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Smooth Non-Bernoulli K-Automorphisms

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§ 1. Introduction

Let M, N be C^∞ compact connected Riemannian manifolds, $g\colon M\to M$ a C^∞ Anosov diffeomorphism which preserves a smooth positive probability measure μ, $\{h_t\}$ a C^∞ flow on N, preserving a smooth positive probability measure λ and ergodic with respect to that measure, φ a real-valued C^∞ function on M.

We shall consider the following skew-product diffeomorphism f acting on the direct product of the manifolds $M\times N$:

$$f(x,y)=(gx, h_{\varphi(x)}y) \tag{1}$$

We are going to give a condition sufficient for such a diffeomorphism f to be a K-automorphism with respect to the measure $\mu\times\lambda$. Our considerations are based on ideas of the theory of partially hyperbolic dynamical systems [1], especially on a modification of the approach which has been used by Brin [2], [3] for the study of group extensions of Anosov diffeomorphisms. Indeed, a diffeomorphism satisfying our condition need not be a partially hyperbolic one. Nevertheless, such a diffeomorphism has two absolutely continuous invariant foliations - contracting and expanding - which form a metrically transitive pair [1].

Let $\pi\colon \hat{M}\to M$ be the universal covering, $\hat{g}\colon \hat{M}\to\hat{M}$ a lifting of g on the universal covering manifold $\hat{M}$ and $\hat{\varphi}=\varphi\circ\pi\colon \hat{M}\to\mathbb{R}$ — the lifting of φ.

Let us denote by $\hat{C}(M)$ the class of all real-valued continuous functions ψ on $\hat{M}$ such that for every two liftings $U_1, U_2\subset\hat{M}$ of any sufficiently small neighborhood $U\subset M$ the difference $(\psi|_{U_1})\circ\pi^{-1}-(\psi|_{U_2})\circ\pi^{-1}$ is a constant on U.

Theorem. 1. *If for every constant c the equation*

$$\psi(\hat{g}x)-\psi(x)=\hat{\varphi}(x)+c \tag{2}$$

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has no continuous solution $\psi \in \hat{C}(M)$, *then* f *is a* K*-automorphism with respect to the measure* $\mu \times \lambda$.

2. *If the flow* $\{h_t\}$ *is weakly mixing and the equation*

$$\psi(gx) - \psi(x) = \varphi(x) - \varphi_0 \tag{3}$$

where $\varphi_0 = \int_M \varphi \, d\mu$ *has no continuous solution* ψ, *then* f *is a* K*-automorphism with respect to* $\mu \times \lambda$.

This theorem will be proved in §2.

Remark 1. Certainly, the lifting of a solution of (3) is a solution of (2). Every function $\psi \in \hat{C}(M)$ defines an element $[\psi]$ of the first cohomology group $H^1(M, \mathbb{R})$. Moreover, $[\psi] = 0$ iff there exists a continuous function ψ_0 on M such that $\psi = \psi_0 \circ \pi$. If $\psi \in \hat{C}(M)$ is a solution of (2) then $g^*[\psi] - [\psi] = 0$ i.e. $[\psi]$ is an invariant vector of the linear operator g^* induced by g in the space $H^1(M, \mathbb{R})$.

All the examples of Anosov diffeomorphisms known up to now are topologically conjugate to hyperbolic infranilmanifold automorphisms [4]. Those automorphisms generate hyperbolic linear operators in one-dimensional cohomologies. Therefore, if $\psi \in \hat{C}(M)$ is a solution of (2) then $[\psi] = 0$ and the corresponding function ψ_0 satisfies the equation

$$\psi_0(gx) - \psi_0(x) = \varphi(x) + c.$$

Integrating over M with respect to μ one can see that $c = -\varphi_0$ and consequently ψ_0 is a solution of (3). Hence, *for every known Anosov diffeomorphism* g *and every* C^∞ *function* φ *equation* (2) *has a solution* $\psi \in \hat{C}(M)$ *iff* (3) *has a continuous solution.*

It seems quite probable that this fact is true for all Anosov diffeomorphisms. It would follow from the positive answer to the following conjecture.

Conjecture. If $g: M \to M$ is an Anosov diffeomorphism then the induced operator $g^*: H^1(M, \mathbb{R}) \to H^1(M, \mathbb{R})$ has no non-zero invariant vectors.

Remark 2. It is proved in [5], that Eq. (3) has continuous solution iff for every periodic trajectory γ of g $\sum_{x \in \gamma} (\varphi(x) - \varphi_0) = 0$. Thus, we can reformulate the condition on φ in assertion 2 of the theorem (and for all known examples of Anosov diffeomorphisms in assertion 1, too) in a following way:

there exists a periodic trajectory γ *of* g *of period* k *such that*

$$\sum_{x \in \gamma} \varphi(x) \neq k\varphi_0 \tag{4}$$

Corollary. *Suppose that* M *is an infranilmanifold,* φ *is a positive* C^∞ *function, the flow* $\{h_t\}$ *is not loosely Bernoulli* (LB) [6] *with respect to due measure* λ, *and condition* (4) *is satisfied. Then* f *defined by* (1) *is a* K*-diffeomorphism which is not* LB *and consequently not Bernoulli.*

Proof. By the theorem and Remarks 1 and 2 f is a K-automorphism with respect to the measure $\mu \times \lambda$.

Let us consider the direct product $\{H_t\} = \{g_t^\varphi\} \times \{h_t\}$ where $\{g_t^\varphi\}$ is the special flow constructed relative to g and φ. The flow $\{g_t^\varphi\}$ acts on the space $M_{\varphi(\cdot)} = \{x, t) \in M \times \mathbb{R}: 0 \leqq t < \varphi(x)\}$. If $\{h_t\}$ is not LB flow then $\{H_t\}$ is also not an LB flow. But the Poincaré map induced by the flow $\{H_t\}$ on the set $(M \times \{0\}) \times N$ coincides with f. So f being a section for non LB flow is not an LB transformation. □

Thus, by Corollary in order to construct a C^∞ diffeomorphism preserving a smooth positive probability measure, which is a K-automorphism with respect to that measure, but not isomorphic to a Bernoulli shift, we need a C^∞ diffeomorphism which is not LB with respect to a smooth positive invariant measure.

By now there are two known methods to produce such a diffeomorphism. The first one which is due to the author is described in [7], § 11 (but not published yet in details). This method allows us to fulfill a version of Feldman's construction [6] of non LB-automorphism in a smooth situation by means of a version of the construction from [8]. It works on every manifold of dimension greater than 1 which possesses a nontrivial action of the circle. I am going to write down that construction in full details because it provides together with the ideas from the present paper and from Brin's paper [9], a way for a construction of a non-Bernoulli K-diffeomorphism on every manifold of dimension greater than 4.

The second method is due to Ratner [10] and gives much more natural and simpler examples. She proves that a Cartesian square of the homocycle flow on a compact surface of constant negative curvature is not LB.

I think that the method of S. Kalikow who solved the so-called $T - T^{-1}$ problem also can be adjusted to the smooth situation. This would provide the third way to construct non LB-diffeomorphisms. Probably, it is possible, using Kalikow's approach to construct non-Bernoulli K-diffeomorphisms directly and avoid the reduction described in the present paper.

§ 2. Proof of the Theorem

Step 1. Let us denote for $x \in M$ and $r > 0$ by $W^s(x, r)$ the stable manifold of x of size r with respect to g. We are going to lift these manifolds to $M \times N$ by an f-invariant way. For this sake, we consider the sequence of functions τ_n, $n = 1, 2 \ldots$ defined on the set $\Gamma_r = \{(x, z) \in M \times M, z \in W^s(x, r)\}$ by means of the formula

$$\tau_n(x, z) = \sum_{k=0}^{n-1} (\varphi(g^k x) - \varphi(g^k z)).$$

Since g is an Anosov diffeomorphism then for some $\beta < 1$ and $c > 0$ the distance $d(g^k z, g^k x)$ between $g^k z$ and $g^k x$ admits the exponential estimation from above:

$$d(g^k z, g^k x) < c \beta^k \tag{5}$$

Moreover, $|\varphi(g^k(z))-\varphi(g^k(x))|\leqq d(g^k z, g^k x)\cdot\|D\varphi\|$, where D is the derivative and $\|\cdot\|$ is a norm induced by the Riemannian metric. Hence, the sequence τ_n converges exponentially and uniformly on Γ_r to a continuous function

$$\tau(x,z)=\sum_{n=0}^{\infty}(\varphi(g^n x)-\varphi(g^n z)), \tag{6}$$

which satisfies the following equation

$$\tau(gx,gz)=\tau(x,z)-\varphi(x)+\varphi(z) \tag{7}$$

We are going to prove that for every $x\in M$ the function $\tau(x,z)$ is continuously differentiable with respect to z. For, let $\xi\in E_z^s$, $\|\xi\|=1$, where $E_z^s\subset T_z M$ is a stable subspace with respect to g. We have

$$(D\tau_n)_{(x,z)}(0,\xi)=-\sum_{k=0}^{n-1}D\varphi_{g^k z}(Dg^k\xi). \tag{8}$$

The norm of the vector $Dg^k\xi$ is estimated exponentially as in (5) so that $\|D\varphi_{g^k z}(Dg^k\xi)\|\leqq\max\|D\varphi\|\,c\cdot\beta^k$. Thus, the series (8) converges exponentially and uniformly on Γ_r to the derivative $(D\tau)_{(x,z)}(0,\xi)$. Moreover, this derivative depends on x continuously.

Let us define for $(x,y)\in M\times N$ the mapping $\Phi_{x,y}\colon W^s(x,r)\to N$ by the formula

$$\Phi_{x,y}(z)=h_{\tau(x,z)}y \tag{9}$$

The differentiability of τ with respect to z and the smoothness of the flow $\{h_t\}$ imply that $\Phi_{x,y}\in C^1(W^s(x,z),N)$ and the map $\Phi_x\colon N\to C^1(W^s(x,z),N)$ defined by $\Phi_x y=\Phi_{x,y}$ is C^1. Moreover, in the natural sense $\Phi_{x,y}$ depends continuously on both its arguments.

Let us denote the graph of the mapping $\Phi_{x,y}$ by $W^s(x,y,r)$. Evidently if $r_1>r_2$ then $W^s(x,y,r_1)\supset W^s(x,y,r_2)$. It follows from the previous arguments that $W^s(x,y,r)$ is a C^1 submanifold of $M\times N$ which depends differentiably on y and continuously on x. These manifolds are glued together to a global manifold $W^s(x,y)$ which form a foliation $\tilde{W}^s$ on $M\times N$. For, if $x_1\in W^s(x,r_1)$, $x_2\in W^s(x_1,r_2)$ then

$$\tau(x,x_2)=\tau(x,x_1)+\tau(x_1,x_2),\quad \text{i.e.}$$
$$W^s(x,y,r_1+r_2)\supset W^s(x_1,\Phi_{x,y}x_1,r_2).$$

Now, let us show that this foliation is f-invariant. For, suppose that $(x_1,y_1)\in W^s(x,y,r)$, i.e. $y_1=h_{\tau(x,x_1)}y$. Using (7), we obtain

$$\begin{aligned}f(x_1,y_1)&=(gx_1,h_{\varphi(x_1)}y_1)=(gx_1,h_{\varphi(x_1)+\tau(x,x_1)}y)\\&=(gx_1,h_{\tau(gx,gx_1)+\varphi(x)}y)=(gx_1,h_{\tau(gx,gx_1)}h_{\varphi(x)}y);\end{aligned}$$

i.e. $f(x_1,y_1)\in W^s(gx,h_{\varphi(x)}y,r_1)$ for some r_1.

The unstable manifolds $W^u(x,y,r)$ and the foliation $\tilde{W}^u$ are constructed by a similar way with the use of the function

$$\sigma(x,z)=\sum_{k=-1}^{-\infty}(\varphi(g^k z)-\varphi(g^k x)) \tag{10}$$

instead of τ. Here $z\in W^u(x,r)$.

Step 2. We shall show that the foliations $\tilde{W}^s$ and $\tilde{W}^u$ are absolutely continuous. Certainly, it is sufficient to consider the first foliation because the both cases are completely similar. For every sufficiently small positive number ε we can choose $\delta>0$ such that for every $x\in M$, $z\in W^s(x,\varepsilon)$, $x_1\in W^u(x,\varepsilon)$ the intersection $W^s(x_1,\delta)\cap W^u(z,\delta)$ contains exactly one point, which we shall denote by $\Pi_{x,z}x_1$. Thus, the map $\Pi_{x,z}$: $W^u(x,\varepsilon)\to W^u(z,\delta)$ is defined. This map is sometimes called the canonical isomorphism. It is absolutely continuous [11].

The natural transversal section for $\tilde{W}^s$ is the product $W^u(x,\varepsilon)\times N$, which we shall denote by $K_{x,\varepsilon}$.

Let us denote for $z\in W^s(x,\varepsilon)$ the mapping from $K_{x,\varepsilon}$ to $K_{z,\varepsilon}$ along the leaves of $\tilde{W}^s$ by $\tilde{\Pi}_{x,z}$. Thus, for $x_1\in W^u(x,\varepsilon)$, $y\in N$ we have

$$\tilde{\Pi}_{x,y}(x_1,y)=W^s(x_1,y,\delta)\cap K_{z,\delta}.$$

Using the definition of $W^s(x,y,\delta)$ we can express $\tilde{\Pi}_{x,z}$ in a following way

$$\tilde{\Pi}_{x,z}(x_1,y)=(\Pi_{x,z}x_1,h_{\tau(x_1,\Pi_{x,z}x_1)}y).$$

Thus, $\tilde{\Pi}_{x,z}$ is a skew product over the absolutely continuous mapping $\Pi_{x,z}$ with smooth maps in leaves, i.e. $\tilde{\Pi}_{x,z}$ is also absolutely continuous.

Step 3. Following the method used by M. Brin [2, 3] in the case of group extensions of Anosov systems we define the transformations of manifolds $\{x\}\times N$, $x\in M$ along the leaves of $\tilde{W}^s$ and $\tilde{W}^u$. Let for $r>0$, $x\in M$, $z\in W^s(x,r)$, $y\in N$

$$\pi_{x,z}(x,y)=(z,h_{\tau(x,z)}y) \tag{11}$$

and similarly for $z\in W^u(x,r)$

$$\pi_{x,z}(x,y)=(z,h_{\sigma(x,z)}y). \tag{12}$$

We shall call an ordered finite system $K=\{x_0,\dots,x_n\}$, $x_0,\dots,x_n\in M$ an admissible system of points if there exists $r>0$ such that $x_{k+1}\in W^s(x_k,r)$ or $x_{k+1}\in W^u(x_k,r)$ for $k=0,1,\dots,n-1$. We let for such a system K:

$$\pi_K=\pi_{x_{n-1}x_n}\circ\pi_{x_{n-2}x_{n-1}}\circ\dots\circ\pi_{x_0x_1}\colon \{x_0\}\times N\to\{x_n\}\times N.$$

In particular, we shall call an admissible system $K=\{x_0,x_1,x_2,x_3,x_0\}$ a local quadrangle if the point x_0 has a local product neighbourhood of size ε and $x_1\in W^s(x_0,\varepsilon)$, $x_3\in W^u(x_0,\varepsilon)$, $x_2=W^u(x_1,\varepsilon)\cap W^s(x_3,\varepsilon)$ or $x_1\in W^u(x_0,\varepsilon)$, $x_3\in W^s(x_0,\varepsilon)$, $x_2=W^s(x_1,\varepsilon)\cap W^u(x_3,\varepsilon)$.

Lemma 1. 1°. *Equation* (2) *has a solution* $\psi\in\hat{C}(M)$ *iff* $\pi_K=\mathrm{id}$ *for any local quadrangle* K.

2°. *Equation* (3) *has a continuous solution* ψ iff $\pi_K=\mathrm{id}$ *for any admissible system* K.

Proof of the Lemma. Let $K=\{x_0, x_1, x_2, x_3, x_0\}$ be a local quadrangle. Suppose for definiteness that $x_1 \in W^s(x_0, \varepsilon)$. From (11) and (12) we have

$$\pi_K(x_0, y)=(x_0, h_{\tau(x_0,x_1)+\sigma(x_1,x_2)+\tau(x_2,x_3)+\tau(x_3,x_0)} y). \tag{13}$$

Now suppose that $\pi_K = \mathrm{id}$ for every local quadrangle K. Let us introduce the following C^0 differential from ω_φ on M:

$$\text{for } \xi=\xi_1+\xi_2 \in T_x M, \quad \xi_1 \in E_x^s, \quad \xi_2 \in E_s^u$$
$$\omega_\varphi(\xi)=D\tau_{(x,x)}(0, \xi_1)+D\sigma_{(x,x)}(0, \xi_2).$$

We shall consider for a local quadrangle K a class of piecewise differentiable closed paths γ: $[0,1] \to K$ such that

$$\gamma\left(\frac{k}{4}\right)=x_k \quad k=0,1,2,3,4, \qquad \gamma\left(\left[0, \frac{1}{4}\right]\right) \subset W^s(x_0, \varepsilon),$$

$$\gamma\left(\left[\frac{1}{4}, \frac{1}{2}\right]\right) \subset W^u(x_1, \varepsilon), \qquad \gamma\left(\left[\frac{1}{2}, \frac{3}{4}\right]\right) \subset W^s(x_2, \varepsilon), \qquad \gamma\left(\left[\frac{3}{4}, 1\right]\right) \subset W^u(x_3, \varepsilon).$$

Since $\pi_K = \mathrm{id}$ then for every such path $\int_\gamma \omega_\varphi = 0$.

It is easy to conclude from this fact that ω_φ is a closed form. Consequently, the lifting $\hat{\omega}_\varphi$ of ω_φ to the universal covering manifold $\hat{M}$ is an exact form i.e. $\hat{\omega}_\varphi = D\psi$, where ψ is a continuous function on $\hat{M}$. We are going to prove that there exists a constant c such that

$$\psi(\hat{g}x)-\psi(x)=\hat{\varphi}(x)+c.$$

Let us choose a point $x \in M$ and such a neighbourhood U of the point x that U and $\hat{g}U$ are projected into M one-to-one and their projections are contained in local product neighbourhoods.

Let z be an arbitrary point in U and $y=W^s(x,\varepsilon) \cap W^u(z,\varepsilon)$. It follows from the definitions of the form ω_φ and function ψ that the restriction of ψ on every stable (unstable) manifold of $\hat{g}$ coincides up to a constant term with the restriction of $\hat{\tau}$(corr. $\hat{\sigma}$) on that manifold. Naturally, by $\hat{\tau}$ and $\hat{\sigma}$ we denote the liftings of functions τ and σ on $\hat{M}$. Thus,

$$\psi(z)=\hat{\tau}(x,y)+\hat{\sigma}(y,z)+c_1,$$
$$\psi(\hat{g}z)=\hat{\tau}(\hat{g}x, \hat{g}y)+\hat{\sigma}(\hat{g}y, \hat{g}z)+c_2.$$

Subtracting the first equality from the second one and using the expressions (6) and (10) for τ and σ we obtain

$$\begin{aligned}\psi(\hat{g}z)-\psi(z)&=\hat{\tau}(\hat{g}x, \hat{g}y)+\hat{\sigma}(\hat{g}y, \hat{g}z)-\hat{\tau}(x,y)-\hat{\tau}(y,z)\\ &\quad +c_2-c_1=\sum_{k=1}^{\infty}(\hat{\varphi}(\hat{g}^k x)-\hat{\varphi}(\hat{g}^k y))-\sum_{k=0}^{-\infty}(\hat{\varphi}(\hat{g}^k x)-\hat{\varphi}(\hat{g}^k y))\\ &\quad +\sum_{k=0}^{\infty}(\hat{\varphi}(\hat{g}^k z)-\hat{\varphi}(g^k y))-\sum_{k=-1}^{-\infty}(\hat{\varphi}(\hat{g}^k z)-\hat{\varphi}(\hat{g}^k x))\\ &\quad +c_2-c_1=\hat{\varphi}(y)-\hat{\varphi}(x)+\hat{\varphi}(z)-\hat{\varphi}(y)+c_2-c_1=\hat{\varphi}(z)+c_3\end{aligned}$$

where $c_3=c_2-c_1-\hat{\varphi}(x)$ is a constant in the neighbourhood U.

Thus, the expression $\psi(\hat{g}z)-\psi(z)-\hat{\varphi}(z)$ is a constant in U and consequently it is constant everywhere on $\hat{M}$, i.e. ψ is a solution of (2). Since $D\psi$ is a form lifted from M, then $\psi\in\hat{C}(M)$. Hence, we have proved the "if" part of assertion 1 of the lemma.

Now let us suppose that $\pi_K=\mathrm{id}$ for every admissible system of points K. Then the form ω_φ is exact, because every closed path on M is homotopic to a path consisting of arcs on stable and unstable manifolds. Thus $\omega_\varphi=D\psi$, where ψ is a continuous function on M and the same computation as above shows that ψ is a solution of (3). Thus the "if" part of assertion 2 is also proved. The remaining part of the assertions of the lemma will not be used in the subsequent arguments. But for the sake of completeness we shall conclude the proof.

Suppose that $\psi\in\hat{C}(M)$ is a solution of (2) and consider points $x\in M$, $z\in W^s(x,\varepsilon)$. Let $x'\in\hat{M}$, $\pi x'=x$ and $U\subset\hat{M}$ be a neighbourhood of the point x' described above. Let, at last, $z'\in U$, $\pi z'=z$. We have

$$\begin{aligned}\tau(x,z)&=\sum_{k=0}^{\infty}(\varphi(g^kx)-\varphi(g^kz))=\sum_{k=0}^{\infty}(\hat{\varphi}(\hat{g}^kx')-\hat{\varphi}(\hat{g}^kz'))\\&=\psi(z')-\psi(x')+\lim_{k\to\infty}(\psi(\hat{g}^kx')-\psi(\hat{g}^kz'))=\psi(z')-\psi(x')\end{aligned}$$

since $\lim_{k\to\infty} d(\hat{g}^k(x'),\hat{g}^k(z'))=0$ and the function ψ is uniformly continuous. Similarly we have for $z\in W^u(x,\varepsilon)$

$$\sigma(x,z)=\psi(z')-\psi(x').$$

Formula (13) shows that $\pi_K=\mathrm{id}$. The "only if" part of assertion 1 is proved.

Similarly, let ψ be a continuous solution of (2.3), $x\in M$, $z\in W^s(x,r)$. Then we obtain as above

$$\begin{aligned}\tau(x,z)&=\sum_{k=0}^{\infty}(\varphi(g^kx)-\varphi(g^kz))\\&=\psi(z)-\psi(x)+\lim_{k\to\infty}(\psi(g^kx)-\psi(g^kz))=\psi(z)-\psi(x)\end{aligned}$$

and similarly $\sigma(x,z)=\psi(z)-\psi(x)$ for $z\in W^u(x,r)$. It is easy to conclude from the definition of π_K that $\pi_K=\mathrm{id}$ for every admissible system of points K. □

Step 4. Let for $x\in M$.

$G_x=\{\pi_K\colon K=\{x,x_1,\dots,x_{n-1},x\}$ is admissible$\}$. The set G_x is a group because for $K'=\{x,x_1',\dots,x_{n-1}',x\}$, $K''=\{x,x_1'',\dots,x_{m-1}'',x\}$ $\pi_{K'}\circ\pi_{K''}=\pi_{K'\vee K''}$ where $K'\vee K''=\{x,x_1',\dots,x_{n-1}',x,x_1'',\dots,x_{m-1}'',x\}$ and $\pi_K^{-1}=\pi_{K^*}$ where for $K=\{x,x_1,\dots,x_{n-1},x\}$ $K^*=\{x,x_{n-1},\dots,x_1,x\}$.

Lemma 2. 1. *If* $\pi_K\neq\mathrm{id}$ *for some local quadrangle* K *then for every point* $x\in M$ *the group* G_x *coincides with the group of all transformations* $\{h_t^x\colon h_t^x(x,y)=(x,h_ty),\ t\in\mathbb{R}\}$.

2. *If* $\pi_K\neq\mathrm{id}$ *for some admissible system* K *then there exists* $t_0\neq 0$ *such that for every* $x\in M$ $G_x\supset\{h^x_{kt_0;\,k\in\mathbb{Z}}\}$.

Proof of the Lemma. Evidently, every element of G_x has the form h_t^x. Let $K=\{x_0,x_1,x_2,x_3,x_0\}$ be a local quadrangle such that $\pi_K\neq\mathrm{id}$, i.e. $\pi_K=h_{t_0}^{x_0}$, $t_0\neq 0$. Without loss of generality we can suppose that $x_1\in W^s(x_0,\varepsilon)$. Let us join the points x_2 and x_0 by a continuous path $x_2(s)$ $0 \leqq s \leqq 1$ inside the local product neighbourhood of x_0 of size ε. Let us denote the point $W^s(x_0, \varepsilon) \cap W^u(x_2(s), \varepsilon)$ by $x_1(s)$, the point $W^s(x_2(s), \varepsilon) \cap W^u(x_0, \varepsilon)$ by $x_3(s)$ and the local quadrangle $\{x_0, x_1(s), x_2(s), x_3(s), x_0\}$ by $K(s)$, $0 \leqq s \leqq 1$.

Since the transformation $\pi_{K(s)}$ depends on s continuously the group G_{x_0} contains all the transformations h_t^x for t between 0 and t_0. Hence $G_{x_0} = \{h_t^{x_0}; t \in \mathbb{R}\}$. Now let us fix an arbitrary point $x \in M$ and choose an admissible system of points $L = \{x, z_1, \ldots, z_{m-1}, x_0\}$. Further, fix $t \in \mathbb{R}$ and choose $K = \{x_0, x_1, \ldots, x_{n-1}, x_0\}$ such that $\pi_K = h_t^{x_0}$. Let $\tilde{K} = \{x, z_1, \ldots, z_{m-1}, x_0, x_1, \ldots, x_{n-1}, x_0, z_{m-1}, \ldots, z_1, x\}$. The direct computation shows that $\pi_{\tilde{K}} = h_t^x$.

This completes the proof of the first assertion of the 1 lemma. The second assertion follows immediately from the same arguments. □

Step 5. We shall denote the non-measurable partitions of $M \times N$ into leaves of the foliations $\tilde{W}^s$ and $\tilde{W}^u$ by the same symbols $\tilde{W}^s$ and $\tilde{W}^u$. The measurable hull of a partition ξ is denoted by $\eta(\xi)$.

The foliation $\tilde{W}^u$ is uniformly expanding with respect to f; the foliation $\tilde{W}^s$ is uniformly expanding with respect to f^{-1}. So the theorem of Sinaj ([12], Theorem 5.1) can be applied. This theorem asserts that $\eta(\tilde{W}^s)$ and $\eta(W^u)$ are refinement of the π-partition $\pi(f)$ or equivalently

$$\pi(f) < \eta(\tilde{W}^s \wedge \tilde{W}^u). \tag{14}$$

Thus, to conclude the proof of our theorem it is sufficient to show that every measurable set $A \subset M \times N$ which consists mod 0 of the whole leaves of $\tilde{W}^s$ and $\tilde{W}^u$ has measure 1 or 0. This fact is an easy consequence of Lemma 2 and the following lemma of Brin ([10], Lemma 2.2).

Lemma 3. *Let $\tilde{M}$ be a smooth fibered space over a manifold $M, g\colon M \to M$ an Anosov diffeomorphism, $f\colon \tilde{M} \to \tilde{M}$ a skew-product diffeomorphism, $\tilde{W}^s$, $\tilde{W}^u$ – two absolutely continuous f – invariant foliations of $\tilde{M}$ which are projected into stable foliation W^s and unstable foliation W^u of g and satisfy the Lipschitz condition along the leaves on the fibration.*

Suppose that the set $A \subset M$ consists mod 0 of the whole leaves of $\tilde{W}^s$ and $\tilde{W}^u$. If $z \in \tilde{M}$ is a point of density for A and W is leaf of $\tilde{W}^s$ containing z then every point $z' \in W$ is a point of density for A.

Conclusion of the Proof

Suppose that the set A is measurable with respect to $\pi(f)$ and $(\mu \times \lambda)(A) > 0$. Then by (14) the set A satisfies the conditions of Lemma 3. Let us denote by A_d the set of all points of density for A. Applying this lemma consequentially we conclude that $\pi_K x \in A_d$ for every point $x \in A_d$ and every admissible system of points K. It is well-known that $A_d = A (\operatorname{mod} 0)$. If Eq. (2) is unsolvable then Lemma 2 implies that the set A_d is invariant with respect to the flow $\mathrm{id} \times \{h_t\}$ and by ergodicity of $\{h_t\}$ we can conclude that A_d consists mod 0 of the whole sets $\{x\} \times N$ i.e. $A_d = A' \times N (\operatorname{mod} 0)$ for some measurable set $A' \subset M$. Lemma 3 implies that A' consists of the whole leaves of W^s. Since this foliation is metrically transitive then $A' = M (\mathrm{mod}\, 0)$ and consequently $A = M \times N (\mathrm{mod}\, 0)$ i.e. $\pi(f)$ is a trivial partition. In other words, f is a K-automorphism. Now suppose that Eq. (3) is unsolvable and the flow $\{h_t\}$ is weakly mixing. Then by Lemma 2 the A_d is invariant with respect to the transformation $\mathrm{id} \times h_{t_0}$ which is ergodic on fibers. The conclusion of the proof in this case is the same as in the previous one. □

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Added in Proof

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