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Differentiability and analyticity of topological entropy for Anosov and geodesic flows

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Summary. In this paper we investigate the regularity of the topological entropy h_{top} for C^k perturbations of Anosov flows. We show that the topological entropy varies (almost) as smoothly as the perturbation. The results in this paper, along with several related results, have been announced in [KKPW].

The authors wish to thank Bernard Shiffman for graciously supplying us with the proof of the key technical theorem used in step 3 of Theorem 1.

Theorem 1. *Let M be a closed n-dimensional manifold and let* $\{\phi_\lambda^t\}$, $-\varepsilon \leqq \lambda \leqq \varepsilon$ *(*ε *sufficiently small) be a* C^ω *(real analytic) perturbation of a* C^ω *Anosov flow* $\phi^t \equiv \phi_0^t$. *Then* $h_{top}(\phi_\lambda^1)$ *is* C^ω.

Theorem 2. *Let M be a closed n-dimensional manifold and let* $\{\phi_\lambda^t\}$, $-\varepsilon \leqq \lambda \leqq \varepsilon$ *(*ε *sufficiently small) be a* C^{k+1} *perturbation of a* C^{k+1} *Anosov flow* $\phi^t \equiv \phi_0^t$, $1 \leqq k \leqq \infty$. *Then* $h_{top}(\phi_\lambda^1)$ *is* C^k.

Since the geodesic flow on a closed Riemannian manifold with negative sectional curvature is Anosov [*A*], the following corollary is an immediate consequence of Theorems 1 and 2:

Corollary. *Let* (M, g) *be a closed n-dimensional Riemannian manifold with negative sectional curvature, and let* $\{g_\lambda\}$, $-\varepsilon \leqq \lambda \leqq \varepsilon$ *(*ε *sufficiently small) be a* C^{k+1} *(or* C^ω*) perturbation of* $g \equiv g_0$, $1 \leqq k \leqq \infty$. *Then* $h_{top}(\phi_\lambda^1)$ *is* C^k *(or* C^ω*), where* ϕ_λ^1 *is the time one map of the geodesic flow on SM.*

Strategy of proof. We give two proofs of Theorem 1. The first proof uses zeta functions and complex analysis and only works for C^ω perturbations. This proof yields valuable insight into how the periods of closed trajectories change when the

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flow is perturbed. We also believe that the techniques in this proof may lead to advances in the study of qualitative properties of holomorphic dynamical systems. The proof of Theorem 2, which also works in the C^ω case and gives a second proof of Theorem 1, involves studying the regularity of the maps obtained from structural stability as a function of the perturbation parameter in the C^α topology, along with studying the regularity of the pressure function.

In Theorem 2, we require a C^2 perturbation of an C^2 Anosov flow to ensure that the topological entropy changes in a C^1 way. Katok et al. [KKW], using different methods, prove a stronger result. They show that a C^1 perturbation of a C^1 Anosov flow results in a C^1 change in entropy. They also find a useful formula for the first derivative of topological entropy.

Background

Anosov flows

Let M be a closed C^∞ Riemannian manifold and let ϕ^t: $M \to M$ be a C^k flow ($k \geqq 1$). We call this flow *Anosov* if there is a continuous splitting $TM = E^0 \oplus E^s \oplus E^u$ into $D\phi$ invariant sub-bundles such that:

(a) E^0 is one dimensional and tangent to the flow.
(b) There exists $C, \lambda > 0$ such that for $t > 0$:

$$\| D\phi^t(v) \| \leqq Ce^{-\lambda t} \| v \|, \; v \in E^s \quad \text{and} \quad \| D\phi^t(v) \| \geqq Ce^{\lambda t} \| v \|, \; v \in E^u$$

The flow is called *transitive* if it contains a dense orbit.

Anosov [A] has shown that Anosov flows form an open subset in the C^1 topology on flows. Anosov also showed that they are *structurally stable*, i.e., if ϕ is Anosov, then there exists a neighborhood $\mathfrak{u}$ of ϕ in the C^1 topology such that every $\psi \in \mathfrak{u}$ is topologically equivalent to ϕ, i.e., there exists a homeomorphism $\theta(\psi)$: $M \to M$ which maps the orbits of ϕ onto orbits of ψ. A subsequent proof by Moser utilizes the Implicit Function Theorem to construct $\theta(\psi)$. This approach has the advantage that it yields information on the regularity of $\psi \mapsto \theta(\psi)$, with the appropriate differential structures.

Topological entropy

Given any ϕ-invariant probability measure μ on M we denote by $h_\mu(\phi)$ the measure theoretic entropy of ϕ^1 with respect to μ. *The topological entropy*, $h_{\text{top}}(\phi)$, can be defined using the variational principle [W] by:

$$h_{\text{top}}(\phi) \equiv \sup \{ h_\mu(\phi) \colon \mu \text{ is a } \phi\text{-invariant probability measure} \} .$$

The topological entropy is always non-negative and finite. For a transitive Anosov flow, the supremum is always realized by a unique measure. This measure was explicitly constructed by Margulis [MA2] and Bowen [B3], and we call it the Bowen-Margulis measure. The Bowen-Margulis measure has the following two characterizations:

(1) It induces measures on the leaves of the unstable (stable) foliation which have a uniform expansion (contraction) property with respect to the flow with coefficient $e^{t h_{top}}$.

(2) The closed orbits are uniformly distributed with respect to the measure.

Let $P(T) \equiv$ number of closed trajectories of ϕ^t with (prime) period $\leqq T$. Bowen [B3] has shown that $h_{top}(\phi) = \lim_{T\to\infty} \frac{1}{T} \log P(T)$.

Geometric example

Let (M, g) be a C^∞ closed Riemannian manifold with negative sectional curvature, and let ϕ^t: $SM \to SM$ be the geodesic flow on the unit tangent bundle. The topological entropy of the geodesic flow has the following two realizations in terms of the geometry of M:

(1) [MA1; B3]. Let $P(T) \equiv$ number of closed geodesics for M of length $\leqq T$. Then $h_{top}(\phi) = \lim_{T\to\infty} \frac{1}{T} \log P(T)$. This is simply a reformulation of property (2) above.

(2) [MA1; MN]. Let $\tilde{M}$ denote the universal cover for M with lifted metric $\tilde{d}$. Then $h_{top}(\phi) = \lim_{T\to\infty} \frac{1}{T} \log \mathrm{Vol}\{x \in \tilde{M} : \tilde{d}(x_0, x) \leqq T\}$ for any fixed point $x_0 \in \tilde{M}$.

Symbolic dynamics

To give a characterization of topological entropy for a flow which does not involve limits, it is useful to introduce a symbolic model for the flow. Bowen [B2] and Ratner [R] have shown that every Anosov flow ϕ^t has a Markov partition $\mathscr{F} = \{T_1, \ldots, T_k\}$ with sections having arbitrarily small diameter.

Let P: $\bigcup_i T_i \to \bigcup_i T_i$ denote the Poincaré map on sections, and let A be the incidence matrix associated with P, i.e., $A(i, j) = 1$ if $P(\mathrm{int}\, T_i) \cap (\mathrm{int}\, T_j) \neq \emptyset$ and 0 otherwise. We can introduce a sequence space: $\sum_A \equiv \{x \in \prod_{i=-\infty}^{\infty} \{1, 2, \ldots, k\}$: $A(x_i, x_{i+1}) = 1\}$ with the metric $d(x, y) \equiv \sum_{i=-\infty}^{\infty} \frac{\delta_{x_i, y_i}}{2^{|i|}}$. With this metric, the shift map σ: $\sum_A \to \sum_A$ defined by $\sigma(\ldots, x_{-1}, x_0, x_1, \ldots) \equiv (\ldots, x_0, x_1, x_2, \ldots)$ is continuous. The map σ: $\sum_A \to \sum_A$ models the Poincaré map P: $\bigcup_i T_i \to \bigcup_i T_i$ in the sense that if we define π: $\sum_A \to \bigcup_i T_i$ by $\pi(x) \equiv \bigcap_{i=-\infty}^{\infty} \overline{P^{-n}(\mathrm{int}\, T_{x_i})}$, then π is Hölder continuous, surjective, bounded-to-one, and $P \circ \pi = \pi \circ \sigma$.

Let r: $\sum_A^+ \to \mathbb{R}^+$, defined by $r(x) \equiv \inf\{t > 0;\ \phi^t \pi(x) \in T_{x_1}\}$, be the symbolic version of the first return time. This map is Hölder continuous and will enable us to characterize the topological entropy of ϕ^t in terms of $\sum_A$ and r.

Survey of known results about regularity of entropy

The study of the regularity of the topological entropy for arbitrary smooth flows (or maps) is very subtle. In general, topological entropy need not be continuous.

This is illustrated by the following simple example:

Let $\bar{\Delta}$ denote the closed unit disk in $\mathbb{C}$ and let f_λ: $\bar{\Delta} \to \bar{\Delta}$ be defined by $f_\lambda(z) \equiv (1-\lambda)z^2$, for $0 \leqq \lambda \leqq 1$. Recalling that the entropy of a map is equal to the entropy of the map restricted to its non-wandering set, it is easy to see that $h_{\text{top}}(f_0) = \log 2$ and $h_{\text{top}}(f_\lambda) = 0$ for $\lambda > 0$.

There are several results concerning the regularity of entropy for diffeomorphisms of compact manifolds. Misiurewicz [MI1] constructed examples to show that h_{top}: $\text{Diff}_\infty(M^n) \to \mathbb{R}$ is not continuous for $n \geqq 4$. His constructions are similar in spirit to the example above. It seems unknown, although unlikely, whether entropy is continuous for $n = 3$. Yomdin and Newhouse [Y1; Y2; N] have proved that h_{top}: $\text{Diff}_\infty(M^n) \to \mathbb{R}$ is upper-semicontinuous for $n \geqq 2$. Katok [K1; K2; KM] has shown that for surfaces, h_{top}: $\text{Diff}_2(M^2) \to \mathbb{R}$ is lower-semicontinuous. By combining these two results, one sees that h_{top}: $\text{Diff}_\infty(M^2) \to \mathbb{R}$ is continuous. This result also holds for C^∞ flows on three dimensional manifolds.

Misiurewicz [MI2] and Margulis (see [Y1] for details) also proved that h_{top}: $\text{Diff}_r(M^n) \to \mathbb{R}$ is not upper-semicontinuous in the C^r topology (r finite) for $n \geqq 2$. It is unknown whether h_{top}: $\text{Diff}_r(M^n) \to \mathbb{R}$ is upper-semicontinuous on a residual set for $n \geqq 3$, or whether h_{top}: $\text{Diff}_\infty(M^n) \to \mathbb{R}$ is continuous on a residual set for $n \geqq 3$.

The structural stability of Anosov diffeomorphisms [A] implies that h_{top} is locally constant.

The structural stability of Anosov flows [A] implies that h_{top} is continuous. However, Misiurewicz [MI2] has shown that for general C^k flows on M^n, $k < \infty$ and $n \geqq 3$, h_{top} need not be continuous.

Theorem 1. *Let M be a closed n-dimensional manifold and let $\{\phi_\lambda^t\}$, $-\varepsilon \leqq \lambda \leqq \varepsilon$ (ε sufficiently small) be a C^ω perturbation of a C^ω Anosov flow $\phi^t \equiv \phi_0^t$. Then $h_{\text{top}}(\phi_\lambda^1)$ is C^ω.*

Proof. If ϕ^t is an Anosov flow on M, we define $d(s) \equiv \prod_\tau (1 - \exp(-sl(\tau)))$, where $l(\tau)$ is the least period of the closed orbit τ, s is a complex variable, and the Euler product is over all closed orbits. $d(s)$ is the reciprocal of the zeta function associated to ϕ^t. It is easy to prove that $d(s)$ defines a non-vanishing holomorphic function for $re(s) > h_{\text{top}}(\phi)$[PP1]. Pollicott [PO1] has shown that $d(s)$ has a holomorphic continuation into a slightly larger half plane with a zero at $s = h_{\text{top}}(\phi)$. To study the regularity of topological entropy under perturbations, we study how the zeros of $d(s)$ vary.

The proof has four main steps:

Step 1. Show that $d(\lambda, s)$ is real analytic in λ for $re(s)$ sufficiently large.

Step 2. For each $\lambda \in (-\varepsilon, \varepsilon)$, apply Pollicott's result to holomorphically continue $d(\lambda, s)$ to a larger half plane. We denote the extended function $\bar{d}(\lambda, s)$.

Step 3. Show that $\bar{d}(\lambda, s)$ is real analytic in λ.

Step 4. Conclude that the zeros $h_{\text{top}}(\lambda)$ of $\bar{d}(\lambda, s)$ vary real analytically in λ.

Step 1

Since Anosov flows are structurally stable, every closed trajectory $\tau = \tau_0$ of $\phi^t = \phi_0^t$ can be naturally identified with some closed trajectory τ_λ of ϕ_λ^t. We let $l_\lambda(\tau) \equiv l_\lambda(\tau_\lambda)$ denote the least period function corresponding to τ, i.e. the function which assigns to each λ, the least period of the closed trajectory of ϕ_λ^t corresponding to τ. The proof of Step 1 begins by holomorphically extending each (real analytic) least period function, $l_\lambda(\tau)$, into an open neighborhood $V \subset \mathbb{C}$, *where V is independent of* τ. We then show that the partial products converge uniformly and conclude that $d(\lambda, s)$ is holomorphic in λ for $\lambda \in V$. The function $d(\lambda, s)$ restricted to $\lambda \in (-\varepsilon, \varepsilon)$ is real analytic. Admittedly, this strategy seems unnatural; we know of no direct proof of this result, even in the geometric case.

Proposition 1.1. *For each τ, the map $\lambda \to l_\lambda(\tau)$ is real analytic and has a holomorphic extension into an open neighborhood $V \subset \mathbb{C}$, where V is independent of τ.*

Proof. The key idea is to employ Markov sections for the flow so that infinitely many closed orbits can be dealt with simultaneously using only a finite number of sections. Choose a family $\mathscr{F} = \{T_1, \ldots, T_k\}$ of Markov sections for the flow ϕ. We do not need to choose Markov sections for the perturbed flows! Since the ambient manifold M is C^ω, we may assume that $T_i \subseteq \operatorname{int} D_i$, where D_i is a C^ω disc transverse to the flow. Let $P\colon \bigcup_i T_i \to \bigcup_i T_i$ be the Poincaré map. A closed orbit τ corresponds to a periodic point $P^n x = x \in T_i$.

Since we are considering C^ω perturbations of a C^ω Anosov flow, it is easy to see that the Poincaré maps, P_λ, and the return times between sections, r_λ, are real analytic functions of λ and x. The closed orbit corresponding to a period point $P^n(x_\lambda) = x_\lambda \in T_i$ has least period $\sum_{j=0}^{n-1} r_\lambda(P_\lambda^j(x_\lambda))$. This expression clearly depends real analytically on λ; hence the least period functions, $l_\lambda(\tau)$, depend real analytically on λ.

It is well known that a real analytic function defined on $(-\varepsilon, \varepsilon)$ has a holomorphic extension into some open neighborhood $(-\varepsilon, \varepsilon) \subset V \subset \mathbb{C}$. Thus, for each τ, $l_\lambda(\tau)$ has a holomorphic extension into an open neighborhood $(-\varepsilon, \varepsilon) \subset V_\tau \subset \mathbb{C}$. However, *a priori*, we cannot choose each V_τ to have uniform size, i.e., we cannot preclude $\bigcap_\tau V_\tau = (-\varepsilon, \varepsilon)$. We will show that each V_τ can indeed be chosen to have uniform size.

For the unperturbed flow, there is a natural local product structure $[\,,]$ on each T_i which extends onto D_i. In particular, each T_i is usually presented in the form $T_i = [U_i, S_i]$, $i = 1, \ldots, k$, where $U_i, S_i \subseteq T_i$ lie in the expanding and contracting submanifolds, respectively. We want to extend this construction in two ways:

(1) We enlarge U_i to $\tilde{U}_i \supseteq \operatorname{int} \tilde{U}_i \supseteq U_i$, S_i to $\tilde{S}_i \supseteq \operatorname{int} \tilde{S}_i \supseteq S_i$, and assume that $\tilde{T}_i = [\tilde{U}_i, \tilde{S}_i] \subseteq \operatorname{int} D_i$.

(2) Using the C^ω charts for M, we can assume that $\tilde{S}_i \subseteq \mathbb{R}^k$, $\tilde{U}_i \subseteq \mathbb{R}^l$ and $\tilde{T}_i \subseteq \mathbb{R}^n$. For small $\delta > 0$ we write:

$$\hat{U}_i = \tilde{U}_i \times i(-\delta, \delta)^k \subseteq \mathbb{R}^k + i\mathbb{R}^k = \mathbb{C}^k$$
$$\hat{S}_i = \tilde{S}_i \times i(-\delta, \delta)^l \subseteq \mathbb{R}^l + i\mathbb{R}^l = \mathbb{C}^l$$
$$\hat{T}_i = \tilde{T}_i \times i(-\delta, \delta)^n \subseteq \mathbb{R}^n + i\mathbb{R}^n = \mathbb{C}^n,$$

where k and l are the dimensions of the expanding and contracting submanifolds, and $n = l + k =$ dimension M. We also identify:

$$U_i \times S_i \leftrightarrow T_i \quad \text{and} \quad \hat{U}_i \times \hat{S}_i \leftrightarrow \hat{T}_i \,.$$

Let A be the incidence matrix associated to $\mathscr{T}$. For $A(i, j) = 1$, we choose $(-\varepsilon, \varepsilon) \subset V_{ij} \subset \mathbb{C}$ and $\delta = \delta_{ij} > 0$ such that $(\hat{P}_\lambda)^{-1}(\hat{U}_j) \subset \hat{T}_i$, where $\hat{P}_\lambda$ is the holomorphic extension of the real analytic map P_λ, which is well defined provided $\delta_{ij} > 0$ is sufficiently small. Let $\delta = \bigcap_{A(i,j)=1} \delta_{ij}$.

For $\lambda \in V_{ij}$, we define π^λ_{ij}: $\hat{U}_j \to \hat{U}_i$ by $\pi^\lambda_{ij} \equiv (p_i, \mathrm{id}) \circ (\hat{P}_\lambda)^{-1}$, where p_i: $\tilde{T}_i \to \tilde{U}_i$ is the canonical projection. In general this map will be only Hölder continuous, since p_i involves projecting along a Hölder continuous foliation. It is convenient to assume that π^λ_{ij} is a contraction. Since p_i is Hölder, we can assume that π^λ_{ij} is a contraction if $(\hat{P}_\lambda)^{-1}|_{\hat{U}_j}$ contracts sufficiently. This can always be effected, possibly by replacing $(\hat{P}_\lambda)^{-1}$ by some iterate. We can similarly define a map ρ^λ_{ij}: $\tilde{S}_j \to \tilde{S}_i$.

A closed orbit τ for ϕ and the corresponding periodic point for $(P_0)^n x = x \in T_{i_0}$ can be coded by a finite sequence $(i_0, i_1, \ldots, i_n)$ with $A(i_j, i_{k+1 (\mathrm{mod}\, n)}) = 1$. Let $V^u = \bigcap_{A(i,j)=1} V_{ij}$. For $\lambda \in V^u$, the sequence $(i_0, i_1, \ldots, i_n)$ gives rise to a contracting map:

$$\hat{U}_{i_0} \xrightarrow{\pi^\lambda_{i_0, i_n}} \hat{U}_{i_n} \xrightarrow{\pi^\lambda_{i_n, i_{n-1}}} \hat{U}_{i_{n-1}} \to \ldots \to \hat{U}_{i_1} \xrightarrow{\pi^\lambda_{i_1, i_0}} \hat{U}_{i_0}$$

The contraction mapping theorem gives a unique fixed point u_λ for this composition of mappings.

Similarly, for $\lambda \in V^s$, the sequence $(i_0, i_1, \ldots, i_n)$ gives rise to a contracting map:

$$\hat{S}_{i_0} \xleftarrow{\rho^\lambda_{i_0, i_n}} \hat{S}_{i_n} \xleftarrow{\rho^\lambda_{i_n, i_{n-1}}} \hat{S}_{i_{n-1}} \leftarrow \ldots \leftarrow \hat{S}_{i_1} \xleftarrow{\rho^\lambda_{i_1, i_0}} \hat{S}_{i_0}$$

and a unique fixed point s_λ.

The point (s_λ, u_λ) guarantees the existence of a periodic point for $(\hat{P}_\lambda)^n$ in $\hat{T}_{i_0}$.

If $V = V^u \cap V^s \supset (-\varepsilon, \varepsilon)$, an application of the Implicit Function Theorem yields that the map $V \to \hat{T}_{i_0}$ defined by $\lambda \mapsto x_\lambda \equiv (s_\lambda, u_\lambda)$ is holomorphic.

Let r_λ: $\bigcup_i T_i \to \mathbb{R}^+$ be the real analytic return times between sections. We denote by $\hat{r}_\lambda$: $\bigcup_i \hat{T}_i \to \mathbb{C}$ the holomorphic extensions of these real analytic functions (defined for δ sufficiently small).

For $\lambda \in (-\varepsilon, \varepsilon)$, the closed orbit corresponding to x_λ has least period $\sum_{j=0}^{n-1} r_\lambda^{i,j+1}(P^j_\lambda(x_\lambda))$, which we denote by $l_\lambda(\tau)$. For $\lambda \in V$, the appropriate complex extension is $\sum_{j=0}^{n-1} \hat{r}_\lambda^{i,j+1}(\hat{P}^j_\lambda(x_\lambda))$. Thus, the map $(-\varepsilon, \varepsilon) \to \mathbb{R}^+$ defined by $\lambda \mapsto l_\lambda(\tau)$ is real analytic and has a holomorphic extension to $\lambda \in V$, independent of the choice of τ. □

Proposition 1.2. *For* re(s) *sufficiently large,* $d(\lambda, s) \equiv \prod_\tau (1 - \exp(-s l_\lambda(\tau))$ *converges uniformly in* $\lambda \in V$ *and hence defines a holomorphic function of* λ.

Proof. We know from elementary complex analysis [*M*] that the uniform convergence of the infinite product $d(\lambda, s)$ is implied by the uniform convergence of the series $\sum_\tau |\exp(-s l_\lambda(\tau))|$. The uniform convergence of this series in $\lambda \in V$ will quickly follow from the next lemma.

Lemma 1.3. *For any $\eta > 0$, there exists a Markov partition for the unperturbed flow and an open set $V' \subseteq V$ (V' open in $\mathbb{C}$ and $(-\varepsilon_0, \varepsilon_0) \subset V'$ for some $\varepsilon_0 > 0$) such that for all $\lambda \in V'$ and for all τ, $|l_\lambda(\tau) - l_0(\tau)| \leqq K(\tau)\eta$, where $K(\tau)$ is the number of Markov sections that τ traverses.*

Proof. Implicit in the statement of this lemma is that for all τ, each curve in $\{\tau_\lambda | \lambda \in V'\}$ has the same coding with respect to the Markov partition of the unperturbed flow. Given $\eta > 0$, we can choose the diameter of our Markov sections sufficiently small, choose $\delta > 0$ used in the definition of $\hat{T}_i$ sufficiently small, and choose V' "sufficiently small" such that we can slightly enlarge our Markov sections, and ensure that for all $\lambda \in V'$, all τ, and all $\hat{T}_i$, $|r_\lambda^{i,i+1}(\tau_\lambda \cap \hat{T}_i) - r_0^{i,i+1}(\tau_0 \cap \hat{T}_i)| < \eta$. This is a simple consequence of the continuity of the return times between the (finitely many) sections and the structural stability of Anosov flows. The lemma follows immediately. □

From definitions, we have:

$$\sum_\tau |\exp(-s l_\lambda(\tau))| = \sum_\tau \exp(-\sigma \operatorname{re}(l_\lambda(\tau))) \exp(\rho \operatorname{im}(l_\lambda(\tau))), \quad \text{where } s = \sigma + i\rho .$$

Lemma 1.3 immediately implies:

(i) $\operatorname{im}(l_\lambda(\tau)) \leqq K(\tau)\eta$ and
(ii) $\operatorname{re}(l_\lambda(\tau)) \geqq l_0(\tau) - K(\tau)\eta$.

Since $K(\tau)$ is the number of Markov sections which τ traverses, it is easy to see that $K(\tau) \sim l_0(\tau)$, i.e., there exists $C = C(\eta) > 0$ such that $K(\tau) \leqq C l_0(\tau)$. Consequently, the sum can be estimated by:

$$\sum_\tau \exp\left(-\sigma l_0(\tau)\left[1 - C\eta\left(1 + \frac{\rho}{\sigma}\right)\right]\right).$$

Let $P(T)$ denote the number of closed trajectories of ϕ^t with prime period $\leqq T$. Margulis has shown that $P(T) \sim \exp(h_{\text{top}} T)/(h_{\text{top}} T)$ [MA1; PP1]. Using this fact, it is easy to check that the above sum converges (uniformly in λ) for $\sigma > \dfrac{h_{\text{top}}(0) + C\eta\rho}{1 - C\eta}$. Since we are only interested in $\rho = \operatorname{im}(s)$ small, say $|\rho| \leqq 1$, this calculation shows that for $\operatorname{re}(s) > \dfrac{h_{\text{top}}(0) + C\eta}{1 - C\eta}$, $\sum_\tau |\exp(-s l_\lambda(\tau))|$ converges uniformly in λ and defines a holomorphic function of λ. □

For $\operatorname{re}(s) > \dfrac{h_{\text{top}}(0) + C\eta}{1 - C\eta}$, $d(s, \lambda)$ restricted to $(-\varepsilon_0, \varepsilon_0) = V' \cap \mathbb{R}$ is real analytic in λ.

Step 2

Pollicott has proved the following theorem for ζ-functions of Axiom A flows restricted to basic sets, although we state it for Anosov flows:

Theorem [PO1]. *Let ϕ: $M \to M$ be a C^1 Anosov flow. There exists $k < h_{\text{top}}(\phi)$, where k depends continuously on:* (i) $h_{\text{top}}(\phi)$, (ii) *the contraction/expansion coefficient*

λ in the definition of Anosov flow, (iii) *the return-time map between sections in the C^0 topology, and* (iv) *the choice of Markov partition for ϕ such that:*

(a) *$d(s)$ is holomorphic and non-vanishing for* $\mathrm{re}(s) > h_{\mathrm{top}}(\phi)$.
(b) *$d(s)$ has a holomorphic extension to* $\mathrm{re}(s) > k$ *with a simple zero at* $s = h_{\mathrm{top}}(\phi)$.

In [PO1], Pollicott gives an explicit characterization of k for a meromorphic extension in terms of the associated symbolic dynamics. The quantities involved in the characterization at the symbolic level can easily be seen to have the desired continuous dependence. All that remains to show is that no poles can occur for $\mathrm{re}(s) > k$ (perhaps after a suitable adjustment to k). It follows from Sects. 5 and 6 in [B2] that the poles for $d(s)$ cannot occur in the meromorphic extension in the region $\mathrm{re}(s) > m$, where $m < h_{\mathrm{top}}(\phi)$ is an upper bound on the topological entropy of certain semi-flows arising from the boundaries of Markov sections. Since nearby flows have conjugate Poincare maps on Markov sections and C^0 close return times, we can again arrange for m to have the same continuous dependence as k. Finally, we replace k by $\max\{k, m\}$. For three dimensional Anosov flows we can choose $m=0$.

The above argument implies that there exists $\kappa > 0$ and $\varepsilon_1 = \varepsilon_1(\kappa) > 0$ such that if $\lambda \in (-\varepsilon_1, \varepsilon_1)$, then $d(\lambda, s)$ has a holomorphic continuation into the half plane $\mathrm{re}(s) > h_{\mathrm{top}}(0) - \kappa$. We denote the extended function by $\bar{d}(\lambda, s)$.

Step 3

In Step 1, we have shown that for $\mathrm{re}(s)$ sufficiently large, $d(\lambda, s)$ is real analytic on $(-\varepsilon_0, \varepsilon_0)$. In Step 2, we have shown that for $\lambda \in (-\varepsilon_1, \varepsilon_1)$, $d(\lambda, s)$ has a holomorphic continuation $\bar{d}(\lambda, s)$ into the half plane $\mathrm{re}(s) > h_{\mathrm{top}}(0) - \kappa$. We need to show that $\bar{d}(\lambda, s)$ is real analytic in λ for $\mathrm{re}(s) > h_{\mathrm{top}}(0) - \kappa$.

This is a "Hartogs type" problem. The problem reduces to the following question in complex analysis, where we are interested in the case $k = \omega$: Suppose that for every $\lambda \in (-1, 1)$, $f(\lambda, *)$ is holomorphic in Δ_1 (the unit disk in $\mathbb{C}$) and has a holomorphic extension to Δ_2 (the disk of radius 2). Furthermore, suppose that for every $z \in \Delta_1$, $f(*, z) \in C^k(-1, 1)$. Is $f(*, z) \in C^k(-1, 1)$ for $z \in \Delta_2$?

The following counterexample exhibits what can go wrong for C^1. This example can be modified to give a counterexample for $1 \leqq k \leqq \infty$.

Example. Let $f(\lambda, z) \equiv \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k \sin(2^k \lambda)$. For every $\lambda \in (-1, 1)$, $f(\lambda, *)$ is holomorphic in $\Delta = \Delta_1$ and has a holomorphic extension to Δ_2. Since $\frac{\partial f}{\partial \lambda}(\lambda, z) = -\sum_{k=0}^{\infty} z^k \cos(2^k \lambda)$, f is C^1 in Δ_1. However, $\left.\frac{\partial f}{\partial \lambda}(\lambda, z)\right|_{\lambda=0} = \frac{-1}{1-z}$, so f is not C^1 in Δ_2.

In [SH2], Shiffman has shown that this question has an affirmative answer in the real analytic case. He proved the following stronger theorem:

Theorem. *Let* $f\colon (-1, 1) \times \Delta_2 \to \mathbb{C}$ *and let* $0 < a < 2$ *be such that* (i) *for every* $x \in (-1, 1)$, $f(x, *)$ *is holomorphic in* Δ_2 (ii) *for every* $z \in \Delta_a$, $f(*, z)$ *is* C^ω *on* $(-1, 1)$. *Then for all* $r < 2$, *there exists an open set* $U \subset \mathbb{C}$ *such that* $U \cap \mathbb{R} = (-1, 1)$, *and a holomorphic function* $\tilde{f}$ *on* $U \times \Delta_r$ *such that* $\tilde{f}|_{(-1, 1) \times \Delta_r} = f$.

This theorem immediately implies that $f(*, z)$ is real analytic on $(-1, 1)$ for all $z \in \Delta_2$. The proof uses pluripotential theory in $\mathbb{C}^n$ (plurisubharmonic extremal functions, analysis of pluripolar sets, approximation theory, etc.) and combines methods from [SH1] with results of Siciak in [S].

Step 4

Since the zeros $h_{\text{top}}(\lambda)$ of $\bar{d}(\lambda, s)$ are simple, we may apply the Implicit Function Theorem to conclude that $h_{\text{top}}(\lambda)$ is real analytic. □

Theorem 2. *Let M be a closed n-dimensional manifold and let* $\{\phi_\lambda^t\}$, $-\varepsilon \leqq \lambda \leqq \varepsilon$ (ε *sufficiently small*) *be a* C^{k+1} *perturbation of a* C^{k+1} *Anosov flow* $\phi^t \equiv \phi_0^t$, $1 \leqq k \leqq \infty$. *Then* $h_{\text{top}}(\phi_\lambda^1)$ *is* C^k.

Proof. Our strategy is to show that the map $(\lambda, x) \mapsto P(-xr_\lambda)$ is C^k, where r_λ denotes the return map between sections for ϕ_λ^t (the height function over $\sum_A$), and $P(f)$ denotes the pressure of f. It is easy to show that $P(-xr_\lambda) = 0$ implies that $x = h_{\text{top}}(\lambda)$. We then apply the Implicit Function Theorem to conclude that $h_{\text{top}}(\lambda)$ is C^k.

Step 1. Show that the map $C^\alpha(\sum_A) \to \mathbb{R}$ defined by $f \mapsto P(f)$ is analytic.

Step 2. Show that the map $(-\varepsilon, \varepsilon) \to C^\alpha(\sum_A)$ defined by $\lambda \mapsto r_\lambda$ is C^k.

Step 1

Given $f \in C^0(\sum_A, \mathbb{R})$, we define the *pressure* $P(f) \in \mathbb{R}$ by:

$$P(f) \equiv \sup\{h_\mu(\sigma) + \textstyle\int f\,d\mu\colon \mu \text{ is a } \sigma\text{-invariant probability measure on } \sum_A\}\,.$$

The case where $f\colon \sum_A \to \mathbb{R}$ is Hölder continuous leads to particularly useful results. For $0 < \alpha < 1$, we denote by $C^\alpha(\sum_A)$ the set of all real valued α-Hölder continuous functions on $\sum_A$. These form a Banach space with norm

$$\|f\| \equiv \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

For $f \in C^\alpha(\sum_A)$, there exists a unique measure μ realizing the above supremum, i.e. $P(f) = h_\mu(\sigma) + \int f\,d\mu$ [B1]. This measure is called the *equilibrium state* for f. The following result indicates some useful properties of pressure:

Proposition 2.1. (a) *If f is a strictly positive Hölder continuous function defined on* $\sum_A$, *let* $\sum_A^f = \{(x, t)\colon x \in \sum_A, 0 \leq t \leq f(x)\}$ *where* $(x, f(x))$ *and* $(\sigma(x), 0)$ *are identified. We define the f suspension flow* $\sigma_f^t\colon \sum_A^f \to \sum_A^f$ *by* $\sigma_f^t(x, r) = (x, r + t)$ *up to identifica-*

tions. Then the map $\mathbb{R} \to \mathbb{R}$ *defined by* $x \to P(-xf)$ *is strictly monotonically decreasing with* $P(-xf) = 0$ *precisely when* $x = h_{\mathrm{top}}(\sigma_f^1)$.

(b) *The map* P: $C^\alpha(\sum_A) \to \mathbb{R}$ *is analytic with derivative* $D_f P = \mu$, *where* μ *is the equilibrium state for* f.

Proof. (a) The monotonicity follows from the definition of pressure since f is strictly positive.

If f is a strictly positive Hölder continuous function defined on $\sum_A$, part (a) implies that there exists $c > 0$ such that

$$P(-cf) = \sup_\mu \{h_\mu(f) - c\int f d\mu\} = \sup_\mu \left(\int f d\mu\right)\left(\frac{h_\mu(f)}{\int f d\mu} - c\right) = 0 .$$

Since $\int f d\mu > 0$ for every σ_A invariant probability measure μ, this implies that

$$c = \sup_\mu \left(\frac{h_\mu(f)}{\int f d\mu}\right) = \sup_{\tilde{\mu}} h_{\tilde{\mu}}(\sigma_f^1) = h_{\mathrm{top}}(\sigma_f^1) ,$$

where $\tilde{\mu}$ is the invariant measure of the suspended flow σ_f induced by μ. The equality in the middle follows from Abramov's theorem for the entropy of the suspended flow.

(b) For $0 < \alpha < 1$, we define $C^\alpha(\sum_A)^+ \equiv \{f \in C^\alpha(\sum_A)$: $f(x) = f(y)$ for $x_n = y_n$, $n \geqq 0\}$. Parry [PA] observed that there is a linear map $C^\xi(\sum_A) \to C^\alpha(\sum_A)^+$ defined by $f \mapsto f^+$ with $P(f) = P(f^+)$ provided that $\alpha > \sqrt{\xi}$. Furthermore, Ruelle introduced a family of linear operators L_{f^+}: $C^\alpha(\sum_A)^+ \to C^\alpha(\sum_A)^+$ such that (i) $f^+ \to L_{f^+}$ is analytic and (ii) L_{f^+} has $e^{P(f)}$ as an isolated eigenvalue. In particular, we know by standard perturbation theory that $L_{f^+} \to e^{P(f)}$ is analytic. Altogether, this shows that $f \to P(f)$ is analytic [RU1].

Finally, once we know that $f \to P(f)$ is differentiable, it is easy to compute the derivative using the variational principle [RU1]. □

Step 2

In this step we analyze how the maps (the conjugating homeomorphisms and the time reparameterizations) obtained from the structural stability of Anosov flows depend on the perturbation parameter λ. It is well known that these maps are always Hölder continuous on the manifold, and that in general they possess no more regularity. It is therefore surprising that for a C^k perturbation of an Anosov flow, these maps depend C^k on the perturbation parameter (in the C^0 topology).

Moser's proof of structural stability for Anosov diffeomorphisms [MO] uses the contracion mapping principle to obtain the conjugating homeomorphism. To prove C^k dependence of the conjugating homeomorphisms for a C^k perturbation, one simply adds a perturbation parameter into Moser's proof. It is easy to check that the contraction mapping depends C^k on the perturbation parameter, and the Implicit Function Theorem guarantees that the fixed points (the conjugating homeomorphisms) depend C^k on the perturbation parameter.

Moser's proof of structural stability for Anosov flows [MO] is incorrect. However, by modifying his argument, a proof based on the Implicit Function Theorem can be given. One can also add a perturbation parameter into this proof and conclude that for C^k perturbations of Anosov flows, the conjugating homeomorphisms and the time reparameterizations depend C^k on the perturbation parameter in the C^0 topology. In Appendix I of [LMM], de la Llave et al. give a detailed proof of this result written in the language of manifolds of mappings.

Theorem [LMM]. *Let* $\{\phi_\lambda^t\}$, $-\varepsilon \leqq \lambda \leqq \varepsilon$ *(*ε *sufficiently small) be a* C^k *perturbation of a* C^k *Anosov flow* $\phi^t = \phi_0^t$, $1 \leqq k \leqq \omega$. *Then there exists a* C^k *map* $\mathfrak{S}:(-\varepsilon, \varepsilon) \to C^0(M, M) \times C^0(M)$ *defined by* $\lambda \mapsto (\theta_\lambda, \rho_\lambda)$, *where* $\theta_\lambda \in C^0(M, M)$ *is the conjugating homeomorphism and* $\rho_\lambda \in C^0(M)$ *is a time reparameterization.*

Remark. $C^\alpha(M, M)$ and $C^\alpha(M)$, $0 \leqq \alpha \leqq 1$, possess C^∞ (or C^ω) Banach manifold structures—assuming that M is a C^∞ (or C^ω) manifold [E].

We need to strengthen this theorem to show that $\mathfrak{S}$ is C^k in the C^α topology. Unfortunately, this can only be assured for C^{k+1} perturbations of C^{k+1} Anosov flows.

Proposition 2.2. *Let* $\{\phi_\lambda^t\}$, $-\varepsilon \leqq \lambda \leqq \varepsilon$ *(*ε *sufficiently small) be a* C^{k+1} *perturbation of a* C^{k+1} *Anosov flow* $\phi^t = \phi_0^t$, $1 \leqq k \leqq \omega$. *Then for* α *sufficiently small, there exists a* C^k *map* $\mathfrak{S}$: $(-\varepsilon, \varepsilon) \to C^\alpha(M, M) \times C^\alpha(M)$ *defined by* $\lambda \mapsto (\theta_\lambda, \rho_\lambda)$, *where* $\theta_\lambda \in C^\alpha(M, M)$ *is the conjugating homeomorphism and* $\rho_\lambda \in C^\alpha(M)$ *is a time reparameterization.*

Proof. The proof is similar to the proof in [LMM] for the C^0 case. The idea is to construct $\mathfrak{S}$ as the implicit solution to an identity on vector fields. This map is guaranteed by the Implicit Function Theorem provided a certain invertibility condition (Lemma 2.3) is satisfied.

Let Φ denote the generating vector field for ϕ^t and $C_\Phi^\alpha(M) \equiv \{u \in C^\alpha(M, M)$ such that for all $p \in M$, $\left.\frac{d}{dt}\right|_{t=0} u \circ \phi^t(p) = Du(p)\Phi$ exists$\}$. We will show that for sufficiently small $\alpha > 0$, there exists a C^{k+1} neighborhood $U(\Phi)$ of Φ, and a C^k map S: $U(\Phi) \to C_\Phi^\alpha(M) \times C^\alpha(M, \mathbb{R})$ defined by $\Psi \to (u, \gamma)$, which solves the structural stability equation $\Psi \circ u = \gamma Du(p)\Phi$. This clearly implies the proposition.

The idea is to use the Implicit Function Theorem to solve $\Psi \circ u = \gamma Du \circ \Phi$. Define G: $V^{k+1}(M) \times C_\Phi^\alpha(M) \times C^\alpha(M, \mathbb{R}) \to C^\alpha(M, TM)$ by

$$G(\Psi, u, \gamma) = \Psi \circ u - \gamma Du \circ \Phi ,$$

where $V^{k+1}(M)$ denotes the space of C^{k+1} vector fields on M. All of the spaces involved in the definition of G are C^∞ (or C^ω) Banach manifolds, and it is an exercise in applying the mean value theorem to show that G is a C^k map.

Remark. It is in this step where we lose a derivative! For G to be a C^k map, we need to know that the perturbations are not just C^k or even C^{k+LIP}, but are C^{k+1}. This is to ensure that the composition operator $(\Psi, u) \to \Psi \circ u$ is C^k. We thank Rafael de la Llave for pointing out to us that a loss of smoothness is inevitable and Albert Fathi for providing us with the precise statement and a C^{K+LIP} counterexample.

It is obvious that $G(\Phi, \text{id}, 1) = 0$. Let $D_{2,3}$ denote the derivative with respect to the second and third variables. $D_{2,3}G(\Phi, \text{id}, 1)$: $V^\alpha_\Phi(M) \times C^\alpha(M, \mathbb{R}) \to V^\alpha(M)$, where $V^\alpha_\Phi(M)$ denotes the space of C^α vector fields such that the directional derivative with respect to Φ exists, i.e. $T_{\text{id}} C^\alpha_\Phi(M) = V^\alpha_\Phi(M)$. One computes:

$$\begin{aligned} D_{2,3}G(\Phi, \text{id}, 1)(v, \gamma) &= D\Phi(v) - Dv(\Phi) - \gamma\Phi \\ &= [\Phi, v] - \gamma\Phi \\ &= \frac{d}{dt}\bigg|_{t=0} D\phi^{-t} v \phi^t - \gamma\Phi \end{aligned}$$

The kernel of the Lie Derivative $L_\Phi(v) = [\Phi, v] = \frac{d}{dt}\Big|_{t=0} D\phi^{-t} v \phi^t$ is $[\Phi]$. Hence, $D_{2,3}G$ is not invertible. To circumvent this problem, we choose a section transverse to Φ, i.e. $\Phi^\perp = \Gamma(E^s) \oplus \Gamma(E^u)$. Now define $V^\alpha_{\Phi,\perp}(M) = V^\alpha_\Phi \cap \Phi^\perp$. We will show in Lemma 2.3 that $L^\perp_\Phi$: $V^\alpha_{\Phi,\perp}(M) \to V^\alpha_{\Phi,\perp}(M)$ is invertible in the C^α topology, with

$$(L^\perp_\Phi)^{-1}(v_u) = -\int_0^\infty D\phi^{-t} v_u \phi^t \, dt, \quad v_u \in \Gamma(E^u)$$

$$(L^\perp_\Phi)^{-1}(v_s) = -\int_0^\infty D\phi^t v_s \phi^{-t} dt, \quad v_s \in \Gamma(E^s)$$

This proves that $(v, \gamma) \to L^\perp_\Phi(v) - \gamma\Phi = D_{2,3}G(\Phi, u, 1)(v, \gamma)$ is invertible. We can now apply the Implicit Function Theorem to the equation $G = 0$, where we replace $C^\alpha_\Phi(M)$ by $\exp(V^\alpha_{\Phi,\perp}(M))$. □

Invertibility condition

Let $\Gamma^\alpha \equiv \Gamma^\alpha(M, TM)$ denote the Banach space of C^α sections σ: $M \to TM$ with norm $\|\sigma\|_\alpha \equiv \|\sigma\|_\infty + \sup\limits_{0<d(x,y)<\beta} \frac{d_{TM}(v(x), v(y))}{d(x, y)^\alpha}$, where $\sigma(x) = (x, v(x))$, and let $TM = E^0 \oplus E^u \oplus E^s$ be the Anosov splitting for the Anosov flow ϕ. The splitting of TM induces a splitting of $\Gamma^0 = \Gamma^0(E^0) \oplus \Gamma^0(E^u) \oplus \Gamma^0(E^s)$ in the obvious way. The operator (the graph transform) ϕ_*: $\Gamma^0 \to \Gamma^0$ defined by $(\phi_*\sigma)(x) \equiv (D\phi_1\sigma)(\phi^{-1}x)$ preserves the Anosov splitting. It is easy to see that the spectrum of $\phi_s = \phi_*|_{\Gamma^0(E^s)}$ is contained in an annulus with inner and outer radii $0 < \mu_1 < \mu_2 < 1$. Similarly, the spectrum of $\phi_u = \phi_*|_{\Gamma^0(E^u)}$ is contained in an annulus with inner and outer radii $1 < \lambda_1 < \lambda_2$. The invertibility condition needed to apply the Implicit Function Theorem is precisely the condition that the spectra of ϕ_s and ϕ_u are disjoint from the unit circle.

Hirsch and Pugh have proved the following theorem:

Theorem [HP]. *If* $\alpha < \min\left\{\frac{\log\lambda_1}{\log\lambda_2}, \frac{\log\mu_1}{\log\mu_2}\right\}$, *then the splitting* $TM = E^0 \oplus E^u \oplus E^s$ *is* C^α. *In particular,* $\Gamma^\alpha = \Gamma^\alpha(E^0) \oplus \Gamma^\alpha(E^u) \oplus \Gamma^\alpha(E^s)$ *is a splitting into closed subspaces.*

The following lemma ensures that we may apply the Implicit Function Theorem in the C^α topology of flows. We thank Rafael de la Llave for suggesting the result to us.

Lemma 2.3 (Hyperbolicity of the graph transform in Hölder norm). *There exists $0 < \alpha < 1$ sufficiently small such that the spectra of ϕ_s: $\Gamma^\alpha(E^s) \to \Gamma^\alpha(E^s)$ and ϕ_u: $\Gamma^\alpha(E^u) \to \Gamma^\alpha(E^u)$ are disjoint from the unit circle.*

Proof. We first consider ϕ_s and show that it is a contraction on $\Gamma^\alpha(E^s)$. As stated above, ϕ_s: $\Gamma^0(E^s) \to \Gamma^0(E^s)$ is a contraction. We can always adapt the metric on M such that ϕ_s: $\Gamma^0(E^s) \to \Gamma^0(E^s)$ is a strict contraction, and hence can assume that $\| D\phi|_{\Gamma^0(E^s)} \| = K < 1$. Let M denote the Lipschitz constant of $D\phi$, N the Lipschitz constant for ϕ^{-1}, and γ the Hölder exponent in the Anosov splitting. Given $\sigma \in \Gamma^\alpha(E^s)$ and $x \neq y$, the triangle inequality gives:

$$d_{TM}(D\phi\, \sigma(\phi^{-1}x), D\phi\, \sigma(\phi^{-1}y)) \leqq d_{TM}(D\phi\ \ \sigma(\phi^{-1}x), D\phi\, v') + d_{TM}(D\phi\, v', D\phi\, \sigma(\phi^{-1}y)),$$

where v' is the projection onto E^s of the parallel translate of $\sigma(\phi^{-1}y)$ to $\phi^{-1}x$.

$$\leqq K\, d_{TM}(\sigma(\phi^{-1}x), v') + M\, d_{TM}(v', \sigma(\phi^{-1}y))$$
$$\leqq K\, d_{TM}(\sigma(\phi^{-1}x), \sigma(\phi^{-1}y)) + MQ\, d(\phi^{-1}x, \phi^{-1}y)^\gamma \|\sigma\|_\infty$$

It immediately follows from the definition of the Hölder norm that:

$$\leqq K(\|\sigma\|_\alpha - \|\sigma\|_\infty)\, d(\phi^{-1}x, \phi^{-1}y)^\alpha + M\,Q\,N^\gamma d(x, y)^\gamma \|\sigma\|_\infty$$
$$\leqq K(\|\sigma\|_\alpha - \|\sigma\|_\infty) N^\alpha d(x, y)^\alpha + M\,Q\,N\, d(x, y)^\gamma\ \|\sigma\|_\infty$$

Choose α, $0 < \alpha \leqq \gamma$ such that $KN^\alpha = P < 1$, $\delta > 0$ such that $\delta < P - K$, and $\beta > 0$ in the definition of the Hölder norm such that $MQN\, d(x, y)^\gamma \leqq \delta\, d(x, y)^\alpha$. This implies that:

$$d_{TM}(D\phi\, \sigma(\phi^{-1}x), D\phi\, \sigma(\phi^{-1}y)) \leqq d(x, y)^\alpha (KN^\alpha(\|\sigma\|_\alpha - \|\sigma\|_\infty) + \delta \|\sigma\|_\infty)$$
$$\leqq d(x, y)^\alpha (P(\|\sigma\|_\alpha - \|\sigma\|_\infty) + \delta \|\sigma\|_\infty)\ .$$

Therefore:

$$\|\phi_s(\sigma)\|_\alpha = \|\phi_s(\sigma)\|_\infty + \sup_{0 < d(x,y) < \beta} \frac{d_{TM}(D\phi\, \sigma(\phi^{-1}x), D\phi\, \sigma(\phi^{-1}y))}{d(x, y)^\alpha},$$

where $\sigma(x) = (x, v(x))$,

$$\leqq K\|\sigma\|_\infty + P\|\sigma\|_\alpha - P\|\sigma\|_\infty + \delta\|\sigma\|_\infty$$
$$\leqq P\|\sigma\|_\alpha, \quad \text{for } 0 < P < 1. \quad \square$$

Let $\mathcal{T}$ be a family of Markov sections for ϕ^t. If $\mathcal{T} = \{T_1, T_2, \cdots, T_k\}$, we can assume that $T_i \subseteq \operatorname{int} D_i$, where D_i is a slightly larger C^∞ transverse section with co-dimension one. Let T_i^λ denote the projection of $\theta_\lambda(T_i)$ onto T_i along the orbits of ϕ_λ^t, where θ_λ is the conjugating homeomorphism from structural stability. This procedure is well defined provided $|\lambda|$ is sufficiently small.

If we use the Markov sections $\mathscr{T}^\lambda = \{T_1^\lambda, T_2^\lambda, \ldots, T_m^\lambda\}$ for ϕ_λ^t, we obtain precisely the same map $\sigma: \sum_A \to \sum_A$ as for ϕ^t. This is because the Poincaré map is oblivious to changes in velocity. The quantity which changed is the return time between sections. We define $r_\lambda \in C^\alpha(\sum_A)$ by:

$$r_\lambda(x) = \int_0^{r(\theta_\lambda \pi(x))} \rho_\lambda(\phi^t(\theta_\lambda \pi(x)))\,dt, \quad \text{where } \pi: \sum_A \to \bigcup_i T_i$$

Proposition 2.4. *The map* $(-\varepsilon, \varepsilon) \to C^\alpha(\sum_A)$, *defined by* $\lambda \mapsto r_\lambda$, *is* C^k.

Proof. This follows immediately from Proposition 2.2. □

Propositions 2.1 and 2.4 imply that the map $(-\varepsilon, \varepsilon) \times \mathbb{R} \to \mathbb{R}$ defined by $(\lambda, t) \mapsto P(-tr_\lambda)$ is C^k. We are interested in solutions $\lambda \mapsto h_{\text{top}}(\lambda)$ to $P(-tr_\lambda) = 0$. Since

$$\left.\frac{dP(-tr_\lambda)}{dt}\right|_{t=t_0} = -t_0 \int_M r_\lambda d_\mu < 0\,,$$

where μ is the unique equilibrium state for $f = -tr_\lambda$ [RU1; PP1], the Implicit Function Theorem shows that $\lambda \to h_{\text{top}}(\lambda)$ is C^k. □

Remark. We have proved that for C^{k+1} perturbations, the topological entropy of Anosov flows varies C^k in the *Gâteaux* sense. We have actually proved more: the topological entropy varies C^k in the *Fréchet* sense. In Proposition 2.1, we show that pressure is analytic in the Fréchet sense and in Proposition 2.2, we show that the maps obtained from structural stability depend C^k on the perturbation parameter in the Fréchet sense. The proof of Theorem 2 is essentially a combination of these two results.

Alternate proof of theorem 2

Although our proof is complete, we shall indicate a (superficially) shorter proof that a C^{k+1} perturbation of an Anosov flow results in a C^k change in topological entropy. It is important to note that all of the following results are derived using techniques described in the last section.

Let τ be a closed orbit for ϕ^t with least period $l(\tau)$. Given $F \in C^\alpha(M)$ with $F > 0$, we can define the "weighted period" of τ to be $l_F(\tau) \equiv \int_0^{l(\tau)} F(\phi^t x_\tau)\,dt$, $x_\tau \in \tau$. We define the *F-weighted zeta function*:

$$\zeta_F(s) \equiv \prod_\tau (1 - \exp(-s\,l_F(\tau))^{-1}\,.$$

This product converges for $re(s)$ sufficiently large.

Let ϕ_λ^t be a C^{k+1} perturbation of an Anosov flow ϕ^t, and let $F_\lambda = \rho_\lambda$ (the time reparameterization from structural stability). It is easy to check that $\zeta_{\rho\lambda}(s) = \zeta(\lambda, s)$.

The following theorem is an immediate consequence of Theorem 2 in [RU2] whose detailed proof has never appeared. It can also be deduced from work in [PA]:

Theorem. (i) *Given $G \in C^{\alpha}(M)$, there exists a neighborhood $\mathscr{U}$ of G and $\varepsilon > 0$ such that the map $\mathscr{R}$: $\mathscr{U} \subset C^{\alpha}(M) \to \mathscr{H}(D_{P(\log G),\varepsilon})$ defined by $F \mapsto d(F, s) = \dfrac{1}{\zeta_F(s)}$ is real analytic as a map into the space of holomorphic functions on the disc D of radius ε about $P(\log G)$.*
(ii) *For each $F \in \mathscr{U}$, $P(\log F) \in D_{P(\log G),\varepsilon}$ and is a simple zero for $d(F, s)$.*

Part (i) along with Proposition 2.2 imply that the composition $\lambda \overset{\mathfrak{S}}{\mapsto} \rho\lambda \overset{\mathscr{R}}{\mapsto} d(\rho\lambda, s)$ is C^k. Using part (ii), we can apply the Implicit Function Theorem to deduce that the zero $s = h_{\text{top}}(\lambda)$ depends C^k on λ. The condition $\left.\dfrac{\partial d(\lambda, s)}{\partial s}\right|_{s = h_{\text{top}}(\phi)} \neq 0$ required to apply the Implicit Function Theorem is an immediate consequence of the simplicity of the zero. □

Remark. If we assume that M is a C^r manifold, then some complications arise with the regularity unless $r \geqq k + 2$. This is because the Banach manifolds $C^{\alpha}(M, M)$ and $C^{\alpha}(M)$ have only C^{r-2} charts. We refer the reader to the paper of Eells [E] for details. Also, if a manifold with negative sectional curvature is C^r, then complications again arise to make the entropy C^k where $k < r$. We refer the reader to the paper of Hirsch and Pugh [HP] for details.

Smoothness of pressure and Gibbs states

The techniques used in this paper can be easily adapted to prove slightly more general results. In particular, everything that we have proved about topological entropy holds true for the pressure relative to some smooth function f: $M \to \mathbb{R}$.

Let $\{\phi^t_\lambda\}$, $-\varepsilon \leqq \lambda \leqq \varepsilon$ (ε sufficiently small) be a C^k perturbation of a C^k Anosov flow $\phi^t = \phi^t_0$, $1 \leqq k \leqq \omega$. For fixed $f \in C^k(M)$, the map $f \to P(f, \lambda)$ is well known to be C^k [RU2].

Proposition. *Let $\{\phi^t_\lambda\}$, $-\varepsilon \leqq \lambda \leqq \varepsilon$ (ε sufficiently small) be a C^{k+1} perturbation of a C^{k+1} Anosov flow $\phi^t = \phi^t_0$, $1 \leqq k \leqq \omega$. For fixed $f \in C^{k+1}(M)$, the map $\lambda \to P(f, \lambda)$ is C^k. Consequently, the map $C^{k+1}(M) \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ defined by $(f, \lambda) \to P(f, \lambda)$ is C^k.*

Let $\mu_{f,\lambda}$ be the ϕ^1_λ Gibbs state for f, i.e., the unique ϕ^1_λ-invariant probability measure characterized by: $P(f, \lambda) = h\mu_{f,\lambda}(\phi^1_\lambda) + \int_M f\,d\mu_{f,\lambda}$. The Variational Principle tells us that $P(f, \lambda) \geqq h_\nu(\phi^1_\lambda) + \int_M f\,d\mu_\nu$ for all other ϕ^1_λ-invariant probability measures. A simple argument (along the lines of the proof of Proposition 2.1) gives that $(D_1 P)_{(f,\lambda)} = \mu_{f,\lambda}$. This fact, together with the above proposition yields:

Corollary 1 (Gibbs measures are weakly smooth). *Let $\{\phi^t_\lambda\}$, $-\varepsilon \leqq \lambda \leqq \varepsilon$ (ε sufficiently small) be a C^{k+1} perturbation of a C^{k+1} Anosov flow $\phi^t = \phi^t_0$, $1 \leqq k \leqq \omega$. For fixed $\rho \in C^{k+1}(M)$, the map $C^{k+1}(M) \times (-\varepsilon, \varepsilon) \to \mathbb{R}$ defined by $(f, \lambda) \to \int_M \rho\,d\mu_{f,\lambda}$ is C^{k-1}.*

The Proposition, Corollary 1, and the Variational Principle immediately imply:

Corollary 2 (Smoothness of entropy for Gibbs states). *Let* $\{\phi_\lambda^t\}$, $-\varepsilon \leqq \lambda \leqq \varepsilon$ *(*ε *sufficiently small) be a* C^{k+1} *perturbation of a* C^{k+1} *Anosov flow* $\phi^t = \phi_0^t$, $1 \leqq k \leqq \omega$. *For fixed* $f \in C^{k+1}(M)$, *the map* $\lambda \to h\mu_{f,\lambda}$ *is* C^{k-1}, *where* $h\mu_{f,\lambda}$ *denotes the measure theoretic entropy with respect to the* ϕ_λ^1 *Gibbs state for* f.

Remark. In [LMM] it was shown that for a C^∞ family of Anosov flows with smooth invariant measures, the measures have a C^∞ weak dependence.

Another direction is a generalization of our results from Anosov flows to Axiom *A* flows. The main ingredients remain the same; the extra element involves the careful extension of the hyperbolic structure to appropriate neighborhoods of the basic sets.

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