C^{\infty}$  and regular inside  $\mathbf{D}^n$  such that  $h^*(\rho\lambda) = h^*(f^*\mu) = \lambda$ . Then

$$(f \circ h)_* \lambda = f_* h_* \lambda = f_* (\rho \lambda) = f_* f^* \mu = \mu$$
 ,

i.e., the mapping  $F = f \circ h$  which obviously satisfies conditions 1.1-1.3 also satisfies condition 1.4.

Now we proceed to the construction of h. First, we construct a diffeomorphism  $h_1: \mathbf{D}^n \to \mathbf{D}^n$  such that  $(h_1^*f^*\mu)(\mathbf{D}_{1/2}^n) = \lambda(\mathbf{D}_{1/2}^n)$ . To do this, we can find r > 0 such that  $(f^*\mu)(\mathbf{D}_r^n) = \lambda(\mathbf{D}_{1/2}^n)$  and let  $h_1$  be an arbitrary diffeomorphism, such that  $h_1(\mathbf{D}_{1/2}^n) = \mathbf{D}_r^n$ . Let  $\rho_1$  be a density of the measure  $h_1^*f^*\mu$  with respect to  $\lambda$ .

Consider the restriction of the measures  $\lambda$  and  $\rho_1\lambda$  to the set  $K^n = \mathbf{D}^n \setminus \operatorname{Int} \mathbf{D}_{1,2}^n$ . Let  $\pi: K^n \to S^{n-1}$  be the radial projection:  $\pi(x_1, \dots, x_n) = (x_1/r, \dots, x_n/r)$  where  $r = (\sum_{i=1}^n x_i^2)^{1/2}$ . Since  $\lambda(K^n) = (\rho_1\lambda)(K^n)$ , we have

(1.2) 
$$(\pi_*\lambda)(S^{n-1}) = (\pi_*(\rho_1\lambda))(S^{n-1}) .$$

The measure  $\pi_*\lambda$  coincides with the Lebesgue measure  $\lambda$  up to a constant factor while the measure  $\pi_*(\rho_1\lambda)$  has the form  $\kappa\lambda$  where

$$\kappa(y_1, \cdots y_n) = \alpha_n \int_{1/2}^1 \rho_1(ry_1, \cdots, ry_n) r^{n-1} dr$$
 ( $\alpha_n$  is a constant).

Since  $\kappa$  is a positive  $C^{\infty}$  function,  $\pi_*(\rho_1\lambda)$  is a positive smooth measure on  $S^{n-1}$ and by the Moser theorem [5] there exists a  $C^{\infty}$  diffeomorphism  $\tilde{h}: S^{n-1} \to S^{n-1}$ such that  $\tilde{h}^*(\pi_*(\rho_1\lambda)) = \pi_*\lambda$ . This diffeomorphism can be lifted to  $K^n$ , for example by the following formula:

 $h_2(x_1, \cdots, x_n) = (\sum_{i=1}^n x_1^2)^{1/2} \widetilde{h} (\pi(x_1, \cdots, x_n)).$ 

Since  $h_2 \circ \pi = \pi \circ \tilde{h}$  we have

(1.3) 
$$\pi_* h_2^*(\rho_1 \lambda) = \tilde{h}^* \big( \pi_*(\rho_1 \lambda) \big) = \pi_* \lambda .$$

Let us denote the density of the measure  $h_2^*(\rho_1\lambda)$  with respect to  $\lambda$  by  $\rho_2$ . Formula (1.3) means that for every  $(y_1, \dots, y_n) \in S^{n-1}$ ,

$$\int_{r=1/2}^{1}\rho_2(ry_1, \cdots, ry_n)r^{n-1}dr = \alpha = \text{const.}$$

Now we want to improve conditional measures on radii. For this purpose we let

$$h_{3}(ry_{1}, \cdots, ry_{n}) = \left(\varphi_{y_{1}\cdots y_{n}}(r)y_{1}, \cdots, \varphi_{y_{1}\cdots y_{n}}(r)y_{n}\right)$$

where  $\varphi_{y_1 \cdots y_n}(r)$  is an inverse function to

$$\psi_{y_1\cdots y_n}(r) = \frac{1}{2} + \left(\frac{1}{2\alpha} \int_{1/2}^r \rho_2(uy_1, \cdots, uy_n) u^{n-1} du\right)^{1/n}$$

It is easy to verify that  $h_3$  is a homeomorphism of  $K^n$  which is  $C^{\infty}$  and regular for  $r \neq 1$  (i.e., on  $K^n \setminus \partial \mathbf{D}^n$ ) and that  $h_3^*(\rho_2 \lambda) = \lambda$ . Thus we have on the set  $K^n$ ,

$$h_3^*h_2^*(\rho_1\lambda) = \lambda$$
.

The homeomorphism  $h_2 \circ h_3 \colon K^n \to K^n$  is  $C^{\infty}$  and regular on  $K^n \langle \partial \mathbf{D}^n$  so that it can be extended to a homeomorphism  $\hat{h} \colon \mathbf{D}^n \to \mathbf{D}^n$  which is  $C^{\infty}$  and regular on Int  $\mathbf{D}^n$ . Denote the density of the measure  $\hat{h}^*(\rho_1\lambda)$  by  $\hat{\rho}$ . This density is a positive  $C^{\infty}$  function which is equal to 1 on  $K^n$ . Moreover,  $\int_{\mathbf{D}^n} \hat{\rho} d\lambda = 1$ , so that we can apply a version of the Moser theorem (see [6], Theorem 1.1) and construct a  $C^{\infty}$  diffeomorphism  $h_4 \colon \mathbf{D}^n \to \mathbf{D}^n$  identical near  $\partial \mathbf{D}^n$  such that

$$h_{ extsf{4}}^{st}(\widehat{
ho}\lambda)=\lambda$$
 .

Finally, let  $h = h_1 \hat{h} h_4$ . Combining the equalities  $h_1^* f^* \mu = \rho_1 \lambda$ ,  $h^*(\rho_1 \lambda) = \hat{\rho} \lambda$ and (1.4), we obtain

$$h^*(f^*\mu) = h^*_*(\hat{h}^*h_{\scriptscriptstyle 1}^*f^*\mu) = h^*_*ig(\hat{h}^*(
ho_{\scriptscriptstyle 1}\lambda)ig) = h^*_*(\hat{
ho}\lambda) = \lambda \; .$$

Proposition 1.2 is proved.

# 2. Construction of Bernoulli diffeomorphisms on the two-dimensional disc

1. THEOREM A. For every admissible sequence of functions  $\rho$  on  $\mathbf{D}^2$ there exists a Bernoulli diffeomorphism  $g \in \text{Diff}_{\rho}^{\infty}(\mathbf{D}^2)$  which preserves the Lebesgue measure  $\lambda$ .

*Remark.* All positive smooth measures are equivalent, so g may have only one invariant ergodic positive probability smooth measure. Consequently the Bernoulli smooth positive measure  $\mu$  coincides with  $\lambda$ .

We begin the construction of such a diffeomorphism g by considering a linear hyperbolic automorphism  $g_0$  of  $T^2$  having positive eigenvalues which leaves the following four points fixed:

(2.1) 
$$x_1 = (0, 0), x_2 = \left(\frac{1}{2}, 0\right), x_3 = \left(0, \frac{1}{2}\right), x_4 = \left(\frac{1}{2}, \frac{1}{2}\right)$$

For example, the automorphism generated by the matrix  $\begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}$  is appropriate.

The construction can be represented by means of the following commutative diagram:

Actually we shall construct the mappings  $g_1, \varphi_1, \varphi_2, \varphi_3$ . The existence of a homeomorphism h that makes the left square of the diagram commutative and transforms the stable and unstable manifolds of  $g_0$  into smooth curves is proved in Section 4. The mappings  $g_2$ ,  $g_3$ , and g (which will prove to be  $C^{\infty}$  diffeomorphisms) are defined by the commutativity of the diagram.

The mapping  $g_1$  will be a homeomorphism which is  $C^{\infty}$  everywhere except for the points  $x_1, x_2, x_3, x_4$  (see (2.1)). It preserves a measure  $\nu = \rho \lambda$ where the density  $\rho$  is a positive function  $C^{\infty}$  everywhere except for the points  $x_i$  with singularities at these points. The mappings  $\varphi_1$  and  $\varphi_2$  will be  $C^{\infty}$  and regular everywhere except for the points  $x_i$ ;  $\varphi_i$  has the same properties everywhere except for  $\varphi_2(x_1)$ .

From the measure-theoretical point of view the squares in the righthand part of the diagram (2.2) can be represented in the following way:

Here the two-way arrows represent automorphisms of measure spaces and the one-way arrows represent non-invertible homomorphisms.

2. Construction of  $g_1$ . It is convenient to use a special coordinate system near each of the points  $x_i$ , i = 1, 2, 3, 4, namely coordinates in which the map  $g_0$  has a diagonal linear form. It will be recalled that all these points are fixed with respect to  $g_0$ . We shall denote these coordinates by  $(s_1, s_2)$ . Let us denote the neighborhood of the point  $x_i$  which is given in these coordinates as  $\{(s_1, s_2), s_1^2 + s_2^2 \leq r^2\}$  by  $D_r^i$ , i = 1, 2, 3, 4. Let us choose numbers  $r_{\scriptscriptstyle 0} > r_{\scriptscriptstyle 1} > r_{\scriptscriptstyle 2} > 0$  so that  $D^i_{r_0} \cap D^j_{r_0} = \oslash$ ,  $i \neq j$ ,  $(g_{\scriptscriptstyle 0} D^i_{r_1} \cup g^{-1} D^i_{r_1}) \subset D^i_{r_0}$ ,  $D^i_{r_2} \subset \operatorname{Int}(g_{\scriptscriptstyle 0} D^i_{r_1}).$ 

The construction of  $g_1$  depends on the choice of a real-valued function w defined on the unit interval and having the following properties:

- 2.1.  $\psi$  is  $C^{\infty}$  except the point 0;
- 2.2.  $\psi(0) = 0;$
- 2.3.  $\psi'(u) \ge 0;$

To provide the differentiability of the mappings  $g_2$ ,  $g_3$ , g and the inclusion  $g \in \operatorname{Diff}_{\rho}^{\infty}(\mathbf{D}^2)$ , the function  $\psi$  must satisfy some extra conditions. Namely, near zero the integral  $\int_{0}^{1} du/\psi(u)$  must converge "very slowly". These conditions are precisely formulated and explained in Section 3.

However, the important dynamical properties of  $g_1$  (and, consequently, those of  $g_2$ ,  $g_3$ , g) do not depend on these additional conditions. Thus we can for the time being construct and study the mapping  $g_1$  using any function  $\psi$  satisfying conditions 2.1-2.5.

The mapping  $g_1$  will coincide with the linear automorphism  $g_0$  outside the set  $\bigcup_{i=1}^{4} D_{r_1}^i$ . In the neighborhood  $D_{r_1}^i$  the automorphism  $g_0$  can be represented as the time-one map generated by the vector field

(2.3) 
$$\begin{aligned} \dot{s}_1 &= \ln \alpha s_1 ,\\ \dot{s}_2 &= -\ln \alpha s_2 . \end{aligned}$$

Here  $\alpha$  is the larger eigenvalue of the matrix generating  $g_0$ . Let us denote by  $\tilde{g}_{\psi}^i$  the time-one map generated by the following vector field  $v_{\psi}$  defined in  $D_{r_0}^i$ :

(2.4) 
$$\begin{aligned} \dot{s}_1 &= \ln \alpha s_1 \psi(s_1^2 + s_2^2) ,\\ \dot{s}_2 &= -\ln \alpha s_2 \psi(s_1^2 + s_2^2) . \end{aligned}$$

The choice of the number  $r_1$  and properties 2.3 and 2.4 imply that the domain of  $\widetilde{g}^i_{\Psi}$  contains  $D^i_{r_1}$  and

$$\widetilde{g}^i_{\psi}(D^i_{r_1}) \subset D^i_{r_0}$$
 .

Further, the choice of  $r_2$  and condition 2.4 guarantee that  $\tilde{g}_{\psi}^i$  coincides with  $g_0$  in some neighborhood of the boundary  $\partial D_{r_1}^i$ , so the following formula defines the homeomorphism  $g_{\psi}: \mathbb{T}^2 \to \mathbb{T}^2$  which is a  $C^{\infty}$  diffeomorphism everywhere except for the points  $x_i$ , i = 1, 2, 3, 4:

$$g_{\psi}(x) = egin{cases} g_{0}x ext{ if } x \in \mathbf{T}^{2}igvee igvee_{i=1}^{4} D_{r_{1}}^{i} \ \widetilde{g}_{\psi}^{i}x ext{ if } x \in D_{r_{1}}^{i} ext{, } i=1,2,3,4 \; . \end{cases}$$

*Remarks.* 1. Actually, to construct  $g_{\psi}$  we do not need the assumption  $\psi(0) = 0$ . Moreover, if  $\psi \in C^{1}([0, 1])$  then  $g_{\psi}$  is a  $C^{1}$  diffeomorphism of  $\mathbf{T}^{2}$ .

2. It is easy to ensure the  $C^1$  differentiability of  $g_{\psi}$  in the case where  $\psi(0) = 0$ . Namely, it is true if

$$\lim_{u\to 0}\psi'(u)\sqrt{u}=0.$$

Denote the local coordinates of the point  $x \in D_{r_1}^i$  by  $(s_1(x), s_2(x))$ . Let

$$ho(x) = egin{cases} rac{1}{\psiig(s^2_1\!(x) + s^2_2\!(x)ig)} & ext{if} \ x \in D^i_{r_1} \ , \qquad i=1,\ 2,\ 3,\ 4 \ 1 & ext{otherwise} \ . \end{cases}$$

The function  $\rho$  so defined is  $C^{\infty}$  everywhere except for the points  $x_i$ , i = 1, 2, 3, 4, and has integrable singularities at these points. The map  $g_{\psi}$  has the invariant probability measure

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 $oldsymbol{
u}=
ho_{\scriptscriptstyle 0}^{\scriptscriptstyle -1}
ho\lambda$  , where  $ho_{\scriptscriptstyle 0}=\int_{oldsymbol{ au}^2}\!
ho d\lambda$  .

We let  $g_1 = g_{\psi}$  for some function  $\psi$  satisfying conditions 2.1-2.5.

The following two propositions contain all of the properties of  $g_{\psi}$  which we need to establish the Bernoulli property for  $g_2$ ,  $g_3$  and g with respect to the Lebesgue measures.

**PROPOSITION 2.1.** The mapping  $g_{\psi} = g_1$  is topologically conjugate to  $g_0$ . Moreover, the conjugating homeomorphism h transforms the stable and unstable manifolds of  $g_0$  into smooth curves.

PROPOSITION 2.2. The larger Lyapunov exponent  $\chi_2(x)$  of  $g_{\psi}$  is positive almost everywhere with respect to any Borel invariant measure  $\mu$  such that  $\mu(\{x_i\}) = 0, i = 1, 2, 3, 4.$ 

These propositions are proved in Section 4.

3. Construction of  $\varphi_1$ . The involution  $J: \mathbf{T}^2 \to \mathbf{T}^2$  given by the formula

$$J(t_1, t_2) = (1 - t_1, 1 - t_2)$$

has four fixed points:  $x_1$ ,  $x_2$ ,  $x_3$ ,  $x_4$  and commutes with  $g_0$ . It is necessary that the mapping  $\varphi_1$ :  $\mathbf{T}^2 \to \mathbf{T}^2$  should be a homeomorphism which is a  $C^{\infty}$  diffeomorphism outside the four fixed points  $x_1$ ,  $x_2$ ,  $x_3$ ,  $x_4$ . Moreover, it must have the following properties:

- 2.6.  $(\varphi_1)_* \nu = \lambda;$
- 2.7.  $\varphi_{\scriptscriptstyle 1} \circ J = J \circ \varphi_{\scriptscriptstyle 1};$

2.8. In a neighborhood of each point  $x_i$ ,

$$(2.5) \qquad \mathcal{P}_{1}(s_{1}, s_{2}) = \left(\frac{\rho_{0}^{-1/2}s_{1}}{\sqrt{s_{1}^{2} + s_{2}^{2}}} \left(\int_{0}^{s_{1}^{2} + s_{2}^{2}} \frac{du}{\psi(u)}\right)^{1/2}, \quad \frac{\rho_{0}^{-1/2}s_{2}}{\sqrt{s_{1}^{2} + s_{2}^{2}}} \left(\int_{0}^{s_{1}^{2} + s_{2}^{2}} \frac{du}{\psi(u)}\right)^{1/2}\right).$$

Such a mapping can be constructed in the following way: Let us choose  $r_3 > 0$  so that  $\rho_0^{-1} \int_0^{r_3} du/\psi(u) < r_1^2$  and define the map  $\tilde{\varphi}_1^i$  in  $D_{r_3}^i$  by the formula (2.5). One can easily verify that  $(\tilde{\varphi}_1^i)_* \nu = \lambda$  and  $\tilde{\varphi}_1^i \circ J = J \circ \tilde{\varphi}_1^i$ . The choice of the number  $r_3$  implies that  $\tilde{\varphi}_1^i D_{r_3}^i \subset D_{r_1}^i$ .

The maps  $\tilde{\varphi}_1^i$  so defined can be extended to a map  $\tilde{\varphi}_1$  which satisfies all the conditions we require from  $\varphi_1$  except 2.6. For example, let  $\tilde{\varphi}_1 = \mathrm{id}$ outside the set  $\bigcup_{i=1}^{i} D_{r_0}^i$  and prolong each of the maps  $\tilde{\varphi}_1^i$  symmetrically along the radii up to a map identical near the boundary. Denote the density of the measure  $(\tilde{\varphi}_{1*})\nu$  with respect to  $\lambda$  by  $\rho_1$ . Obviously  $\rho_1 \circ J = \rho_1$  and  $\rho_1 = 1$  in a neighborhood of each point  $x_i$ . Now we can construct a diffeomorphism  $\chi: \mathbf{T}^2 \to \mathbf{T}^2$ , identical in neighborhoods of  $x_i$ , i = 1, 2, 3, 4, such that  $\chi \circ J = J \circ \chi$  and  $\chi_*(\rho_1 \lambda) = \lambda$ . To do this, let us consider the submanifold

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 $M = \mathbf{T}^2 \setminus \operatorname{Int} (\bigcup_{i=1}^i D_{r_4}^i)$ , where  $r_4$  is a sufficiently small positive number, as a double covering over its factor space  $M|_J = N$  with projection  $\pi: M \to N$ . By Theorem 1.1 of [6] there exists a diffeomorphism  $\tilde{\chi}: N \to N$ , homotopic to the identity and identical near the boundary  $\partial N$ , such that  $(\tilde{\chi})_* \pi_*(\rho_1 \lambda) = \pi_* \lambda$ . To obtain the required diffeomorphism  $\chi$ , let us lift  $\tilde{\chi}$  to the covering and extend it identically to the remaining part of the torus.

The map  $\varphi_1 = \chi \circ \widetilde{\varphi}_1$  satisfies all our requirements.

4. Construction of  $\varphi_2$ . The factor space  $\mathbf{T}^2|_J$  is homeomorphic to the 2-sphere  $S^2$  and admits the natural smooth structure. This structure is induced from the torus everywhere except for the points  $x_i$ . The coordinate mapping  $\theta$  in a neighborhood of the point  $x_i$  is defined as follows:  $\theta(s_1, s_2) = (\tau_1, \tau_2)$  where

$$au_{_1}=rac{m{s}_1^2-m{s}_2^2}{\sqrt{m{s}_1^2+m{s}_2^2}}$$
 ,  $au_{_2}=rac{2m{s}_1m{s}_2}{\sqrt{m{s}_1^2+m{s}_2^2}}$  .

The image of the Lebesgue measure  $\lambda$  on  $\mathbf{T}^2$  is a positive smooth measure on the factor space which coincides with the measure  $d\tau_1 d\tau_2$  in a neighborhood of each point  $x_i$ .

These facts imply the existence of a map  $\varphi_2: \mathbf{T}^2 \to S^2$  with the following properties;

2.9.  $\varphi_2$  is a double branched covering which is regular and  $C^{\infty}$  everywhere except for points  $x_1, x_2, x_3, x_4$  and branches at these points;

2.10. 
$$\varphi_2 \circ J = \varphi_2;$$

2.11.  $(\varphi_2)_* \lambda = \lambda;$ 

2.12. There exist the local coordinates  $(\tau_1, \tau_2)$  in a neighborhood of each point  $p_i = \varphi_2(x_i)$ , i = 1, 2, 3, 4, such that locally

(2.6) 
$$\varphi_2(s_1, s_2) = \left(\frac{s_1^2 - s_2^2}{\sqrt{s_1^2 + s_2^2}}, \frac{2s_1s_2}{\sqrt{s_1^2 + s_2^2}}\right),$$

where  $(s_1, s_2)$  are the local coordinates near the point  $x_i$  which we usually employ.

We shall not construct such a map explicitly although this is not a difficult problem.

5. Construction of  $\varphi_3$ . The mapping  $\varphi_3$  is a  $C^{\infty}$  diffeomorphism between  $S^2 \setminus \{p_1\}$  and Int  $\mathbf{D}^2$  with the following properties:

2.13.  $(\varphi_3)_*\lambda = \lambda$ .

2.14. In a neighborhood of the point  $p_1$ , the map  $\varphi_3$  has the following form

(2.7) 
$$\mathcal{P}_{3}(\tau_{1}, \tau_{2}) = \left(\frac{\tau_{1}\sqrt{1-\tau_{1}^{2}-\tau_{2}^{2}}}{\sqrt{\tau_{1}^{2}+\tau_{2}^{2}}}, \frac{\tau_{2}\sqrt{1-\tau_{1}^{2}-\tau_{2}^{2}}}{\sqrt{\tau_{1}^{2}+\tau_{2}^{2}}}\right)$$

The mapping  $g: \mathbf{D}^2 \to \mathbf{D}^2$  is defined by

$$g(x_1, x_2) = egin{cases} arphi_3 g_3^{-1} arphi_3^{-1}(x_1, x_2) & ext{if} \quad (x_1, x_2) \in ext{Int } \mathbf{D}^2 \ (x_1, x_2) & ext{if} \quad (x_1, x_2) \in \partial \mathbf{D}^2 \ . \end{cases}$$

6. Let us suppose that Propositions 2.1 and 2.2 are proved and that the choice of the function  $\psi$  can ensure the  $C^{\infty}$  differentiability of the mappings  $g_2$ ,  $g_3$ , g and the inclusion  $g \in \text{Diff}_{\rho}^{\infty}(\mathbf{D}^2)$ . We show how to conclude the proof of the theorem.

Proposition 2.1 implies that the homeomorphism h transforms the unstable foliation of  $g_0$  into a continuous mod  $O(\delta(x), 1)$  foliation of  $T^2$  in the sense of Definition 7.1 [3]. Naturally,  $g_1$  preserves this foliation. Further, the images of this foliation under the action of the mappings  $\varphi_1, \varphi_2\varphi_1, \varphi_3\varphi_2\varphi_1$ are continuous mod  $O(\delta(x), 1)$  foliations of  $T^2$ ,  $S^2$ ,  $D^2$  which are invariant with respect to the diffeomorphisms  $g_2, g_3, g$  respectively. Proposition 2.2 and the regularity of the mappings  $\varphi_1, \varphi_2, \varphi_3$  almost everywhere imply that the larger Lyapunov exponents  $\chi_2$  for  $g_2, g_3, g$  are positive almost everywhere with respect to the Lebesgue measures. Since in each of these cases Lebesgue measure is invariant, the smaller Lyapunov exponent  $\chi_1$  is equal to  $-\chi_2$ . Consequently, for almost every point y there exists a one-dimensional local unstable manifold  $V^-(y)$  (see [3], Theorem 4.1 and Definition 4.1). It is easy to see that  $V^-(y)$  belongs to the leaf of a corresponding foliation described above.

Let M mean  $\mathbf{T}^2$ ,  $S^2$  or  $\mathbf{D}^2$  and G mean  $g_2$ ,  $g_3$  or g. For every point  $z \in V^-(y)$  the distance between  $G^{-n}z$  and  $G_y^{-n}$  tends to 0 as  $n \to \infty$ . On the other hand the leaf of the corresponding foliation coincides with the set of all points z for which this distance tends to zero. The diffeomorphism G is topologically transitive. Thus we can apply Theorems 7.7 and 7.8 of [3]. It follows from Proposition 2.2 that  $\bigcup_{s=1}^{\infty} \Lambda_{s,1} = M \pmod{0}$  so by these theorems G is ergodic. Theorem 8.1 [3] shows that G is metrically isomorphic to the direct product of a Bernoulli shift and a cyclic permutation of a finite set. But conditions of Theorem 7.7 hold for every power  $G^m$ ,  $m \neq 0$ . Consequently, the cyclic component is trivial and G itself is a Bernoulli diffeomorphism.

#### 3. Smoothness of mappings $g_2$ , $g_3$ , $g_4$

In this section we explain how to ensure the differentiability of the mappings  $g_2$ ,  $g_3$ , g and the inclusion  $g \in \text{Diff}_{\rho}^{\infty}(\mathbf{D}^2)$  by the choice of the function  $\psi$ . All our considerations will be local because the problem concerns only neighborhoods of points  $x_1$ ,  $x_2$ ,  $x_3$ ,  $x_4$ . The vector field (2.3) is Hamiltonian with respect to the Lebesgue measure  $\lambda$  with the Hamiltonian function

 $H_1(s_1, s_2) = \ln \alpha s_1 \cdot s_2$ . The vector field  $v_{\psi}$  (see (2.4)) is obtained from this vector field by a time change, so it is Hamiltonian with respect to the measure  $\nu$  with the same Hamiltonian function  $H_1$ .

The mappings  $g_1$ ,  $g_2$ ,  $g_3$ , g are locally the time-one maps generated by the vector fields  $v_{\psi}$ ,  $(\varphi_1)_* v_{\psi}$ ,  $(\varphi_2 \varphi_1)_* v_{\psi}$  and  $(\varphi_3 \varphi_2 \varphi_1)_* v_{\psi}$ . Since  $(\varphi_1)_* \nu = \lambda$ ,  $(\varphi_2)_* \lambda = (\varphi_3)_* \lambda = \lambda$ , the last three vector fields are Hamiltonian with respect to the Lebesgue measures with the Hamiltonian functions  $H_2 = H_1 \circ \varphi_1^{-1}$ ,  $H_3 = 2H_2 \circ \varphi_2^{-1}$  and  $H = H_3 \circ \varphi_3^{-1}$  respectively. Coefficient 2 arises because  $\varphi_2^* \lambda = 2\lambda$ .

Let us denote the function inverse to

$$\gamma(u)=
ho_{_0}^{_{-1/2}}\sqrt{\int_{_0}^u\!rac{d au}{\psi( au)}}$$

by  $\beta(u)$ . Using the local explicit expressions of the mappings  $\varphi_1$ ,  $\varphi_2$  and  $\varphi_3$  (see (2.5), (2.6), (2.7)), we obtain the expressions of the functions  $H_2$  and  $H_3$  near the origin and H near  $\partial \mathbf{D}^2$ , namely:

$$egin{aligned} H_2(s_1,\,s_2) &= rac{\lnlpha\cdot s_1s_2ig(eta(\sqrt{s_1^2+s_2^2})ig)}{s_1^2+s_2^2} \,\,, \ H_3( au_1,\, au_2) &= rac{\lnlpha\cdot au_2ig(eta(\sqrt{ au_1^2+ au_2^2})ig)}{\sqrt{ au_1^2+ au_2^2}} \,\,, \ H(x_1,\,x_2) &= rac{\lnlpha\cdot x_2ig(eta(\sqrt{ au_1-x_1^2- au_2^2})ig)}{\sqrt{ au_1^2+ au_2^2}} \,\,. \end{aligned}$$

If the constant  $\rho_0$  is fixed, the function  $\psi$  can be found from  $\gamma$ , namely  $\psi(u) = \rho_0^{-1}/2\gamma(u)\gamma^1(u)$  and, consequently, from  $\beta$ .

Let us see how conditions 2.1-2.5 can be satisfied by the choice of the function  $\beta$ . If  $\beta(u)$  is  $C^{\infty}$ , then condition 2.1 holds. Condition 2.2 follows from the tendency of  $(\gamma^{-2}(u))'$  to  $\infty$  when  $u \to 0$ , and condition 2.3 from the convexity of the function  $\gamma^2$ . Both these properties hold if  $\beta$  decreases sufficiently fast near 0. Condition 2.4 is provided by an appropriate extension of the function  $\psi$ . Finally, condition 2.5 holds automatically. All the derivatives of the functions  $H_2$ ,  $H_3$  and H can be expressed from the derivatives of  $\beta$ , so that if the derivatives of  $\beta$  decrease near 0 sufficiently fast, then the derivatives for  $H_2$  and  $H_3$  near the origin and H near  $\partial \mathbf{D}^2$  will also decrease at any prescribed speed. Thus we have proved the following fact.

PROPOSITION 3.1. For any sequence  $\kappa$  of admissible germs near the origin in  $\mathbf{R}^2$  and any sequence  $\rho$  of admissible functions on  $\mathbf{D}^2$  there exists a sequence  $\theta$  of admissible germs near 0 in  $\mathbf{R}^+$  such that, if  $\beta \in C^{\infty}_{\theta}(\mathbf{R}^+, \mathbf{0})$ , then  $H_2$ ,  $H_3 \in C^{\infty}_{\kappa}(\mathbf{R}^2, \mathbf{0})$  and  $H \in C^{\infty}_{\rho}(\mathbf{D}^2)$ .

We shall not give a detailed description of translation from Hamiltonian

functions and vector fields to time-one maps. The decrease of the derivatives of solutions of differential equations near 0 and near  $\partial \mathbf{D}^2$  may be estimated from the decrease of the right sides and, consequently, through the Hamiltonian functions. Thus, an appropriate choice of the function  $\beta$  guarantees that the diffeomorphisms  $g_2$ ,  $g_3$ , g are  $C^{\infty}$  and  $g \in \text{Diff}_{\rho}^{\infty}(\mathbf{D}^2)$ .

#### 4. Proof of Propositions 2.1 and 2.2

The given function  $\psi$  satisfying conditions 2.1-2.5 can be joined with the function equal to 1 identically through  $C^1$  functions satisfying all of these conditions except 2.2. To be more definite, let us fix the following continuous family of functions joining  $1 = \psi_0$  and  $\psi = \psi_1$ :

$$\psi_ au(u)=1- au+rac{ au}{1- au} \int_0^{1- au} \psi(u+s) ds \; .$$

To simplify our notations, we let  $g_{\psi_{\tau}} = g_{\varepsilon}$ . We are going to prove that  $g_{\varepsilon}$ is an Anosov diffeomorphism for every  $\tau: 0 \leq \tau < 1$  and that  $g_1$  is of the "almost Anosov" type. Let  $(\xi_1, \xi_2)$  be the natural coordinates in each tangent space  $T_x \mathbf{T}^2$  such that the linear map  $Dg_0$  has the form  $Dg_0(\xi_1, \xi_2) = (\alpha \xi_1, \alpha^{-1} \xi_2)$ , and let us denote the cone  $\{(\xi_1, \xi_2) \in T_x \mathbf{T}^2, |\xi_2| \leq |\xi_1|\}$  by  $K_x^+$  and the cone  $\{(\xi_1, \xi_2) \in T_x \mathbf{T}^2, |\xi_1| \leq |\xi_2|\}$  by  $K_x^-$ .

**PROPOSITION 4.1. 1.** For every  $\tau \in [0, 1]$  and  $x \in \mathbf{T}^2$  the families of cones  $K_x^+$  and  $K_x^-$  are semi-invariant; i.e.,

$$Dg_{z}(K_{x}^{+}) \subset K_{g_{z}x}^{+}$$
,  $D(g_{z})^{-1}(K_{x}^{-}) \subset K_{g_{z}^{-1}x}^{-}$ .

2. For every  $x \in \mathbf{T}^2$  and  $\tau \in [0, 1]$  except when  $\tau = 1$  and  $x = x_i$ , i = 1, 2, 3, 4, the intersections

$$E_x^{u,x} = igcap_{\pi^2} Dg^n_{ au} K^+_{g^{-n}_{ au^*}} and E_x^{s, au} = igcap_{\pi^2} Dg^n_{ au} K^-_{g^{-n}_{ au^*}}$$

are one-dimensional subspaces of  $T_x \mathbf{T}^2$ .

*Proof.* We restrict ourselves to the case of the cones  $K_x^+$  because the other case is completely similar. Obviously, the family  $K^+ = \{K_x^+, x \in \mathbf{T}^2\}$  is semi-invariant outside the neighborhoods  $D_r^i$ . To check this property inside  $D_{r_0}^i$  we use the method of [7]. The linear part of the vector field  $v_{\psi_{\tau}}$  (cf. (2.4)) has the form

(4.1) 
$$\dot{\xi}_1 = \ln lpha ig( (\psi_ au + 2 s_1^2 \psi_1') \xi_1 + 2 s_1 s_2 \psi_1' \xi_2 ig) , \ \dot{\xi}_2 = -\ln lpha ig( 2 \psi_1' s_1 s_2 \xi_1 + (\psi_ au + 2 s_2^2 \psi_1') \xi_2 ig) .$$

The equation for the tangent  $\eta = \xi_2/\xi_1$  is

(4.2) 
$$\frac{d\eta}{dt} = -2\ln\alpha((\psi_{z} + (s_{1}^{2} + s_{2}^{2})\psi_{z}')\eta + s_{1}s_{2}\psi_{z}'(\eta^{2} + 1)).$$

Substitution of  $\eta = 1$  and  $\eta = -1$  in (4.2) gives

(4.3) 
$$\frac{d\eta}{dt} = -2\ln\alpha(\psi_{\tau} + (s_1 + s_2)^2\psi_{\tau}') \leq 0$$

and

(4.4) 
$$\frac{d\eta}{dt} = 2\ln\alpha(\psi_{\tau} + (s_1 - s_2)^2\psi_{\tau}') \ge 0$$

Moreover, these inequalities are strict everywhere for  $\tau < 1$  and everywhere, except the origin, for  $\tau = 1$ . Inequalities (4.3) and (4.4) imply the first assertion of the proposition.

To prove the second assertion we shall estimate from below the decrease of the angle between two arbitrary lines inside the cone  $K_x^+$  for  $x \in D_{r_0}^i$ . We shall obtain two estimations of that kind. The first of these deals with the change of the angle during a unit time. This estimation implies assertion 2 in the case of  $\pi < 1$ . To finish the proof we shall consider the change of the angle along the whole segment of trajectory of the vector field  $v_{\psi_{\tau}}$  inside  $D_r^i$  for a sufficiently small r. Let  $\eta_{s_1^0, s_2^0}(t, \alpha_0)$  be the solution of equation (4.2) with the initial conditions t = 0 and  $\alpha = \alpha_0$  along the trajectory of  $v_{\psi_{\tau}}$  with the initial conditions  $(s_1^o, s_2^o) \in D_{r_1}^i$ . We are going to estimate from above the ratio

$$\frac{|\eta_{s_1^0,s_2^0}(1,\eta_1)-\eta_{s_1^0,s_2^0}(1,\eta_2)|}{|\eta_1-\eta_2|}$$

for all initial conditions  $\eta_1$ ,  $\eta_2$  such that  $|\eta_1| \leq 1$ ,  $|\eta_2| \leq 1$ . To do this, we introduce the function

(4.5) 
$$\hat{\eta}_{s_1^0,s_2^c}(t) = \exp -2\ln \alpha \int_0^t (\psi_{\tau} + (s_1^2 + s_2^2)\psi_{\tau}') du$$

For the sake of brevity we omit here and below the dependence of  $s_1$ ,  $s_2$ ,  $\psi_z$  etc. on t,  $s_1^{\gamma}$ ,  $s_2^{\gamma}$ . Moreover, we shall omit the indices in the notations for the functions  $\eta$  and  $\hat{\eta}$ . The following equality is verified by direct comparison:

(4.6) 
$$\eta(t,\eta_0) = \eta_0 \widehat{\eta}(t) - 2\ln\alpha \widehat{\eta}(t) \int_0^t \frac{s_1 s_2 \psi'(\eta^2(u,\eta_0)+1)}{\widehat{\eta}(u)} du .$$

It follows from (4.3) and (4.4) that the inequality  $|\eta_0| \leq 1$  implies that  $|\eta(t, \eta_0)| \leq 1$  for every t > 0 so that

$$\frac{\left|\eta(t,\,\eta_{\scriptscriptstyle 1})-\eta(t,\,\eta_{\scriptscriptstyle 2})\right|}{\hat{\eta}(t)} \leq |\eta_{\scriptscriptstyle 1}-\eta_{\scriptscriptstyle 2}|\,+\,4\ln\alpha\int_{\scriptscriptstyle 0}^t \frac{|s_1s_2|\,\psi_{\scriptscriptstyle \tau}'|\,\eta(u,\,\eta_{\scriptscriptstyle 1})-\eta(u,\,\eta_{\scriptscriptstyle 2})|}{\hat{\eta}(u)}du\;.$$

The Gronwall inequality (cf. [8]) gives

$$rac{|\eta(t,\,\eta_{\scriptscriptstyle 1})-\eta(t,\,\eta_{\scriptscriptstyle 2})|}{\widehat{\eta}(t)} \leq |\eta_{\scriptscriptstyle 1}-\eta_{\scriptscriptstyle 2}|\exp4\lnlpha {\int_{\scriptscriptstyle 0}^t} |s_{\scriptscriptstyle 1}\cdot s_{\scriptscriptstyle 2}|\psi_{\scriptscriptstyle 0}'du|$$

Substituting t = 1 and using the expression (4.5) for  $\hat{\eta}(t)$  we obtain

(4.7) 
$$|\eta(1, \eta_1) - \eta(1, \eta_2)| \leq |\eta_1 - \eta_2| \exp -2\ln \alpha \int_0^1 (\psi_z + \psi'_z (|s_1| - |s_2|)^2) du$$
.

If  $\tau < 1$ , then the integrand  $\psi_{\tau} + \psi'_{\tau}(|s_1| - |s_2|)^2$  is strictly positive in the set  $D'_{r_1} \cup g_{\tau} D^i_{r_1}$ . Let  $L_{\tau} > 0$  be the minimum value of this function. Inequality (4.7) implies that

$$ig|\eta(1,\,\eta_{\scriptscriptstyle 1})-\eta(1,\,\eta_{\scriptscriptstyle 2})ig| \leq rac{1}{lpha^{{\scriptscriptstyle 2}L_{ au}}}ig|\eta_{\scriptscriptstyle 1}-\eta_{\scriptscriptstyle 2}ig|$$
 ,

so that every cone inside  $K_x^+$  contracts uniformly. This concludes the proof of the proposition for  $\tau < 1$ .

It remains to consider the most important case, namely, au=1.

Let us fix positive numbers  $\varepsilon_0$ ,  $\beta$ ,  $\delta$  and denote the region  $\{(s_1, s_2): |s_1s_2| \leq \varepsilon_0, |s_1| \leq \beta, |s_2| \leq \beta\}$  by  $Q_{\varepsilon_0}^{\beta}$ . Now we are going to estimate the solution of equation (4.2) along the segment of the hyperbola  $\{s_1s_2 = \varepsilon, 0 \leq s_1 \leq \beta, 0 \leq s_2 \leq \beta\}$  for each  $\varepsilon < \varepsilon_0$ . Let us consider  $s_1$  as a parameter on such a hyperbola. Equations (2.4) and (4.2) imply that

(4.8) 
$$\frac{d\eta}{ds_1} = -\left(\frac{2}{s_1} + \frac{2(s_1^2 + s_2^2)}{s_1}\frac{\psi'_z}{\psi}\right)\eta + 2s_2\frac{\psi'_z}{\psi}(\eta^2 + 1) .$$

Let  $\eta(s_1, \varepsilon/\beta, \eta_0)$  be the solution of this equation with the initial conditions  $s_1 = \varepsilon/\beta$ ,  $\eta = \eta_0$ ,  $|\eta_0| < 1$ . The Gronwall inequality and the inequality  $|\eta(s_1, \varepsilon/\beta, \eta_0)| < 1$  imply that for every  $\eta_1, \eta_2, |\eta_1| \leq 1, |\eta_2| \leq 1$ ,

$$igg| \left| \eta \Big(eta, rac{arepsilon}{eta}, \eta_1 \Big) - \eta \Big(eta, rac{arepsilon}{eta}, \eta_2 \Big) 
ight| \, \leq |\eta_1 - \eta_2| \exp -2 \! \int_{\epsilon/eta}^{eta} \! \Big( rac{1}{s_1} + rac{\psi_{ au}'(s_1^2 - arepsilon)^2}{s_1^3} \Big) ds_1 
onumber \ = |\eta_1 - \eta_2| \cdot rac{arepsilon^2}{eta^4} \exp -2 \! \int_{\epsilon/eta}^{eta} \! rac{\psi_{ au}'}{\psi_{ au}} rac{(s_1^2 - arepsilon)^2}{s_1^3} ds_1 \leq |\eta_1 - \eta_2| rac{arepsilon^2}{eta^4} \, .$$

The same inequalities are true in other quadrants, i.e., for  $\eta(-\beta, -\varepsilon/\beta, \eta_0)$  etc.

Let  $\{x, g_{\tau}x, \dots, g_{\tau}^{n-1}x\}$  be a segment of a trajectory of  $g_{\tau}$  lying inside  $Q_{\varepsilon_0}^{\beta}$ . This implies that all these points belong to  $Q_{\varepsilon_0}^{\beta}$ , but  $g_{\tau}^{-1}x$  and  $g_{\tau}^{n}x$  do not belong to  $Q_{\varepsilon_0}^{\beta}$ . Suppose that  $x = (s_1^0, s_2^0)$ . Then for every  $\eta_1, \eta_2, |\eta_1| < 1, |\eta_2| < 1$  the following inequality holds:

$$ig|\eta_{s_1^0,s_2^0}\!(n,\,\eta_1)-\eta_{s_1^0,s_2^0}\!(n,\,\eta_2)ig|\leq |\eta_1-\eta_2|rac{M(s_1^0s_2^0)^2}{eta^4}\leq |\eta_1-\eta_2|rac{Marepsilon_0^2}{eta^4}$$

Here M is a uniform constant for all sufficiently small  $\varepsilon_0$  and  $\beta$ . It means that every angle inside the cone  $K_x^+$  is contracted under the action of  $Dg_1^n$  by at least  $M\varepsilon_0^2/\beta^4$  times. Let us choose  $\varepsilon_0 > 0$  so small that

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$$\frac{M\varepsilon_0^2}{\beta^4} < \frac{1}{2} \ .$$

Estimation (4.7) is uniform for every  $\tau$  including  $\tau = 1$ , outside every neighborhood of the origin. Consequently, we can find a constant  $\mu < 1$ such that every angle inside  $K_x^+$  is contracted at least  $\mu$  times if x does not belong to the region  $Q_{\beta^0}^{\epsilon_0}$  near one of the points  $x_i$ . Thus we have proved that for every  $x \in \mathbf{T}^2$  such that  $g_i^n x$  does not tend to one of the points  $x_i$  as  $n \to -\infty$ , the intersection

$$(4.10) \qquad \qquad \bigcap_{n \ge 0} Dg_1^n(K_{g^-n_x})$$

must be a single line.

Finally, let us consider a point x such that  $\lim_{n\to\infty} g_1^n x = x_i$ . The local unstable manifold (non hyperbolic) of the point  $x_i$  is a segment of the  $s_2 = 0$ . Consequently, equation (4.8) on this line is reduced to the linear equation

(4.11) 
$$\frac{d\eta}{ds_1} = -\left(\frac{2}{s_1} + \frac{2s_1\psi_{\tau}'}{\psi}\right)\eta \; .$$

Let us fix the number  $\lambda > 0$  and consider the solution of (4.11) with the initial conditions  $s_1 = \varepsilon$ ,  $\eta = 1$  along the segment  $[\varepsilon, \lambda]$ :

$$\eta(\lambda,\,arepsilon,\,ar{1})=\exp\,-\!\int_{arepsilon}^{\imath}\!\Bigl(rac{2}{s_1}+rac{2s_1\psi_ au^{\prime}}{\psi}\Bigr)ds_1\leqrac{arepsilon^2}{\lambda^2}\;.$$

Consequently,  $\eta(\lambda, \varepsilon, 1) \to 0$  as  $\varepsilon \to 0$  and by the linearity of the equation,  $\eta(\lambda, \varepsilon, -1) \to 0$  too. Thus, for the point  $x = (\lambda_0, 0)$  intersection (4.10) is also a single line. Proposition 4.1 is proved.

COROLLARY 4.1. The subspaces  $E_x^{u,\tau}$  and  $E_x^{s,\tau}$  depend on x continuously for every  $\tau < 1$ . The subspaces  $E_x^{u,1}$  and  $E_x^{s,1}$  depend on x continuously except, maybe, for the points  $x = x_i$ , i = 1, 2, 3, 4.

*Proof.* The family of cones  $K_{x,n}^{+,\tau} = Dg_{\tau}^{n}K_{g_{\tau}^{-n}x}^{+-n}$  forms a closed subset of the tangent bundle  $T\mathbf{T}^{2}$ . Consequently, the same is true for the family of intersections  $E_{x}^{u,\tau} \cap_{n\geq 0} K_{x,n}^{+\tau}$ .

COROLLARY 4.2. For every  $\tau < 1$ ,  $g_{\tau}$  is an Anosov diffeomorphism.

*Proof.* This follows immediately from Proposition 4.1 and the fact that  $g_{\epsilon}$  preserves a smooth positive measure.

COROLLARY 4.3. The homeomorphism  $g_1$  is topologically conjugate to  $g_0$ .

*Proof.* It is proved in [7,  $n^{\circ}3$ ] that  $g_1$  is an expansive homeomorphism. Corollary 4.2 implies that  $g_1$  belongs to the  $C^{\circ}$  closure of the set of all Anosov

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diffeomorphisms. Thus the theorem from [7] implies that  $g_1$  is topologically conjugate to  $g_0$ .

Denote the homeomorphism (homotopic to the identity) which conjugates  $g_0$  and  $g_1$  by h. Each point  $x \in \mathbf{T}^2$  has local stable and unstable manifolds  $W_x^s$  and  $W_x^u$  which are images of the local stable and unstable manifolds of  $g_0$ .

LEMMA 4.1. The sets  $W_x^s$  and  $W_x^u$  are smooth curves and  $T_x W_x^s = E_x^{s,1}$ ,  $T_x W_x^u = E_x^{u,1}$ .

*Proof.* We restrict ourselves to the case of unstable manifolds. First, let us consider the unstable manifold  $W_{x_i}^u$  of the point  $x_i$ . Locally this manifold coincides with a segment of the line  $s_2 = 0$ . The proof of Proposition 4.1 shows that at each point z of this segment the space  $E_z^{u,1}$  is the line  $\xi_2 = 0$ , i.e.,  $TW_z^u = E_z^{u,1}$ . This fact remains true for every point z belonging to the global unstable manifold of the point  $x_i$ . Thus, for such a point

$$(4.12) TW_z^u = E_z^{u,1} \subset \operatorname{Int} K_z^+ .$$

The global unstable manifold  $\widetilde{W}_{x_i}^u$  of the point  $x_i$  is everywhere dense. Now let  $z \in \mathbf{T}^2$  and  $z_n \in \widetilde{W}_{x_i}^u$ ,  $z_n \to z$ . Let us translate our local coordinate system from a neighborhood of the point  $x_1$  into some neighborhood U of the point z and denote the local coordinates of the points  $z_n$  by  $(s_{1,n}, s_{2,n})$ . Fix a compact segment  $I_n$  on the manifold  $W_{z_n}^u$  in such a way that  $I_n \subset U$ and  $I_n$  tend in a topological sense to a compact segment I of the manifold  $W_z^u$ . By (4.12),

$$I_n \setminus \{ \boldsymbol{z}_n \} \subset \left\{ (s_1, s_2) \in U, \ \left| rac{s_1 - s_{1,n}}{s_2 - s_{2,n}} 
ight| < 1 
ight\} \, .$$

Consequently,

$$(4.13) I \setminus \{z\} \subset \left\{ (s_1, s_2) \in U, \frac{s_1}{s_2} < 1 \right\} .$$

Let us denote the set of all limit tangent vectors for 1 at the point z by  $\prod_z \subset T_z T^2$ .  $\prod_z$  is a nonempty closed set. Inclusion (4.13) implies that  $\prod_z \subset K_z^+$ . Obviously,  $Dg_1 \prod_z = \prod_{g_{1^z}}$ . Thus, Proposition 4.1 implies that  $\prod_z = E_z^{u,1}$ . The lemma is proved.

Proposition 2.1 follows immediately from Corollary 4.3 and Lemma 4.1.

To finish the proof of Proposition 2.2 we need a simple fact about Lyapunov characteristic exponents. We shall consider a purely measuretheory situation, namely an automorphism T of a Lebesque space  $(X, \mu)$  and some linear extension  $\hat{T}: X \times \mathbb{R}^m \to X \times \mathbb{R}^m$  of T. Such an extension is given by a matrix function on  $X: \hat{T}(x, t) = (Tx, A_x t)$ , where  $A_x \in \mathrm{GL}(m, \mathbb{R})$ . The multiplicative ergodic theorem of Oseledec [4] deals with such extensions satisfying the additional assumption  $\max\left(\ln^+||A_x||,\ln^+||A_x^{-1}||\right)\in L^{\scriptscriptstyle 1}(X,\mu).$ 

Let  $E \subset X$  be a subset of positive measure. Then we can consider the induced map  $T_B$  and the corresponding induced extension  $\hat{T}_B$ . By the Birkhoff ergodic theorem the following limit exists almost everywhere:

$$\chi(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \chi_B(T_x^i)$$
.

**PROPOSITION 4.2.** The Lyapunov exponents  $\chi_i^B(x)$ ,  $i = 1, \dots, m$ , of the extension  $\hat{T}_B$  are equal to  $\chi_i(x)/\chi(x)$ , where  $\chi_1(x) \leq \chi_2(x) \leq \dots \chi_m(x)$  are the Lyapunov exponents of the extension  $\hat{T}$ . We count each exponent with its multiplicity.

This is an easy consequence of the Birkhoff ergodic theorem and Theorem 4 from [4].

COROLLARY 4.4. If almost every trajectory of T intersects the set B and  $\chi_i^{\scriptscriptstyle B}(x) \neq 0$  almost everywhere, then  $\chi_i(x) \neq 0$  almost everywhere.

Proof of Proposition 2.2. Let us denote  $\mathbf{T}^{\mathbf{z}} \setminus \bigcup_{i=1}^{4} Q_{i_0}^{\beta,i}$  by B, where  $Q_{i_0}^{\beta,i}$  is the set  $Q_{i_0}^{\beta}$  constructed near the point  $x_i$  and  $\beta$  and  $\varepsilon_0$  are chosen so that (4.9) holds. We shall consider the induced map  $(g_1)_B$  and linear extension  $D_{g_1}^{B}$ :  $TB \to TB$ . Let us fix orientation for the one-dimensional subbundles  $E^{u,1}$  and  $E^{s,1}$  of TB and denote positive rays of  $E_x^{u,1}$  and  $E_x^{s,1}$  by  $\hat{E}_x^{u}$  and  $\hat{E}_x^{s}$ . We introduce a new continuous Riemannian metric on TB by choosing the orthonormal basis  $(e_1(x), e_2(x))$  in each space  $T_x\mathbf{T}^2$  for  $x \in B$ . That is, let  $e_1(x) \in \hat{E}_x^{u}, e_2(x) \in \hat{E}_x^{s}, e_1(x) + e_2(x) \in \partial K_x^{+}$ , and  $\omega_{\nu}(e_1(x), e_2(x)) = 1$ , where  $\omega_{\nu}$  is a 2-form which generates the smooth positive invariant measure  $\nu$  (see § 2,  $n^{\circ}2$ ). It is easy to see that the Euclidean norm of  $e_1(x)$  is bounded from below. Proposition 4.1 implies that

$$D^{\scriptscriptstyle B}_{g_1} e_{\scriptscriptstyle 1}(x) = \gamma_{\scriptscriptstyle 1}(x) e_{\scriptscriptstyle 1}ig((g_{\scriptscriptstyle 1})_{\scriptscriptstyle B}(x)ig) \;, \quad D^{\scriptscriptstyle B}_{g_1} e_{\scriptscriptstyle 2}(x) = \gamma_{\scriptscriptstyle 2}(x) e_{\scriptscriptstyle 2}ig((g_{\scriptscriptstyle 1})_{\scriptscriptstyle B}(x)ig) \;,$$

where  $\gamma_2(x)/\gamma_1(x) < K < 1$ . On the other hand,  $\gamma_1(x)\gamma_2(x) = 1$  because the form  $\omega_{\nu}$  is invariant. Consequently

$$(4.14)$$
  $\gamma_{_2}(x) < K^{_{1/2}} < 1$  ,  $\gamma_{_1}(x) > K^{_{-1/2}} > 1$  .

Thus, the Lyapunov exponent for the vector  $e_1(x)$  with respect to  $D_{g_1}^B$  must be positive.

Since every trajectory of  $g_1$ , except for the fixed points  $x_i$ , intersects the set *B*, Corollary 4.4 implies that for  $Dg_1$  the larger Lyapunov exponent is positive almost everywhere. Proposition 2.2 is proved.

#### 5. Main theorem

Now we are able to prove our final result.

THEOREM B. Let M be an arbitrary compact connected two-dimensional manifold (possibly with boundary),  $\mu$  a smooth positive measure on M. There exists a  $C^{\infty}$  Bernoulli diffeomorphism  $\hat{f}: M \to M$  which preserves the measure  $\mu$ .

Proof. Let  $F: \mathbf{D}^2 \to M$  be a mapping which satisfies the assertions of Proposition 1.2 and  $\rho$  be an admissible sequence of function on  $\mathbf{D}^2$  which satisfies the assertion of Proposition 1.1 for F chosen in this way. By Theorem A we can construct a Bernoulli diffeomorphism  $g \in \text{Diff}_{\rho}^{\infty}(\mathbf{D}^n)$ . Proposition 1.1 guarantees that the mapping  $\hat{f}$  defined by (1.1) is a  $C^{\infty}$  diffeomorphism of M. This mapping preserves the measure  $\mu$  and  $\hat{f}$  considered as an automorphism of the Lebesgue space  $(M, \mu)$  is metrically isomorphic to  $g: (\mathbf{D}^2, \lambda) \to (\mathbf{D}^2, \lambda)$ , i.e.,  $\hat{f}$  is a Bernoulli diffeomorphism. The theorem is proved.

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