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MONOTONE EQUIVALENCE IN ERGODIC THEORY

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Abstract. A class of monotonely equivalent automorphisms (standard automorphisms), which includes all ergodic automorphisms with discrete spectrum and most of the well-known examples of automorphisms with zero entropy, is studied. The basic results are two necessary and sufficient conditions for standardness: the first in terms of periodic approximation and the second in terms of the asymptotic properties of "words" arising from a coding of most trajectories by a finite partition. Also certain monotone invariants are defined and their properties discussed.

Bibliography: 39 titles.

Introduction

1. Abstract ergodic theory studies the action of groups of automorphisms of measure spaces or, as is sometimes said, groups of measure-preserving transformations. For such actions there is a natural, from the intrinsic viewpoint, notion of isomorphism called metric isomorphism. Namely, actions $\{T_g\}$ and $\{S_g\}, g \in G$, of a group G on measure spaces (X, μ) and (Y, ν) are called *metrically isomorphic* if there is an isomorphism $R:(X, \mu) \longrightarrow (Y, \nu)$ such that $RT_g = S_gR, g \in G$. In other words, we can say that under a metric isomorphism the invariant measure of the action $\{T_g\}$ is transformed into the invariant measure of $\{S_g\}$, and each (more precisely, almost each) trajectory of $\{T_g\}$ is mapped onto a trajectory of $\{S_g\}$ with preservation of the group structure on the trajectory.

It turns out that the complete classification up to metric isomorphism of the actions of groups, excluding the trivial case of compact groups, is a hopeless problem. In attempting such classification two kinds of difficulty arise: the invariants turn out to be too many, and many invariants are difficult or impossible to calculate. A typical case is the group $\mathbf{Z}$; that is, the group consisting of the powers of a single automorphism (in this case one usually talks simply of automorphisms), which has been studied most intensively. Here there are remarkable partial results—the classification of automorphisms with discrete spectrum (von Neumann; see [1]) and Bernoulli shifts (Ornstein [2]). However, even on passing to broader special classes, for example to automorphisms with simple spectrum, or to K-automorphisms, the problem of metric classification becomes immense.

2. Entirely in the spirit of modern mathematics one tries to replace metric isomorphism by some weaker equivalence relation in the hope that the new classification problem

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will prove to be interesting and at the same time solvable completely or in important parts. One of these equivalence relations, which is prompted by the definition of metric isomorphism, is that of *trajectory isomorphism*, for which it is required that there exist a mapping R, transforming the measure μ to ν and the trajectories of $\{T_g\}$ to the trajectories of $\{S_g\}$, but without preserving the group structure on the trajectories. However, this equivalence relation, at least for sufficiently simple groups G, turns out to be almost vacuous: it preserves only the partition into ergodic components of the action (see [3] and [4]). Therefore we must look for an equivalence relation weaker than metric isomorphism but stronger than trajectory isomorphism.

There are some "exotic" groups for which such equivalence relations arise in an obvious way. There is the question of quasi-cyclic groups of the form $G = \bigcup_{n=1}^{\infty} \mathbf{Z}_{q_n}$, where $\mathbf{Z}_{q_1} \hookrightarrow \mathbf{Z}_{q_2} \hookrightarrow \ldots$ is an increasing sequence of cyclic groups. The subgroups $\mathbf{Z}_{q_n}$ form a filtration in G, and it is natural to consider, for actions of G, trajectory isomorphism which preserves not only the trajectory partition of G but also the decreasing sequence of partitions $\{\xi_n\}$ into the trajectories of the actions of $\mathbf{Z}_{q_n}$. The problem of classification of such sequences was first taken up by Veršik [5]. In a subsequent paper [6], the class of standard sequences was selected and necessary and sufficient conditions for standardness were formulated in terms of the so-called universal projection operator. In the same place there are some results on the connection between metric properties of the action of quasi-cyclic groups and the properties of the decreasing sequence of partitions generated by these actions; in particular, the nonstandardness of the sequence generated by a Bernoulli action and the standardness of the sequence generated by an ergodic action with discrete spectrum (cf. with our Corollary 8.2). We note further the work of Stepin [7], in which it is proved that for sequences q_n bounded in growth, the entropy of the action of quasi-cyclic groups is the same for two actions generating isomorphic sequences of partitions, and consequently, is an invariant for trajectory isomorphism with preservation of filtration.

3. The general idea, which is prompted by the case of quasi-cyclic groups, consists of the following. Let the group G have some structure (a filtration, a partial order, a topology, smoothness, etc.). We will regard the actions $\{T_g\}$ and $\{S_g\}$ of G on measure spaces (X, μ) and (Y, ν) as equivalent if there exists an isomorphic mapping $R\colon X \longrightarrow Y$, transforming the trajectories of $\{T_g\}$ onto the trajectories of $\{S_g\}$ and preserving the induced structure on almost each trajectory. It should be noted, in contrast to the case of quasi-cyclic groups, that in many cases it is not necessary to require that R should be an isomorphism of measure spaces. Thus in the case of the group $\mathbf{Z}$ the natural structure is the order relation, and if in the above scheme we require that R be one-to-one and transform μ to ν, then the ensuing equivalence relation will coincide with metric isomorphism. But if we limit the requirement so that R transforms μ to a measure absolutely continuous relative to ν, then there arises an interesting equivalence relation which we call *monotone equivalence* (see Definition 2.2). This equivalence relation was first introduced more than thirty years ago by Kakutani [9], in connection with describing the different special representations of a flow (see Proposition 2.4). The study of ergodic automorphisms from the viewpoint of monotone equivalence is the main topic of this article. In at least two respects monotone equivalence turns out to be a more visible equivalence relation than metric isomorphism.

First, there is a class of monotonely equivalent automorphisms which includes the majority of the natural examples of automorphisms with zero entropy. Taking account of the analogy with the case of quasi-cyclic groups, we call the automorphisms of this class *standard* (see Definition 2.5). Second, between the classes of monotonely equivalent automorphisms we can define a transitive binary relation of majorization (see Definition 2.4); moreover, the class of standard automorphisms turns out to be the unique one which is majorized by all the classes (Theorem 1).

We note that the analogous definition of monotone equivalence in the case of flows (actions of **R**; see Definition 2.1) appears more natural, since in that case we may take R to be one-to-one and transforming μ to a measure equivalent to ν.(1) However, the construction of a special representation for flows allows us to essentially reduce this case to the case of automorphisms (see Proposition 2.4). Technically, even, the case of automorphisms is much simpler, since in this case many difficulties, connected with the necessity to consider sets of measure zero, are absent.

4. We will systematically use the fact that monotone equivalence for ergodic automorphisms is precisely the strongest equivalence relation which allows metric isomorphism and also the passage to any derived and special automorphism (Proposition 2.4). This idea, due to Kakutani [9], has for a long time not been seriously developed. It is true that from time to time there have appeared papers in which these and other metric invariants of the passage to derived and special automorphisms have been studied. The most important result of this kind is the formula of Abramov [10] for the entropy of a derived automorphism, from which it follows that the property of the entropy of an automorphism being zero, a positive number or infinity is an invariant of monotone equivalence. Other results are of a negative character; they show that automorphisms with various metric properties may belong to any monotone equivalence class. For the property of weak mixing this was proved by Chacon [11], for mixing by Ornstein and Friedman [12], for the property of having a proper function with a given proper value $\lambda \in S^1$ by Hansel [13]. There is a more detailed survey of similar results in [8], Chapter 6. These results show that the traditional metric invariants (except entropy) bear no relation to the study of monotone equivalence.

5. In this paper we start a systematic study of monotone equivalence. We introduce a series of new ideas, the basic one of which is the metric ρ^M (see (4.2) and (4.3)) in the space of finite words of elements of a finite alphabet. With the help of this metric are defined the notions of M-triviality (Definitions 9.2 and 9.3) and invariants of entropy type, and also the d^M-metric for random processes, which has been introduced independently by Sataev [14] and Feldman [37]. The ideas mentioned form the basis for the creation of a new technique of working with automorphisms, adapted for the study of monotone equivalence. In this paper we are engaged in the study of standard automorphisms. As explained in subsection 3, this class has great significance, and any detailed investigation of monotone equivalence must include an analysis of the notion of standardness.

6. The paper consists of 11 sections. The first 4 are of an introductory nature. In §2 we give the basic definitions and establish the equivalence of various forms of these

(1)In addition, in the case of flows there is a natural (and fruitful) analogue of monotone equivalence in the theory of continuous and smooth dynamical systems. This question is discussed in more detail in Chapter 6 of the survey [8].

definitions. In §4 we give the definitions of the metrics ρ^H and ρ^M (ρ^H is the well-known Hamming metric) and list the properties of these metrics which are used later. In §§5–7 the first–approximation–criterion for standardness is proved. In §§8–10 we prove the most important criterion for standardness (Theorem 4), which is that sufficiently long segments of the trajectories of most points must, under coding, transform to words which are close in the metric ρ^M (the property of M-triviality; see Definitions 9.2 and 9.3). If we consider monotone equivalence as the analogue, for actions of $\mathbf{Z}$, of trajectory isomorphism with preservation of filtration for actions of quasi-cyclic groups, then this criterion can be interpreted as the analogue of Veršik's criterion mentioned in subsection 2. In §10 we also give some immediate corollaries of Theorem 4. Finally, §11 is of the nature of a survey. In it we list some less immediate corollaries of the standardness criteria and also mention recent works of other authors and unsolved problems.

The results of this paper were announced in [16] and [17].

The author discussed many questions relating to this work with A. V. Kočergin and E. A. Sataev. These discussions assisted in the simplification of a number of proofs. A summary of lectures given by the author in a seminar at the V. A. Steklov Institute of Mathematics in the Academy of Sciences of the USSR, made by M. I. Brin, greatly facilitated the writing of §§8–10 of this paper. In addition, M. I. Brin made a number of useful remarks of an editorial naturs.

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§1. Notation and necessary information from ergodic theory

1. *Lebesgue spaces and measurable partitions.* All the measure spaces considered here will be assumed to be Lebesgue spaces with a continuous normalized measure. This means, from the viewpoint of abstract measure theory, that these spaces are indistinguishable from the interval [0, 1] with Lebesgue measure. All the necessary information on Lebesgue spaces and their measurable partitions is presented in §1 of Rohlin's article [18]. Therefore we will limit ourselves to notation and the description of certain constructions.

The standard notation for a Lebesgue space will be (X, μ), where X is a set in which there is given a measure, and μ is that measure. We denote by $\mathfrak{A}(X, \mu)$ the σ-algebra of all measurable sets in X; and, for a measurable partition ξ, $\mathfrak{A}(\xi)$ denotes the subalgebra of $\mathfrak{A}(X, \mu)$ consisting of all sets which mod 0 consist of elements of ξ (for an explanation of the term mod 0, see [18]). The quotient space of X by the partition ξ is denoted $X|_\xi$.

If $\xi_1, \xi_2, \ldots$ are measurable partitions, then we denote the product of the first n of these partitions by $\xi_1 \cdot \ldots \cdot \xi_n$ or $\bigvee_1^n \xi_i$, and the product of all the partitions correspondingly by $\xi_1 \cdot \xi_2 \cdot \ldots$ or $\bigvee_1^\infty \xi_n$; ϵ_X, or simply ϵ, denotes the partition of X coinciding mod 0 with the partition into individual points; ν_X, or ν, denotes the trivial partition in which the measure of one of the elements is equal to 1. The notation $\xi \leqslant \eta$ denotes that $\mathfrak{A}(\xi) \subset \mathfrak{A}(\eta)$.

A sequence of measurable partitions $\xi_1, \xi_2, \ldots$ is called *exhaustive* if $\overline{\mathfrak{A}(\xi_n)} = \mathfrak{A}(X, \mu)$, where the closure is taken in the metric ρ: for $A, B \subset \mathfrak{A}(X, \mu)$, $\rho(A, B) = \mu(A \Delta B)$. A subset $\mathfrak{B} \subset \mathfrak{A}(X, \mu)$, dense in this metric, is called a basis of the σ-algebra $\mathfrak{A}(X, \mu)$. The notation $\xi_n \longrightarrow \epsilon$ means that ξ_n is an exhaustive sequence of partitions. If, moreover, $\xi_1 \leqslant \xi_2 \leqslant \ldots$, then we will write $\xi_n \nearrow \epsilon$.

The direct product of spaces, measures and partitions will be denoted by the symbol $\times$.

The space of nonnegative integer-valued integrable functions on X will be denoted by $L^1(X, \mu, \mathbf{Z}_0^+)$. Let $m \in L^1(X, \mu, \mathbf{Z}_0^+)$, $m \not\equiv 0$. We denote

$$X_{m(\cdot)} = \{(x, s) : x \in X, s \in \{1, \ldots, m(x)\}\}. \tag{1.1}$$

Further, let $A \subset X_{m(\cdot)}$, where for any nautral number s the set $A_s = \{x \in X : (x, s) \in A\}$ is measurable, put

$$\mu_{m(\cdot)}(A) = \frac{\sum_{s=1}^{\infty} \mu(A_s)}{\int_X m\,d\mu} \tag{1.2}$$

and denote by $\mathfrak{A}(X_{m(\cdot)}, \mu_{m(\cdot)})$ the set of all such A.

Let $m \in L^1(X, \mu, \mathbf{Z}_0^+)$ and $n \in L^1(X_{m(\cdot)}, \mu_{m(\cdot)}, \mathbf{Z}_0^+)$. For $x \in X$ we denote

$$(m * n)(x) = \sum_{s=1}^{m(x)} n(x, s). \tag{1.3}$$

Obviously $m * n \in L^1(X, \mu, \mathbf{Z}_0^+)$.

We will identify $X \times \{1\}$ with X, and therefore will sometimes speak of intersections $X \cap X_{m(\cdot)}$.

Let ξ be a measurable partition of X. We denote by $\xi_{m(\cdot)}$ the partition of $X_{m(\cdot)}$ into all possible sets c_s, $c \in \xi$, $s = 1, 2, \ldots$.

We will sometimes call a function $m \in L^1(X, \mu, \mathbf{Z}_0^+)$ a *specfunction.*

Let $A \subset \mathfrak{A}(X, \mu)$, $\mu(A) > 0$ and χ_A the characteristic function of A. In this case we identify $X_{\chi_A(\cdot)}$ with A, and instead of $\mu_{\chi_A(\cdot)}$ and $\xi_{\chi_A(\cdot)}$ we will write μ_A and ξ_A.

We will denote the space of almost everywhere positive integrable functions on X by $L^1_+(X, \mu)$. For $\varphi \in L^1_+(X, \mu)$, we denote

$$X^{\varphi} = \{(x, t), x \in X, t \in \mathbf{R}, 0 \leqslant t \leqslant \varphi(x)\}.$$

Further we denote by μ^{φ} the normalized measure induced on X^{φ} by the measure $\mu \times \lambda$, where λ is Lebesgue measure on the line.

Let (X, μ) and (Y, ν) be Lebesgue spaces, and $R: X \longrightarrow Y$ a mapping such that $R^{-1}A \subset \mathfrak{A}(X, \mu)$ for $A \subset \mathfrak{A}(Y, \nu)$. Then we can define a measure $R_*\mu$ on $\mathfrak{A}(Y, \nu)$ by putting

$$R_*\mu(A) = \mu(R^{-1}A).$$

2. *Automorphisms of Lebesgue spaces.* The basic definitions regarding automorphisms can be found in §3 of [18]. We, as a rule, will denote automorphisms of Lebesgue spaces by the symbols T and S with various indices. If T and S are metrically isomorphic we will write $T \sim S$.

A measurable partition ξ is called an *invariant partition* for T if $T\mathfrak{A}(\xi) = \mathfrak{A}(\xi)$. In this case there is a quotient automorphism defined on $X|_{\xi}$ which we denote by $T|_{\xi}$. Let k be a natural number. The partition $\bigvee_{i=0}^{k-1} T^i\xi$ will be denoted ξ_T^k, or simply ξ^k, and the partition $\bigvee_{-\infty}^{\infty} T^i\xi$ will be denoted by ξ_T. A partition ξ is called a *generator* (or *generating*) if $\xi_T = \epsilon$.

To each automorphism T: $(X, \mu) \longrightarrow (X, \mu)$ and specfunction $m \in L^1(X, \mu, \mathbf{Z}_0^+)$ there corresponds an automorphism

$$T_{m(\cdot)} : (X_{m(\cdot)}, \quad \mu_{m(\cdot)}) \to (X_{m(\cdot)}, \mu_{m(\cdot)}),$$

which is defined in the following way:

$$T_{m(\cdot)}(x, s) = \begin{cases} (x, s+1), & \text{if} \quad s < m(x), \\ (T^{i(x)}x, 1), \text{ where } i(x) = \min\{i > 0 : m(T^i x) > 0\}, & \text{if} \quad s = m(x). \end{cases} \quad (1.4)$$

Obviously, for $m \in L^1(X, \mu, \mathbf{Z}_0^+)$ and $n \in L^1(X_{m(\cdot)}, \mu_{m(\cdot)}, \mathbf{Z}_0^+)$ we have $(T_{m(\cdot)})_{n(\cdot)} = T_{m*n(\cdot)}$, where $m*n$ is defined by (1.3). If $m = \chi_A$, where $A \subset \mathfrak{A}(X, \mu)$, then instead of $T_{\chi_A(\cdot)}$ we will write T_A. We may suppose that T_A acts on A and, by virtue of (1.4), $T_A x = T^{i(x)}x$, where $i(x) = \min\{i > 0: T^i x \in A\}$. The automorphism T_A is called the *derived* (or *induced*) *automorphism* of T on the set A.

If $m(x) > 0$ for almost all $x \in X$, then the automorphism $T_{m(\cdot)}$ is called the *special automorphism* over T constructed relative to m.

We quote two classical results of ergodic theory which will be used repeatedly in what follows.

THE HALMOS-ROHLIN LEMMA ON UNIFORM APPROXIMATION (see [1], p. 75). *Let T: $(X, \mu) \longrightarrow (X, \mu)$ be an aperiodic automorphism; that is, the measure of the set of periodic points of T is zero. Then for any natural number n and any $\epsilon > 0$ there is a measurable set $A = A_{n,\epsilon}$ such that $A \cap T^iA = \varnothing$, $i = 1, \ldots, n-1$, and $\mu(\bigcup_{i=0}^{n-1} T^iA) > 1 - \epsilon$.*

THE ERGODIC THEOREM. *Let $A \subset \mathfrak{A}(X, \mu)$, and let T be an automorphism. The sequence of functions*

$$\frac{1}{n}\sum_{i=0}^{n-1} \chi_A(T^i x)$$

converges to some function

a) *in measure,*

b) *almost everywhere.*

Assertion a) is a simple fact and follows, for example, from the statistical ergodic theorem of von Neumann (see [1], p. 16). Assertion b) is one of the forms of Birkhoff's individual ergodic theorem (see [1], p. 18), which has proved to be considerably more complex than the von Neumann theorem. Everywhere in this article, where the ergodic theorem is applied, only the convergence in measure is used. However, for the proof of standardness of group extensions of standard automorphisms (see §11), the almost everywhere convergence is used.

An automorphism T is called *ergodic* if any measurable set invariant relative to T has measure 0 or 1.

3. *Automorphisms with discrete spectrum.* For $f \in L^2(X, \mu)$ we denote

$$(U_T f)(x) = f(Tx).$$

Obviously U_T is a unitary operator on $L^2(X, \mu)$. All the spectral characteristics of U_T are ascribed to T; that is, we speak of proper functions, proper values, spectral types of automorphisms, of automorphisms with discrete spectrum, etc.

We denote by $PV(T)$ the set of proper values of T. If T is ergodic, then $PV(T)$ is a subgroup of the circle $\mathbf{S}^1 = \{\lambda \in \mathbf{C}, |\lambda| = 1\}$. Let Λ be a subgroup of $PV(T)$. We denote by $G(\Lambda)$ the group of all proper functions, of modulus 1, with proper values in Λ. The linear hull of $G(\Lambda)$ consists of all functions constant on the elements of a certain partition, which we denote by $\eta(\Lambda)$. If

$$\Lambda = \mathbf{Z}_q = \left\{\exp \frac{2\pi i p}{q}, p \in \mathbf{Z}\right\},$$

then

$$\eta(\Lambda) = \{\Delta_1, \ldots, \Delta_{q-1}, \Delta_q = \Delta_0\}, \quad T\Delta_i = \Delta_{i+1}, \quad i = 0, 1, \ldots, q-1.$$

VON NEUMANN'S THEOREM on automorphisms with discrete spectrum (see [1], pp. 46–50). *Let Γ be a countable subgroup of $\mathbf{S}^1$. There is a unique, up to metric isomorphism, ergodic automorphism T with discrete spectrum for which $PV(T) = \Gamma$. As such an automorphism we may take a transformation of the character group Γ^* of the discrete group Γ, preserving Haar measure, and consisting of multiplication of each character $g \in \Gamma^*$ by the character i_Γ, where $i_\Gamma(\gamma) = \gamma, \gamma \in \Gamma$.*

Let $\hat{\mathbf{Q}}$ be the group of all roots of 1; that is,

$$\hat{\mathbf{Q}} = \left\{\exp \frac{2\pi i p}{q}, p, q \in \mathbf{Z}\right\}.$$

An ergodic automorphism with discrete spectrum for which $PV(T) \subset \hat{\mathbf{Q}}$ will be called an *automorphism with rational spectrum*. Since any infinite subgroup $\Gamma \subset \hat{\mathbf{Q}}$ has the form $\Gamma = \bigcup_1^\infty \mathbf{Z}_{q_n}$, where $q_n = r_1 \cdot \ldots \cdot r_n$ (such a representation is clearly not unique), it follows that the group Γ^*, which we denote by $\mathbf{Z}_{\{r_n\}}$, is an inverse (projective) limit of groups $\mathbf{Z}^*_{q_n} \sim \mathbf{Z}_{q_n}$. In particular, the group $\mathbf{Z}_{\{r\}}$ corresponding to the sequence $r_n \equiv r$ is the multiplicative group of r-adic integers.

We denote $\mathbf{D}_{\{r_n\}}: \mathbf{Z}_{\{r_n\}} \longrightarrow \mathbf{Z}_{\{r_n\}}$, where $\mathbf{D}_{\{r_n\}} g = i_{\cup \mathbf{Z}_{q_n}} \cdot g$, and let $\pi_n: \mathbf{Z}_{\{r_n\}} \longrightarrow \mathbf{Z}^*_{q_n}$ be the natural projection. Obviously the partition $\eta_n = \pi_n^{-1}(\epsilon)$ coincides with $\eta(\mathbf{Z}_{q_n})$; moreover, the sequence of partitions $\{\eta_n\}$, $n = 1, 2, \ldots$, is increasing and exhaustive.

PROPOSITION 1.1. *Let $T: (X, \mu) \longrightarrow (X, \mu)$ be an ergodic automorphism. Suppose that $\{\eta_n\}$, $n = 1, 2, \ldots$, is an increasing exhaustive sequence of invariant partitions for T, where η_n consists of $q_n = r_1 \cdot \ldots \cdot r_n$ elements of positive measure. Then $T \sim \mathbf{D}_{\{r_n\}}$ and $\eta_n = \eta(\mathbf{Z}_{q_n})$.*

PROOF. From the ergodicity of T and the invariance of η_n it follows that T cyclically permutes the elements of η_n, and consequently all these elements have the same measure. Therefore the proper values of U_T, in the invariant subspace of functions constant on the elements of η_n, form the group $\mathbf{Z}_{q_n}$. Since the sequence η_n is exhaustive, T has discrete spectrum and $PV(T) = \bigcup_1^\infty \mathbf{Z}_{q_n}$. By von Neumann's theorem on discrete spectrum, $T \sim \mathbf{D}_{\{r_n\}}$.

We note that the proof of the proposition could be completed without reference to von Neumann's theorem, since, in the case of rational spectrum, metric isomorphism can be quite simply deduced from spectral isomorphism.

We denote by $\mathbf{D}$ the automorphism with rational spectrum for which $PV(\mathbf{D}) = \hat{\mathbf{Q}}$. The character group of $\hat{\mathbf{Q}}$ is called the *group of integer ideals*. We will denote the group by $\hat{\mathbf{Z}}$. It is easy to show that $\hat{\mathbf{Z}} = \hat{\mathbf{Z}}_{(n,\ )} = \mathbf{Z}_{\{p_n\}}$, where p_n is the product of the first n primes. It is indeed the latter representation we will have in mind later when we consider the partitions η_n for $\mathbf{D}$.

4. *Flows and special representations*. For the basic definitions and results concerning flows, see [19], §1. By flow we will always mean measurable flow on a Lebesgue space. Flows will as a rule be denoted by the symbols $\{T_t\}$ or $\{S_t\}$, with various additional indices; metric isomorphism of flows $\{T_t\}$ and $\{S_t\}$ will be denoted $\{T_t\} \sim \{S_t\}$. Let T: $(X, \mu) \longrightarrow (X, \mu)$ be an automorphism and $\varphi \in L^1_+(X, \mu)$. In the Lebesgue space (X^φ, μ^φ) there is defined a flow called the *special flow constructed relative to T and φ* (see [19]). A flow $\{T_t\}$ is called *ergodic* if any set invariant mod 0 relative to each automorphism T_t, $t \in \mathbf{R}$, has measure 0 or 1. The theorem on special representations for ergodic flows asserts that any such flow is metrically isomorphic to a special flow constructed relative to some ergodic automorphism T and some function φ.

In this article we barely touch on the case of flows. Using Proposition 2.4, the reader can easily deduce corollaries, relating to flows, of the results on automorphisms proved in this paper, even if this is not done in the text.

§2. Monotone equivalence

1. Definition 2.1. Flows $\{T_t\}$ and $\{S_t\}$, acting on Lebesgue spaces (X_1, μ_1) and (X_2, μ_2) respectively, are called *monotonely equivalent* if there is a one-to-one mod 0 measurable mapping $R: X_1 \longrightarrow X_2$ with the following properties:

2.1.1. The measure $R_*\mu_1$ is equivalent to μ_2.

2.1.2. For almost all $x \in X_1$ and all $t \in \mathbf{R}$

$$RT_t x = S_{\varphi(t,x)} Rx$$

where $\varphi(t, x)$ is a monotone increasing function of t.

If flows $\{S_t\}$ and $\{T_t\}$ are monotonely equivalent, we will write $\{S_t\} \overset{M}{\sim} \{T_t\}$.

Property 2.1.2 means that the trajectories of $\{T_t\}$ are mapped onto the trajectories of $\{S_t\}$ with preservation of the natural order relation on the trajectories. Since $T_{t_1+t_2} = T_{t_1} \cdot T_{t_2}$, for all $t_1, t_2 \in \mathbf{R}$ and almost all $x \in X_1$, $\varphi(t, x)$ satisfies

$$\varphi(t_1 + t_2, x) = \varphi(t_1, x) + \varphi(t_2, T_{t_1} x). \tag{2.1}$$

For $y \in X_2$ and $t \in \mathbf{R}$ we denote $\psi(t, y) = \varphi(t, Rx)$. Then the flow $\{\hat{S}_t\} = \{RT_tR^{-1}\}$ on X_2, preserving $R_*\mu_1$ and metrically isomorphic to $\{T_t\}$, has the same trajectories as $\{S_t\}$ and can be written in the form $\hat{S}_t y = S_{\psi(t,y)} y$. It follows from (2.1) that, for almost all y, ψ satisfies

$$\psi(t_1 + t_2, y) = \psi(t_1, y) + \psi(t_2, S_{\psi(t_1,y)} y). \tag{2.2}$$

It can be shown that ψ for almost all y (which means also φ for almost all x) is absolutely continuous relative to t. Therefore we say that $\{\hat{S}_t\}$ is obtained from $\{S_t\}$ by an absolutely continuous time change. Therefore Definition 2.1. may be formulated as follows:

Flows $\{T_t\}$ and $\{S_t\}$ are monotonely equivalent if $\{T_t\}$ is metrically isomorphic to a flow which is obtained from $\{S_t\}$ by an absolutely continuous time change.

The following assertion obviously follows from Definition 2.1.

PROPOSITION 2.1. *The relation of monotone equivalence is reflexive, symmetric and transitive.*

PROPOSITION 2.2. *Special flows $\{T_t^{\varphi_1}\}$ and $\{T_t^{\varphi_2}\}$ over the same automorphism are monotonely equivalent.*

PROOF. Let $(y, s) \in X^{\varphi_1}$. Put $R(y, s) = (y, s\varphi_2(y)/\varphi_1(y))$. Obviously $RX^{\varphi_1} = X^{\varphi_2}$, $R_*\mu^{\varphi_1} = (\varphi_1/\varphi_2)\mu^{\varphi_2}$, and condition 2.1.2 is satisfied. Moreover, φ is piecewise smooth relative to t.

The partition into ergodic components is preserved under monotone equivalence. In order to avoid minor complications, due to the fact that nonergodic flows cannot always be represented in the form of a special flow over an automorphism of a space with finite measure, we will in what follows restrict ourselves to the case of ergodic flows.

PROPOSITION 2.3. *Monotonely equivalent ergodic flows are metrically isomorphic to special flows over the same automorphism. If the ergodic flows $\{T_t\}$ and $\{S_t\}$ are monotonely equivalent, then the mapping R may be chosen so that, for almost all $x_0 \in X$, the restriction $\varphi(x, t)|_{\{x_0\}\times\mathbf{R}}$ gives a diffeomorphism $\mathbf{R} \longrightarrow \mathbf{R}$.*

PROOF. 1°. By the theorem on special representations $\{S_t\}$ is isomorphic to a special flow $\{T_t^\psi\}$ over some automorphism T. Then $\{T_t\} \overset{M}{\sim} \{T_t^\psi\}$. We consider the measurable, relative to μ^ψ and $R_*\mu_1$ (R is the mapping of X_1 to X^ψ which establishes the isomorphism of $\{S_t\}$ and $\{T_t^\psi\}$), partition ξ_ψ of X^ψ into segments $c_x^\psi = \{x\} \times [0, \psi(x)]$, $x \in X$. By property 2.1.2 almost every segment c_x^ψ is a segment of a trajectory of $\{\hat{T}_t\} = \{RT_tR^{-1}\}$; moreover, c_{Tx}^ψ immediately follows c_x^ψ in the trajectory. The length $\tau(x)$ of a segment c_x^ψ of a trajectory of $\{\hat{T}_t\}$ is defined by the equation $\varphi(\tau(x), R^{-1}x) = \psi(x)$, and therefore is a measurable function. We define a mapping U: $X^\psi \longrightarrow X^\tau$, putting $U(x, s) = (x, \theta(x, s))$, where $\theta(x, s)$ is the solution of $\varphi(\theta(x, s), R^{-1}x) = s$, $0 \leqslant s < \varphi(x)$. Obviously $U\hat{T}_t = T_t^\tau U$, and consequently $\{URT_t(UR)^{-1}\} = \{T_t^\tau\}$. It remains only to prove that $(UR)_*\mu_1 = \mu^\tau$. By the ergodicity of $\{T_t\}$ and $\{T_t^\tau\}$ it is sufficient to prove the equivalence of these measures. The measures induced by these measures in $X^\tau|_{\xi_\tau}$ are equivalent by 2.1.1 and the definition of U. The conditional measures of both measures on each segment c_x^τ have the form $ds/\tau(x)$, since both measures are invariant relative to $\{T_t^\tau\}$. Thus the first part of the proposition is proved.

2°. It is obviously sufficient to prove the second part of the proposition for special flows $\{T_t^{\varphi_1}\}$ and $\{T_t^{\varphi_2}\}$ over an ergodic automorphism T. We may suppose that $\varphi_1, \varphi_2 \geqslant C > 0$ almost everywhere, since otherwise $\{T_t^{\varphi_1}\}$ and $\{T_t^{\varphi_2}\}$ may be represented as special flows over the induced automorphism T_{A_C}, where $A_C = \{x \in X\colon \varphi_1(x) \geqslant C, \varphi_2(x) \geqslant C\}$, and C is chosen so that $\mu(A_C) > 0$.

Let $\rho(t, \alpha, \beta)$ be a C^∞-function defined for $t, \alpha, \beta \in \mathbf{R}$, $\alpha \geqslant C$, $\beta \geqslant C$ and $0 \leqslant t \leqslant \alpha$, and having the following properties:

1. $\rho(0, \alpha, \beta) = 0$.
2. $\rho(\alpha, \alpha, \beta) = \beta$.
3. $\dfrac{\partial\rho(t, \alpha, \beta)}{\partial t} > \dfrac{2\beta - C}{200\alpha - 100C}$.
4. $\partial\rho(t, \alpha, \beta)/\partial t = 1$ for $0 \leqslant t \leqslant C/4$ and $\alpha - C/4 \leqslant t \leqslant \alpha$.

It is not difficult to explicitly construct such a function. Fix α and β, and denote by $\kappa_{\alpha,\beta}(t)$ the function inverse to $\rho(t, \alpha, \beta)$. For $(x, s) \in X^{\varphi_1}$ put

$$R(x, s) = (x, \rho(s, \varphi_1(x), \varphi_2(x))).$$

By properties 1, 2 and 3, R is a one-to-one mapping of X^{φ_1} onto X^{φ_2}; moreover,

$$R_*\mu^{\varphi_1} = \frac{\partial\varkappa_{\varphi_1(x),\varphi_2(x)}(s)}{\partial s}\,\mu^{\varphi_2}.$$

By properties 3 and 4, each trajectory of $\{T_t^{\varphi_1}\}$ is diffeomorphically mapped onto a trajectory of $\{T_t^{\varphi_2}\}$.

2. DEFINITION 2.2. Ergodic automorphisms T and S of Lebesgue spaces (X_1, μ_1) and (X_2, μ_2), respectively, are called *monotonely equivalent* if there exists a measurable mapping $R\colon X_1 \longrightarrow X_2$ with the following properties:

2.2.1. For almost all $x \in X_2$, the inverse image $R^{-1}x$ consists of not more than a finite number of points.

2.2.2. The measure $R_*\mu_1$ is absolutely continuous relative to μ_2.

2.2.3. For almost all $x \in X_1$, $RT(x) = S^{n(x)}Rx$, where $n(x)$ is a nonnegative measurable function.

As in the case of flows, monotone equivalence of T and S will be denoted $T \overset{M}{\sim} S$. A property of automorphisms or flows will be called *monotonely invariant*, or a *monotone invariant*, if all monotonely equivalent automorphisms or flows either simultaneously possess the property or do not possess it.

Properties 2.2.1 and 2.2.3 mean that the trajectories of T are mapped to the trajectories of S with preservation of order in the trajectory; moreover, "gluing", when a finite segment of the trajectory of T is transformed to one point, and "gaps", when nothing is mapped onto some segment of the trajectory of S, are possible.

PROPOSITION 2.4. *Let T and S be ergodic automorphisms. The following three assertions are equivalent*:

1. $T \overset{M}{\sim} S$.
2. *There is a function $m \in L^1(X_2, \mu_2, \mathbf{Z}_0^+)$ such that $T \sim S_{m(\cdot)}$.*
3. *There are positive functions $\varphi_i \in L^1(X_i, \mu_i)$, $i = 1, 2$, such that $\{T_t^{\varphi_1}\} \sim \{S_t^{\varphi_2}\}$.*(2)

PROOF. We derive 2 from 1. Let $T \overset{M}{\sim} S$. For $y \in X_2$ we put $m(y) = |R^{-1}y|$, and we shall prove that $T \sim S_{m(\cdot)}$. Indeed, we define a mapping $\hat{R}\colon X_1 \longrightarrow (X_2)_{m(\cdot)}$ by putting for $x \in X_1$

(2)The equivalence of 2 and 3 was first proved by Kakutani [9].

$$\hat{R}x = (Rx, \max\{i : RT^{-i+1}x = Rx\}).$$

It follows from the definition of $m(y)$ that $\hat{R}$ is a one-to-one mod 0 mapping. From 2.2.3 and (1.4) it follows that $S_{m(\cdot)}\hat{R} = \hat{R}T$. Finally from 2.2.2 and the ergodicity of S it follows that $\hat{R}_*\mu_1 = (\mu_2)_{m(\cdot)}$.

We derive 1 from 2. For this it is sufficient to show that $S_{m(\cdot)} \overset{M}{\sim} S$ for S: $(X, \mu) \longrightarrow (X, \mu)$. For $(x, s) \in X_{m(\cdot)}$ we put $R(x, s) = x$. Condition 2.2.1 follows from Poincaré's recurrence theorem. Condition 2.2.2 is obvious from the fact that $m \in L^1(X, \mu)$; finally, condition 2.2.3 follows from the definition of R.

We derive 3 from 2. If $m(x) \geqslant 1$, then obviously

$$\{S_t^{m(\cdot)}\} \sim \{(S_{m(\cdot)})_t^1\}.$$

The general case reduces to this, since the automorphisms S and $S_{m(\cdot)}$ are special over a common derived automorphism $S_{X\setminus m^{-1}(0)}$.

Finally, let 3 be satisfied. By Propositions 2.2 and 2.3, any special flow over T is isomorphic to some special flow over S; in particular, $\{T_t^1\} \sim \{S_t^\psi\}$.

Let U: $X_1^1 \longrightarrow X_2^\psi$ and $UT_t^1 = S_t^\psi U$, $t \in \mathbf{R}$. For almost all $s \in [0, 1]$ the image $A_s \in X_2^\psi$ of the segment $\{s\} \times X_1$ is well defined, and for almost all $y \in X_2$ the number $m(y)$ of points of intersection of A_s with c_y^ψ is well defined. Obviously $m(y)$ is a measurable function and $m(y) \leqslant \psi(y) + 1$; therefore $m \in L^1(X_2, \mu_2, \mathbf{Z}_0^+)$. Let

$$A_s \cap c_y^\psi = \{(y, \rho_1(y)), \ldots, (y, \rho_{m(y)}(y))\},$$

where $\rho_1(y) < \cdots < \rho_{m(y)}(y)$. Obviously, $\rho_i(y)$ ($i = 1, \ldots, m(y)$) are measurable functions defined on measurable subsets of X_2. On the set A_s there is a successor mapping, induced by $\{S_t^\psi\}$ and preserving the induced measure. This mapping is isomorphic both to T (by the well-defined restriction $U^{-1}|_{A_s}$), and to $S_{m(\cdot)}$ by the mapping V: $A_s \longrightarrow X_{m(\cdot)}$, where $V(y, \rho_i(y)) = (y, i)$. The proposition is proved.

COROLLARY 2.1. *The relation of monotone equivalence is reflexive, symmetric and transitive.*

Our basic working criterion for monotone equivalence will be property 2. Property 3 is very useful for deducing corollaries for flows from results about automorphisms. However, both these properties are attached to a specific one-dimensional situation (actions of $\mathbf{Z}$ or $\mathbf{R}$) and cannot be immediately carried over to the case of more general group actions (for example $\mathbf{Z}^m$ or $\mathbf{R}^m$). On the other hand, property 1 can be generalized. We limit ourselves to the case of $\mathbf{Z}^m$ (lattices) and give the corresponding definition, which, in particular, allows the introduction in this case of the notions of "derived" and "special" actions.

DEFINITION 2.3. Let $\{T_1^{k_1} \cdot \ldots \cdot T_m^{k_m}\}$ and $\{S_1^{k_1} \cdot \ldots \cdot S_m^{k_m}\}$, $(k_1, \ldots, k_m) \in \mathbf{Z}^m$, be two ergodic actions of $\mathbf{Z}^m$ by automorphisms of Lebesgue spaces (X_1, μ_1) and (X_2, μ_2) respectively. These actions are called *monotonely equivalent* if there exists a measurable mapping R: $X_1 \longrightarrow X_2$ satisfying conditions 2.2.1, 2.2.2 and

2.3.1. For almost all $x \in X_1$, $R(T_ix) = S_1^{n_1^i(x)} \cdots S_m^{n_m^i(x)}Rx$, where the $n_j^i(x)$, $i, j = 1, \ldots, m$, are nonnegative measurable functions.

If we do not make a distinction between metrically isomorphic actions, then we can say that the action $\{S_1^{k_1} \cdots S_m^{k_m}\}$ is derived from the action $\{T_1^{k_1} \cdots T_m^{k_m}\}$ (respectively, special over this action), if mod 0 we have $RX_1 = X_2$ (respectively, mod 0, $R^{-1}\epsilon_{X_2} = \epsilon_{X_1}$).

Whereas, in condition 2.2.3, $n(x)$ may be an arbitrary measurable function with given nonnegative values, the functions $n_j^i(x)$ in 2.3.1 are connected by the following relations which follow from the commutativity of the automorphisms T_i:

$$n_k^i(T_j x) - n_k^i(x) = n_k^j(T_i x) - n_k^j(x),$$
$$i, j, k = 1, 2, \ldots, m.$$

We will not stop to give any more details on the actions of $\mathbf{Z}^m$ in this paper.

3. Between classes of monotonely equivalent automorphisms we can introduce a binary relation which, in a sense, compares the "complexity" of automorphisms.

DEFINITION 2.4. An automorphism T *majorizes* an automorphism S if there is an automorphism $T_1 \overset{M}{\approx} T$ having an invariant measurable partition ξ such that $T_1|_\xi \overset{M}{\approx} S$. In this case we will write $T \succ S$. Obviously if $T_1 \overset{M}{\approx} T_2$, $S_1 \overset{M}{\approx} S_2$ and $T_1 \succ S_1$, then $T_2 \succ S_2$.

PROPOSITION 2.5. *If $T \succ S$, then there exists an automorphism $T_2 \overset{M}{\approx} T$ with an invariant measurable partition η such that $T_2|_\eta \sim S$.*

PROOF. Let T_1 and S act in spaces X_1 and X_2, respectively. We have the following diagram:

$$\begin{tikzcd}
X_1 \arrow[r, "T_1"] \arrow[d, "\pi"'] \arrow[rrrdd, dashed, "\hat{k}"] & X_1 \arrow[d, "\pi"] & & & \\
X_1|_\xi \arrow[r, "T_1|_\xi"] \arrow[dr, "R"'] & X_1|_\xi \arrow[dr, "R"'] & & & \\
& (X_2)_{m(\cdot)} \arrow[r, "S_{m(\cdot)}"] & (X_2)_{m(\cdot)} \arrow[r, "k"] & \mathbf{Z}_0^+ &
\end{tikzcd}$$

Here $S = (S_{m(\cdot)})_{k(\cdot)}$. Put $\hat{k} = k \cdot R \cdot \pi$. Obviously $(T_1)_{\hat{k}(\cdot)}$ has a quotient automorphism isomorphic to $(S_{m(\cdot)})_{k(\cdot)} = S$.

The relation $\succ$ is transitive, but unfortunately it is unknown whether it is a partial order relation on monotone equivalence classes. If this is so, then it will follow that weakly isomorphic (in the sense of Sinaĭ [20]) automorphisms are monotonely equivalent.

PROPOSITION 2.6. *Any ergodic automorphism T majorizes any automorphism $\mathbf{D}_{\{r_n\}}$ (see §1.3).*

PROOF. We will apply the Halmos-Rohlin Lemma (see §1.2) inductively. First we construct a set A_1 such that

$$T^i A_1 \cap A_1 = \varnothing, \quad i = 1, \ldots, r_1 - 1, \quad \mu(A_1) > \frac{1 - \frac{1}{4}}{r_1},$$

and put

$$B_1 = \bigcup_{i=0}^{r_1-1} T^i A_1.$$

Then we apply the Halmos-Rohlin lemma to T_{A_1}, to construct a set $A_2 \subset A_1$ such that

$$T^i_{A_1} A_2 \cap A_2 = \varnothing, \quad i = 1, \ldots, r_1 r_2 - 1, \quad \mu(A_2) \geqslant \frac{1 - \frac{1}{4} - \frac{1}{16}}{r_1 r_2},$$

and put

$$B_2 = \bigcup_{i=0}^{r_1 r_2 - 1} T^i A_2$$

etc. In the limit we obtain a set $B = \bigcap_1^\infty B_n$, $\mu(B) \geqslant \frac{1}{2}$, and an increasing sequence of invariant, relative to T_B, partitions $\eta_k = \{T^i A_k \cap B, i = 0, 1, \ldots, r_1 \cdot \ldots \cdot r_k - 1\}$. We denote $\eta = \lim \eta_k$. By Proposition 1.1,

$$T_B|_\eta \sim \mathbf{D}_{\{r_n\}}.$$

4. Definition 2.5. An automorphism T is called *standard* if $T \overset{M}{\sim} \mathbf{D}$. A flow $\{T_t\}$ is called *standard* if $\{T_t\} \overset{M}{\sim} \{\mathbf{D}^1_t\}$.

The standard automorphisms are the basic objects of study in this paper. The following theorem shows, from the viewpoint of "complexity" presented in the previous subsection, that the standard automorphisms, and only they, are very simple.

Theorem 1. *In order that an automorphism S be standard it is necessary and sufficient that any ergodic automorphism majorizes S.*

The proof rests upon the following assertion, which follows from Theorem 4 (see §10).

Corollary 10.1. *A quotient automorphism of a standard automorphism relative to any infinite invariant measurable partition is a standard automorphism.*

Proof of the Theorem. By Proposition 2.6 any ergodic automorphism majorizes any standard automorphism. Now let S be majorized by any ergodic automorphism. Then, in particular, $\mathbf{D} \succ S$, and by Proposition 2.5 there is a standard automorphism T such that S is metrically isomorphic to a quotient automorphism of T. But then S is standard by the corollary of Theorem 10.1.

Definition 2.4, Propositions 2.5 and 2.6, and Theorem 1 carry over in a natural and rather obvious way to the case of flows.

§3. Homological equations

1. There are important sufficient conditions for metric isomorphism for monotonely equivalent automorphisms and flows. These conditions correspond to the case when the isomorphism preserves the trajectories of the automorphism on the base.

Proposition 3.1. *Let T: $(X, \mu) \longrightarrow (X, \mu)$ be an automorphism (not necessarily ergodic), and let $m_1, m_2 \in L^1(X, \mu, \mathbf{Z}_0^+)$. If there is an integer-valued measurable function h*

such that

$$m_2(x) - m_1(x) = h(Tx) - h(x), \tag{3.1}$$

then $T_{m_1(\cdot)} \sim T_{m_2(\cdot)}$.

PROOF. Let $x \in X$ and $m_1(x) > 0$. We denote

$$n_1(x) = \min\{n > 0 : m_1(T^n x) \neq 0\}. \tag{3.2}$$

It is obvious that almost everywhere $n_1(x) < \infty$. Further, let

$$p(x) = T_{m_1(\cdot)}^{h(x)}(x, 1).$$

By (3.2),

$$(T^{n_1(x)}x, 1) = T_{m_1(\cdot)}^{m_1(x)}(x, 1),$$

and from (3.2) and (3.1) it follows that

$$m_1(x) + h(T^{n_1(x)}x) = \sum_{i=0}^{n_1(x)-1} m_2(T^i x) + h(x).$$

Thus

$$p(T^{n_1(x)}x) = T_{m_1(\cdot)}^{\sum_{i=0}^{n_1(x)-1} m_2(T^i x)} p(x). \tag{3.3}$$

We denote by $J(x)$ the segment of the trajectory of $T_{m_1(\cdot)}$ from $p(x)$ to $T_{m_1(\cdot)}^{-1} p(T^{n_1(x)}x)$. It follows from (3.3) that the sets $J(x)$ are pairwise disjoint and cover almost all $X_{m_1(\cdot)}$, and that the partition of $X_{m_1(\cdot)}$ into these sets is measurable.

We denote

$$n_2(x) = \min\{n \geqslant 0 : m_2(T^n x) \neq 0\}$$

and define a mapping U: $X_{m_1(\cdot)} \longrightarrow X_{m_2(\cdot)}$ by putting for $x \in X$ in the set $J(x)$

$$UT_{m_1(\cdot)}^j p(x) = T_{m_2(\cdot)}^j (T^{n_2(x)}x, 1),$$
$$j = 0, \ldots, \left(\sum_{j=0}^{n_1(x)-1} m_2(T^i x)\right) - 1. \tag{3.4}$$

It follows from (3.3) and (3.4) that $UT_{m_1(\cdot)} = T_{m_2(\cdot)}U$. The measurability of U is obvious from the fact that all the functions in the definition of $J(x)$ and in (3.4) are measurable. Finally, it is not difficult to show that $U_* \mu_{m_1(\cdot)} = \mu_{m_2(\cdot)}$.

PROPOSITION 3.2. *Let T: $(X, \mu) \longrightarrow (X, \mu)$ be an automorphism, and let $\varphi_1, \varphi_2 \in L_+^1(X, \mu)$. If there exists a real measurable function h such that*

$$\varphi_2(x) - \varphi_1(x) = h(Tx) - h(x), \tag{3.5}$$

then $\{T_t^{\varphi_1}\} \sim \{T_t^{\varphi_2}\}$.

PROOF. Since this fact is well known we will restrict ourselves to the construction of a mapping U: $X^{\varphi_1} \longrightarrow X^{\varphi_2}$ establishing an isomorphism of the special flows. Namely, for $x \in X$ put

$$J(x) = \bigcup_{t=0}^{\varphi_2(x)} \{T^{\varphi_1}_{h(x)+t}(x, 0)\}.$$

By virtue of (3.5), the end of $J(x)$ coincides with the beginning of $J(Tx)$. Put

$$UT^{\varphi}_{h(x)+t}(x, 0) = (x, t), \quad x \in X, \quad 0 \leqslant t \leqslant \varphi_2(x).$$

Obviously $UT^{\varphi_1} = T^{\varphi_2}U$. The verification of one-to-oneness, measurability and preservation of measures for U we leave to the reader.

If for functions $m_1, m_2 \in L^1(X, \mu, \mathbf{Z}_0^+)$ (respectively $\varphi_1, \varphi_2 \in L^1_+(X, \mu)$) the equality (3.1) is satisfied for some integer-valued (respectively (3.5) with some real-valued) measurable function h, then we will say that m_1 and m_2 are **Z**-*homologous* (respectively, φ_1 and φ_2 are **R**-*homologous*) over T.

Propositions 3.1 and 3.2 play an important role in the study of monotone equivalence, since they show how, without loss of generality, one can replace a specfunction by another which is simpler or, in some sense, more convenient.

A. V. Kočergin, at the request of the author, has proved a series of assertions on equations of the form (3.1) and (3.5), from which follows the possibility of certain changes of such a kind. We quote some of his results (see [21]).

PROPOSITION 3.3. *Let $m_1 \in L^1(X, \mu, \mathbf{Z}_0^+)$ and $\beta = \int_X m_1\, d\mu$, and let T: $(X, \mu) \longrightarrow (X, \mu)$ be an automorphism.*

1) *If β is not an integer, then m_1 is* **Z**-*homologous over T to a function taking the values $[\beta]$ and $[\beta] + 1$.*

2) *If β is an integer, then for any $\epsilon > 0$ there is a function m_2 taking the values $\beta - 1, \beta, \beta + 1$, and such that $\|m_2 - \beta\|_{L_1} < \epsilon$, which is* **Z**-*homologous over T to m_1.*

3) *If $m \in L^1(X, \mu, \mathbf{Z}_0^+)$, $A \subset X$ and $\int_A m\, d\mu < \beta$, then m_1 is* **Z**-*homologous over T to a bounded function m_2, coinciding with m on A.*

PROPOSITION 3.4. *Let $\varphi_1 \in L^1_+(X, \mu)$ and $\beta = \int_X m_1\, d\mu$, and let T: $(X, \mu) \longrightarrow (X, \mu)$ be an automorphism.*

1) *For any $\epsilon > 0$, φ_1 is* **R**-*homologous over T to some function φ_2 such that*

$$\operatorname{ess\,sup}_X |\varphi_2 - \beta| < \varepsilon.$$

2) *If $\varphi \in L^1(X, \mu)$, $A \subset X$ and $\int_A \varphi\, d\mu < \beta$, then φ_1 is* **R**-*homologous over T to some function φ_2, bounded outside A, and coinciding with φ on A.*

3) *If X is a compact metric space, μ Borel, A closed and φ continuous on A, then φ_2 in* 1) *and* 2) *can be chosen to be continuous.*

DEFINITION 3.1. Let T and S be ergodic automorphisms, and let $\beta > 0$. The automorphism S is *β-monotonely connected* with T if $S \sim T_{m(\cdot)}$, where $\int m\, d\mu = \beta$.

Similarly, an ergodic flow $\{S_t\}$ is *β-monotonely connected* with $\{T_t\}$ if $\{S_t\} \sim \{T^{\varphi}_t\}$ and $\{T_t\} \sim \{T^{\psi}_t\}$, where $\int \varphi = \beta \int \psi$.

Propositions 3.1 and 3.3 imply the following:

COROLLARY 3.1. *Let S be β-monotonely connected with T.*

1. *If $\beta < 1$, then S is metrically isomorphic to a derived automorphism of T on some set of measure β.*

2. *If $\beta > 1$, then S is metrically isomorphic to a special automorphism over T, where the specfunction m takes the values $[\beta]$ and $[\beta] + 1$ if β is not an integer, and $\beta - 1$, β and $\beta + 1$ if β is an integer; and $\int m\,d\mu = \beta$.*

3. *If $\beta = 1$, then for any $\epsilon > 0$ there is a function m such that $S \sim T_{m(\cdot)}$, where m takes the values $0, 1, 2$ and $\|m - 1\|_{L^1} < \epsilon$.*

§4. Metrics in sequence spaces

1. Let N and n be natural numbers. We denote

$$\Omega_{N,n} = \{(\omega_0, \ldots, \omega_{n-1}) : \omega_i \in \{1, 2, \ldots, N\},\ i = 0, 1, \ldots, N-1\},$$
$$\Omega_N = \{(\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) : \omega_i \in \{1, \ldots, N\},\ i \in \mathbf{Z}\},$$

$\pi_n \colon \Omega_N \longrightarrow \Omega_{N,n}$ is the natural projection, $\pi_n(\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) = (\omega_0, \ldots, \omega_{n-1})$. For $\omega \in \Omega_{N,n}$ and $j = 1, \ldots, N$, put

$$n_j(\omega) = \sum_{j=0}^{n-1} \delta_{j\omega_j},$$

where δ_{kl} is the Kronecker delta.

We define two metrics, ρ^H and ρ^M, in the spaces $\Omega_{N,n}$ which (particularly the latter) will play an important role later. Let $\omega = (\omega_0, \ldots, \omega_{n-1})$ and $\bar\omega = (\bar\omega_0, \ldots, \bar\omega_{n-1})$, $\omega, \bar\omega \in \Omega_{N,n}$. Put

$$\rho^H(\omega, \bar\omega) = \frac{1}{n} \sum_{i=0}^{n-1} (1 - \delta_{\omega_i \bar\omega_i}). \tag{4.1}$$

In other words, $\rho^H(\omega, \bar\omega)$ is equal to the frequency of noncoincident coordinates in ω and $\bar\omega$. Formula (4.1) gives a metric on $\Omega_{N,n}$. This is known in probability and information theory as the Hamming metric, and has also been widely used in ergodic theory (see, for example, [6] and [22]).

For $\omega \in \Omega_{N,n}$ we denote by $\mathfrak{B}(\omega)$ the subset of $\Omega_{N,n-1} \cup \Omega_{N,n+1}$ consisting of those elements of $\Omega_{N,n-1}$ which can be obtained from ω by crossing out a coordinate, and those elements of $\Omega_{N,n+1}$ which can be obtained from ω by adding one of the symbols $1, \ldots, N$, in any position (front, middle or end). For $\omega \in \Omega_{N,n}$ and $\bar\omega \in \Omega_{N,m}$, put

$$O^M(\omega, \bar\omega) = \min\{k : \text{ there exist } \omega^{(0)}, \ldots, \omega^{(k)} \in \bigcup_{l=1}^{\infty} \Omega_{N,l}\}. \tag{4.2}$$

In other words, if we call the *deletion or insertion of a symbol in* ω an elementary M-operation, then $O^M(\omega, \bar\omega)$ is equal to the minimal number of sequences of elementary M-operations which it is necessary to apply in order to obtain ω from $\bar\omega$. Obviously (4.2) gives a metric on $\bigcup_{n=1}^{\infty} \Omega_{N,n}$. Normalizing this metric, we obtain a convenient metric on each $\Omega_{N,n}$. For $\omega, \bar\omega \in \Omega_{N,n}$, we put

$$\rho^M(\omega, \bar\omega) = \frac{1}{2n} O^M(\omega, \bar\omega). \tag{4.3}$$

We note that by a similar procedure we can also describe ρ^H, if we say that an elementary H-operation is a *replacement of any coordinate* ω_i *by any of the symbols* $1, \ldots, N$. Obviously $\rho^M(\omega, \bar\omega) \leqslant \rho^H(\omega, \bar\omega)$.

For $r > 0$ and $\omega \in \Omega_{N,n}$, $B_r^H(\omega)$ and $B_r^M(\omega)$ denote the closed spheres in $\Omega_{N,n}$ of radius r and center ω in the metrics ρ^H and ρ^M respectively.

Let $\omega = (\omega_0, \ldots, \omega_{n-1}) \in \Omega_{N,n}$ and $\bar\omega = (\bar\omega_0, \ldots, \bar\omega_{m-1}) \in \Omega_{N,m}$. We denote

$$\omega * \bar\omega = (\omega_0, \ldots, \omega_{n-1}, \bar\omega_0, \ldots, \bar\omega_{m-1}) \in \Omega_{N,\, n+m}.$$

The following inequalities follow immediately from the definitions:

If $\omega^{(1)}, \omega^{(2)} \in \Omega_{N,n}$ and $\bar\omega^{(1)}, \bar\omega^{(2)} \in \Omega_{N,m}$, then

$$\rho^H(\omega^{(1)} * \bar\omega^{(1)}, \omega^{(2)} * \bar\omega^{(2)}) = \frac{n}{m+n} \rho^H(\omega^{(1)}, \omega^{(2)}) + \frac{m}{m+n} \rho^H(\bar\omega^{(1)}, \bar\omega^{(2)}), \tag{4.4}$$

$$\rho^M(\omega^{(1)} * \bar\omega^{(1)}, \omega^{(2)} * \bar\omega^{(2)}) \leqslant \frac{n}{m+n} \rho^M(\omega^{(1)}, \omega^{(2)}) + \frac{m}{m+n} \rho^M(\bar\omega^{(1)}, \bar\omega^{(2)}); \tag{4.5}$$

If $\omega^{(i)} \in \Omega_{N,n_i}$ and $\bar\omega^{(i)} \in \Omega_{N,m_i}$, $i = 1, 2$, then

$$O^M(\omega^{(1)} * \bar\omega^{(1)}, \omega^{(2)} * \bar\omega^{(2)}) \leqslant O^M(\omega^{(1)}, \omega^{(2)}) + O^M(\bar\omega^{(1)}, \bar\omega^{(2)}), \tag{4.6}$$

$$O^M(\omega^{(1)} * \bar\omega^{(1)}, \omega^{(2)} * \bar\omega^{(2)}) = \min_{\omega, \bar\omega \,:\, \omega * \bar\omega = \omega^{(2)} * \bar\omega^{(2)}} (O^M(\omega^{(1)}, \omega) + O^M(\bar\omega^{(1)}, \bar\omega)). \tag{4.7}$$

Let $\omega \in \Omega_{N,n}$, and k be a natural number. We denote

$$\omega^k = \underbrace{\omega * \ldots * \omega}_{k \text{ times}}.$$

Let $k < n$. Using the lexicographic ordering in $\Omega_{N,k}$, we define a mapping $Q_n^k \colon \Omega_{N,n} \to \Omega_{N^k, n-k+1}$,

$$Q_n^k(\omega_0, \ldots, \omega_{n-1}) = ({}^{(k)}\omega_0, \ldots, {}^{(k)}\omega_{n-k}),$$

where

$${}^{(k)}\omega_i = \omega_i + \sum_{j=1}^{k-1} N^j (\omega_{i+j} - 1), \quad i = 0, \ldots, n-k. \tag{4.8}$$

Each elementary H-operation or M-operation over $\omega \in \Omega_{N,n}$ generates a certain transformation of $Q_n^k\omega$. In the case of H-operations this transformation can be represented in the form of not more than k elementary H-operations in $\Omega_{N^k, n-k+1}$, and in the case of M-operations, in the form of a composition of not more than $2k+1$ elementary M-operations. Hence the following inequalities hold:

$$\rho^H(Q_n^k\omega, Q_n^k\bar\omega) \leqslant \frac{nk}{n-k+1} \rho^H(\omega, \bar\omega), \tag{4.9}$$

$$O^M(Q_n^k\omega, Q_m^k\bar{\omega}) \leqslant (2k+1)\, O^M(\omega, \bar{\omega}). \tag{4.10}$$

$$\rho^M(Q_n^k\omega, Q_n^k\bar{\omega}) \leqslant \frac{n(2k+1)}{n-k+1}\rho^M(\omega, \bar{\omega}). \tag{4.11}$$

We define a mapping $K_N^k\colon \Omega_{N^k,n} \longrightarrow \Omega_{N,kn}$. If

$$\omega = (\omega_0, \ldots, \omega_{n-1}) \in \Omega_{N^k,n}, \quad \omega_i = \alpha_{i,0} + \sum_{j=1}^{k-1} N^j(\alpha_{i,j} - 1),$$

we put

$$K_N^k(\omega) = (\alpha_{0,0}, \ldots, \alpha_{0,k-1}, \alpha_{1,0}, \ldots, \alpha_{1,k-1}, \ldots, \alpha_{n,0}, \ldots, \alpha_{n,k-1}).$$

Obviously

$$\rho^H(K_N^k\omega, K_N^k\bar{\omega}) \leqslant \rho^H(\omega, \bar{\omega}), \tag{4.12}$$

$$\rho^M(K_N^k\omega, K_N^k\bar{\omega}) \leqslant \rho^M(\omega, \bar{\omega}). \tag{4.13}$$

2. Let $\xi = \{c_1, \ldots, c_{|\xi|}\}$ be a finite ordered measurable partition of (X, μ), and T an automorphism of (X, μ). Let $T^ix \in c_{k_i(x)}$, $i \in \mathbf{Z}$.

We define a mapping $\varphi_{T,\xi}\colon X \longrightarrow \Omega_{|\xi|}$ by putting

$$\varphi_{T,\xi}x = (\ldots, k_{-1}(x), k_0(x), k_1(x), \ldots). \tag{4.14}$$

The mapping $\varphi_{T,\xi}$ is sometimes called a *coding* of the automorphism by ξ, and the pair (T, ξ) itself is called a *random process*. We also denote $\varphi_{T,\xi}^n = \pi_n \circ \varphi_{T,\xi}$. The image $\varphi_{T,\xi}^n x$ is called the (T, ξ)-*name* of x. We denote

$$\mu_{T,\xi} = (\varphi_{T,\xi})_*\mu, \quad \mu_{T,\xi}^n = (\varphi_{T,\xi}^n)_*\mu.$$

Obviously

$$\mu_{T,\xi}^n\{(\omega_0, \ldots, \omega_n)\} = \mu(c_{\omega_0} \cap T^{-1}c_{\omega_1} \cap \ldots \cap T^{-n+1}c_{\omega_{n-1}}).$$

We will point out some simple relations between the codings of an automorphism relative to different partitions. Let the elements of ξ^k be lexicographically ordered. Then

$$\varphi_{T,\xi^k}^{n-k+1} = Q_n^k \circ \varphi_{T,\xi}^n. \tag{4.15}$$

If we amalgamate (4.15) with (4.9) and (4.11), we obtain without difficulty that

$$\rho^H(\varphi_{T,\xi^k}^{n-k+1}x, \varphi_{T,\xi^k}^{n-k+1}y) \leqslant \frac{nk}{n-k+1}\rho^H(\varphi_{T,\xi}^n x, \varphi_{T,\xi}^n y), \tag{4.16}$$

$$\rho^M(\varphi_{T,\xi^k}^{n-k+1}x, \varphi_{T,\xi^k}^{n-k+1}y) \leqslant \frac{n(2k+1)}{n-k+1}\rho^M(\varphi_{T,\xi}^n x, \varphi_{T,\xi}^n y). \tag{4.17}$$

Let $|\xi| = |\eta|$, $\eta = \{d_1, \ldots, d_{|\xi|}\}$. We denote $B = \bigcup_1^{|\xi|}(c_i \,\Delta\, d_i)$. It is obvious that for $x \in X$

$$\rho^H(\varphi_{T,\xi}^n x, \varphi_{T,\eta}^n x) = \frac{1}{n}\sum_{i=0}^{n-1}\chi_B(T^ix). \tag{4.18}$$

LEMMA 4.1. *Let T be ergodic. Then for almost all $x \in X$*

$$\lim_{n\to\infty} \rho^H(\varphi^n_{T,\xi}x, \varphi^n_{T,\eta}x) = \mu(B).$$

The lemma follows at once from (4.18) and the Birkhoff ergodic theorem.

We denote by $R\colon \Omega_{N,n} \to \Omega_{N,n}$ the cyclic shift

$$R(\omega_0, \omega_1, \ldots, \omega_{n-1}) = (\omega_1, \ldots, \omega_{n-1}, \omega_0).$$

Obviously for $s \in \mathbf{Z}$, $\omega \in \Omega_{N,n}$ and $x \in X$

$$\rho^M(R^s\omega, \omega) \leqslant \frac{|s|}{n}, \tag{4.19}$$

$$\rho^H(\varphi^n_{T,\xi}(T^s x), R^s\omega) \leqslant \rho^H(\varphi^n_{T,\xi}x, \omega) + \frac{|s|}{n}. \tag{4.20}$$

From (4.19) and (4.20) we obtain

$$\rho^M(\varphi^n_{T,\xi}(T^s x), \varphi^n_{T,\xi}x) \leqslant \frac{|s|}{n}. \tag{4.21}$$

Suppose we have a mapping

$$\sigma : \Omega_{N_1,1} \to \bigcup_{n=0}^{K} \Omega_{N_2,n},$$

where $\Omega_{N,0} = \varnothing$, This mapping induces a mapping

$$\bar\sigma : \bigcup_{n=1}^{\infty} \Omega_{N_1,n} \to \bigcup_{n=0}^{\infty} \Omega_{N_2,n}$$

by the following rule:

$$\sigma(\omega_1, \ldots, \omega_n) = \sigma(\omega_1) * \sigma(\omega_2) * \ldots * \sigma(\omega_n).$$

If $O^M(\omega^{(1)}, \omega^{(2)}) = 1$, then, obviously,

$$O^M(\bar\sigma(\omega^{(1)}), \bar\sigma(\omega^{(2)})) \leqslant K$$

and therefore, by induction,

$$O^M(\bar\sigma(\omega^{(1)}), \bar\sigma(\omega^{(2)})) \leqslant K O^M(\omega^{(1)}, \omega^{(2)}). \tag{4.22}$$

Let $\xi = \{c_1, \ldots, c_N\}$, and let $m \in L^1(X, \mu, \mathbf{Z}_0^+)$ be constant on the elements of ξ and take value m_i on c_i, where $\max_{1\leqslant i\leqslant N} m_i \leqslant K$. Then the passage from the coding of T by ξ to the coding of $T_{m(\cdot)}$ by

$$\xi_{m(\cdot)} = \{c_1\times\{1\}, \ldots, c_1\times\{m_1\}, c_2\times\{1\}, \\ \ldots, c_2\times\{m_2\}, \ldots, c_N\times\{1\}, \ldots, c_N\times\{m_N\}\}$$

is described by the mapping $\bar\sigma$, where

$$\sigma(i)=\left(1+\sum_{j=1}^{i-1} m_j, \ \ldots, \ \sum_{j=1}^{i} m_j\right), \quad i=1, \ldots, N,$$

$$\sigma: \Omega_{N,1} \to \bigcup_{k=0}^{K} \Omega_{N',k}, \quad N'=\sum_{j=1}^{N} m_j.$$

§5. Automorphisms with rational spectrum

In the next three sections we successively prove the standardness of automorphisms satisfying three progressively weaker properties of periodic approximation—rational spectrum, good approximation and monotone approximation. The basic result is that the last of these properties is necessary and sufficient for standardness. First we will prove the monotone equivalence of an automorphism with rational spectrum and the automorphism $\mathbf{D}$ (§5), then the monotone equivalence of any automorphism admitting a good approximation and some automorphism with rational spectrum (§6), and, finally, the monotone equivalence of any automorphism admitting a monotone approximation and some automorphism admitting a good approximation (§7). In §7 we also prove the monotone invariance of the property of monotone approximation.

PROPOSITION 5.1. 1. *If a set $A \subset \mathbf{Z}_{\{p_n\}}$ consists of elements of some partition η_m, then the derived automorphism $\mathbf{D}_A$ is metrically isomorphic to $\mathbf{D}$.*

2. *If n: $\mathbf{Z}_{\{p_n\}} \to \mathbf{Z}^+$ is constant on the elements of some partition η_m, then the special automorphism $\mathbf{D}_{n(\cdot)}$ is metrically isomorphic to $\mathbf{D}$.*

PROOF. 1. Let A consist of k elements of η_m. Then, by Proposition 1.1, $\mathbf{D}_A \sim \mathbf{D}_{\{r_n\}}$, where $r_1 = k$ and $r_n = p_{n+m-1}$, $n = 2, 3, \ldots$. We will prove that $PV(\mathbf{D}_A) = \hat{\mathbf{Q}}$. In fact,

$$\hat{\mathbf{Q}} \supset PV(\mathbf{D}_A) \supset \bigcup_{n=1}^{\infty} \mathbf{Z}_{k \cdot p_{m+1} \cdots p_{m+n}} \supset \bigcup_{n=1}^{\infty} \mathbf{Z}_{p_{m+1} \cdots p_{m+n}} \supset \bigcup_{n=1}^{\infty} \mathbf{Z}_{p_1 \cdots p_n} = \hat{\mathbf{Q}}.$$

2. We denote

$$k = p_1 \cdot \ldots \cdot p_m \int_{\mathbf{Z}_{\{p_n\}}} n(x)\, d\chi.$$

Obviously k is an integer. By Proposition 1.1, $\mathbf{D}_{n(\cdot)} \sim \mathbf{D}_{\{r_n\}}$, where, as above, $r_1 = k$ and $r_n = p_{n+m-1}$, $n = 2, 3, \ldots$. The proposition is proved.

PROPOSITION 5.2. *Let $\{r_n\}$, $n = 1, 2, \ldots$, be any sequence of natural numbers greater than* 1, *and let $0 < \beta < 1$. Then the following assertions are true*:

1. *There is a set $B \subset \mathbf{Z}_{\{p_n\}}$ such that $\mu(B) = \beta$ and $\mathbf{D}_B \sim \mathbf{D}_{\{r_n\}}$.*
2. *There is a function n: $\mathbf{Z}_{\{p_n\}} \to \mathbf{Z}^+$ such that $\int_{\mathbf{Z}_{\{p_n\}}} n(x)d\chi = \beta^{-1}$ and $\mathbf{D}_{n(\cdot)} \sim \mathbf{D}_{\{r_n\}}$.*

PROOF. We limit ourselves to a detailed proof of the first assertion, since the second assertion is proved similarly.

We will construct B in the form $B = \bigcap_0^\infty B_n$, where $B_0 \supset B_1 \supset B_2 \supset \cdots$. The sets B_n will be constructed inductively so that the following conditions are satisfied:

(1_n) There exist integers $k_n \geqslant n$ and s_n, a multiple of $p_1 \cdot \ldots \cdot p_n$, such that the set B_n consists of $q_{k_n} = r_1 \cdot \ldots \cdot r_{k_n}$ elements of $\eta(\mathbf{Z}_{s_n})$.

(2_n) $\beta < \mu(B_n) \leqslant \beta + (1/2^n)(1 - \beta)$.

(3_n) For $n > 0$ the restriction to B_n of a proper function of $\mathbf{D}_{B_{n-1}}$ with proper value $\lambda \in \mathbf{Z}_{q_{k_{n-1}}}$ is a proper function of $\mathbf{D}_{B_n}$ with proper value λ.

We will show that the satisfaction of these conditions for $n = 0, 1, 2, \ldots$, implies the first assertion of the proposition.

In fact, from (2_n) it follows that $\mu(B) = \beta$; from (3_n) we obtain that

$$PV(\mathbf{D}_B) \supset \bigcup_{n=1}^{\infty} \mathbf{Z}_{q_{k_n}} = \bigcup_{n=1}^{\infty} \mathbf{Z}_{q_n}.$$

It also follows from (3_n) that the partition $\eta(\mathbf{Z}_{q_{k_n}})$ for $\mathbf{D}_B$ coincides with the restriction to B of the partition $\eta(\mathbf{Z}_{s_n})$ for $\mathbf{D}$. But by (1_n), for $\mathbf{D}$ we have $\eta(\mathbf{Z}_{s_n}) > \eta(\mathbf{Z}_{p_1 \cdots p_n})$, and therefore for $\mathbf{D}_B$ we have $\eta(\mathbf{Z}_{q_{k_n}}) \nearrow \epsilon$. Hence $\mathbf{D}_B$ is an automorphism with rational spectrum, and

$$PV(\mathbf{D}_B) = \bigcup_{n=1}^{\infty} \mathbf{Z}_{q_n}.$$

We now construct the sets B_n. Put $B_0 = \mathbf{Z}\{p_n\}$, $s_0 = 1$ and $k_0 = 0$. Suppose we have already constructed $B_0, B_1, \ldots, B_{n-1}$. Choose k_n so large that

$$q_{k_n} > \frac{2^{n+2} p_n q_{k_{n-1}}}{(1-\beta)}. \tag{5.1}$$

Obviously $k_n \geqslant k_{n-1} + 1$, and therefore $k_n \geqslant n$ if $k_{n-1} \geqslant n - 1$. We will prove that for such a choice of k_n there is a natural number l such that

$$\frac{\beta}{\mu(B_{n-1})} < \frac{q_{k_n}}{q_{k_{n-1}} l p_n} < \frac{\beta + 2^{-n}(1-\beta)}{\mu(B_{n-1})}. \tag{5.2}$$

In fact, (5.2) is equivalent to

$$\frac{\mu(B_{n-1})}{\beta + 2^{-n}(1-\beta)} < l \frac{p_n q_{k_{n-1}}}{q_{k_n}} < \frac{\mu(B_{n-1})}{\beta}. \tag{5.3}$$

But the length of the interval

$$\left(\frac{\mu(B_{n-1})}{\beta + 2^{-n}(1-\beta)}, \frac{\mu(B_{n-1})}{\beta} \right)$$

is larger than $2^{-n-1}(1 - \beta)$. Therefore inside this interval there is a number which is a multiple of $p_n q_{k_{n-1}} / q_{k_n}$, which, by (5.1), is less than $2^{-n-2}(1 - \beta)$.

We now put $s_n = s_{n-1} l p_n$, where l satisfies (5.2). Let $\eta(\mathbf{Z}_{q_{k_{n-1}} p_n l})$ for $\mathbf{D}_{B_{n-1}}$ consist of elements $\Delta_1^{n-1}, \ldots, \Delta_{q_{k_{n-1}} p_n l}^{n-1}$, where $\Delta_i^{n-1} = \mathbf{D}_{B_{n-1}}^{i-1} \Delta_1^{n-1}$. Then put

$$B_n = \bigcup_{i=1}^{q_{k_n}} \Delta_i^{n-1}.$$

We will verify (1_n), (2_n) and (3_n) for this set. Condition (1_n) is satisfied by construction since the partition $\eta(\mathbf{Z}_{q_{k_{n-1}} p_n l})$ for $\mathbf{D}_{B_{n-1}}$ coincides with the restriction to B_{n-1} of $\eta(\mathbf{Z}_{s_{n-1} p_n l})$ for $\mathbf{D}$.

Condition (2_n) follows from (5.2), since

$$\mu(B_n) = \mu(B_{n-1}) \cdot \frac{q_{k_n}}{q_{k_{n-1}} l p_n}.$$

Finally, condition (3_n) follows from the fact that for any element Δ of $\eta(\mathbf{Z}_{q_{k_{n-1}}})$, for $\mathbf{D}_{B_{n-1}}$ we have

$$\mathbf{D}_{B_n}(\Delta \cap B_n) = \mathbf{D}_{B_{n-1}} \Delta \cap B_n.$$

The proposition is proved.

COROLLARY 5.1. *Let $\beta > 0$, and let T and S be standard automorphisms. Then S is metrically isomorphic to some derived automorphism T_A on some set A of measure β.*

PROOF. In view of Corollary 3.1 and Proposition 5.2 it is sufficient to prove that any standard automorphism T is β-monotonely connected with $\mathbf{D}$.

Let $T \sim \mathbf{D}_{m(\cdot)}$, where $\int_{\hat{\mathbf{Z}}} m = \gamma$. Choose a specfunction m_1 such that $\mathbf{D}_{m_1(\cdot)} \sim \mathbf{D}$ and $\int m_1 = \beta/\gamma$. Let $R\colon \hat{\mathbf{Z}} \longrightarrow \hat{\mathbf{Z}}_{m_1(\cdot)}$ implement the isomorphism between $\mathbf{D}$ and $\mathbf{D}_{m_1(\cdot)}$. For $x \in \hat{\mathbf{Z}}_{m_1(\cdot)}$ we denote $\hat{m}(x) = m(R^{-1}x)$. We have

$$T \sim \mathbf{D}_{m(\cdot)} \sim (\mathbf{D}_{m_1(\cdot)})_{\hat{m}(\cdot)} \sim \mathbf{D}_{(m_1 * \hat{m})(\cdot)}.$$

Since $\int m_1 * \hat{m} = (\beta/\gamma) \cdot \gamma = \beta$, it follows that T is β-monotonely connected with $\mathbf{D}$.

§6. Good approximation

DEFINITION 6.1. An automorphism T of a Lebesgue space (X, μ) *admits a good approximation by periodic transformations (good a.p.t.)* if there is a sequence of finite partitions $\xi_n = \{c_n^1, \ldots, c_n^{q_n} = c_n^0, d_n\}$, $n = 1, 2, \ldots$, with the following properties:

6.1.1. $\mu(c_n^1) = \cdots = \mu(c_n^{q_n})$.

6.1.2. ξ_n is an exhaustive sequence.

6.1.3. $\lim q_n \cdot \sum_{i=0}^{q_n-1} \mu(Tc_n^i \, \Delta \, c_n^{i+1}) = 0$.

The property of good a.p.t. is equivalent to the property called in [23] a strong approximation by partitions (see the beginning of the proof of Proposition 6.2), and takes an intermediate position between the weaker property of approximation by partitions (see [24] or [8]) and the property of cyclic a.p.t. with speed $o(1/n)$ of [25], which corresponds to the case $d_n = \varnothing$ or, equivalently, $q_n \, \mu(d_n) \longrightarrow 0$ as $n \longrightarrow \infty$. A slight modification of the proofs of Theorems 3.1 and 3.3 of [25] allows one to prove the following assertion.

PROPOSITION 6.1. *An automorphism admitting a good a.p.t. has simple singular spectrum and is not mixing.*

PROPOSITION 6.2.(3) *Let T admit a good a.p.t. Then there exists a sequence of partitions $\eta_n = \{c_n, Tc_n, \ldots, T^{s_n-1}c_n, f_n\}$, $n = 1, 2, \ldots$, with the following properties:*

(3)There is a similar assertion in [26].

1. $\eta_n \nearrow \epsilon$ *as* $n \longrightarrow \infty$.
2. $f_1 \supset f_2 \supset \cdots$.
3. $s_n \cdot \mu(T^{s_n} c_n \cap c_n) \longrightarrow 1$ *as* $n \longrightarrow \infty$.

PROOF. Put

$$c_n' = \bigcap_{i=1}^{q_n} T^{-i+1} c_n^i, \quad d_n' = X \setminus \bigcup_{i=0}^{q_n-1} T^i c_n'.$$

Obviously, $c_n' \cap T^i c_n' = \varnothing$, $i = 1, \ldots, q_n - 1$; therefore the sets $c_n', Tc_n', \ldots, T^{q_n-1}c_n'$ and d_n' form a partition, which we denote by ξ_n'. Since

$$\mu(c_n^1 \setminus c_n') \leqslant \sum_{i=1}^{q_n-1} \mu(c_n^{i+1} \setminus Tc_n^i), \tag{6.1}$$

we have

$$\mu(d_n') \leqslant \mu(d_n) + \frac{q_n}{2} \sum_{i=1}^{q_n-1} \mu(Tc_n^i \triangle c_n^{i+1}). \tag{6.2}$$

Since $\xi_n \longrightarrow \epsilon$, by (6.1) and (6.2) $\xi_n' \longrightarrow \epsilon$. In addition

$$\mu(c_n' \triangle T^{q_n} c_n') < 2\mu(c_n^1 \setminus c_n') + \mu(T^{q_n} c_n^1 \triangle c_n^1) \leqslant 2 \sum_{i=0}^{q_n-1} \mu(Tc_n^i \triangle c_n^{i+1}).$$

Thus the sequence ξ_n' satisfies the conditions of the definition of a strong approximation by partitions [23].

We choose from ξ_n' a subsequence

$$\xi_{k_n}' = \eta_n^n = \{c_n^n, Tc_n^n, \ldots, T^{s_n-1} c_n^n, l_n^n\}$$

(the notation is somewhat modified) such that for each $n = 1, 2, \ldots$, there is a family of natural numbers

$$I_n \subset \{1, \ldots, s_n - s_{n-1}\}, \tag{6.3}$$

for which

$$\mu\Big(c_n^n \triangle \big(\bigcup_{i \in I_n} T^i c_{n+1}^{n+1}\big)\Big) < \frac{1}{s_n^3 2^{s_n}}, \tag{6.4}$$

where for each $i \in I_n$

$$\mu(T^i c_{n+1}^{n+1} \cap c_n^n) > \frac{3}{4 s_{n+1}}. \tag{6.5}$$

From (6.5) it follows that

$$I_n \cap (I_n + \{i\}) = \varnothing, \quad i = 1, \ldots, s_n - 1. \tag{6.6}$$

Denote

$$c_n^{n+1} = \bigcup_{i \in I_n} T^i c_{n+1}^{n+1}$$

and, inductively, for $k = n - 1, n - 2, \ldots, 1$

$$c_k^{n+1} = \bigcup_{i \in I_k} T^i c_{k+1}^{n+1}. \tag{6.7}$$

It follows from (6.3) and (6.6) that for $k = 1, \ldots, n$ and $i = 1, \ldots, s_k - 1$ we have $T^i c_k^{n+1} \cap C_k^{n+1} = \varnothing$, and consequently the sets

$$c_k^{n+1},\ Tc_k^{n+1},\ \ldots,\ T^{s_k-1} c_k^{n+1},\ X \setminus \bigcup_{i=0}^{s_k-1} T^i c_k^{n+1}$$

form a partition, which we denote by η_k^{n+1}. Obviously

$$\eta_{n+1}^{n+1} > \eta_n^{n+1} > \ldots > \eta_1^{n+1}.$$

The definition of the c_k^{n+1} and (6.4) imply that

$$\mu\,(c_k^{n+1} \triangle c_k^n) < \frac{1}{s_n^2 2^{s_n}}.$$

Therefore, as $n \longrightarrow \infty$ and for fixed k, the sequence of sets c_n^k tends to a limit, which we denote c_k, where $T^i c_k \cap c_k = \varnothing$, $i = 1, \ldots, s_k - 1$. We introduce the partition

$$\eta_k = \left\{c_k,\ Tc_k,\ \ldots,\ T^{s_k-1} c_k,\quad f_k = X \setminus \bigcup_{i=0}^{s_k-1} T^i c_k\right\}.$$

Passing to the limit as $n \longrightarrow \infty$ in (6.7), we obtain

$$c_k = \bigcup_{i \in I_k} T^i c_{k+1}$$

and consequently the sequence η_k is monotonely increasing, and $f_{k+1} \subset f_k$. Since $\eta_n^n \longrightarrow \epsilon$ as $n \longrightarrow \infty$ and

$$\mu\,(c_n \triangle c_n^n) < \sum_{l=1}^{\infty} (c_n^{n+l} \triangle c_n^{n+l-1}) \leqslant \sum_{l=1}^{\infty} \frac{1}{(s_{n+l})^2\, 2^{s_{n+l}}} < \frac{2}{s_{n+1}^2},$$

it follows that $\eta_n \nearrow \epsilon$ as $n \longrightarrow \infty$. Finally,

$$\mu\,(T^{s_n} c_n \triangle c_n) \leqslant 2\mu\,(c_n \triangle c_n^n) + \mu\,(T^{s_n} c_n^n \triangle c_n^n).$$

The first term has already been estimated. By (6.1) the second term is also $o(1/s_n)$. The proposition is proved.

COROLLARY 6.1. *An automorphism T admitting a good a.p.t. is standard.*

PROOF. By Proposition 5.2 it is sufficient to prove that T has a derived automorphism which is metrically isomorphic to some $\mathbf{D}\{r_n\}$. But this follows at once from Proposition 6.2. In fact, fix any k and put $A_k = \bigcup_{i=0}^{s_k-1} T^i c_k$. Obviously, for $n \geqslant k$ the restriction of η_n to A_k is invariant relative to T_{A_k}. Since $(\eta_n)_{A_k} \nearrow \epsilon_{A_k}$, by Proposition 1.1 T_{A_k} is an ergodic automorphism with rational spectrum.

§7. Monotone approximation

Let $q \in \mathbf{Z}^+$, $N_q = \{0, 1, \ldots, q-1\}$, $X_q = [0, 1] \times N_q$, $X_q^i = [0, 1] \times \{i\}$, $i = 0, 1, \ldots, q-1$, and let λ_q be the direct product of Lebesgue measure on $[0, 1]$ and the measure μ_q on N_q, where $\mu_q(\{i\}) = 1/q$, $i = 0, 1, \ldots, q-1$. Let T: $(X, \mu) \longrightarrow (X, \mu)$ be an automorphism, $F \subset X$ a measurable set of positive measure and φ: $F \longrightarrow X_q$ a measurable mapping.

DEFINITION 7.1. We will say that the pair (F, φ) defines a *partial monotone structure for* T if mod 0 the following conditions are satisfied:

7.1.1. φ is an injective mapping with constant Jacobian; that is, $\lambda_q(\varphi(A)) = \alpha(\varphi)\mu(A)$ for $A \subset F$, where $\alpha(\varphi) > 0$ is independent of A.

7.1.2. If $x_1, x_2 \in F$, $\varphi(x_1) = (t, m) \in X_q$ and $\varphi(x_2) = (t, m + k)$, $k > 0$, then $x_2 = T^l x_1$ for some $l > 0$.

7.1.3. If $x \in F$ and $\varphi(x) = (t, m)$, then either $\varphi(T_F x) = (t, m + k)$, $k > 0$, or $\varphi(F) \cap \{t\} \times \{m + 1, \ldots, q-1\} = \varnothing$.

For a partial monotone structure (F, φ) we put

$$F^i = \varphi^{-1}(X_q^i), \quad i = 0, 1, \ldots, q-1,$$

where $\xi(F, \varphi)$ is the partition of X into $F^0, \ldots, F^{q-1}$ and $X \backslash F$. Finally we denote by π_1 and π_2 the natural projections:

$$\pi_1 : X_q \to [0, 1], \quad \pi_2 : X_q \to N_q.$$

REMARK. Obviously properties 7.1.1–7.1.3 depend only on the derived automorphism T_F. Therefore if (X', μ') is another Lebesgue space with $F \subset X \cap X'$, $\mu(F) > 0$, and T': $X' \longrightarrow X'$ an automorphism such that $(T')_F = T_F$, then (F, φ) is a partial monotone structure for T', where the corresponding constant $\alpha'(\varphi)$ is equal to $\alpha(\varphi) \cdot \mu(F)/\mu'(F)$.

DEFINITION 7.2. A sequence of partial monotone structures (F_n, φ_n) for T is called *exhaustive* if, as $n \longrightarrow \infty$,

7.2.1 $\xi(F_n, \varphi_n) \longrightarrow \epsilon$, and

7.2.2. $\alpha(\varphi_n) \longrightarrow 1$.

REMARK. Obviously 7.2.1 implies that $\mu(F_n) \longrightarrow 1$.

DEFINITION 7.3. An automorphism T admits a *monotone approximation* if there exists an exhaustive sequence of partial monotone structures for T.

We will now state an approximation criterion for standardness.

THEOREM 2. *The following two properties of an automorphism T are equivalent*:
T is a standard automorphism;
T admits a monotone approximation.

Theorem 2 follows from Corollary 6.1 and Propositions 7.1 and 7.2 proved below.

PROPOSITION 7.1. *The property of admitting a monotone approximation is monotonely invariant.*

PROOF. 1°. We first describe a certain reconstruction of X_q.

Given a function k: $N_q \longrightarrow \mathbf{Z}_0^+$, we denote $q' = \Sigma_{i=0}^{q-1} k(i)$. We define a mapping

$$\psi_{k(\cdot)}: (X_q)_{k(\cdot)} \to X_{q'},$$

putting for $t \in [0, 1]$, $i \in N_q$ and $0 \leqslant j < k(i)$

$$\psi_{k(\cdot)}(t, i, j) = \left(t, \sum_{l=0}^{i-1} k(l) + j \right).$$

Obviously $\psi_{k(\cdot)}$ is an isomorphism (we recall that the measures in $(X_q)_{k(\cdot)}$ and $X_{q'}$ are normalized).

2°. Now let (F, φ) be a partial monotone structure for T and let $\bar{m} \in L^1(X, \mu, \mathbf{Z}_0^+)$ be constant on the elements of $\xi(F, \varphi)$. We define a partial monotone structure $(F_{\bar{m}(\cdot)}, \varphi_{\bar{m}(\cdot)})$ for $T_{\bar{m}(\cdot)}$ by putting $\varphi_{\bar{m}(\cdot)} = \psi_{k(\cdot)} \circ \kappa$, where $k: X_q \to \mathbf{Z}_0^+$, $k = \bar{m} \circ \varphi^{-1}$, $\kappa: F_{\bar{m}(\cdot)} \to (X_q)_{k(\cdot)}$ and $\kappa(x, i) = (\varphi(x), i)$. Obviously,

$$\xi(F_{\bar{m}(\cdot)}, \varphi_{\bar{m}(\cdot)}) = (\xi(F, \varphi))_{\bar{m}(\cdot)}, \quad \alpha(\varphi_{\bar{m}(\cdot)}) = \alpha(\varphi) \int \bar{m} d\mu \cdot q \cdot (q')^{-1}.$$

3°. Let T have an exhaustive sequence of partial monotone structures (F_n, φ_n), and let $m \in L^1(X, \mu, \mathbf{Z}_0^+)$. Fix $\delta > 0$ and choose $N(\delta)$ so that for any $n \geqslant N(\delta)$ there is a function $\bar{m}_n \in L^1(X, \mu, Z_0^+)$, constant on the elements of $\xi(F_n, \varphi_n)$ and such that

$$\begin{gathered}\int_X |m(x) - \bar{m}_n(x)| \, d\mu(x) < \delta, \\ \int_X (1 - \delta_{m(x), \bar{m}_n(x)}) \, m(x) \, d\mu(x) < \delta.\end{gathered} \tag{7.1}$$

Here δ_{ij} is the Kronecker symbol. We will now omit the index n, since we will only be considering one partial monotone structure.

Put $\bar{F} = \{(x, i) \in X_{m(\cdot)}: x \in F, m(x) = \bar{m}(x)\}$. By (7.1), $\mu(F \setminus \bar{F}) < \delta / \int_X m \, d\mu$. Obviously, $\bar{F} \subset X_{m(\cdot)} \cap X_{\bar{m}(\cdot)}$ and $(T_{m(\cdot)})_{\bar{F}} = (T_{\bar{m}(\cdot)})_{\bar{F}}$; therefore, by the remark after Definition 7.1, $(\bar{F}, \varphi_{\bar{m}(\cdot)}|_{\bar{F}})$ is a partial monotone structure for $T_{m(\cdot)}$, where

$$\alpha(\varphi_{\bar{m}(\cdot)}|_{\bar{F}}) = \alpha(\varphi_{\bar{m}(\cdot)}) \frac{\int_X \bar{m} d\mu}{\int_X m d\mu},$$

and $\xi(\bar{F}, \varphi_{\bar{m}(\cdot)}|_{\bar{F}})$ on $\bar{F}$ coincides with $(\xi(F, \varphi))_{m(\cdot)}$.

We apply this procedure to each partial monotone structure (F_n, φ_n) for $n \geqslant N(\delta)$. Since δ is arbitrary we thus obtain an exhaustive sequence of partial monotone structures for $T_{m(\cdot)}$. The proposition is proved.

PROPOSITION 7.2. *Let T: $(X, \mu) \to (X, \mu)$ admit a monotone approximation. Then there is a function $m \in L^1(X, \mu, \mathbf{Z}_0^+)$ such that $T_{m(\cdot)}$ admits a good a.p.t.*

PROOF. 1°. Let (F, φ) be a partial monotone structure for T. We introduce some notation.

For $x \in F$ we put $i(x)$ equal to the index of the set F_i to which x belongs. In other words, $i(x) = \pi_2(\varphi(x))$. Further, let $j(x)$ be the number of elements in the set

$$\varphi(F) \cap (\{\pi_1(\varphi(x))\} \times \{0, \ldots, i(x)\}) \subset X_q.$$

For $t \in [0, 1]$ we denote by $s(t)$ the number of elements in the set $\varphi(F) \cap (\{t\} \times N_q)$. We call $t \in [0, 1]$ *ϵ-saturated* if $s(t) \geqslant (1-\epsilon)q$. We denote by $A_\epsilon = A_\epsilon(F, \varphi) \subset [0, 1]$ the set of all ϵ-saturated points, and set $F_\epsilon = \varphi^{-1}(A_\epsilon \times N_q)$. Since $\lambda_q(\varphi(F)) = \alpha(\varphi) \cdot \mu(F)$, we have

$$\mu(A_\varepsilon) \geqslant 1 - \frac{1-\alpha(\varphi)\mu(F)}{\varepsilon}, \tag{7.2}$$

$$\mu(F_\varepsilon) \geqslant \frac{1-\varepsilon}{\alpha(\varphi)}\left(1 - \frac{1-\alpha(\varphi)\mu(F)}{\varepsilon}\right). \tag{7.3}$$

2°. The basis of the proof is the following construction of a "reconstruction" of an automorphism. Let $t \in A_\epsilon$. The inverse image $\varphi^{-1}(\{t\} \times N_q)$ consists of $s(t)$ points, which we denote $y_1(t), \ldots, y_{s(t)}(t)$, where $j(y_i(t)) = i$, $i = 1, \ldots, s(t)$.

We observe that, by conditions 7.1.2 and 7.1.3, $y_{i+1}(t) = T_{F_\epsilon} y_i(t)$ for $i = 1, \ldots, s(t) - 1$.

DEFINITION 7.4. *An ϵ-reconstruction of T, corresponding to a partial monotone structure* (F, φ), will be a passage from T to $T_{m_\epsilon(\cdot)}$, where $m_\epsilon = m_{\epsilon,F,\varphi} \in L^1(X, \mu, \mathbf{Z}_0^+)$ is defined in the following way: for $x \in X$

$$m_{\varepsilon,F,\varphi}(x) = \begin{cases} i(T_{F_\varepsilon}(x)) - i(x), & \text{if } x \in F_\varepsilon,\ j(x) \neq s(\pi_1(\varphi(x))), \\ q - i(x) + i(T_{F_\varepsilon}(x)), & \text{if } x \in F_\varepsilon,\ j(x) = s(\pi_1(\varphi(x))), \\ 0, & \text{if } x \notin F_\varepsilon. \end{cases} \tag{7.4}$$

The function m_ϵ is measurable by the measurability of F_ϵ and the functions $i(x), j(x)$ and $s(t)$. We now define a mapping $i\colon X_{m_\epsilon(\cdot)} \to \{0, 1, \ldots, q-1\}$, putting for $y = (x, k) \in X_{m_\epsilon(\cdot)}$

$$i(y) = i(x) + k - 1 \pmod q. \tag{7.5}$$

A basic property of $T_{m_\epsilon(\cdot)}$, obvious in view of (7.4) and (7.5), is

$$i(T_{m_\varepsilon(\cdot)} y) = i(y) + 1.$$

Thus the sets $c_k = i^{-1}(k)$, $k = 0, 1, \ldots, q-1$, have equal measure and are cyclically permuted under the action of $T_{m_\epsilon(\cdot)}$.

3°. We will show that

$$\sum_{k=0}^{q-1} \mu_{m_\varepsilon(\cdot)}(c_k \setminus (F^k \cap X_{m_\varepsilon(\cdot)})) \leqslant \varepsilon. \tag{7.6}$$

Obviously, $F^k \cap F_\epsilon = F^k \cap X_{m_\epsilon(\cdot)} \subset c_k$. Therefore

$$\bigcup_{k=0}^{q-1} (c_k \setminus F^k) = X_{m_\varepsilon(\cdot)} \setminus F_\varepsilon. \tag{7.7}$$

Furthermore,

$$\mu_{m_\varepsilon(\cdot)}(F_\varepsilon) \geqslant 1 - \varepsilon. \tag{7.8}$$

The inequality follows from the fact that, by the construction of $X_{m_\epsilon(\cdot)}$, to each set $\varphi^{-1}(\{t\} \times N_q)$, $t \in A_\epsilon$, consisting of $s(t) \geqslant (1-\epsilon)q$ points, we add $q - s(t)$ points (we count all points of $X_{m_\epsilon(\cdot)}$ lying over the $y_i(t)$, $i = 1, \dots, s(t)-1$, $q - i(y_{s(t)}(t))$ of the points lying over $y_{s(t)}(t)$, and $i(y_1(t))$ of the points lying over $(T_{F_\epsilon})^{-1}y_1(t)$). (7.6) follows from (7.7) and (7.8).

4°. We construct an automorphism $T_{m(\cdot)}$ admitting a good a.p.t. as follows: $m(x) = \lim_{n\to\infty} m^{(n)}(x)$, where convergence holds almost everywhere and in L^1 and $m^{(n)} = m^{(n-1)} * \widetilde{m}^{(n)}$. The automorphism $T_{m^{(n)}(\cdot)}$ is obtained from $T_{m^{(n-1)}(\cdot)}$ with the help of an ϵ_n-reconstruction corresponding to some partial monotone structure.

We put $m^{(0)} \equiv 1$ and suppose that $m^{(1)}, \dots, m^{(n)}$ are already constructed. In each of the spaces $X_{m^{(k)}(\cdot)}$, $k = 0, 1, \dots, n$, we fix bases $\{A_l^k\}$, $l = 1, 2, \dots$, of the σ-algebras $\mathfrak{A}(X_{m^{(k)}(\cdot)}, \mu_{m^{(k)}(\cdot)})$. Since each $T_{m^{(k)}(\cdot)}$ is obtained by ϵ_k-reconstruction, it has an invariant finite partition into the sets c(see 2°), which we denote by ξ_k. Let $|\xi_k| = q_k$.

By Proposition 7.1 there is an exhaustive sequence of partial monotone structures for $T_{m^{(n)}(\cdot)}$. We choose a partial monotone structure $(F^{(n+1)}, \varphi^{(n+1)})$ and an $\epsilon_{n+1} > 0$ so that the automorphism $T_{m^{(n+1)}(\cdot)}$, obtained from $T_{m^{(n)}(\cdot)}$ by an ϵ_{n+1}-reconstruction, satisfies the following conditions:

(1_n).

$$\mu_{m^{(n)}(\cdot)}(X_{m^{(n)}(\cdot)} \cap X_{m^{(n+1)}(\cdot)}) > 1 - \frac{1}{2^{n+1}q_n^3},$$

$$\mu_{m^{(n+1)}(\cdot)}(X_{m^{(n)}(\cdot)} \cap X_{m^{(n+1)}(\cdot)}) > 1 - \frac{1}{2^{n+1}q_n^3}.$$

This is possible by (7.3) and (7.8).

(2_n). The sets $A_l^k \cap X_{m^{(n)}(\cdot)}$, $k, l = 0, \dots, n$, are approximable within $1/2^{n+10}$ by sets consisting of elements of $\xi(F^{(n+1)}, \varphi^{(n+1)})$.

The inequalities (1_n) imply the convergence of $m^{(n)}$ as $n \to \infty$ in $L^1(X, \mu)$ and almost everywhere to a function m. We will show that $T_{m(\cdot)}$ admits a good a.p.t. The inequalities (1_{n+l}), $l = 0, 1, \dots$, imply

$$\begin{gathered}\mu_{m^{(n)}(\cdot)}(X_{m^{(n)}(\cdot)} \cap X_{m(\cdot)}) > 1 - \frac{1}{2^n q_n^3},\\ \mu_{m(\cdot)}(X_{m^{(n)}(\cdot)} \cap X_{m(\cdot)}) > 1 - \frac{1}{2^n q_n^3}.\end{gathered} \tag{7.9}$$

Hence it follows that $T_{m(\cdot)}$ and $T_{m^{(n)}(\cdot)}$ coincide on the set G_n, both measures ($\mu_{m(\cdot)}$ and $\mu_{m^{(n)}(\cdot)}$) of which are greater than $1 - 1/2^{n-1}q_n^3$. We denote

$$F_n = \bigcap_{i=0}^{q_n} T_{m(\cdot)}^{-i} G_n.$$

We have

$$\mu_{m(\cdot)}(F_n) > 1 - \frac{1}{2^{n-1}q_n^2},$$

and if $x \in F_n$, then $T^i_{m(\cdot)}x = T^i_{m^{(n)}(\cdot)}x$, $i = 0, 1, \ldots, q_n - 1$. Choose a $c \in \xi_n$ and put $\bar{c}_n = c \cap F_n$. Obviously

$$T^i_{m(\cdot)}\bar{c}_n \cap \bar{c}_n = \varnothing, \quad i = 1, \ldots, q_n - 1, \quad \mu_{m(\cdot)}(\bar{c}_n) > \frac{1}{q_n} - \frac{1}{2^{n-1}q_n^2}$$

and

$$\mu_{m(\cdot)}(\bar{c}_n \cap T^{q_n}_{m(\cdot)}\bar{c}_n) > \frac{1}{q_n} - \frac{1}{2^{n-2}q_n^2}.$$

Consider the partition

$$\bar{\xi}_n = \{\bar{c}_n, T_{m(\cdot)}\bar{c}_n, \ldots, T^{q_n-1}_{m(\cdot)}\bar{c}_n, X_{m(\cdot)} \setminus \bigcup_{i=0}^{q_n-1} T^i_{m(\cdot)}\bar{c}_n\}.$$

From (7.6), (2_n) and (7.9) it follows that $\bar{\xi}_n$ is an exhaustive sequence. Thus the sequence $\bar{\xi}_n$ satisfies conditions 6.1.1–6.1.3.

§8. Generalized almost periodicity

We now pass to the proof of the second and most important criterion for standardness—Theorem 4. This theorem will be proved—more precisely, reduced to Corollary 6.1—in this and the next two sections. The assertion of Theorem 4 says that an automorphism is standard if and only if for any finite partition (or, equivalently, for a generating partition, or an exhaustive sequence of partitions) sufficiently long pieces of the trajectories of most points are coded into sequences which are close in the metric ρ^M (the property of M-triviality; see Definitions 9.1–9.3). The scheme of the proof of Theorem 4 is this. First we prove Theorem 3, a similar criterion for the existence of a good a.p.t. (H-almost periodicity; see Definition 8.3), which is also of independent interest; then we prove the equivalence of the condition of M-triviality for all partitions, for generating partitions and for exhaustive sequences of partitions (Propositions 9.2 and 9.3), then the monotone invariance of the first of these conditions (Proposition 9.4) and, finally, the monotone equivalence of an automorphism satisfying this condition with some H-almost periodic automorphism (§10). Theorem 4 then follows at once from Theorem 3 and Corollary 6.1.

Throughout this section p and q are natural numbers, $\epsilon > 0$, ξ is a finite ordered partition of (X, μ) and T is an automorphism of (X, μ).

DEFINITION 8.1. A random process (T, ξ) is *(p, q, ϵ)-periodic* if there is an element $\omega^{(0)} \in \Omega_{pq,|\xi|}$ such that $R^q\omega^{(0)} = \omega^{(0)}$ and

$$\mu^{pq}_{T,\xi}\left(\bigcup_{s=0}^{q-1} B^H_\varepsilon(R^s\omega^{(0)})\right) > 1 - \varepsilon.$$

DEFINITION 8.2. A random process (T, ξ) is *H-almost periodic* if for some sequences $p_n \to \infty$, $q_n \to \infty$ and $\epsilon_n \to 0$ the process is (p_n, q_n, ϵ_n)-periodic.

DEFINITION 8.3. An automorphism T is *H-almost periodic* if (T, ξ) is H-almost periodic for any partition ξ.

REMARK. It obviously follows from the definition that a quotient automorphism of an H-almost periodic automorphism relative to an infinite invariant partition is an H-almost periodic automorphism.

PROPOSITION 8.1. *Let $\{\eta_n\}$, $n = 1, 2, \ldots$, be an exhaustive sequence of finite partitions of (X, μ). If there are sequences $p_n \longrightarrow \infty$, $q_n \longrightarrow \infty$ and $\epsilon_n \longrightarrow 0$ such that the processes (T, η_n) are (p_n, q_n, ϵ_n)-periodic, then T is H-almost periodic.*

PROOF. Fix $\xi = \{c_1, \ldots, c_m\}$, $\epsilon > 0$ and a natural number N. Choose n so that there is a partition $\xi_n = \{c_1', \ldots, c_m'\}$, $\xi_n < \eta_n$, for which

$$\sum_{i=1}^{m} \mu(c_i \triangle c_i') < \frac{\varepsilon^2}{4}, \tag{8.1}$$

and, in addition, $\epsilon_n < \epsilon/2$, $p_n > N$ and $q_n > N$. Then (T, ξ_n) is obviously (p_n, q_n, ϵ_n)-periodic. Let $\omega^{(0)} \in \Omega_{m,p_nq_n}$ be such that $R^{q_n}\omega^{(0)} = \omega^{(0)}$ and

$$\mu_{T,\xi_n}^{p_nq_n}\left(\bigcup_{s=0}^{q_n-1} B_{\frac{\varepsilon}{2}}^{H}\left(R^{(s)}\omega^{(0)}\right)\right) > 1 - \frac{\varepsilon}{2}. \tag{8.2}$$

Integrating (4.18), we obtain

$$\int_X \rho^H\left(\varphi_{T,\xi}^{p_nq_n}(x),\ \varphi_{T,\xi_n}^{p_nq_n}(x)\right) d\mu = \mu\left(\bigcup_{i=1}^{m}(c_i \triangle c_i')\right).$$

Denote

$$A_r = \{x : \rho^H\left(\varphi_{T,\xi}^{p_nq_n}(x),\ \varphi_{T,\xi_n}^{p_nq_n}(x)\right) > r\}.$$

(8.1) implies

$$\mu\left(A_{\frac{\varepsilon}{2}}\right) < \frac{\varepsilon}{2}. \tag{8.3}$$

Thus if $x \notin A_{\epsilon/2}$ and

$$\varphi_{T,\xi_n}^{p_nq_n}(x) \in \bigcup_{s=0}^{q_n-1} B_{\frac{\varepsilon}{2}}^{H}\left(R^s(\omega^{(0)})\right),$$

then

$$\varphi_{T,\xi}^{p_nq_n}(x) \in \bigcup_{s=0}^{q_n-1} B_{\varepsilon}^{H}\left(R^s(\omega^{(0)})\right),$$

whence by (8.2) and (8.3) we have

$$\mu_{T,\xi}^{p_nq_n}\left(\bigcup_{s=0}^{q_n-1} B_{\varepsilon}^{H}\left(R^s(\omega^{(0)})\right)\right) > 1 - \varepsilon.$$

Since N, ϵ and ξ are arbitrary, the proposition is proved.

PROPOSITION 8.2. *If (T, ξ) is H-almost periodic, so is $T|_{\xi_T}$.*

PROOF. Passing from T to the quotient automorphism $T|_{\xi_T}$, we may suppose that ξ is a generating partition; that is, $\xi_T = \epsilon$. Since $\varphi^n_{T,T^k\xi}(x) = \varphi^n_{T,\xi}(T^k x)$, the H-almost periodicity of $(T, T^k\xi)$, for any integer k, follows from the H-almost periodicity of (T, ξ). Furthermore, from Definition 8.2 and (4.16) it follows that (T, ξ^k) is H-almost periodic. Since $T^{-k}\xi^{2k-1} = \bigvee_{-k}^{k} T^i\xi = \eta_k$, it follows that (T, η_k) is H-almost periodic.

But the sequence $\{\eta_k\}$, $k = 1, 2, \dots$, is exhaustive, and consequently Proposition 8.2 follows from Proposition 8.1.

THEOREM 3. *An automorphism T of a Lebesgue space is H-almost periodic if and only if it admits a good a.p.t.*

PROOF 1°. We will derive H-almost periodicity from the property of good a.p.t. Let $\{\xi_n\}$, $n = 1, 2, \dots$, be the sequence of partitions of Definition 6.1. By virtue of Proposition 8.1 it is sufficient to verify that for some sequences $p_n \longrightarrow \infty$ and $\xi_n \longrightarrow 0$ the random processes (T, ξ_n) are (p_n, q_n, ϵ_n)-periodic.

We denote

$$q_n \sum_{i=0}^{q_n-1} \mu(Tc_n^i \triangle c_n^{i+1}) = a_n, \quad \mu(d_n) = b_n$$

and put

$$p_n = [a_n^{-1/2}] - 1, \quad c = \bigcap_{k=0}^{p_n} \bigcap_{i=0}^{q_n-1} T^{-kq_n-i} c_n^1.$$

Obviously

$$\mu(c_n^1 \setminus c) < \frac{p_n a_n}{q_n} \leqslant \frac{a_n^{1/2}}{q_n}$$

and consequently

$$\mu\left(\bigcup_{i=0}^{q_n-1} T^i c\right) > 1 - b_n - a_n^{1/2}.$$

If $x \in T^i c$, $i = 0, 1, \dots, q_n - 1$, then

$$\varphi_{T,\xi}^{p_n q_n}(x) = (i, i+1, \dots, q_n, \underbrace{1, \dots, q_n, \dots, 1, \dots, q_n}_{p_n-1 \text{ times}}, 1, \dots, i-1) = R^i \omega^{(0)},$$

where

$$\omega^{(0)} = (\underbrace{1, \dots, q_n, \dots, 1, \dots, q_n}_{p_n \text{ times}}).$$

Putting $\epsilon_n = b_n + a_n^{1/2}$, we obtain the result.

2°. We now derive the property of good a.p.t. from H-almost periodicity. A sequence of partitions satisfying the conditions of Definition 6.1 is easy to construct by a diagonal process, if for any finite partition $\eta = \{b_1, \dots, b_L\}$ and any $\delta > 0$ we construct $\eta^{(\delta)} = \{d_1, \dots, d_r = d_0, d\}$ with the following properties:

$$\mu(d) < \delta, \tag{8.4}$$

$$\mu(d_1) = \mu(d_2) = \dots = \mu(d_r). \tag{8.5}$$

For each $i = 1, \dots, r$, there is a $j_i \in [1, \dots, L]$ such that

$$\sum_{i=1}^{r} \mu(d_i \setminus b_{l_i}) < \delta, \tag{8.6}$$

$$\sum_{i=0}^{r-1} \mu(Td_i \triangle d_{i+1}) < \frac{\delta}{r}. \tag{8.7}$$

Thus, let η and δ, $0 < \delta < 1/10$, be fixed, and let T be H-almost periodic. Choose $p > 2\delta^{-2}$ and q so that for some $\omega^{(0)} \in \Omega_{L,pq}$

$$\mu_{T,\eta}^{pq}\left(\bigcup_{s=0}^{q-1} B_{\delta/2}^{H}(R^s\omega^{(0)})\right) > 1 - \frac{\delta}{2},$$

and denote for convenience

$$A(s, \varepsilon) = (\varphi_{T,\eta}^{pq})^{-1}(B_{\varepsilon}^{H}(R^s\omega^{(0)}))$$

in order to do the subsequent construction in X and not in $\Omega_{L,pq}$.

By (4.21), $TA(s, \epsilon) \subset A(s + 1, \epsilon + 1/pq)$.

Fix $N > 100\delta^{-1}pq$, and construct a set $B \subset X$ so that $T^jB \cap B = \varnothing$, $j = 1, \ldots, N - 1$, and $\mu(\bigcup_{j=0}^{N-1} T^jB) > 1 - \delta^2/2$. Let κ_0 denote the partition consisting of the intersections of the elements of $\eta \cdot T\eta \cdot \cdots \cdot T^{-N+1}\eta$ with B, and $X \setminus B$, and further let

$$\varkappa = \varkappa_0 \cdot T\varkappa_0 \cdot \ldots \cdot T^{N-1}\varkappa_0.$$

The partition $\eta^{(\delta)}$ satisfying (8.4)–(8.7) will be a refinement of κ.

Fix $c \in \kappa_0$. Denote

$$l_0(c) = \min\left\{l : 0 \leqslant l \leqslant N - pq,\ T^lc \in A\left(0, \frac{\delta}{2} + \frac{1}{p}\right)\right\}$$

and by induction set

$$l_k(c) = \min\left\{l : l_{k-1}(c) + pq \leqslant l \leqslant N - pq,\ T^lc \in A\left(0, \frac{\delta}{2} + \frac{1}{p}\right)\right\}$$

for as long as $l_k(c)$ can be defined. We denote by $k(c)$ the maximal k for which $l_k(c)$ is defined.

Put

$$d_i = \bigcup_{c \in \varkappa_0} \bigcup_{k=0}^{k(c)} \bigcup_{j=0}^{p-1} T^{l_k(c)+jq+i}c, \quad i = 0, 1, \ldots, q-1.$$

Obviously, $Td_i = d_{i+1}$, $i = 0, \ldots, q - 2$, and consequently (8.5) is satisfied. Furthermore,

$$(Td_{q-1} \cap d_0) \supset \left(\bigcup_{c \in \varkappa_0} \bigcup_{k=0}^{k(c)} \bigcup_{j=1}^{p-1} T^{l_k(c)+jq}c\right)$$

and therefore

$$\mu(Td_{q-1} \triangle d_0) < \frac{3}{p}\mu(d_0) < \frac{\delta}{q};$$

that is, (8.7) is also satisfied.

It remains to verify the "smallness" of $\eta^{(\delta)}$; that is, (8.4) and (8.6). Condition (8.4) easily follows from the next lemma.

LEMMA 8.1. *Let $c \in \kappa_0$, $l_{k+1}(c) > l > l_k(c) + (p+1)q$ and $l < N - pq$. Then*

$$T^l c \subset X \setminus \bigcup_{s=0}^{q-1} A\left(s, \frac{\delta}{2}\right).$$

PROOF. First, it is obvious that $T^l c$ either is wholly contained in one of the sets $A(s, \delta/2)$ or does not intersect one of these sets. Let $T^l c \subset A(s, \delta/2)$. Then by (4.21)

$$T^{l-s} \subset A\left(0, \frac{\delta}{2} + \frac{s}{pq}\right) \subset A\left(0, \frac{\delta}{2} + \frac{1}{p}\right)$$

and consequently $l_{k+1}(c) \leqslant l - s < l$, which contradicts the conditions of the lemma.

Lemma 8.1 implies that

$$d = X \setminus \bigcup_{i=0}^{q-1} d_k \subset \left(\bigcup_{c \in \kappa_0} \bigcup_{i=0}^{k(c)} \bigcup_{s=1}^{q} T^{l_i(c)+pq+s}\right)$$

$$\cup \left(\bigcup_{r=1}^{pq} T^{N-r} B\right) \cup \left(X \setminus \bigcup_{j=0}^{N-1} T^j B\right) \cup \left(X \setminus \bigcup_{s=0}^{q-1} A\left(s, \frac{\delta}{2}\right)\right).$$

Therefore $\mu(d) < 1/p + pq/N + \delta^2/2 + \delta^2/2 < \delta$.

Finally we verify (8.6). We denote by j_s the coordinate of $\omega^{(0)}$ with index $jq + s$, $j = 0, \ldots, p-1$, $s = 0, 1, \ldots, q-1$. Let $c \in \kappa_0$ and $k = 0, 1, \ldots, k(c)$. Since

$$T^{l_k(c)} c \subset A\left(0, \frac{\delta}{2} + \frac{1}{p}\right) \subset A(0, \delta),$$

amongst the pq sets of equal measure $T^{l_k(c)+jq+s}c$, $j = 0, \ldots, p-1$, $s = 0, \ldots, q-1$, there are not more than $(1-\delta)pq$ such that $T^{l_k(c)+jq+s}c \subset b_{j_s}$. Summing over $k = 0, 1, \ldots, k(c)$ and over $c \in \kappa_0$, we obtain

$$\sum_{i=1}^{q-1} \mu(d_i \setminus b_{j_i}) < \delta.$$

Theorem 3 is proved.

COROLLARY 8.1. *A quotient automorphism $T|_\xi$ of an automorphism T admitting a good a.p.t., relative to an infinite invariant measurable partition ξ, admits a good a.p.t.*

The property of H-almost periodicity, if stated in the language of functions, and not partitions, turns out to be more general than the property of almost periodicity of functions which arise as the trajectories of automorphisms with discrete spectrum, similar to the various classical notions of almost periodicity. We will not discuss the situation in detail but limit ourselves to a proof of the following assertion.

PROPOSITION 8.3. *An ergodic automorphism T with discrete spectrum is H-almost periodic.*

PROOF. By von Neumann's theorem on discrete spectrum (see §1.3) we may suppose that T is an ergodic (relative to Haar measure), and hence topologically transitive, translation on a compact commutative group G. On G, obviously, there exists a metric ρ invariant relative to all translations. Proposition 8.3 follows from the next lemma.

LEMMA 8.2. *Let X be a compact metric space with an infinite number of elements, T: $X \longrightarrow X$ a topologically transitive isometry of X, and μ a continuous Borel measure, positive on any open set and invariant relative to T. Then T is an H-almost periodic automorphism of the Lebesgue space (X, μ).*

PROOF. We will show that there is an exhaustive sequence of finite partitions such that for any partition η from this sequence

$$\mu\left(\bigcup_{c\in\eta}(\partial c)\right)=0. \tag{8.8}$$

Fix $\delta > 0$ and some finite cover of X by spheres of radius $\delta/4$. Obviously by increasing the radius of each sphere by a factor of not more than 2, we can obtain that the measure of the boundary of the new spheres is equal to zero. Let $U_1, \ldots, U_k$ be the elements of the new cover, η_i the partition of X into U_i and $X\backslash U_i$, and $\eta = \eta_1 \cdot \ldots \cdot \eta_k$. The diameter of all the elements of η is less than δ, and (8.8) is satisfied.

Thus by Proposition 8.1 it is sufficient to prove H-almost periodicity of (T, η) for any η satisfying (8.8). Fix a natural number p and $\epsilon > 0$. Let $\partial\eta = \bigcup_{c\in\eta} \partial c$, and let $U_\delta(A)$ be the δ-neighborhood of $A \subset X$. Choose $\delta > 0$ so that $\mu(U_\delta(\partial\eta)) < \epsilon^2/4$. By the topological transitivity of T there is a point $x_0 \in X\backslash U_\delta(\partial\eta)$ and a number r such that the points $x_0, Tx_0, \ldots, T^{r-1}x_0$ form a $\delta/2$-net in X. Further there is an arbitrarily large q (we choose $\epsilon^2 q/4 \geqslant r$) such that $d(x_0, T^q x_0) < \delta/2p$ (d is the metric in X). Denote

$$\omega^{(0)} = (\omega_0^{(0)}, \ldots, \omega_{pq-1}^{(0)}) \in \Omega_{|\eta|,pq},$$

where

$$\omega_{jq+i}^{(0)} = \varphi_{T,\eta}^{pq}(T^i x_0), \quad i=0, \ldots, q-1, \quad j=0, \ldots, p-1.$$

Let $x \in X$. We will find $k(x)$, depending measurably on x, so that $0 \leqslant k(x) < r$ and $d(T^{k(x)}x_0, x) < \delta/2$, and estimate

$$\int_X \rho^H(\varphi_{T,\eta}^{pq}(x), R^{k(x)}\omega^{(0)})\, d\mu.$$

Obviously

$$\begin{aligned} &d(T^{jq+i}x, T^{k(x)+i}x_0) \\ &\leqslant d(T^{jq+i}x, T^{jq+j+k(x)}x_0) + d(T^{k(x)+i}x_0, T^{jq+i+k(x)}x_0) \leqslant \frac{\delta}{2} + \frac{j\delta}{2p} \leqslant \delta. \end{aligned}$$

Therefore $T^{jq+i}x$ and $T^{k(x)+i}x_0$ can only lie in different elements of η if $T^{jq+i}x \in U_\delta(\partial\eta)$. Hence it follows (see (4.18) and (4.20)) that

$$\int_X \rho^H(\varphi_{T,\eta}^{pq}(x), R^{k(x)}\omega^{(0)})\, d\mu \leqslant \mu(U_\delta(\partial\eta)) + \frac{r}{q} < \frac{\varepsilon^2}{2},$$

and this means

$$\mu_{T,\eta}^{pq}\left(\bigcup_{j=0}^{r} B_{\varepsilon}^{H}\left(R^{j}\omega^{(0)}\right)\right) > 1-\varepsilon \tag{8.9}$$

(we recall that $r \leqslant q$).

Thus (T, η) is (p, q, ϵ)-periodic. Since p and q can be chosen arbitrarily large and $\epsilon > 0$ can be chosen arbitrarily small, the lemma is proved.

Theorem 3 and Proposition 8.3 immediately imply

COROLLARY 8.2. *An ergodic automorphism with discrete spectrum is standard.*

§9. M-trivial processes

Let n be a natural number, $\xi = \{c_1, \ldots, c_{|\xi|}\}$ a finite ordered measurable partition of (X, μ), T an automorphism of (X, μ), $\omega \in \Omega_{|\xi|,n}$ and $A(T, \xi, n, \epsilon, \omega) = (\varphi_{T,\xi}^{n})^{-1} B_{\epsilon}^{M}(\omega)$.

DEFINITION 9.1. A random process (T, ξ) is called *(n, ϵ)-trivial* if there is an $\omega \in \Omega_{|\xi|,n}$ such that

$$\mu\left(A\left(T, \xi, n, \varepsilon, \omega\right)\right) = \mu_{T,\xi}^{n}\left(B_{\varepsilon}^{M}(\omega)\right) \geqslant 1-\varepsilon. \tag{9.1}$$

The element ω is called an *n-standard* for the process (T, ξ).

The following lemma is a direct corollary of Čebyšev's inequality.

LEMMA 9.1. *If (T, ξ) is (n, ϵ)-trivial, then*

$$\int_X \rho^M\left(\varphi_{T,\xi}^{n}x, \omega\right) d\mu \leqslant 2\varepsilon.$$

If $\int_X \rho^M(\varphi_{T,\xi}^{n}x, \omega) = \beta$, then (T, ξ) is $(n, \beta^{1/2})$-trivial.

PROOF. For brevity we write $A(T, \xi, n, \gamma, \omega) = A_\gamma$. For $0 < \gamma < 1$ we have

$$0 \leqslant \int_{A_\gamma} \rho^M\left(\varphi_{T,\xi}^{n}x, \omega\right) \leqslant \gamma\mu\left(A_\gamma\right),$$

$$\gamma\left(1-\mu\left(A_\gamma\right)\right) \leqslant \int_{X\setminus A_\gamma} \rho^M\left(\varphi_{T,\xi}^{n}x, \omega\right) \leqslant 1-\mu\left(A_\gamma\right).$$

Putting $\gamma = \epsilon$ and taking the right-hand inequalities together with the assumption $\mu(A_\gamma) \geqslant 1 - \epsilon$, we obtain

$$\int_X \rho^M\left(\varphi_{T,\xi}^{n}x, \omega\right) \leqslant \varepsilon\left(1-\varepsilon\right) + \varepsilon < 2\varepsilon.$$

Putting $\gamma = \beta^{1/2}$ and taking the left-hand inequalities, we obtain

$$\beta^{1/2}\left(1-\mu\left(A_{\beta^{\frac{1}{2}}}\right)\right) \leqslant \beta, \quad \mu\left(A_{\beta^{\frac{1}{2}}}\right) \geqslant 1-\beta^{1/2}.$$

PROPOSITION 9.1. *If (T, ξ) is (n, ϵ)-trivial and $m = kn + r$, where $k \geqslant 1$ and $0 \leqslant r \leqslant n - 1$, then (T, ξ) is $(m, (2\epsilon + r/m)^{1/2})$-trivial; moreover, as an m-standard we may choose an element of the form $\omega^k * \alpha$, where $\omega \in \Omega_{|\xi|,n}$ and $\alpha \in \Omega_{|\xi|,r}$.*

PROOF. We first consider the case $r = 0$. By (4.6), for $x \in X$ we have

$$\rho^M(\varphi_{T,\xi}^{kn}x,\ \omega^k) \leqslant \frac{1}{k}\sum_{i=0}^{k-1} \rho^M(\varphi_{T,\xi}^n(T^{in}x),\ \omega).$$

Since

$$\int_X \rho^M(\varphi_{T,\xi}^n(T^{in}x),\ \omega)\,d\mu = \int_X \rho^M(\varphi_{T,\xi}^n x,\ \omega)\,d\mu,$$

it follows that

$$\int_X \rho^M(\varphi_{T,\xi}^{kn}x,\ \omega^k)\,d\mu \leqslant \int_X \rho^M(\varphi_{T,\xi}^n x,\ \omega)\,d\mu. \tag{9.2}$$

Applying Lemma 9.1, we obtain the proposition for $r = 0$. In the general case let $\bar{\omega} = \omega^k * \alpha \in \Omega_{|\xi|,m}$, where $\alpha \in \Omega_{|\xi|,r}$ is any element. Obviously

$$\rho^M(\varphi_{T,\xi}^m x,\ \bar{\omega}) \leqslant \rho^M(\varphi_{T,\xi}^{kn},\ \omega^k) + \frac{r}{m}. \tag{9.3}$$

Comparing (9.2) and (9.3) and applying Lemma 9.1, we obtain the desired result.

DEFINITION 9.2. A random process (T, ξ) is called *M-trivial* if for any $\epsilon > 0$ there is an $n_0(\epsilon)$ such that for $n > n_0(\epsilon)$ the process (T, ξ) is (n, ϵ)-trivial.

Proposition 9.1 immediately implies

COROLLARY 9.1. *If there are sequences k_n and $\epsilon_n \longrightarrow 0$ such that (T, ξ) is (k_n, ϵ_n)-trivial, then (T, ξ) is M-trivial.*

PROPOSITION 9.2. *Let $\{\eta_n\}, n = 1, 2, \ldots$, be an exhaustive sequence of finite partitions of X. If the random processes (T, η_n) are (k_n, ϵ_n)-trivial, where $k_n \longrightarrow \infty$ and $\epsilon_n \longrightarrow 0$, then (T, ξ) is M-trivial for any partition $\xi = \{c_1, \ldots, c_m\}$.*

PROOF. Fix $\epsilon > 0$. Choose n so that for some partition $\xi_n = \{c_1^n, \ldots, c_m^n\}$ we have $\xi_n < \eta_n$ and

$$\sum_{i=1}^m \mu(c_i \triangle c_i^n) < \frac{\varepsilon^2}{4};$$

in addition, (T, η_n), and consequently (T, ξ_n), is $(l_n, \epsilon/2)$-trivial for some l_n. Repeating the proof of Proposition 8.1, we obtain that (T, ξ) is (l_n, ϵ)-trivial. By Corollary 9.1 the process (T, ξ) is M-trivial.

PROPOSITION 9.3. *Let (T, ξ) be M-trivial and $\eta < \xi_T$ a finite partition. Then (T, η) is M-trivial.*

PROOF. Passing to the quotient automorphism $T|_{\xi_T}$, we will suppose that ξ is a generator. By Proposition 9.2 it is sufficient to prove that the processes (T, ξ^k) are M-trivial (compare with the proof of Proposition 8.2). But this follows from the definition of M-triviality, and from (4.15) and (4.17).

COROLLARY 9.2. *An M-trivial random process (T, ξ) is ergodic.*

PROOF. If $A \in \mathfrak{A}(\xi_T)$ is an invariant set and $\eta = \{A, X \backslash A\}$, then the conditions of

Definition 9.2 are obviously not satisfied. Thus the ergodicity of (T, ξ) follows from Proposition 9.3.

PROPOSITION 9.4. *If T is an automorphism such that (T, ξ) is M-trivial for any finite ξ, $m \in L^1(X, \mu, \mathbf{Z}_0^+)$ and η is any finite partition of $X_{m(\cdot)}$, then $(T_{m(\cdot)}, \eta)$ is M-trivial.*

PROOF. Proposition 3.3 implies that m may be supposed to be bounded. By Proposition 9.2 it is sufficient to prove the M-triviality of the processes $(T_{m(\cdot)}, \xi_{m(\cdot)})$, where ξ is a finite partition of X and m is constant on the elements of ξ. Denote

$$\beta = \int_X m\, d\mu, \qquad K = \max_X m(x).$$

Fix $\epsilon > 0$ and denote

$$D_{n,\varepsilon} = \left\{ x \in X : \left| \frac{1}{n} \sum_{i=0}^{n-1} m(T^i x) - \beta \right| < \frac{\varepsilon\beta}{100} \right\}.$$

By the ergodicity of T (Corollary 9.2) we can choose $n_0(\epsilon)$ so that for $n > n_0(\epsilon)$

$$\mu(D_{n,\varepsilon}) > 1 - \frac{\varepsilon\beta}{100K}. \tag{9.4}$$

Let $n > \max(2/\beta, 100K/\epsilon\beta, n_0(\epsilon))$ be chosen so that (T, ξ) is $(n, \epsilon\beta/100K)$-trivial. In this case from (9.1), (9.4) and the inequality $\mu_{m(\cdot)}(E_{m(\cdot)}) \leqslant K\mu(E)/\beta$ for any $E \subset X$, it follows that

$$\begin{aligned} &\mu_{m(\cdot)}\left[\left(D_{n,\varepsilon} \cap A\left(T, \xi, n, \frac{\varepsilon\beta}{100K}, \omega\right)\right)_{m(\cdot)}\right] \\ &\geqslant 1 - \frac{K}{\beta}\left[\left(\frac{\varepsilon\beta}{100K}\right) + \left(\frac{\varepsilon\beta}{100K}\right)\right] \geqslant 1 - \frac{\varepsilon}{50}. \end{aligned} \tag{9.5}$$

Denote $m_n(x) = \Sigma_{i=0}^{n-1} m(T^i x)$. Let $y_1 = (x_1, k_1), y_2 = (x_2, k_2), y_1, y_2 \in X_{m(\cdot)}$, and

$$x_1, x_2 \in D_{n,\varepsilon} \cap A\left(T, \xi, n, \frac{\varepsilon\beta}{100K}, \omega\right). \tag{9.6}$$

Since $y_1 = T^{k_1}_{m(\cdot)}(x_1, 1)$ and $y_2 = T^{k_2}_{m(\cdot)}(x_2, 1)$, $k_1, k_2 \leqslant K$, by the choice of n we get

$$\begin{aligned} &\rho^M\left(\varphi^{[n\beta]}_{T_{m(\cdot)}, \xi_{m(\cdot)}} y_1, \varphi^{[n\beta]}_{T_{m(\cdot)}, \xi_{m(\cdot)}} y_2\right) \\ &\leqslant \frac{\varepsilon}{100} + \rho^M\left(\varphi^{[n\beta]}_{T_{m(\cdot)}, \xi_{m(\cdot)}}(x_1, 1), \varphi^{[n\beta]}_{T_{m(\cdot)}, \xi_{m(\cdot)}}(x_2, 1)\right); \end{aligned} \tag{9.7}$$

therefore for the proof of $([n\beta], \epsilon)$-triviality of $(T_{m(\cdot)}, \xi_{m(\cdot)})$ it is sufficient to estimate the second term on the right side of (9.7). By (4.22) and the remark at the end of §4 we have

$$O^M\left(\varphi^{m_n(x_1)}_{T_{m(\cdot)}, \xi_{m(\cdot)}}(x_1, 1), \varphi^{m_n(x_2)}_{T_{m(\cdot)}, \xi_{m(\cdot)}}(x_2, 1)\right) \leqslant K O^M\left(\varphi^n_{T,\xi} x_1, \varphi^n_{T,\xi} x_2\right). \tag{9.8}$$

We pass to the estimate:

$$\rho^M(\varphi^{[n\beta]}_{T_{m(\cdot)},\xi_{m(\cdot)}}(x_1, 1), \varphi^{[n\beta]}_{T_{m(\cdot)},\xi_{m(\cdot)}}(x_2, 1))$$
$$\leqslant \frac{|m_n(x_1)-[n\beta]|+|m_n(x_2)-[n\beta]|+\frac{1}{2}O^M(\varphi^{m_n(x_1)}_{T_{m(\cdot)},\xi_{m(\cdot)}}(x_1, 1), \varphi^{m_n(x_2)}_{T_{m(\cdot)},\xi_{m(\cdot)}}(x_2, 1))}{[n\beta]}$$
$$\leqslant \frac{2+\frac{n\varepsilon\beta}{50}+\frac{1}{2}KO^M(\varphi^n_{T,\xi}x_1, \varphi^n_{T,\xi}x_2)}{[n\beta]} \leqslant \frac{\varepsilon}{25}+\frac{\varepsilon}{25}+\frac{\varepsilon}{50}<\frac{\varepsilon}{5}. \tag{9.9}$$

The first inequality follows from the definition of ρ^M, the second from the definition of $D_{n,\epsilon}$ and (9.8), and the third from the conditions on n and (9.6). It follows from (9.6), (9.7) and (9.9) that $(T_{m(\cdot)}, \xi_{m(\cdot)})$ is $([n\beta], \epsilon)$-trivial.

LEMMA 9.2. *An H-almost periodic random process is M-trivial.*

PROOF. Let (T, ξ) be (p, q, ϵ)-periodic, and let

$$\varphi^{pq}_{T,\xi}x \in \bigcup_{s=0}^{q-1} B^H_\varepsilon(R^s\omega^{(0)}).$$

By (4.20)

$$\rho^M(\varphi^{pq}_{T,\xi}x, \omega^{(0)}) \leqslant \varepsilon+\frac{1}{p};$$

that is, (T, ξ) is $(pq, \epsilon + 1/p)$-trivial. The lemma now follows from Proposition 9.2.

Lemma 9.2, Theorem 3 and Propositions 9.3 and 9.4 imply

COROLLARY 9.3. *If T is standard, then for any finite partition ξ the random process (T, ξ) is M-trivial.*

§10. M-triviality and standardness

THEOREM 4. *The following assertions are equivalent:*

1. *The automorphism T is standard.*
2. *For any finite ordered measurable partition ξ the random process (T, ξ) is M-trivial.*
3. *For an exhaustive sequence of partitions $\{\xi_n\}$, $n = 1, 2, \ldots$, the random processes (T, ξ_n) are M-trivial.*
4. *For any generating partition ξ the random process (T, ξ) is M-trivial.*

PROOF. The equivalence of properties 2, 3 and 4 has already been proved (Propositions 9.2 and 9.3). By Corollary 9.3, 2 follows from 1. Thus it remains only to derive 1 from 2. For this we construct an H-almost periodic automorphism monotonely equivalent to an automorphism satisfying 2, and then use Theorem 3.

LEMMA 10.1. *Let $T: (X, \mu) \longrightarrow (X, \mu)$ be an automorphism, $\xi = \{b_1, \ldots, b_{|\xi|}\}$ a partition of X, p and q natural numbers, and $\epsilon > 0$. If (T, ξ) is (pq, ϵ)-trivial with a pq-standard of the form ω^p, where $\omega \in \Omega_{|\xi|,q}$, then there is a function $\hat{m} \in L^1(X, \mu, \mathbf{Z}_0^+)$ and a partition $\hat{\xi} = \{\hat{b}_1, \ldots, \hat{b}_{|\xi|}\}$ of $X_{\hat{m}(\cdot)}$ such that*

$$\|\hat{m}-1\|_{L^1} < 4\varepsilon, \tag{10.1}$$

$$\mu\left(\bigcup_{i=1}^{|\xi|}(b_i \cap \hat{b}_i)\right) > 1-4\varepsilon, \tag{10.2}$$

the random process $(T_{\hat{m}(\cdot)}, \hat{\xi})$ *is* $(p, q, 0)$*-periodic.* (10.3)

PROOF. Fix $N > 100pq/\epsilon$ and construct a set $B \subset X$ such that $T^jB \cap B = \emptyset$, $j = 1, \ldots, N-1$ and $\mu(\bigcup_{j=0}^{N-1} T^jB) > 1 - \epsilon/100$. As in the proof of Theorem 3, we introduce the partition

$$\varkappa_0 = \left\{c \cap B : c \in \bigvee_{i=0}^{N-1} T^{-i}\xi;\ X\setminus B\right\} \text{ and } \varkappa = \bigvee_{i=0}^{N-1} T^i\varkappa_0.$$

For brevity we set $A(T, \xi, pq, \epsilon, \omega^p) = A_\epsilon$. Let $c \in \kappa_0$, $c \subset B$. Put

$$l_0(c) = \min\{l : 0 \leq l \leq N-pq : T^lc \subset A_\varepsilon\} \tag{10.4}$$

and inductively

$$l_k(c) = \min\{l : l_{k-1}(c) + pq \leq l \leq N-pq,\ T^lc \subset A_\varepsilon\} \tag{10.5}$$

for as long as $l_k(c)$ can be defined, say to $k = k(c)$. Fix c and i and consider the set $T^{l_i(c)+j}c$, $j = 0, \ldots, pq-1$. Let $\varphi^{pq}_{T,\xi}(T^{l_i(c)}c) = \omega^{(0)}$ (we omit the dependence on c, i, T, ξ and pq in the notation). Since $T^{l_i(c)}c \subset A_\epsilon$, it follows that $O^M(\omega^{(0)}, \omega^p) \leqslant 2\epsilon pq$. Consider a minimal sequence of $2s \leqslant 2\epsilon pq$ elementary M-operations, transforming $\omega^{(0)}$ to ω^p. These elementary M-operations can be realized as follows: first we insert into $\omega^{(0)}$, in suitable positions, s new coordinates and obtain successively $\omega^{(1)} \in \Omega_{|\xi|,pq+1}, \ldots, \omega^{(s)} \in \Omega_{|\xi|,pq+s}$; and then we delete a certain s coordinates from $\omega^{(s)}$ and obtain successively $\omega^{(s+1)} \in \Omega_{|\xi|,pq+s-1}, \ldots, \omega^{(2s)} = \omega^p$. Moreover, if the sequence of M-operations is minimal, then none of the inserted coordinates is deleted.

For each $k = 0, 1, \ldots, 2s-1$ we will construct a space $X^{(k)}$, a system of sets $c_j^{(k)} \subset X^{(k)}$, $j = 0, 1, \ldots, pq-1+\min(k, 2s-k)$, an automorphism $T^{(k)}: X^{(k)} \longrightarrow X^{(k)}$ and a partition $\xi^{(k)} = \{b_1^{(k)}, \ldots, b_{|\xi|}^{(k)}\}$ such that $T^{(k)}c_j^{(k)} = c_{j+1}^{(k)}$, and $c_j^{(k)} \in b_{\omega_j^{(k)}}$, where $\omega_j^{(k)}$ is the jth coordinate of $\omega^{(k)}$.

For $k = 0$, put $X^{(0)} = X$ and $c_j^{(0)} = T^{l_i(c)+j}c$, $j = 0, \ldots, pq-1$, and let $T^{(0)}$ and $\xi^{(0)}$ coincide with T and ξ on $\bigcup_{j=0}^{pq-1} c_j^{(0)}$. We describe the passage from k to $k+1$. If $k < s$, then $\omega^{(k+1)}$ is obtained from $\omega^{(k)}$ by the insertion of some, say the r_kth, coordinate equal to $\omega_{r_k}^{(k+1)}$. Let $X^{(k+1)} = X^{(k)}_{m_k(\cdot)}$, where for $r_k \neq 0$

$$m_k(x) = \begin{cases} 2, & x \in c_{r_k-1}^{(k)}, \\ 1 & \text{for other } x, \end{cases}$$

and for $r_k = 0$

$$m_k(x) = \begin{cases} 2, & x \in (T^{(k)})^{-1}c_0^{(k)}, \\ 1 & \text{for other } x. \end{cases}$$

Further, we put $T^{(k+1)} = T^{(k)}_{m_k(\cdot)}$, $c_j^{(k+1)} = c_j^{(k)}$, $j = 0, 1, \ldots, r_k - 1$, $c_{r_k}^{(k+1)} = X^{(k+1)}\setminus X^{(k)}$, $c_j^{(k+1)} = c_{j-1}^{(k)}$, $j = r_k+1, \ldots, pq+k$, $b_l^{(k+1)} = b_l^{(k)}$ if $l \neq \omega_{r_k}^{(k+1)}$, and

$$b^{(k+1)}_{\omega^{(k+1)}_{r_k}} = b^{(k)}_{\omega^{(k+1)}_{r_k}} \cup c^{(k+1)}_{r_k}.$$

For $k \geqslant s$, $\omega^{(k+1)}$ is obtained from $\omega^{(k)}$ by striking out a coordinate whose index, as before, we denote by r_k. In this case we put $X^{(k+1)} = X^{(k)} \backslash c^{(k)}_{r_k}$, $T^{(k+1)} = T^{(k)}_{X^{(k+1)}}$, $c_j^{(k+1)} = c_j^{(k)}$, $j = 0, 1, \ldots, r_k - 1$, $c_j^{(k+1)} = c^{(k)}_{j+1}$, $j = r_k, \ldots, pq + 2s - k - 2$, $\xi^{(k+1)} = \xi^{(k)}|_{X^{(k+1)}}$. By construction $(T^{(2s)}, \xi^{(2s)})$ has the following property:

$$\varphi^{pq}_{T^{(2s)}, \xi^{(2s)}} (c_0^{(2s)}) = \omega^p. \tag{10.6}$$

Further, $T^{(2s)}$ has the form $T_{\tilde{m}(\cdot)}$, and, since at each stage we add or remove a set of measure (in X) equal to the measure of c,

$$\| \tilde{m} - 1 \|_{L^1} \leqslant 2s\mu(c). \tag{10.7}$$

From the same considerations we have

$$\mu \left(\bigcup_{i=1}^{|\xi|} (b_i^{(2s)} \cap b_i) \right) \geqslant 1 - 2s\mu(c). \tag{10.8}$$

Denote

$$\hat{A}_\varepsilon = \bigcup_{c \in \varkappa_0} \bigcup_{i=0}^{k(c)} \bigcup_{j=0}^{pq-1} T^{l_i(c)+j} c. \tag{10.9}$$

We pass to the derived automorphism $T_{\hat{A}_\varepsilon}$, and repeat the above procedure for each $c \in \kappa_0$ and $i = 0, \ldots, k(c)$. As a result of these reconstructions we obtain an automorphism $T_{\hat{m}(\cdot)}$ and a partition $\hat{\xi} = \{\hat{b}_1, \ldots, \hat{b}_{|\xi|}\}$. By virtue of (10.4)–(10.6) and (10.9), for any $x \in X_{\hat{m}(\cdot)}$ we have

$$\varphi^{pq}_{T_{\hat{m}(\cdot)}, \hat{\xi}} (x) = R^s \omega^p$$

for some $s \in \{0, 1, \ldots, q - 1\}$; that is, $(T_{\hat{m}(\cdot)}, \hat{\xi})$ is $(p, q, 0)$-periodic. We note that, in general, $\hat{\xi} \neq \xi_{\hat{m}(\cdot)}$. We will estimate $\|\hat{m} - 1\|_{L^1}$. By (10.4), (10.5) and (10.9)

$$\hat{A}_\varepsilon \supset \left(A_\varepsilon \cap \left(\bigcup_{j=0}^{N-pq} T^j B \right) \right).$$

Therefore by the choice of N and B it follows that

$$\mu(\hat{A}_\varepsilon) > 1 - 2\varepsilon. \tag{10.10}$$

Summing (10.7) over all $c \in \kappa_0$ and $i = 0, 1, \ldots, k(c)$, taking account of (10.9) and the inequality $s \leqslant \epsilon pq$, we obtain that on $\hat{A}_\epsilon$

$$\| \hat{m} - 1 \|_{L^1} \leqslant \varepsilon. \tag{10.11}$$

(10.1) follows from (10.10) and (10.11). (10.2) follows in the same way from (10.8) and (10.10). The lemma is proved.

LEMMA 10.2. *Let T: $(X, \mu) \longrightarrow (X, \mu)$ be an automorphism, $\xi = \{c_1, \ldots, c_{|\xi|}\}$ a partition of X, $m \in L^1(X, \mu, \mathbf{Z}_0^+)$, $\|m - 1\|_{L^1} = \beta$, and $\eta = \{d_1, \ldots, d_{|\xi|}\}$ a partition of $X_{m(\cdot)}$, where*

$$\mu\left(\bigcup_{i=1}^{|\xi|}(c_i \cap d_i)\right)=1-\gamma. \tag{10.12}$$

If (T, ξ) is (p, q, ϵ)-periodic, then $(T_{m(\cdot)}, \eta)$ is $(p, q, (1-\beta)^{-1}(\epsilon + 2pq\beta + \gamma^{1/2}))$-periodic.

PROOF. Let $K \subset X$, $K = \{x: m(T^i x) = 1, i = 0, 1, \ldots, 2pq - 1\}$. Obviously $\mu(K) \geqslant 1 - 2pq\beta$, and for $x \in K$

$$\varphi^{pq}_{T,\xi}x = \varphi^{pq}_{T_{m(\cdot)},\xi_{m(\cdot)}}(x, 1). \tag{10.13}$$

Since $\xi_{m(\cdot)} = \xi$ on K, by (10.12) and (4.18) we have

$$\int_K \rho^H(\varphi^{pq}_{T,\xi}x, \varphi^{pq}_{T_{m(\cdot)},\eta}(x, 1))\, d\mu \leqslant \gamma. \tag{10.14}$$

The assertion of the lemma now follows from the (p, q, ϵ)-periodicity of (T, ξ), (10.13), (10.14) and the inequality $\mu_{m(\cdot)}(A) \leqslant (1-\beta)^{-1}\mu(A)$ for any $A \subset X \cap X_{m(\cdot)}$.

The proof of Theorem 4 is now completed by an inductive application of Lemma 10.1, taking account of Lemma 10.2. Let T satisfy 2. For $n = 1, 2, \ldots$, we will construct a function $m_n \in L^1(X, \mu, \mathbf{Z}_0^+)$, natural numbers p_n, q_n and a partition $\xi_{(n)}$ of $X_{m_n(\cdot)}$ satisfying the following conditions:

$$\text{the random process } (T_{m_n(\cdot)}, \xi_{(n)}) \text{ is } (p_n, q_n, 0)\text{-periodic;} \tag{10.15}$$

$$\| m_{n+1} - m_n \|_{L^1} \leqslant \frac{1}{100 \cdot 2^n p_n q_n}; \tag{10.16}$$

$$p_n \geqslant 2^n, \quad q_n \geqslant 2^n. \tag{10.17}$$

Let $m = \lim_{n\to\infty} m_n$. There is a basis $\{A_s\}$, $s = 1, 2, \ldots$, of $\mathfrak{A}(X_{m(\cdot)})$ such that the sets $A_s \cap X_{m_n(\cdot)}$, $s = 1, \ldots, n$, are approximated up to $1/2^n$ by sets of $\mathfrak{A}(\xi_{(n)})$: (10.18)

We will deduce from these conditions the H-almost periodicity of $T_{m(\cdot)}$. Let $T_{m(\cdot)} = (T_{m_n(\cdot)})_{\overline{m}_n(\cdot)}$. It follows from (10.16) that

$$\| \overline{m}_n - 1 \|_{L^1} < \frac{1}{10 \cdot 2^n p_n q_n}; \tag{10.19}$$

therefore (10.15) and Lemma 10.2 imply that $(T_{m(\cdot)}, (\xi_{(n)})_{\overline{m}_n(\cdot)})$ is $(p_n, c_n, 2^{-n})$-periodic.

Finally, (10.18) and (10.19) imply that $(\xi_{(n)})_{\overline{m}_n(\cdot)}$ is an exhaustive sequence of partitions of $X_{m(\cdot)}$, and consequently, by Proposition 8.1, T is H-almost periodic.

It only remains for us to construct m_n, p_n, q_n and $\xi_{(n)}$. For $n = 1$ we can use Lemma 10.1 directly. We carry out the passage from n to $n + 1$. We will suppose that for $k = 1, \ldots, n$ we have chosen bases $\mathfrak{A}_k = \{A^k_s\}$, $s = 1, 2, \ldots$, in the σ-algebras of measurable sets of the spaces $X_{m_k(\cdot)}$. By Proposition 9.4, for any partition η of $X_{m_n(\cdot)}$, $(T_{m_n(\cdot)}, \eta)$ is M-trivial. Choose a partition $\eta_{(n+1)}$ of $X_{m_n(\cdot)}$ so fine that the sets $A^k_s \cap X_{m_n(\cdot)}$, $k, s = 1, \ldots, n$, are approximated by sets of $\mathfrak{A}(\eta_{(n+1)})$ up to $1/100 \cdot 2^n$. We use Proposition 9.1 and choose p_{n+1} and q_{n+1}, satisfying (10.17), so that $(T_{m_n(\cdot)}, \eta_{(n+1)})$ is $(p_{n+1}, q_{n+1}, 1/800 \cdot 2^{n+1} p_n q_n)$-trivial with standard of the form $\omega^{p_{n+1}}$, $\omega \in \Omega_{|\eta_{(n+1)}|, q_{n+1}}$. With the help of Lemma 10.1 we will construct a function $\hat{m}_n$ on $X_{m_n(\cdot)}$ and a partition $\xi_{(n+1)}$ of $(X_{m_n(\cdot)})_{\hat{m}_n(\cdot)}$. We denote $m_{n+1} = m_n * \hat{m}_n$. By Lemma 10.1, conditions (10.15) and (10.16) are satisfied for $n+1$. By (10.2), any set $A_s^k \cap X_{m_{n+1}(\cdot)}$, $k, s = 1, \ldots, n$, is approximated by a set from $\mathfrak{A}(\xi_{(n+1)})$ up to $1/10 \cdot 2^n$. We denote $B_s^k = A_s^k \cap X_{m(\cdot)}$, and let A_s be the sth element of the sequence

$$B_1^1, B_1^2, B_1^2, \ldots, B_1^n, B_2^{n-1}, \ldots, B_n^1, \ldots .$$

The sets A_s, $s = 1, 2, \ldots$, form a basis of $\mathfrak{A}(X_{m(\cdot)})$, and, by (10.19), condition (10.18) is satisfied for this basis. Theorem 4 is proved.

We mention some immediate corollaries of Theorem 4.

COROLLARY 10.1. *A quotient automorphism of a standard automorphism relative to any infinite invariant measurable partition is a standard automorphism.*

PROOF. Condition 2 of Theorem 4 is clearly invariant under the passage to a quotient automorphism.

COROLLARY 10.2. *A projective limit of a sequence of standard automorphisms is a standard automorphism.*

PROOF. Let $X = \lim X_n$, $T_n: X_n \longrightarrow X_n$ and $T = \lim T_n$. Since $\mathfrak{A}(X_n) \hookrightarrow \mathfrak{A}(X)$ and $\overline{\bigcup \mathfrak{A}(X_n)} = \mathfrak{A}(X)$, condition 3 is satisfied for T.

COROLLARY 10.3. *Let T and S be automorphisms of (X, μ), $S^k = T$, $k \in \mathbf{Z}$ and T standard. Then S is standard.*

PROOF. For $k = -1$ the assertion follows immediately from the definition of a standard automorphism. In fact, if $T \sim \mathbf{D}_{m(\cdot)}$, then $T^{-1} \sim (\mathbf{D}^{-1})_{\bar{m}(\cdot)}$, where $\bar{m}(x) = m(T^{-1}x)$. Since $\mathbf{D}^{-1} \sim \mathbf{D}$, it follows that $T^{-1} \overset{M}{\sim} \mathbf{D}$. Therefore it is sufficient to consider the case $k > 1$. Fix $\xi = \{c_1, \ldots, c_{|\xi|}\}$ and consider ξ_S^k with lexicographic ordering. We have

$$\varphi_{S,\xi}^{kn} = K_{|\xi|}^k \cdot \varphi_{T,\xi_S^k}^n. \tag{10.20}$$

Choose n so that (T, ξ_S^k) is (n, ϵ)-trivial with standard $\omega \in \Omega_{|\xi|^k,n}$. Then from (10.20) and (4.13) it follows that (S, ξ) is $(k \cdot n, \epsilon)$-trivial with standard $K_{|\xi|}^k \omega \in \Omega_{|\xi|,kn}$. By Theorem 4, S is standard.

§11. Concluding remarks

1. From what we have proved in this paper it follows that the class of standard automorphisms is sufficiently broad. Thus, even amongst automorphisms admitting a good approximation, there are automorphisms with such varied and sometimes "pathological" metric properties (see [25] and [27]) that the metric classification of such automorphisms is impossible. Corollaries 10.1–10.3 show that the class of standard automorphisms, which is defined as the minimal class of ergodic automorphisms containing $\mathbf{D}$ and closed relative to the operations of passing to an induced or special automorphism, is in fact closed relative to some other natural operations. Further, based on Theorem 4, we have succeeded in proving that the class of standard automorphisms is closed relative to the passage to any

ergodic compact group extension (4) or finite extension (see [17]). From these statements we may easily deduce the standardness (in almost any ergodic component) of a rational power of a standard automorphism, and any, except the identity, automorphism of a standard flow; also, the standardness of classes of concrete automorphisms and flows, in particular, ergodic automorphisms with quasi-discrete spectrum [28], ergodic nilflows [29], and minimal distal automorphisms and flows [30].

2. The first example of a nonstandard ergodic automorphism with zero entropy was constructed by Feldman [37]. His construction is inductive and includes in each inductive step a number of parameters. Let $f(n)$, $n = 1, 2, \ldots$, be any (arbitrarily quickly increasing) sequence of positive numbers. It turns out that by choosing a suitable form of parameters in Feldman's construction, it is possible to guarantee that the automorphism admits an a.p.t. I or a.p.t. II with speed $f(n)$ (see Definition 1.1 of [25]). Of course these approximations will not be cyclic.

Feldman's construction (more precisely, a slight modification) admits a smooth realization. Let M^m be a connected manifold of class C^∞ (not necessarily compact and possibly with boundary) in which there is a nontrivial C^∞ action $\{T_t\}$, $0 \leqslant t \leqslant 1$, $T_1 = \mathrm{id}$, of S^1, $m = \dim M^m > 1$, and let μ be a finite measure on M^m given in any coordinate neighborhood of positive measure by a smooth density of class C^∞, and invariant relative to $\{T_t\}$. Then in any C^r-neighborhood of any diffeomoephism T_α it is possible to construct a C^∞-diffeomoephism, preserving μ, ergodic relative to μ and metrically isomorphic to an automorphism obtained by Feldman's construction. The construction of such a diffeomorphism can be realized by a version of the construction in §3 of [31]. The basic difference is in the replacement of the Second Step, which resulted in the existence of a cyclic a.p.t. with high speed, by some other condition.

Further, using the methods of [32] and [33], we can construct on any compact smooth manifold of dimension 3 or higher a C^∞-flow, preserving a given smooth positive measure, ergodic relative to this measure, with zero entropy and which is nonstandard. From the metric viewpoint such a flow is isomorphic to a special flow over the automorphism described in the previous paragraph.

From the purely metric aspect, Feldman's construction also admits various modifications and generalizations. Thus Rudolph and the author have independently constructed a family consisting of a continuum of ergodic automorphisms with zero entropy no two of which are monotonely equivalent or even connected by the majorizing relation (§2). Rudolph's construction [38] is based on a nonsymmetry of Feldman's construction, in each inductive step, relative to time inversion. Our construction uses some general invariants of monotone equivalence, which we call invariants of entropy type.

Let T be an automorphism, ξ a partition (that is, as always above, a finite ordered measurable partition), ϵ and δ positive numbers, and n a natural number. We denote by $N^M(T, \xi, \epsilon, \delta, n)$ the minimal number of spheres of radius ϵ in the metric ρ^M on $\Omega_{|\xi|,n}$ whose union has $\mu^n_{T,\xi}$-measure not less than $1 - \delta$. From (4.15) and (4.17) it follows that for any natural number k

(4)This has also been proved by B. Weiss.

$$N^M\left(T, \xi^k, \frac{nk}{n-k+1}\varepsilon, \delta, n-k+1\right) \leqslant N^M(T, \xi, \varepsilon, \delta, n). \tag{11.1}$$

Further, let $\xi = (c_1, \ldots, c_m)$, $\eta = (d_1, \ldots, d_m)$ and $\Sigma_1^m \mu(c_i \Delta d_i) = \beta$. Then

$$N^M(T, \eta, \varepsilon + \sqrt{\beta}, \delta + \sqrt{\beta}, n) \leqslant N^M(T, \xi, \varepsilon, \delta, n). \tag{11.2}$$

Let $m \in L^1(X, \mu, \mathbf{Z}_0^+)$ be a bounded function, where $\int_X m\, d\mu = \gamma$ and $\max_X m(x) = K$. Then, if (T, ξ) is ergodic, there is a sequence $\alpha(n) \longrightarrow 0$ such that

$$N^M\left(T_{m(\cdot)}, \xi_{m(\cdot)}, \frac{2K}{\gamma}\varepsilon, \delta + \alpha(n), [\gamma n]\right) \leqslant N^M(T, \xi, \varepsilon, \delta, n). \tag{11.3}$$

Now suppose that we have fixed some sequence of natural numbers n_k, $k = 1, 2, \ldots$. We call a sequence m_k *equivalent* to n_k if

$$0 < \underline{\lim} \frac{m_k}{n_k} \leqslant \overline{\lim} \frac{m_k}{n_k} < \infty.$$

We now consider the asymptotic character (in k) of the behavior of the values $N^M(T, \xi, \delta, m_k)$ for all possible sequences m_k equivalent to a given n_k as $\epsilon, \delta \longrightarrow 0$. From (11.1)–(11.3) it follows that this character does not depend on the choice of generator ξ and is invariant relative to monotone equivalence. On passage to a quotient automorphism the asymptotics can only become "slower". The monotone invariants obtained from these asymptotics are called *invariants of entropy type.* Certainly the definition of these invariants demands concretization, which may depend on the situation in which they are applied. We will explain how to carry out this concretization for the construction of the examples mentioned above. We construct a certain sequence n_k such that

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty,$$

then partition this sequence into a countable set of subsequences n_k^i, $i = 1, 2, \ldots$. Further, we construct a sequence l_k^i such that $l_k^i \longrightarrow \infty$ as $k \longrightarrow \infty$, and for each sequence $\Lambda = (\lambda_1, \lambda_2, \ldots)$, $0 < \lambda_i \leqslant 1$, an automorphism $T^{(\Lambda)}$ and a generator $\xi^{(\Lambda)}$ such that for $i = 1, 2, \ldots$, for any $\epsilon, \delta > 0$, and for any sequence m_k^i equivalent to n_k^i

$$\lim \frac{N(T^{(\Lambda)}, \xi^{(\Lambda)}, \varepsilon, \delta, m_k^i)}{(l_k^i)^{\lambda_i}} = 1.$$

From (11.1)–(11.3) it is simple to deduce that the automorphisms $T^{(\Lambda)}$ corresponding to different sequences Λ are not monotonely equivalent. Further, if $T^{(\Lambda_1)} \prec T^{(\Lambda_2)}$, then each term in the sequence Λ_1 does not exceed the corresponding term of Λ_2.

In connection with invariants of entropy type we note the following result of Sataev and the author [34]: if for some sequence $n_k \longrightarrow \infty$

$$N(T, \xi, \varepsilon, \delta, n_k) < c(T, \xi, \varepsilon, \delta),$$

then (T, ξ) is M-trivial. From this follows the standardness of the interval shifting automorphisms (see [8], Chapter 4, §3) relative to any continuous ergodic Borel invariant measure, and also of the C^1-flows on two-dimensional surfaces having a finite number of fixed points and separatrices relative to ergodic invariant measures whose support contains an open set.

Returning to nonstandard automorphisms, we note that Ornstein and Rudolph have constructed an example of a standard automorphism T for which the cartesian square $T \times T$ is ergodic but nonstandard. This example disproves certain conjectures on standard automorphisms which appeared to be quite natural.

We list a series of unsolved problems on the connection between standardness and the commutativity properties of automorphisms.

Let T and S be standard automorphisms, $TS = ST$, and let TS be ergodic. Is it standard? Let T be standard, S ergodic and $TS = ST$. Is S standard? Let T be standard and $f: S^1 \longrightarrow S^1$ a Borel function, where the operator $f(U_T)$ is generated by some automorphism S. Is S in almost every ergodic component either periodic or standard? Will T be standard if S is standard? From the above, if $f(\lambda) = \lambda^n$, then the answer to this question is positive.

Let $\{T_n\}$, $n = 1, 2, \ldots$, be a sequence of commuting standard automorphisms weakly converging to T. Is T in almost every ergodic component either periodic or standard?

3. Sataev [14] and, simultaneously and independently, Feldman [37] have defined a class of random processes (MVWB proceses, of LB processes in the terminology of Feldman) and the connected monotonely invariant class (WMB automorphisms). Sataev proved that a WMB automorphism is metrically isomorphic to a quotient of an automorphism monotonely equivalent to a Bernoulli automorphism, and in the case of zero entropy MVWB processes are M-trivial (the latter result was also found by Feldman), and consequently, by Theorem 4, WMB automorphisms with zero entropy are standard. Weiss has proved [39] that a WMB automorphism with positive entropy is monotonely equivalent to a Bernoulli automorphism.

The theorem on standardness of a quotient with zero entropy of an automorphism monotonely equivalent to a Bernoulli automorphism allows the construction of many new examples of K-automorphisms not isomorphic to, and even not monotonely equivalent to, a Bernoulli automorphism. The existence of such K-automorphisms follows from this theorem and the results of Juzvinskiĭ [35]. Concrete examples are to be found in Feldman [37] and Rudolph [38] (cf. [36]). We note that an approach from the viewpoint of monotone equivalence allows one to positively answer the question of existence of smooth non-Bernoulli K-automorphisms, put by Ornstein in the book [22]. The construction uses a nonstandard ergodic flow with zero entropy, as discussed above.

Let $S: M \longrightarrow M$ be a y-diffeomorphism with a smooth invariant measure μ, $\{T_t\}$ an ergodic flow with a smooth invariant measure ν on a manifold N, f a real positive C^∞-function on M, and $f_0 = \int_M f\,d\mu$.

Consider a diffeomorphism $R: M \times N \longrightarrow M \times N$ preserving $\mu \times \nu$:

$$R(x, y) = (Sx, T_{f(x)}y). \tag{11.4}$$

If M is a torus, then we can suppose there is a constant $C > 0$, depending only on S, such that if $f < C$ and there is no continuous function h for which

$$h(Sx) - h(x) = f(x) - f_0, \tag{11.5}$$

then R is a K-automorphism. In the general case we must replace (11.5) by a somewhat stronger condition. The result is proved by the methods of the theory of partially hyperbolic dynamical systems constructed in [15].

As is easily seen, R is a section of $\{S_t^f \times T_t\}$ (recall that $\{S_t^f\}$ is the special flow over S). We represent $\{T_t\}$ as a special flow over some automorphism T. Then $\{S_t^f \times T_t\}$ has another section, in which some extension of T acts, and consequently R is monotonely equivalent to this extension. If $\{T_t\}$ has zero entropy and is not standard, T also has these properties. If R were monotonely equivalent to a Bernoulli automorphism, then no automorphism monotonely equivalent to it could have a nonstandard factor with zero entropy.

4. The relation of majorization, defined in §2, allows the introduction of additional structure in the set of classes of monotonely equivalent automorphisms. Unfortunately it is unknown whether this relation is a partial order. By Proposition 2.5 this would follow from a positive answer to the following question.

Let ξ and η, $\xi \geqslant \eta$, be two invariant partitions of an ergodic automorphism T, and let $T \overset{M}{\sim} T|_\eta$. Is it true that $T \overset{M}{\sim} T|_\xi$?

There are two interesting monotone invariants associated with the majorizing relation. The height of an automorphism T, denoted $B(T)$, is the least upper bound of the power of ordered sets I such that there is a system of automorphisms $\{T_i\}$, $i \in I$, with $T \succ T_i$ for all $i \in I$ and, if $i, j \in I$ and $i > j$, then T_i majorizes T_j, and T_i is not monotonely equivalent to T_j. The capacity of T, denoted $E(T)$, is the power of the set of classes of monotonely equivalent automorphisms which are majorized by T but do not coincide with the class of T. By Theorem 1 the conditions $B(T) = 0$ or $E(T) = 0$ are equivalent to standardness. It is obvious that $B(T) \leqslant E(T)$; and, by Theorem 1, $B(T) = 1$ implies $E(T) = 1$. Automorphisms for which $E(T) = 1$ naturally may be regarded as the simplest after the standards from the viewpoint of monotone equivalence. For a WMB automorphism T we have $B(T) = E(T) = 1$ if $0 < h(T) < \infty$, and $B(T) = E(T) = 2$ for $h(T) = \infty$. It is of interest to know whether there are automorphisms with zero entropy for which $E(T) = 1$. If such automorphisms exist, then possibly they admit a visible classification up to monotone equivalence. It is of interest to know what values the invariants $B(T)$ and $E(T)$ can take.

5. As was mentioned in the Introduction and in §2, monotone equivalence can be defined not only for $\mathbf{R}$ and $\mathbf{Z}$ but also for more general groups. It would be interesting to translate the results of this paper to $\mathbf{Z}^m$ (see Definition 2.3). The definition of standard actions of $\mathbf{Z}^m$ presents no difficulty. The role of $\mathbf{D}$ can be played by the action on $\mathbf{Z}_{\{2\}} \times \cdots \times \mathbf{Z}_{\{l_m\}}$ generated by $\mathbf{D}_{\{2\}}, \mathbf{D}_{\{3\}}, \ldots, \mathbf{D}_{\{l_m\}}$ (l_i is the ith prime number, $i = 1, \ldots, m$), acting coordinatewise.

Using in $\mathbf{R}^m$ the partial order $(x_1, \ldots, x_m) \leqslant (y_1, \ldots, y_m)$ if $x_i \leqslant y_i$, we can, by analogy with Definitions 2.1 and 2.3, define monotone equivalence for actions of $\mathbf{R}^n$. However, because of a lack of an analogue of the theorem on special representations for $m > 1$ the results for $\mathbf{R}^m$ cannot be obtained as direct corollaries of the results for $\mathbf{Z}^m$. We must therefore directly define the notions of monotone approximation, M-triviality, MVWB, etc. directly for "time" $\mathbf{R}^m$ and then carry over to this case the results for $\mathbf{Z}$ and $\mathbf{R}$.

The case of quasi-cyclic time $G = \bigcup_{n=1}^{\infty} \mathbf{Z}_{q_n}$, mentioned in the Introduction, merits extensive detailed consideration. We mention that Stepin has constructed a dyadic sequence of partitions $\epsilon = \xi_1 < \xi_2 < \cdots$ (see [5] and [6]) such that the subsequence ξ_2, ξ_4, ξ_6, $\ldots$, is isomorphic to the corresponding subsequence of a standard sequence $\{\eta_n\}$, but the sequences $\{\xi_n\}$ and $\{\eta_n\}$ are not isomorphic. It would be interesting to classify sequences with such a property.

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BIBLIOGRAPHY

1. Paul R. Halmos, *Lectures on ergodic theory*, Math. Soc. Japan, Tokyo, 1956.
2. Donald S. Ornstein, *Bernoulli shifts with the same entropy are isomorphic*, Advances in Math. **4** (1970), 337–352.
3. H. A. Dye, *On groups of measure preserving transformations*. I, Amer. J. Math. **81** (1959), 119–159.
4. ———, *On groups of measure preserving transformations*. II, Amer. J. Math. **85** (1963), 551–576.
5. A. M. Veršik, *Theorem on lacunary isomorphisms of monotonic sequences of partitions*, Funkcional. Anal. i Priložen. **2** (1968), no. 3, 17–21; English transl. in Functional Anal. Appl. **2** (1968).
6. ———, *Decreasing sequences of measurable partitions and their applications*, Dokl. Akad. Nauk SSSR **193** (1970), 748–751; English transl. in Soviet Math. Dokl. **11** (1970).
7. A. M. Stepin, *On entropy invariants of decreasing sequences of measurable partitions*, Funkcional. Anal. i Priložen. **5** (1971), no. 3, 80–84; English transl. in Functional Anal. Appl. **5** (1971).
8. A. B. Katok, Ja. G. Sinaĭ and A. M. Stepin, *The theory of dynamical systems and general groups of transformations with an invariant measure*, Itogi Nauki: Mat. Anal., vol. 13, VINITI, Moscow, 1975, pp. 129–262; English transl. in J. Soviet Math. 7 (1977).
9. Shizuo Kakutani, *Induced measure preserving transformations*, Proc. Imp. Acad. Tokyo **19** (1943), 635–641.
10. L. M. Abramov, *The entropy of a derived automorphism*, Dokl. Akad. Nauk SSSR **128** (1959), 647–650. (Russian)
11. R. V. Chacon, *Change of velocity in flows*, J. Math. Mech. **16** (1966), 417–431.
12. N. A. Friedman and D. S. Ornstein, *Ergodic transformations induce mixing transformations*, Advances in Math. **10** (1973), 147–163.
13. G. Hansel, *Automorphismes induits et valeurs propres*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **25** (1972/73), 155–157.
14. E. A. Sataev, *An invariant of monotone equivalence defining quotients of automorphisms monotonely equivalent to a Bernoulli shift*, Fourth Internat. Sympos. Information Theory, Abstracts of Reports, Part 1, Moscow, 1976. (Russian)
15. M. I. Brin and Ja. B. Pesin, *Partially hyperbolic dynamical systems*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 170–212; English transl. in Math. USSR Izv. **8** (1974).
16. A. B. Katok, *Time change, monotone equivalence and standard dynamical systems*, Dokl. Akad. Nauk SSSR **223** (1975), 789–792; English transl. in Soviet Math. Dokl. **16** (1975).
17. ———, *Properties of standard dynamical systems*, Fourth Internat. Sympos. Information Theory, Abstracts of Reports, Part 1, Moscow, 1976. (Russian)
18. V. A. Rohlin, *Lectures on the entropy theory of measure-preserving transformations*, Uspehi Mat. Nauk **22** (1967), no. 5 (137), 3–56; English transl. in Russian Math. Surveys **22** (1967).
19. ———, *New progress in the theory of measure-preserving transformations*, Uspehi Mat. Nauk **15** (1960), no. 4 (94), 3–26; English transl. in Russian Math. Surveys **15** (1960).
20. Ja. G. Sinaĭ, *Weak isomorphism of transformations with invariant measure*, Mat. Sb. **63** (105) (1964), 23–42; English transl. in Amer. Math. Soc. Transl. (2) **57** (1966).
21. A. V. Kočergin, *On additive homology equations over dynamical systems*, Fourth Internat. Sympos. Information Theory, Abstracts of Reports, Part 1, Moscow, 1976. (Russian)
22. Donald S. Ornstein, *Ergodic theory, randomness, and dynamical systems*, Yale Univ. Press, New Haven, Conn., 1974.
23. R. V. Chacon and T. Schwartzbauer, *Commuting point transformations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **11** (1969), 277–287.
24. T. Schwartzbauer, *Automorphisms that admit an approximation by periodic translations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **15** (1970), 239–248.

25. A. B. Katok and A. M. Stepin, *Approximations in ergodic theory*, Uspehi Mat. Nauk **22** (1967), no. 5 (137), 81–106; English transl. in Russian Math. Surveys **22** (1967), no. 5.

26. J. R. Baxter, *A class of ergodic transformations having simple spectrum*, Proc. Amer. Math. Soc. **27** (1971), 275–279.

27. V. I. Oseledec, *Two nonisomorphic dynamical systems with the same simple continuous spectrum*, Funkcional. Anal. i Priložen. **5** (1971), no. 3, 75–79; English transl. in Functional Anal. Appl. **5** (1971).

28. L. M. Abramov, *Metric automorphisms with quasi-discrete spectrum*, Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962), 513–530; English transl. in Amer. Math. Soc. Transl. (2) **39** (1964).

29. L. Auslander, L. Green and F. Hahn, *Flows on homogeneous spaces*, Princeton Univ. Press, Princeton, N. J., 1963.

30. I. U. Bronšteĭn, *Extensions of minimal transformation groups*, Štiinca, Kishinev, 1975. (Russian)

31. D. V. Anosov and A. B. Katok, *New examples in smooth ergodic theory. Ergodic diffeomorphisms*, Trudy Moskov. Mat. Obšč. **23** (1970), 3–36; English transl. in Trans. Moscow Math. Soc. **23** (1970).

32. ———, *New examples of ergodic diffeomorphisms of smooth manifolds*, Uspehi Mat. Nauk **25** (1970), no. 4 (154), 173–174. (Russian)

33. D. V. Anosov, *Existence of smooth ergodic flows on smooth manifolds*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 518–545; English transl. in Math. USSR Izv. **8** (1974).

34. A. B. Katok and E. A. Sataev, *Standardness of rearrangement automorphisms of segments and flows on surfaces*, Mat. Zametki **20** (1976), 479–488; English transl. in Math. Notes **20** (1976).

35. S. A. Juzvinskiĭ, *Distinction of K automorphisms by the scale*, Funkcional. Anal. i Priložen. **7** (1973), no. 4, 70–75; English transl. in Functional Anal. Appl. **7** (1973).

36. Donald S. Ornstein and Paul C. Shields, *An uncountable family of K-automorphisms*, Advances in Math. **10** (1973), 63–88.

37. J. Feldman, *New K-automorphisms and a problem of Kakutani*, Israel J. Math. **24** (1976), 16–38.

38. D. Rudolph, *Nonequivalence of measure preserving transformations*, Preprint, 1976.

39. B. Weiss, *Equivalence of measure preserving transformations*, Preprint, 1976.

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