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MEASURE AND COCYCLE RIGIDITY FOR CERTAIN NONUNIFORMLY HYPERBOLIC ACTIONS OF HIGHER-RANK ABELIAN GROUPS

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ABSTRACT. We prove absolute continuity of "high-entropy" hyperbolic invariant measures for smooth actions of higher-rank abelian groups assuming that there are no proportional Lyapunov exponents. For actions on tori and infranilmanifolds the existence of an absolutely continuous invariant measure of this kind is obtained for actions whose elements are homotopic to those of an action by hyperbolic automorphisms with no multiple or proportional Lyapunov exponents. In the latter case a form of rigidity is proved for certain natural classes of cocycles over the action.

1. INTRODUCTION

In this paper we continue the program of studying hyperbolic measures for actions of higher-rank abelian groups first alluded to in [7, Part II] and started in earnest [8, 10, 13]. We refer to those papers for basic definitions and standard facts concerning those actions.

Specifically, we extend some of the main results of [8, 10, 13] from maximal-rank actions ($\mathbb{Z}^k$ actions on $k+1$-dimensional manifolds and $\mathbb{R}^k$ actions on $2k+1$-dimensional manifolds, $k \geq 2$) to a class of actions where dimension and rank are not related, except of the standard assumption of rank being at least 2. Thus we partially realize the "low rank and high dimension" program of [10, Section 8.3]. While we use the general methods and some specific results from the previous papers as well as heavy machinery of smooth ergodic theory, we introduce three important new ingredients that make these advances possible. These new elements are:

- The *holonomy-invariance* that appears in the proof of Theorem 2.10. It is an extension to general nonlinear and nonuniformly hyperbolic actions of certain arguments which appeared in [7] for the study of invariant measures for linear actions on the torus.

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- A new *entropy inequality*, Lemma 6.1, that is crucial in the proof of Theorem 2.5, by allowing to show that all Lyapunov hyperplanes for the linear action persist for the nonlinear one.
- The *index argument* in the uniqueness proof (Section 7.2, Lemma 7.5) that replaces the argument in [13], which depends on the existence of elements with codimension-one stable foliations.

Our second goal is to prove cocycle rigidity for actions on the torus satisfying our measure rigidity results. For the case of maximal-rank actions those results have been announced in [9].

In this paper we restrict ourselves to the case where all Lyapunov exponents are simple and there are no proportional Lyapunov exponents. This allows us to avoid extra technical complications that appear in the presence of multiple or proportional exponents. In this situation the coarse Lyapunov foliations are one-dimensional and invariant geometric structures on their leaves are affine. Our approach extends to certain cases where multiple or positively proportional Lyapunov exponents are allowed (totally nonsymplectic condition, TNS for short). In this case one needs to use a version of the theory of nonstationary normal forms (see [7, Section 6]) to produce invariant geometric structures on the leaves of coarse Lyapunov foliations. Those structures may be more complicated than affine if there are resonances between Lyapunov exponents. We discuss this more general situation in the last section. A detailed treatment will appear in a separate paper.

2. FORMULATION OF RESULTS

Let α be a $C^{1+\theta}$ $\mathbb{R}^k$-action ($k \geq 2$, $\theta > 0$) on an n-dimensional manifold M[1]. Let μ be an invariant ergodic probability measure for α and let $\chi_1, \dots, \chi_n : \mathbb{R}^k \to \mathbb{R}$ be the Lyapunov exponents (linear functionals) associated to μ. Recall that an ergodic invariant measure μ for a smooth locally free $\mathbb{R}^k$ action α is said to be *hyperbolic* if all nontrivial Lyapunov exponents χ_i, $i = 1, \dots, l$, are nonzero linear functionals on $\mathbb{R}^k$. The kernels of nonzero Lyapunov exponents are called *Lyapunov hyperplanes*. Vectors in $\mathbb{R}^k$ which do not lie on any of the Lyapunov hyperplanes are said to be *regular*. Connected components of the sets of regular vectors are called *Weyl chambers*.

Recall that in the absence of positively proportional Lyapunov exponents every Lyapunov distribution E_i integrates to an invariant family of smooth manifolds $\mathscr{W}^i$ defined μ a.e., which is customarily called the Lyapunov foliation. Leaves of of those foliations are intersections of stables manifolds of properly chosen elements of the action. See Section 3 for details.

2.1. **Strongly simple actions.**

[1] Traditionally one assumes M to be compact. Note however that the basic constructions and results of smooth ergodic theory only require uniform Hölder bounds for the derivative of the action, which is sufficient in our setting.

DEFINITION 2.1. Let μ be a hyperbolic measure for an action α. We say that (α,μ), or simply α if μ is understood, is *strongly simple* if all Lyapunov exponents are simple and there are no proportional Lyapunov exponents.

We say that (α,μ) satisfies the *full-entropy condition* if the entropy function is not differentiable at Lyapunov hyperplanes.

THEOREM 2.2. *Let μ be an ergodic invariant measure for a strongly simple action α. Then for any element $\mathbf{t}$ of the action such that the entropy $h_\mu(\mathbf{t}) > 0$, there exists a Lyapunov exponent χ such that $\chi(\mathbf{t}) < 0$ and the conditional measures on the Lyapunov foliation $\mathscr{W}$ corresponding to χ are equivalent to Lebesgue measure.*

Since all Lyapunov exponents change sign for the inverse transformation while the entropy remains the same, Theorem 2.2 immediately implies the following result.

COROLLARY 2.3. *Let μ be an ergodic invariant measure for a strongly simple action α. If $h_\mu(\mathbf{t}) > 0$ for some $\mathbf{t} \in \mathbb{R}^k$ then there are at least two Lyapunov foliations such that the corresponding conditional measures are equivalent to Lebesgue measure.*

It is probable that one can strengthen Theorem 2.2 in the following way.

CONJECTURE. *The conditional measures on unstable manifolds of the action elements are equivalent to Lebesgue measures on certain smooth submanifolds that are obtained by integrating those Lyapunov foliations for which the conditional measures are Lebesgue measure.*

The full-entropy condition leads to a stronger assertion.

THEOREM 2.4. *Let μ be an ergodic invariant measure for an action α. Assume that*

1. (α,μ) *is strongly simple, and*
2. (α,μ) *satisfies the full-entropy condition.*

Then, μ is absolutely continuous with respect to the smooth measure class on M.

2.2. **Actions on tori and nilmanifolds.** Let N be a simply connected nilpotent Lie group and A a group of affine transformations of N acting freely that contains a finite-index subgroup Γ of translations that is a lattice in N. Then the orbit space N/A is a compact manifold that is called an *infranilmanifold*. An automorphism of N that maps orbits of A onto orbits of A generates a diffeomorphism of N/A that is called an infranilmanifold-automorphism.

An action α_0 of $\mathbb{Z}^k$ by automorphisms of an infranilmanifold M is an Anosov action if induced linear action on the Lie algebra $\mathfrak{N}$ of N has nonzero Lyapunov exponents.

Now let α be an action of $\mathbb{Z}^k$ by diffeomorphisms of M such that its elements are homotopic to elements of an Anosov action by automorphisms. We will say that α has *homotopy data* α_0. There may exist *affine* actions with homotopy data α_0 that are not isomorphic to α_0. This happens when α_0 has more than one fixed point and affine action interchanges those fixed points. Note that any affine action with homotopy data α_0 coincides with α_0 on a finite-index subgroup $A \subset \mathbb{Z}^k$. There exists an affine map $\tilde{\alpha}$ with homotopy data α_0 and a continuous map $h\colon M \to M$ homotopic to identity such that $h \circ \alpha = \tilde{\alpha} \circ h$. Hence, for $\gamma \in A$

$$h \circ \alpha(\gamma) = \alpha_0(\gamma) \circ h. \tag{2.1}$$

This map is customarily said to be a semiconjugacy between α and $\tilde{\alpha}$. There are finitely many semiconjugacies that differ by some translations by elements of the fixed point group of $\tilde{\alpha}$. [2]

An α-invariant Borel probability measure μ is said to be *large* if the push-forward $h_*\mu$ is Haar measure. The following theorem is the extension of Theorems 1.3 and 1.7 in [8] from actions on a torus with Cartan homotopy data to our case of actions on infranilmanifolds with strongly simple homotopy data.

THEOREM 2.5. *Let α_0 be a strongly simple Anosov action of $\mathbb{Z}^k$ by automorphisms of an infranilmanifold and let α be a smooth action with homotopy data α_0. Let μ be an ergodic large invariant measure μ for α. Then,*

(a) *μ is absolutely continuous,*
(b) *The Lyapunov characteristic exponents of the action α with respect to μ are equal to the Lyapunov characteristic exponents of the action α_0.*

CONJECTURE. *Under the assumptions of Theorem 2.5, the large invariant measure is unique.*

We can prove this under a stronger assumption on α_0 that forces the group N to be abelian and hence M to have a torus as a finite cover.

A relation between different Lyapunov exponents $\chi_1,\dots,\chi_l$ of a $\mathbb{Z}^k$ action of the form $\chi_1 = \sum_{i=2}^{l} m_i\chi_i$, where $m_2,\dots,m_l$ are positive integers, is called a *resonance*. A resonance with $l=2$ is called a *double resonance*.

In this setting, we may assume that $N = \mathbb{R}^m$ and $\Gamma = \mathbb{Z}^m$. The factor of the torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ by a finite group of fixed-point free affine maps is called an *infratorus*. The following theorem is a generalization of the main result of [13, Corollary 2.2].

THEOREM 2.6. *Let α_0 be a linear strongly simple $\mathbb{Z}^k$ action without resonances on an infratorus. Any action α with homotopy data α_0 has unique large invariant measure μ. Furthermore, the semiconjugacy h is bijective μ-a.e. and effects a measurable isomorphism between (α,μ) and an affine action $\tilde{\alpha}$ with homotopy data α_0 with Haar measure.*

Indeed we have smoothness of the semiconjugacy on large sets.

[2] In [8, Lemma 1.2] and [13, (1.1)], it is erroneously claimed that (2.1) holds for the whole actions α and α_0. This unfortunate but minor error does not affect the results of these papers, the only necessary change being the replacement of α_0 by $\tilde{\alpha}$ in the statements and proofs.

THEOREM 2.7. *If the action α is C^r, $r > 1$, then for every $\varepsilon > 0$ there is a set of measure larger than $1-\varepsilon$ such that the restriction of the semiconjugacy to this set is $C^{r-\varepsilon}$ in the Whitney sense,* i.e., *it has a $C^{r-\varepsilon}$ extension.*

REMARK 1. The difference between the Cartan condition of [8] and our strongly simple condition is quite considerable. Cartan actions are maximal-rank actions on the torus by hyperbolic automorphisms; thus the rank is equal to the dimension minus one. On the other hand, strongly simple actions may have any rank starting from two: for example, the restriction of a Cartan action to any $\mathbb{Z}^2$ subgroup is strongly simple. Also strongly simple actions may be reducible; *e.g.,* the product of two Cartan actions is usually strongly simple.

REMARK 2. Actions by automorphisms of nonabelian simply connected Lie groups always have resonances that appear from the nontrivial bracket relations in the Lie algebra. While the easiest standard examples have double resonances there are strongly simple actions on some compact nilmanifolds, see for example [12, Example 2.2.35].

Generally speaking, cocycle rigidity means that one-cocycles of certain regularity over a group action are cohomologous to constant cocycles via transfer functions of certain (often lower) regularity. Cocycle rigidity is prevalent in hyperbolic and partially hyperbolic actions of higher-rank abelian groups, see [11, 14, 15, 3]. The method of [11] for uniformly hyperbolic actions satisfying the TNS condition (no negatively proportional Lyapunov exponents) can be used in the nonuniformly hyperbolic case and is in particular applicable in the settings considered in the present paper.

One can apply this method to smooth or Hölder-continuous cocycles but there are reasons to consider two broader classes of cocycles which are in general defined only almost everywhere with respect to a hyperbolic absolutely continuous invariant measure:

(i) *Lyapunov Hölder* cocycles: Hölder-continuous with respect to a properly defined *Lyapunov metric* which is equivalent to a smooth metric on Pesin sets and changes slowly along the orbits, and
(ii) *Lyapunov smooth* cocycles: smooth along invariant foliations at the points of Pesin sets with a similar slow change condition.

See Section 8 for a more detailed formal description of those classes of cocycles. Note that the most important intrinsically defined cocycles, the logarithms of the Jacobians along invariant foliations (Lyapunov, stable, and likewise), are Lyapunov smooth.

THEOREM 2.8. *For any action α of $\mathbb{Z}^k$ on an infratorus as in Theorem 2.6 any Lyapunov Hölder (resp. Lyapunov smooth) cocycle is cohomologous to a constant cocycle via a Lyapunov Hölder (resp. Lyapunov smooth) transfer function.*

REMARK 3. The assertion of this theorem is likely to hold in the more general setting of Theorem 2.5. The difficulties in the proof are of a technical nature and have to do with a proper application of a version of the Hopf argument in the resonance case when the invariant geometric structures on stable foliations are not affine.

2.3. **Technical results and overall structure of proofs.** The starting point in the proofs of all three results formulated above, Theorems 2.2, 2.4, and 2.5, is the main technical theorem we proved in [10, Theorem 4.1] joint with B. Kalinin. So, let us recall it.

THEOREM 2.9. *Let μ be a hyperbolic ergodic invariant measure for a locally free $C^{1+\theta}$, $\theta > 0$, action α of $\mathbb{R}^k$, $k \geq 2$, on a compact smooth manifold M. Suppose that a Lyapunov exponent χ is simple and there are no other exponents proportional to χ. Let E be the one-dimensional Lyapunov distribution corresponding to the exponent χ and $\mathcal{W}$ the corresponding Lyapunov foliation. Then, the conditional measures of μ on $\mathcal{W}$ are either atomic a.e., or equivalent to Lebesgue measure a.e.*

Let α and χ be as in the hypothesis of Theorem 2.9. Let us fix a generic singular element $\mathbf{t} \in \mathbb{R}^k$, *i.e.*, an element such that $\chi(\mathbf{t}) = 0$ but $\chi_j(\mathbf{t}) \neq 0$ for any other Lyapunov exponent, and an element $\mathbf{s}$ close to $\mathbf{t}$ such that $\chi(\mathbf{s}) < 0$ but it is still the biggest negative Lyapunov exponent for $\mathbf{s}$. Then $\mathcal{W}^s_{\alpha(\mathbf{t})} \subset \mathcal{W}^s_{\alpha(\mathbf{s})}$ and in fact $E^s_{\alpha(\mathbf{s})} = E^s_{\alpha(\mathbf{t})} \oplus E_\chi$.

The second principal technical result that appears in the proofs of Theorems 2.4 directly and 2.5 (through Theorem 2.11) is new. It shows that if we have atomic conditional measures along some $\mathcal{W}^i$ direction then the conditional measures along stable manifolds containing the $\mathcal{W}^i$ direction fit in a lower-dimensional submanifold.

THEOREM 2.10. *If χ, $\mathbf{t}$, and $\mathbf{s}$ are as above and the conditional measure along $\mathcal{W}$ is atomic for a.e. point, then the conditional measure along $\mathcal{W}^s_{\alpha(\mathbf{s})}(x)$ has its support inside $\mathcal{W}^s_{\alpha(\mathbf{t})}(x)$ for a.e. x.*

While the proof of Theorem 2.10 does not use Theorem 2.9 directly, it relies on the the principal technical construction that appears in the proof of the latter result: the synchronizing time change, see Section 3.3.

Theorems 2.9 and 2.10 are also used in the proof of the following result that is used in the proof of Theorem 2.4.

THEOREM 2.11. *If χ is as in Theorem 2.9, then the conditional measure along $\mathcal{W}$ is Lebesgue measure if the entropy function is not differentiable at the Lyapunov hyperplane* $\ker\chi$.

In addition to the innovations listed in the introduction, other additional major ingredients that appear in the proofs include:

(i) The Ledrappier–Young entropy formula, which is reviewed in Section 3.4 and used in the proofs of Theorems 2.11 and 2.4;
(ii) an extension of H. Hu's result on linearity of the entropy functions inside Weyl chambers, see Corollary 3.9. It is used in the proof of Theorem 2.11 to treat the case when a Lyapunov exponent is proportional to the difference of two others, which may appear in strongly simple actions.

REMARK 4. It is probable that the converse to the statement of Theorem 2.11 is also true. This would follow from an extension of the Ledrappier–Young formula, see Statement A in Section 3.4.

3. PRELIMINARIES

3.1. **Lyapunov exponents and Pesin sets.** For a smooth $\mathbb{R}^k$ action α on a manifold M and an element $\mathbf{t} \in \mathbb{R}^k$ we denote the corresponding diffeomorphism of M by $\alpha(\mathbf{t})$. Sometimes we will omit α and write, for example, $\mathbf{t}x$ in place of $\alpha(\mathbf{t})x$ and $D\mathbf{t}$ in place of $D\alpha(\mathbf{t})$ for the derivative of $\alpha(\mathbf{t})x$.

PROPOSITION 3.1. *Let α be a locally free $C^{1+\theta}$, $\theta > 0$, action of $\mathbb{R}^k$ on a manifold M preserving an ergodic invariant measure μ. There are linear functionals χ_i, $i = 1,\dots,l$, on $\mathbb{R}^k$ and an α-invariant measurable splitting of the tangent bundle TM, called the* Lyapunov decomposition, *(or sometimes the* Oseledets decomposition*), $TM = T\mathscr{O} \oplus \bigoplus_{i=1}^{l} E_i$ over a set of full measure $\mathfrak{Re}$, where $T\mathscr{O}$ is the distribution tangent to the $\mathbb{R}^k$ orbits, such that for any $\mathbf{t} \in \mathbb{R}^k$ and any nonzero vector $v \in E_i$ the Lyapunov exponent of v is equal to $\chi_i(\mathbf{t})$,* i.e.,

$$\lim_{n\to\pm\infty} n^{-1} \log \| D(n\mathbf{t})\, v\| = \chi_i(\mathbf{t}),$$

where $\|\cdot\|$ is any continuous norm on TM. Any point $x \in \mathfrak{Re}$ is said to be a regular point.

Furthermore, for any $\varepsilon > 0$, there exist positive measurable functions $C_\varepsilon(x)$ and $K_\varepsilon(x)$ such that for all $x \in \mathfrak{Re}$, $v \in E_i(x)$, $\mathbf{t} \in \mathbb{R}^k$, and $i = 1,\dots,l$,

1. $C_\varepsilon^{-1}(x) e^{\chi_i(\mathbf{t}) - \varepsilon\|\mathbf{t}\|/2} \|v\| \le \|D\mathbf{t}\, v\| \le C_\varepsilon(x) e^{\chi_i(\mathbf{t}) + \varepsilon\|\mathbf{t}\|/2} \|v\|$;
2. *Angles* $\angle(E_i(x), T\mathscr{O}) \ge K_\varepsilon(x)$ *and* $\angle(E_i(x), E_j(x)) \ge K_\varepsilon(x)$, $i \ne j$;
3. $C_\varepsilon(\mathbf{t}x) \le C_\varepsilon(x) e^{\varepsilon\|\mathbf{t}\|}$ *and* $K_\varepsilon(\mathbf{t}x) \ge K_\varepsilon(x) e^{-\varepsilon\|\mathbf{t}\|}$.

The stable and unstable distributions $E^s_{\alpha(\mathbf{t})}$ and $E^u_{\alpha(\mathbf{t})}$ of an element $\alpha(\mathbf{t})$ are defined as the sums of the Lyapunov distributions corresponding to the negative and the positive Lyapunov exponents for $\alpha(\mathbf{t})$ respectively. Note that stable and unstable distributions are the same within a Weyl chamber, and conversely, the set of vectors with given stable and unstable distributions, if nonempty, is a Weyl chamber. A minimal nonzero intersection of stable distributions for various elements of the action is called a *coarse Lyapunov distribution*. Equivalently, any coarse Lyapunov distribution is the sum of Lyapunov distributions corresponding all Lyapunov exponents positively proportional to each other. Note that in the absence of positively proportional Lyapunov exponents, in particular, for the strongly simple case considered in this paper, coarse Lyapunov distributions coincide with Lyapunov distributions.

3.2. **Invariant manifolds and affine structures.** We will use standard material on invariant manifolds corresponding to the negative and positive Lyapunov exponents (stable and unstable manifolds) for $C^{1+\theta}$ measure-preserving diffeomorphisms of compact manifolds, see for example [1, Chapter 4]. In particular, stable distributions and hence their transverse intersections are always Hölder-continuous (see, for example, [2]). Here is a summary of some of those results adapted to the case of an $\mathbb{R}^k$ action.

PROPOSITION 3.2. *Let α be a $C^{1+\theta}$, $\theta > 0$, action of $\mathbb{R}^k$ as in Proposition 3.1. Suppose that a Lyapunov distribution E is the intersection of the stable distributions of some elements of the action. Then E is Hölder-continuous on any Pesin set*

$$\mathfrak{Re}_\varepsilon^l = \left\{x \in \mathfrak{Re} : C_\varepsilon(x) \le l, K_\varepsilon(x) \ge l^{-1}\right\} \tag{3.1}$$

with Hölder constant which depends on l and Hölder exponent $\delta > 0$ which depends on the action α only.

Furthermore, on those sets the size of local stable manifolds for any element of α is bounded away from zero.

We will denote by $\mathscr{W}_\alpha(\mathbf{t})^s(x)$ the (global) stable manifold for $\alpha(\mathbf{t})$ at a regular point x. This manifold is an immersed Euclidean space tangent to the stable distribution $E_\alpha(\mathbf{t})^s$. The unstable manifold $\mathscr{W}_\alpha(\mathbf{t})^u(x)$ is defined as the stable one for $\alpha(-\mathbf{t})$ and thus has similar properties. Local stable/unstable manifolds will be denoted by $W_\alpha(\mathbf{t})^s(x)$ and $W_\alpha(\mathbf{t})^u(x)$ correspondingly.

Intersections of stable manifolds for different elements of the action are integral manifolds for the coarse Lyapunov distributions and they form *coarse Lyapunov foliations*. While in general the Lyapunov foliations may not be uniquely integrable, this is obviously the case in the absence of positively proportional Lyapunov exponents. Hence everywhere in this paper we will talk about *Lyapunov foliations*. These foliations are defined for any Lyapunov distribution E as in Proposition 3.2. We will denote the Lyapunov foliation corresponding to the exponent χ by $\mathscr{W}$ and its local leaf at a regular point x by $W(x)$.

As a comment on terminology, let us emphasize that it is customary to use the words "distributions" and "foliations" in this setting, although these objects are correspondingly measurable families of tangent spaces defined a.e. and measurable families of smooth manifolds which fill a set of full measure.

Let us also recall the existence of affine structures. Let α be an action as in Theorem 2.9. The following proposition provides α-invariant affine parameters on the leaves of the Lyapunov foliation $\mathscr{W}$.

PROPOSITION 3.3 ([8, Proposition 3.1, Remark 5]). *There exist a unique family of $C^{1+\theta}$ smooth α-invariant affine parameters on the leaves $\mathscr{W}(x)$. Moreover, they depend uniformly continuously in the $C^{1+\theta}$ topology on x in any Pesin set.*

3.3. **Lyapunov metrics and synchronizing time change.** We fix a smooth Riemannian metric $\langle\cdot,\cdot\rangle$ on M. Given $\varepsilon > 0$ and a regular point $x \in M$ we define the *standard ε-Lyapunov scalar product* (or *metric*) $\langle\cdot,\cdot\rangle_{x,\varepsilon}$ as follows. For any $u, v \in E(x)$ we define

$$\langle u, v\rangle_{x,\varepsilon} = \int_{\mathbb{R}^k} \langle (D\mathbf{s})u, (D\mathbf{s})v\rangle \exp(-2\chi(\mathbf{s}) - 2\varepsilon\|\mathbf{s}\|)\, d\mathbf{s}. \tag{3.2}$$

We shall need to use the time change we introduced with B. Kalinin in [10] in the context of Theorem 2.9. Let $L=\ker\chi$, fix a vector $\mathbf{w}\in\mathbb{R}^k$ normal to L with $\chi(\mathbf{w})=1$ and take $\varepsilon>0$ small such that $\varepsilon\|\mathbf{w}\|$ is also small, in particular less than 1/2.

PROPOSITION 3.4 ([10, Propositions 6.2, 6.3])**.** *For μ-a.e. x and any $\mathbf{t}\in\mathbb{R}^k$, there exists $g(x,\mathbf{t})\in\mathbb{R}^k$ such that the function $\mathbf{g}(x,\mathbf{t})=\mathbf{t}+g(x,\mathbf{t})\mathbf{w}$ satisfies the equality*

$$\|D_x^E\alpha(\mathbf{g}(x,\mathbf{t}))\|_\varepsilon=e^{\chi(\mathbf{t})}.$$

The function $g(x,\mathbf{t})$ is measurable and is Hölder-continuous on Pesin sets, is C^1 in $\mathbf{t}$ and $|g(x,\mathbf{t})|\le 2\varepsilon\|\mathbf{t}\|$. Moreover, the formula $\beta(\mathbf{t},x)=\alpha(\mathbf{g}(x,\mathbf{t}))x$ defines an $\mathbb{R}^k$ action β on M which is a measurable time change of α. The action β is measurable and continuous on Pesin sets for α and preserves a measure ν which is equivalent to μ.

Now we describe invariant "foliations" for β whose leaves are not smooth but still these objects have properties close to those of true invariant foliations for smooth actions. Let us denote by $\mathcal{N}$ the orbit foliation of the one-parameter subgroup $\{r\mathbf{w}\}$.

PROPOSITION 3.5 ([10, Proposition 6.7])**.** *For any element $\mathbf{s}\in\mathbb{R}^k$ there exists a stable "foliation" $\tilde{\mathcal{W}}^s_{\beta(\mathbf{s})}$ that is contracted by $\beta(\mathbf{s})$ and invariant under the new action β. It consists of "leaves" $\tilde{\mathcal{W}}^s_{\beta(\mathbf{s})}(x)$ defined for every x. The "leaf" $\tilde{\mathcal{W}}^s_{\beta(\mathbf{s})}(x)$ is a measurable subset of the leaf $(\mathcal{N}\oplus\mathcal{W}^s_{\alpha(\mathbf{s})})(x)$ of the form*

$$\tilde{\mathcal{W}}^s_{\beta(\mathbf{s})}(x)=\left\{\alpha(\phi_x(y)\mathbf{w})y : y\in\mathcal{W}^s_{\alpha(\mathbf{s})}(x)\right\},$$

where $\phi_x\colon\mathcal{W}^s_{\alpha(\mathbf{s})}(x)\to\mathbb{R}$ is an almost-everywhere defined measurable function. For x in a Pesin set, the ϕ_x is Hölder-continuous on the intersection of this Pesin set with any ball of fixed radius in $\mathcal{W}^s_{\alpha(\mathbf{s})}(x)$ with Hölder exponent γ and constant that depends on the Pesin set and radius.

We will use the fact for any $\mathbf{s}\in\mathbb{R}^k$ the partition into global stable manifolds $\tilde{\mathcal{W}}^s_{\beta(\mathbf{s})}(x)$ refines the partition into ergodic components of $\beta(\mathbf{s})$.

3.4. **Ledrappier–Young entropy formula.** Let $f\colon M\to M$ be $C^{1+\theta}$ diffeomorphism and let μ be an ergodic invariant measure. Let $\chi_1>\chi_2>\cdots>\chi_r$ be its Lyapunov exponents and let $TM=E_1\oplus\cdots\oplus E_r$ be the corresponding Lyapunov decomposition. Let $u=\max\{i:\chi_i>0\}$ and for $1\le i\le u$ let us define

$$V^i(x)=\left\{y\in M:\limsup_{n\to\infty}\frac{1}{n}\log d\big(f^{-n}(x),f^{-n}(y)\big)\le-\chi_i\right\}.$$

For a.e. x, $V^i(x)$ is a smooth manifold tangent to $\bigoplus_{j\le i}E_j$ and we have the flag $V^1\subset V^2\subset\cdots\subset V^u$ with $V^u=W^u$, the unstable manifold. We can build partitions ξ^i subordinate to V^i as the ones built in [18] and consider conditional measures μ^i_x. Let $B^i(x,\varepsilon)$ be the ε ball in $V^i(x)$ centered in x with respect to the induced Riemannian metric. Then

$$\delta_i = \delta_i(f) = \lim_{\varepsilon\to 0} \frac{\log \mu_x^i B^i(x,\varepsilon)}{\log\varepsilon}$$

exists a.e. and does not depend on x. Moreover, writing $\gamma_i = \gamma_i(f) = \delta_i - \delta_{i-1}$ we have the Ledrappier–Young entropy formula (see [18, Theorem C])

$$h_\mu(f) = \sum_{1\le j\le u} \gamma_j \chi_j. \tag{3.3}$$

In fact, a more precise statement is true. Given a measure μ and two measurable partitions α, β, the conditional entropy is defined by

$$H(\alpha|\beta) = -\int \log \mu_x^\beta(\alpha(x))\, d\mu.$$

Let $T: X \to X$ be a measure-preserving transformation. Given a measurable partition α, define the entropy of T with respect to α by $h(\alpha, T) = H(T^{-1}\alpha|\alpha^+)$ where $\alpha^+ = \bigvee_{n\ge 0} T^n\alpha$. We have

$$h_\mu(\xi_i, f) = \sum_{1\le j\le i} \gamma_j\chi_j$$

for every $1 \le i \le u$.

The following addition to the Ledrappier–Young formula is needed for the proof of the converse to Theorem 2.11.

STATEMENT A. *If $\gamma_i = 0$ in the Ledrappier–Young formula* (3.3), i.e.,

$$h_\mu(\xi_i, f) = h_\mu(\xi_{i-1}, f),$$

then the conditional measure on almost every leaf of V^i is supported on a single leaf of V^{i-1}.

Added in Proof: This statement was proved recently by F. Ledrappier and J.-S. Xie [16]. Thus, the converse to Theorem 2.11 holds.

3.5. **Linearity of entropies.** Given two commuting diffeomorphisms f and g preserving a measure μ with coinciding unstable manifolds, H. Hu [6] built a partition subordinate to this unstable manifold which is increasing for both maps. The same construction can be carried out for partitions subordinate to simultaneous fastest directions, following the same lines, to get the following analog of Proposition 8.1 in [6].

Let f and g be two diffeomorphisms preserving a measure μ. Assume both maps preserve a measurable bundle $F \subset TM$ such that the Lyapunov exponents associated to F, for both f and g, are larger than any other Lyapunov exponents. Hence F is tangent to a "foliation" V which is exactly the fastest foliation associated to the first dim(F) exponents.

PROPOSITION 3.6. *There is a measurable partition η on M with the following properties.*

1. *η is subordinate to V.*
2. *η is increasing for f and g,* i.e., *$f\eta < \eta$ and $g\eta < \eta$.*

3. $\bigvee_{n\geq 0} f^{-n}\eta$ *and* $\bigvee_{n\geq 0} g^{-n}\eta$ *are the partitions into points.*
4. *The biggest* σ*-algebra contained in* $\bigcap_{n\geq 0}\bigcap_{k\geq 0} f^{-n}g^{-k}\eta$ *is the* σ*-algebra of sets saturated by leaves of* V.

As always, (3) and (4) follow from (1) and (2). Also, as in [17], the entropy of f and g with respect to this partition does not depend on the partition as long as the partition is subordinate to V.

LEMMA 3.7. *If* η *and* $\hat{\eta}$ *are two partitions subordinate to* V *as in Proposition 3.6, then* $h_\mu(\eta, f) = h_\mu(\hat{\eta}, f)$. *The same holds for* f, g, *and* $f\circ g$.

Thus, we can call $h_\mu(V, f) = h_\mu(\eta, f)$. This yields the analog of [6, Proposition 9.1], which gives that entropy is linear.

PROPOSITION 3.8. *Let* f, g, μ, V, *and* η *be as above. Then,*

$$h_\mu(V, f\circ g) = h_\mu(V, f) + h_\mu(V, g).$$

Proof. Since the proof is rather short, we repeat it here. Let us write fg for $f\circ g$.

$$\begin{aligned} h_\mu(\eta, fg) = H(\eta|fg\eta) = H(\eta\vee g\eta|fg\eta) &= H(g\eta|fg\eta) + H(\eta|g\eta\vee fg\eta)\\ &= H(\eta|f\eta) + H(\eta|g\eta)\\ &= h_\mu(\eta, f) + h_\mu(\eta, g) \end{aligned}$$

□

COROLLARY 3.9. *If* C *is the cone where* F *is still the fastest bundle, then the map* $\mathbf{s}\mapsto h_\mu(V, \alpha(\mathbf{s}))$ *is linear for* $\mathbf{s}\in C$.

4. PROOFS OF THEOREMS 2.10 AND 2.11

4.1. **Holonomy-invariance of conditional measures: A model case.** Before we consider the situation in Theorem 2.10, we discuss another case of holonomy-invariance of conditional measures along stable directions that is of independent interest and that has a similar but simpler proof.

Let f be a diffeomorphism preserving a measurable foliation $\mathscr{F}$ with smooth leaves. Assume that $Df|T\mathscr{F}$ is an isometry. Let μ be an f-invariant measure and let $\mu_x^{\mathscr{F}}$ be the conditional measure along $\mathscr{F}(x)$. Recall that those measures are defined up to a scalar multiple. We use the normalization $\mu_x^{\mathscr{F}}(B^{\mathscr{F}}(x,1)) = 1$, where $B^{\mathscr{F}}(x,1)$ is the the ball in the leaf $\mathscr{F}(x)$ centered in x of radius 1. Given a regular point x and $y\in W^s(x)$ let us denote by $h_{xy}\colon \mathscr{F}(x)\to\mathscr{F}(y)$ the holonomy along the stable manifolds. If it is clear from the context, we shall omit the lower index xy.

PROPOSITION 4.1. *Let* f, $\mathscr{F}$, *and* μ *be as above. Then for* μ*-a.e.* x *and* μ_x^s*-a.e.* y *in* $W^s(x)$, *we have that* $h_*\mu_x^{\mathscr{F}} = \mu_y^{\mathscr{F}}$.

Proof. If x and y are in the same leaf of $\mathscr{F}$ then $\mu_x^{\mathscr{F}} = c_{xy}\mu_y^{\mathscr{F}}$ for some positive constant c_{xy}. Also, by invariance of μ and $\mathscr{F}$ and that f restricted to $\mathscr{F}$-leaves is an isometry we get $f_*\mu_x^{\mathscr{F}} = \mu_{f(x)}^{\mathscr{F}}$.

Consider now $\mu_x^{\mathscr{F},1} = \mu_x^{\mathscr{F}}|B^{\mathscr{F}}(x,1)$ the restriction of $\mu_x^{\mathscr{F}}$ to $B^{\mathscr{F}}(x,1)$. By invariance of conditional measures and our choice of normalizations, we have $f_*\mu_x^{\mathscr{F},1} = \mu_{f(x)}^{\mathscr{F},1}$. We will prove that

$$h_*\mu_x^{\mathscr{F},1} = \mu_y^{\mathscr{F},1}$$

for μ-a.e. x and for μ_x^s-a.e. $y \in W^s(x)$. Take a sequence $n_i \to \infty$ such that $f^{n_i}(x) \to z$, and hence $f^{n_i}(y) \to z$. Since f restricted to the leaves of $\mathscr{F}$ is an isometry we can take a subsequence such that $f^{n_i}|\mathscr{F}(x)$ converges uniformly on compact sets to an isometry $g_{xz}\colon \mathscr{F}(x) \to \mathscr{F}(z)$. Similarly, $f^{n_i}|\mathscr{F}(y)$ converges uniformly on compact sets to an isometry $g_{yz}\colon \mathscr{F}(y) \to \mathscr{F}(z)$ and $h = g_{yz}^{-1} \circ g_{xz}$ since stable manifolds are contracted in the future. Thus, it is sufficient to prove that that

$$(g_{xz})_*\mu_x^{\mathscr{F},1} = \mu_z^{\mathscr{F},1} \qquad \text{and} \qquad (g_{yz})_*\mu_y^{\mathscr{F},1} = \mu_z^{\mathscr{F},1}.$$

But $f_*^{n_i}\mu_x^{\mathscr{F},1} = \mu_{f^{n_i}(x)}^{\mathscr{F},1}$ and $f^{n_i} \to g_{xz}$ and $f^{n_i}(x) \to z$. Hence,

$$f_*^{n_i}\mu_x^{\mathscr{F},1} \to (g_{xz})_*\mu_x^{\mathscr{F},1} \qquad \text{and} \qquad f_*^{n_i}\mu_y^{\mathscr{F},1} \to (g_{yz})_*\mu_y^{\mathscr{F},1}.$$

So, we need to prove that

$$\mu_{f^{n_i}(x)}^{\mathscr{F},1} \to \mu_z^{\mathscr{F},1} \qquad \text{and} \qquad \mu_{f^{n_i}(y)}^{\mathscr{F},1} \to \mu_z^{\mathscr{F},1}.$$

To this end, we need to use some kind of continuity of the map $x \to \mu_x^{\mathscr{F},1}$. This map is only measurable so we need to do something to guarantee some kind of continuity. We can apply Luzin's Theorem and obtain continuity on a compact set of an arbitrarily large measure, so we need to pick the iterates (and hence z) inside this set. The problem is to pick *the same* iterates of x and y in this large set and we will now explain how to achieve that.

We have that $x \to \mu_x^{\mathscr{F},1}$ is a measurable map and so we have by Luzin's Theorem an increasing sequence of compact sets K_n, $\mu(K_n) \to 1$, such that the map restricted to K_n is continuous. Consider the forward Birkhoff average $\widetilde{\mathbf{1}}_{K_n}$ of the characteristic function $\mathbf{1}_{K_n}$ of K_n. Take R_n to be the set of points where $\widetilde{\mathbf{1}}_{K_n} > 1/2$, then $\mu(R_n) \to 1$ since $\mu(K_n) \to 1$. Since the partition into stable manifolds refines the partition into ergodic components of f, for μ-a.e. point $x \in R_n$, μ_x^s-a.e. point in $W^s(x)$ is inside R_n. Take one of these typical points $x \in R_n$ and $y \in W^s(x) \cap R_n$. Let $L(x) = \{n \geq 0 : f^n(x) \in K_n\}$ and similarly $L(y) = \{n \geq 0 : f^n(y) \in K_n\}$. We have from the choice of R_n and since $x, y \in R^n$

$$\frac{\#(L(x) \cap [0,n])}{n} \to \widetilde{\chi}_{K_n}(x) > \frac{1}{2}$$

and the same is true for y. Hence both sets $L(x)$ and $L(y)$ have asymptotic density greater than $1/2$ and hence they should intersect in a set of positive asymptotic density, in particular $L(x) \cap L(y)$ is an infinite set. So we take the sequence n_i inside $L(x) \cap L(y)$ and we get then that $f^{n_i}(x)$ and $f^{n_i}(y)$ are inside K_n and hence their limit z is also in K_n. Now by continuity of $x \to \mu_x^{\mathscr{F},1}$ restricted to K_n we get $\mu_{f^{n_i}(x)}^{\mathscr{F},1} \to \mu_z^{\mathscr{F},1}$ and $\mu_{f^{n_i}(y)}^{\mathscr{F},1} \to \mu_z^{\mathscr{F},1}$. □

4.2. **Proof of Theorem 2.10.** Now let us return to our case. We want to prove the same invariance by holonomy of the conditional measures in a less uniform but more specialized setting.

Let α, μ, and χ satisfy the assumptions of the technical Theorem 2.9. Let us fix a generic singular element $\mathbf{t} \in \mathbb{R}^k$, *i.e.,* an element such that $\chi(\mathbf{t}) = 0$ but $\chi_j(\mathbf{t}) \neq 0$ for any other Lyapunov exponent, and an element $\mathbf{s}$ close to $\mathbf{t}$ such that $\chi(\mathbf{s}) < 0$ but it is still the biggest negative Lyapunov exponent for $\mathbf{s}$. Then $\mathcal{W}^s_{\alpha(\mathbf{t})} \subset \mathcal{W}^s_{\alpha(\mathbf{s})}$ and in fact $E^s_{\alpha(\mathbf{s})} = E^s_{\alpha(\mathbf{t})} \oplus E_\chi$. We have the invariant foliation $\mathcal{W}$ associated to χ tangent to E_χ and we consider the conditional measures $\mu^{\mathcal{W}}$ associated to this foliation that we normalize in a certain convenient way. Given x and $y \in \mathcal{W}^s_\alpha(\mathbf{t})(x)$ we define the holonomy map $h_{xy} \colon \mathcal{W}(x) \to \mathcal{W}(y)$ by sliding along $\mathcal{W}^s_\alpha(\mathbf{t})$ manifolds; we omit the lower index xy if it is understood from the context.

Theorem 2.10 is an immediate corollary of the following holonomy invariance property of conditional measures.

PROPOSITION 4.2. *Let χ, $\mathbf{t}$, and μ be as above. Then for μ-a.e. x and μ^s_x-a.e. $y \in \mathcal{W}^s_\alpha(\mathbf{t})(x)$, there is a scalar measurable function $c(x,y)$ such that $h_*\mu^{\mathcal{W}}_x = c(x,y)\mu^{\mathcal{W}}_y$ where h is holonomy along $\mathcal{W}^s_\alpha(\mathbf{t})$.*

REMARK 5. Both Propositions 4.1 and 4.2 assert that the system of conditional measures, defined affinely, is holonomy invariant. The difference is that in the former case there is a normalization that makes normalized conditional measures invariant.

Proof. We will argue as in the proof of Proposition 4.1 but we need to address the problem that the dynamics of $\alpha(t)$ along $\mathcal{W}$ is not an isometry although the Lyapunov exponent along $\mathcal{W}$ is equal to zero.

By Proposition 3.3 there is a measurable α-invariant family of affine parameters $H_x \colon \mathbb{R} \to \mathcal{W}(x)$. We normalize $\mu^{\mathcal{W}}_x$ in such a way that $\mu^{\mathcal{W}}_x(H_x(-1,1)) = 1$, and then define $\mu^{\mathcal{W},1}_x := \mu^{\mathcal{W}}_x | H_x(1-,1)$.

We shall use the time change β introduced in Proposition 3.4. Using Luzin's Theorem, let K_n be an increasing sequence of compact sets (which we take also inside Pesin sets for α) with $\nu(K_n) \to 1$ for the β-invariant measure ν (and hence $\mu(K_n) \to 1$) such that each of the following holds on the set K_n:

1. the time change is continuous;
2. the map $x \mapsto H_x$ is continuous (with respect to the $C^1(\mathbb{R}, M)$-topology for affine structures);
3. the map $x \mapsto \mu^{\mathcal{W},1}_x$ is continuous with respect to the weak*-topology in measures on $[-1,1]$.

Take $f = \beta(t)$ and consider $\widetilde{\mathbf{1}}_{K_n}$ the forward Birkhoff average of the characteristic function $\mathbf{1}_{K_n}$. Let as before R_n be the set of points where $\widetilde{\mathbf{1}}_{K_n} > 1/2$. Since $\nu(K_n) \to 1$, we also have $\nu(R_n) \to 1$ (and hence $\mu(R_n) \to 1$). Also, since the partition into $\tilde{\mathcal{W}}^s_{\beta(\mathbf{t})}$ stable "manifolds" refines the partition into ergodic components (see the remark after Proposition 3.5) we have that for ν-a.e. point x in R_n, ν^s_x-a.e. point in $\tilde{\mathcal{W}}^s_{\beta(\mathbf{t})}(x)$ is inside R_n. Take one of these typical points $x \in R_n$ and

$y \in \tilde{\mathcal{W}}^s_{\beta(\mathbf{t})}(x) \cap R_n$. As in the proof of Proposition 4.1 we can take a sequence of iterates n_i such that $f^{n_i}(x)$ and $f^{n_i}(y)$ are inside K_n and hence their limit z is also in K_n. Now by continuity of $x \to \mu^{\mathcal{W},1}_x$ restricted to K_n we get that $\mu^{\mathcal{W},1}_{f^{n_i}(x)} \to \mu^{\mathcal{W},1}_z$ and $\mu^{\mathcal{W},1}_{f^{n_i}(y)} \to \mu^{\mathcal{W},1}_z$.

On the other hand, if we write $\mathbf{a}_i = \mathbf{g}(x, n_i\mathbf{t})$ and $\mathbf{b}_i = \mathbf{g}(y, n_i\mathbf{t})$, we have that $\|D^E_x\alpha(\mathbf{a}_i)\|_\varepsilon = 1$ and $\|D^E_y\alpha(\mathbf{b}_i)\|_\varepsilon = 1$. Hence, using the affine parameters from Proposition 3.3 we obtain

$$\alpha(\mathbf{a}_i) \circ H_x = H_{\alpha(\mathbf{a}_i)(x)} \qquad \text{and} \qquad \alpha(\mathbf{b}_i) \circ H_y = H_{\alpha(\mathbf{b}_i)(y)}. \tag{4.1}$$

The holonomy $\tilde{h}\colon \mathcal{W}(x) \to \mathcal{W}(y)$ along $\tilde{\mathcal{W}}^s_{\beta(\mathbf{t})}$ equals

$$\lim_{i\to\infty} \big(\alpha(\mathbf{b}_i)|\mathcal{W}(y)\big)^{-1} \circ P_i \circ \big(\alpha(\mathbf{a}_i)|\mathcal{W}(x)\big),$$

where P_i is a sequence of smooth maps from $\mathcal{W}(\alpha(\mathbf{a}_i(x))$ to $\mathcal{W}(\alpha(\mathbf{b}_i(y))$ converging to the identity. Using (4.1) and property 2. above, we obtain

$$\lim_{i\to\infty} \alpha(\mathbf{a}_i)|\mathcal{W}(x) = \lim_{i\to\infty} H_{\alpha(\mathbf{a}_i)(x)} \circ H_x^{-1} = H_z \circ H_x^{-1} =: g_{xz}$$

since $\alpha(\mathbf{a}_i)(x) \to z$. Similarly

$$\lim_{i\to\infty} \alpha(\mathbf{b}_i)|\mathcal{W}(y) = \lim_{i\to\infty} H_{\alpha(\mathbf{b}_i)(y)} \circ H_y^{-1} = H_z \circ H_y^{-1} =: g_{yz}$$

since $\alpha(\mathbf{b}_i)(y) \to z$ also. So, $\tilde{h} = g_{yz}^{-1} \circ g_{xz} = H_y \circ H_x^{-1}$. Using (4.1) and the definition of $\mu^{\mathcal{W},1}$ we get

$$\alpha(\mathbf{a}_i)_*\mu^{\mathcal{W},1}_x = \mu^{\mathcal{W},1}_{\alpha(\mathbf{a}_i)(x)} \qquad \text{and} \qquad \alpha(\mathbf{b}_i)_*\mu^{\mathcal{W},1}_y = \mu^{\mathcal{W},1}_{\alpha(\mathbf{b}_i)(y)}.$$

Putting all of this together and sending i to infinity, we get $(g_{xz})_*\mu^{\mathcal{W},1}_x = \mu^{\mathcal{W},1}_z$ and $(g_{yz})_*\mu^{\mathcal{W},1}_y = \mu^{\mathcal{W},1}_z$ which gives $\tilde{h}_*\mu^{\mathcal{W},1}_x = \mu^{\mathcal{W},1}_y$. So, $\tilde{h}_*\mu^{\mathcal{W},1}_x = \mu^{\mathcal{W},1}_y$ for a.e. x and for ν^s_x-a.e. y in $\tilde{\mathcal{W}}^s_{\beta(\mathbf{t})}(x)$.

So, if we prove that ν^s_x-a.e. point corresponds to μ^s_x-a.e. point under the projection from $\tilde{\mathcal{W}}^s_{\beta(\mathbf{t})}(x)$ to $\mathcal{W}^s_{\alpha(\mathbf{t})}(x)$ then for μ-a.e. point x and for μ^s_x-a.e. point $y \in \mathcal{W}^s_{\alpha(\mathbf{t})}(x)$ we get that $\tilde{y} = \alpha(\phi_x(y)\mathbf{w})(y)$ is a typical point for ν^s_x in $\tilde{\mathcal{W}}_{\beta(\mathbf{t})}(x)$ and hence the holonomy $\tilde{h}\colon \mathcal{W}(x) \to \mathcal{W}(\tilde{y})$ makes $\tilde{h}_*\mu^{\mathcal{W},1}_x = \mu^{\mathcal{W},1}_{\tilde{y}}$. So the proof is finished since $h\colon \mathcal{W}(x) \to \mathcal{W}(y)$ equals $h = (\alpha(\phi_x(y)\mathbf{w}))^{-1} \circ \tilde{h}$ and $\mu^{\mathcal{W}}$ is α-invariant modulo multiplication by a constant.

Let us see that ν^s_x-a.e. point corresponds to μ^s_x-a.e. point under the projection from $\tilde{\mathcal{W}}^s_{\beta(\mathbf{t})}(x)$ to $\mathcal{W}^s_{\alpha(\mathbf{t})}(x)$. To this end we will consider the foliation $(\mathcal{N} \oplus \mathcal{W}^s_{\alpha(\mathbf{t})})(x)$, tangent to the orbit direction plus the stable direction of $\alpha(\mathbf{t})$, see Proposition 6.7 in [10], here $\mathcal{N}$ stands for the orbit foliation. The conditional measures of μ and ν along $\big(\mathcal{N} \oplus \mathcal{W}^s_{\alpha(\mathbf{t})}\big)(x)$, μ^{os}_x and ν^{os}_x, are equivalent since μ and ν are equivalent measures. On the other hand, since μ and ν are invariant under α, and β respectively and β is smooth along the orbits of α (which are the same as the orbits for β), the conditional measures of μ and ν along $\mathcal{N}$, μ^o_x and ν^o_x are equivalent to Lebesgue measure. On the other hand, holonomies are absolutely continuous both for α and β-stable foliations, hence μ^{os}_x and ν^{os}_x

are locally equivalent to a product measure (see the proof of [10, Lemma 7.1]). These product measures are Lebesgue on the orbit direction and μ_x^s on $\mathcal{W}^s_{\alpha(\mathbf{t})}(x)$ for μ_x^{os} and Lebesgue on $\mathcal{N}$ and ν_x^s on $\tilde{\mathcal{W}}^s_{\beta(\mathbf{t})}(x)$ for μ_x^{os}. But this implies that the saturations by the orbit foliation of μ_x^s-zero sets are the same as the saturations by the orbit foliation of ν_x^s-zero sets. This gives that ν_x^s-a.e. point corresponds to μ_x^s-a.e. point by the holonomy from $\tilde{\mathcal{W}}^s_{\beta(\mathbf{t})}(x)$ to $\mathcal{W}^s_{\alpha(\mathbf{t})}(x)$ along the orbit. □

4.3. **Proof of Theorem 2.11.** Take $\mathbf{t}$ and $\mathbf{s}$ as in Theorem 2.10. Take now a neighborhood of $\mathbf{t}$ such that positive Lyapunov exponents other than χ in this neighborhood are all bigger than χ. For $\mathbf{s}$ in this neighborhood, we denote by V^{u-1} the strong unstable foliation associated to the positive Lyapunov exponents different from χ; this does not depend on $\mathbf{s}$. Pick $\mathbf{s}$ in this neighborhood and observe that if $\mathbf{s}$ is on one side of $\ker\chi$, where $\chi(\mathbf{s}) > 0$, (denote this side L^+) there is only one more positive Lyapunov exponent, *i.e.*, χ, and no new exponent appears to the other side L^-, where $\chi(\mathbf{s}) < 0$. So, for $\mathbf{s} \in L^- \cup \ker\chi$, $V^{u-1} = \mathcal{W}^u_{\alpha(\mathbf{s})}$ and for $\mathbf{s} \in L^+$, $V^{u-1} \subsetneq \mathcal{W}^u_{\alpha(\mathbf{s})}$.

By Corollary 3.9 we see that the map $\mathbf{s} \mapsto h(\xi_{u-1}, \alpha(\mathbf{s}))$ is linear in this neighborhood. On the other hand, by the Ledrappier–Young entropy formula we have that for $\mathbf{s} \in L^+$,

$$h_\mu(\alpha(\mathbf{s})) = h_\mu(\xi_u, \alpha(\mathbf{s})) = h(\xi_{u-1}, \alpha(\mathbf{s})) + \gamma_u \chi(\mathbf{s}),$$

where ξ_u is any partition subordinate to $\mathcal{W}^u_{\alpha(\mathbf{s})}$ and γ_u does not depend on $\mathbf{s}$ by its definition and the assumption on $\mathbf{s}$, see the definition of γ_u in Section 3.4.

Finally, for $\mathbf{s} \in L^- \cup \ker\chi$ with $\mathbf{s}$ close to $\mathbf{t}$, we have $h_\mu(\alpha(\mathbf{s})) = h(\xi_{u-1}, \alpha(\mathbf{s}))$. Hence, $h_\mu(\alpha(\mathbf{s}))$ is linear on L^+, and $h_\mu(\alpha(\mathbf{s}))$ is also linear on L^-. So, in order for this h_μ be differentiable, it is necessary and sufficient that these two linear maps coincide. But since $\mathbf{s} \to h(\xi_{u-1}, \alpha(\mathbf{s}))$ is linear in the whole neighborhood, this is the same as asking that $\gamma_u = 0$. Hence, h_μ is differentiable at $\mathbf{t}$ if and only if $\gamma_u = 0$.

Now, if the conditional measures along $\mathcal{W}$ are atomic, then by Theorem 2.10 the conditional measure along V^u is supported in V^{u-1} and hence $\delta^u = \delta^{u-1}$ which gives $\gamma^u = 0$. Thus, h_μ is differentiable at $\mathbf{t}$. □

5. PROOFS OF THEOREMS 2.2 AND 2.4

5.1. **Proof of Theorem 2.2.** By Theorem 2.9 we know that the conditional measures along the Lyapunov directions are either Lebesgue measure or atomic. We will show that if all conditionals are atomic, then the conditional measures along stable manifolds of $\alpha(\mathbf{t})$ are atomic and hence $h_\mu(\mathbf{t}) = 0$.

Observe that without loss of generality we can assume $\mathbf{t}$ to be a regular element since for every $\mathbf{t}$ one can find a regular element $\mathbf{t}'$ whose stable foliation contains that of $\mathbf{t}$. Hence if $h_\mu(\mathbf{t}) > 0$ then $h_\mu(\mathbf{t}') > 0$.

We shall build a sequence of nested subfoliations $\widetilde{\mathcal{V}}_0 \supset \widetilde{\mathcal{V}}_1 \supset \cdots \supset \widetilde{\mathcal{V}}_n$, where $\mathcal{W}^s_{\alpha(\mathbf{t})} = \widetilde{\mathcal{V}}_0$, $\widetilde{\mathcal{V}}_n(x) = \{x\}$ and each $\widetilde{\mathcal{V}}_i$ is either equal to $\widetilde{\mathcal{V}}_{i-1}$ or has one less dimension for a.e. x. We shall prove also that the conditional measures on the leaves $\widetilde{\mathcal{V}}_i$ are supported by single leaves of $\widetilde{\mathcal{V}}_{i+1}$, and the theorem will follow.

Since $\mathbf{t}$ is regular it belongs to a Weyl chamber, which we denote by C_0. Let γ be a curve which begins at $\mathbf{t}$, passes through every Lyapunov hyperplane and crosses each Lyapunov hyperplane only once and at a point that does not lie on any other Lyapunov hyperplane. An example of such a curve is the half-circle in a two-dimensional plane through $\mathbf{t}$ that is in general position, *i.e.,* intersects all Lyapunov hyperplanes along different lines.

Let us number $C_0, C_1, \ldots, C_n$ the Weyl chambers and $\chi_1, \ldots, \chi_n$ the Lyapunov exponents, in the order they appear.

The stable foliation does not change within a Weyl chamber so we shall denote by $\mathcal{W}^s_{C_i}$ the stable foliation associated to this Weyl chamber, also the sign of a Lyapunov exponent does not change so we can denote this sign by $\chi_j(C_i)$. So, we define $\widetilde{\mathcal{V}}_0 = \mathcal{W}^s_{C_0}$ and $\widetilde{\mathcal{V}}_i = \widetilde{\mathcal{V}}_{i-1} \cap \mathcal{W}^s_{C_i}$. Clearly $\widetilde{\mathcal{V}}_i \subset \widetilde{\mathcal{V}}_{i-1}$ and $\widetilde{\mathcal{V}}_i$ is a nice foliation since it is an intersection of stable foliations.

When passing from C_{i-1} to C_i, χ_i is the only Lyapunov exponent that changes sign. So, if $\chi_i(C_i) < 0$ then $\chi_i(C_{i-1}) > 0$ so $\mathcal{W}^s_{C_i} \supset \mathcal{W}^s_{C_{i-1}}$ and hence $\widetilde{\mathcal{V}}_i = \widetilde{\mathcal{V}}_{i-1}$. On the other hand, if $\chi_i(C_i) > 0$ then $\chi_i(C_{i-1}) < 0$ and hence $\mathcal{W}^s_{C_i} \subsetneq \mathcal{W}^s_{C_{i-1}}$ and in this case $\widetilde{\mathcal{V}}_i \subsetneq \widetilde{\mathcal{V}}_{i-1}$, in fact $\mathcal{W}^i$ in no more in $\widetilde{\mathcal{V}}_i$, *i.e.,* $\mathcal{W}^i(x) \cap \widetilde{\mathcal{V}}_i(x) = \{x\}$. Moreover, if we take an element inside C_{i-1} but close to $\ker \chi_i$ we have that $\mathcal{W}^i$ is the slowest direction in $\mathcal{W}^s_{C_{i-1}}$ while $\widetilde{\mathcal{V}}_i$ is inside the fast direction in $\mathcal{W}^s_{C_{i-1}}$ (which is exactly $\mathcal{W}^s_{C_i}$).

Let us fix measurable partitions η_i subordinate to $\mathcal{W}^s_{C_i}$ whose elements are open subsets of the leaves $\mod 0$, *i.e.,* the conditional measures of the boundaries are equal to zero.

We shall chose these partitions in such a way that if $\mathcal{W}^s_{C_i} \supset \mathcal{W}^s_{C_{i-1}}$ then $\eta_i < \eta_{i-1}$ and if $\mathcal{W}^s_{C_i} \subset \mathcal{W}^s_{C_{i-1}}$ then $\eta_i > \eta_{i-1}$. Let $\xi_0 = \eta_0$ and define inductively the measurable partitions $\xi_i = \xi_{i-1} \vee \eta_i$. It is sufficient to prove that $\mu^x_{\xi_0}(\xi_n(x)) > 0$ for a.e. x since $\xi_n = \varepsilon$ by construction. To this end we shall argue inductively and prove that $\mu^x_{\xi_{i-1}}(\xi_i(x)) > 0$ for a.e. x. Once we know this, we have that for any measurable set A

$$\mu^x_{\xi_i}(A) = \frac{\mu^x_{\xi_{i-1}}(A \cap \xi_i(x))}{\mu^x_{\xi_{i-1}}(\xi_i(x))}.$$

Hence

$$\mu^x_{\xi_{i-1}}(\xi_n(x)) = \mu^x_{\xi_i}(\xi_n(x))\mu^x_{\xi_{i-1}}(\xi_i(x)),$$

which gives

$$\mu^x_{\xi_0}(\xi_n(x)) = \prod_{i=1}^{n} \mu^x_{\xi_{i-1}}(\xi_i(x)) > 0.$$

When $\mathscr{W}^s_{C_i} \supset \mathscr{W}^s_{C_{i-1}}$ we have $\xi_i = \xi_{i-1}$ and hence $\mu^x_{\xi_{i-1}}(\xi_i(x)) > 0$ trivially. So, let us assume that $\mathscr{W}^s_{C_i} \subsetneq \mathscr{W}^s_{C_{i-1}}$ and hence $\eta_i > \eta_{i-1}$. Since the conditionals along $\mathscr{W}^i$ are atomic, by Theorem 2.10, we conclude that $\mu^x_{\eta_{i-1}}(\eta_i(x)) > 0$ for a.e x.

Finally, we prove that $\mu^x_{\xi_{i-1}}(\xi_i(x)) > 0$ for a.e. x using the following lemma with $X = \eta_{i-1}(x)$, $\eta = \eta_i$, $\xi_1 = \xi_{i-1}$ and $\xi_2 = \xi_i$.

LEMMA 5.1. *Let ξ_1, ξ_2, and η be three measurable partitions of a Lebesgue space X. Let us assume that $\xi_2 = \xi_1 \vee \eta$. Let μ be a measure on X. If there is an x such that $\mu(\eta(x)) > 0$, then $\mu^y_{\xi_1}(\xi_2(y)) = \mu^y_{\xi_1}(\eta(x))$ and $\mu^y_{\xi_1}(\eta(x)) > 0$ for μ-a.e. y in $\eta(x)$, and hence $\mu^y_{\xi_1}(\xi_2(y)) > 0$ for a.e. $y \in \eta(x)$.*

Proof. The first equality is trivial since $\xi_2(y) = \xi_1(y \cap \eta(y))$ and $\eta(x) = \eta(y)$ for $y \in \eta(x)$. For the inequality, let $D = \{y \in X : \mu^y_{\xi_1}(\eta(x)) = 0\}$. Observe that D is ξ_1-saturated. Let $B = D \cap \eta(x)$. We want to show that $\mu(B) = 0$. We have

$$\mu(B) = \int \mu^y_{\xi_1}(B)\, d\mu = \int_D \mu^y_{\xi_1}(B)\, d\mu + \int_{D^c} \mu^y_{\xi_1}(B)\, d\mu.$$

The first integral on the right-hand side is 0 since $B \subset \eta(x)$ and $\mu^y_{\xi_1}(\eta(x)) = 0$ for $y \in D$. The second is zero since for $y \in D^c$ we have that $\xi_1(y) \subset D^c$ and hence, since $B \subset D$ we get that $\xi_1(y) \cap D = \emptyset$. □

5.2. **Proof of Theorem 2.4.** Theorem 2.4 is an immediate corollary of Theorem 2.11 and the following general criterion of absolute continuity.

THEOREM 5.2. *Let $f: M \to M$ be a $C^{1+\theta}$ diffeomorphism preserving an ergodic measure μ. Let $TM = E^u \oplus E^c \oplus E^s$ be the Oseledets splitting associated to μ. Let us assume that:*

1. *E^c is tangent to a smooth foliation $\mathscr{O}$, $Df|E^c$ is an isometry with respect to to the standard metric in M, and conditional measures along $\mathscr{O}$ are Lebesgue measure;*
2. *$E^u = E_1 \oplus \cdots \oplus E_u$ and $E^s = E_s \oplus \cdots \oplus E_r$, where $\chi_i < \chi_j$ if $i < j$;*
3. *each E_i is tangent to an absolutely continuous Lyapunov foliation $\mathscr{W}^i$ and the conditional measures along $\mathscr{W}^i$ are absolutely continuous with respect to Lebesgue measure for a.e. point.*

Then, μ is absolutely continuous with respect to Lebesgue measure.

Proof. The proof reduces to seeing that the conditional measures along stables and unstables are absolutely continuous. We shall argue by induction on the flag tangent to $E_1, E_1 \oplus E_2, \ldots, E_1 \oplus \cdots \oplus E_u = E^u$. So, let us call V_i the "foliation" tangent to $E_1 \oplus \cdots \oplus E_i$. That conditional measures along $V_1 = \mathscr{W}^1$ are absolutely continuous is by assumption. Let us see that conditionals along V_2 are absolutely continuous and then the general step will follows as well.

Let R be the set of regular points. Take a regular point x and a zero Lebesgue measure set $A \subset V_2(x)$. We want to see that $\mu^x_{V_2}(A) = 0$ also. Taking $A \cap R$ we have $\mu^x_{V_2}(A) = \mu^x_{V_2}(A \cap R)$ and $\mathrm{Leb}_{V_2(x)}(A \cap R) = 0$. So, we can assume without loss of generality that A consists of regular points (indeed arguing similarly we can also assume A is inside some Pesin set if necessary). Now, since the foliation $\mathscr{W}^1 = V_1$ is absolutely continuous and indeed it is also absolutely continuous when restricted top $V_2(x)$, we can saturate the set A by V_1 leaves and get a set of zero $\mathrm{Leb}_{V_2(x)}$-measure which is V_1-saturated and which contains A. Let us call this set by B and let us see that $\mu^x_{V_2}(B) = 0$.

Now, if $\mu^x_{V_2}(B) > 0$, then there should be a regular point $z \in V_2(x)$ such that $\mu^z_{\mathscr{W}^2}(B) > 0$. However, since $\mu^z_{\mathscr{W}^2}$ is equivalent to Lebesgue measure, this will imply that $\mathrm{Leb}_{\mathscr{W}^2(z)}(B) > 0$ and again, absolute continuity of V_1 and the fact that B is V_1 saturated will imply that $\mathrm{Leb}_{V_2(x)}(B) > 0$, which is a contradiction. □

6. PROOF OF THEOREM 2.5

6.1. **General facts about entropy.** We shall make use of the following standard facts. Given measurable partitions ξ, η and ζ we have

1. $H(\xi \vee \eta|\zeta) = H(\xi|\zeta) + H(\eta|\zeta \vee \xi)$
2. If $\xi > \zeta$, then $H(\xi|\eta) \geq H(\zeta|\eta)$ and $H(\eta|\xi) \leq H(\eta|\zeta)$
3. If $\xi_n \uparrow \xi$, then $H(\xi_n|\eta) \uparrow H(\xi|\eta)$
4. If $\xi_n \downarrow \xi$ and $H(\xi_1|\eta) < \infty$, then $H(\xi_n|\eta) \downarrow H(\xi|\eta)$
5. If $\eta_n \uparrow \eta$ and $H(\xi|\eta_1) < \infty$, then $H(\xi|\eta_n) \downarrow H(\xi|\eta)$
6. If $\eta_n \downarrow \eta$, then $H(\xi|\eta_n) \uparrow H(\xi|\eta)$
7. For every $n \in \mathbb{Z}$, $h(\xi \vee \eta, T) = h(\xi \vee T^n\eta, T)$

Given a measurable partition η we write $\eta_T = \bigvee_{n\in\mathbb{Z}} T^n\eta$ and $\eta^+ = \bigvee_{n=0}^{\infty} T^n\eta$.

LEMMA 6.1. *Given two measurable partitions ξ and η,*

$$h(\xi \vee \eta, T) \leq h(\eta, T) + h(\xi \vee \eta_T, T).$$

Proof. First of all

$$\begin{aligned} h(\xi \vee \eta, T) &= H(T^{-1}\xi \vee T^{-1}\eta|\xi^+ \vee \eta^+) \\ &= H(T^{-1}\eta|\xi^+ \vee \eta^+) + H(T^{-1}\xi|\xi^+ \vee T^{-1}\eta^+) \end{aligned}$$

Then

$$\begin{aligned} h(\xi \vee T^{-n}\eta, T) &= H(T^{-1-n}\eta|\xi^+ \vee T^{-n}\eta^+) + H(T^{-1}\xi|\xi^+ \vee T^{-1-n}\eta^+) \\ &= H(T^{-1}\eta|T^n\xi^+ \vee \eta^+) + H(T^{-1}\xi|\xi^+ \vee T^{-1-n}\eta^+) \\ &\leq H(T^{-1}\eta|\eta^+) + H(T^{-1}\xi|\xi^+ \vee T^{-1-n}\eta^+) \end{aligned}$$

On one hand, the last term is bounded by $H(T^{-1}\xi|\xi^+) = h(\xi, T)$ and also $\xi \vee T^{-1-n}\eta^+ \uparrow \xi \vee \eta_T$ hence this last term $\downarrow H(T^{-1}\xi|\xi^+ \vee \eta_T)$. On the other hand, since η_T is a T-invariant partition we get easily that $h(\xi \vee \eta_T, T) = H(T^{-1}\xi|\xi^+ \vee \eta_T)$ and hence the inequality. □

Finally we have the standard formula: given an invariant partition ζ (*i.e.*, $p^{-1}\zeta = \zeta$) we have

$$h(T) = h(T|\zeta) + \sup_{\xi} h(\xi \vee \zeta, T),$$

where $h(T|\zeta) = \sup_{\eta<\zeta} h(\eta, T)$.

PROPOSITION 6.2. *Let us consider $T\colon (X,\mu)\to(X,\mu)$ and $S\colon (Y,\lambda)\to(Y,\lambda)$, and assume that S is a factor of T via a measure-preserving map $p\colon (X,\mu)\to(Y,\lambda)$. Let ξ be a full-entropy partition for T and η a partition such that $\eta_S=\varepsilon:=$ the partition into points and $p^{-1}\eta<\xi$. Then η is a full-entropy partition for S.*

Proof. Let us call ζ the partition into preimages of p. We have on one hand

$$h(T)=h(T|\zeta)+\sup_\gamma h(\gamma\vee\zeta,T)=h(S)+\sup_\gamma h(\gamma\vee\zeta,T),$$

and on the other hand, since $(p^{-1}\eta)_T=p^{-1}(\eta_S)=p^{-1}\varepsilon=\zeta$,

$$\begin{aligned}h(T)&=h(\xi,T)=h(\xi\vee p^{-1}\eta,T)\\&\le h(p^{-1}\eta,T)+h(\xi\vee(p^{-1}\eta)_T,T)=h(\eta,S)+h(\xi\vee\zeta,T),\end{aligned}$$

where the inequality follows from Lemma 6.1. Thus $h(\eta,S)=h(S)$, *i.e.*, η is a full-entropy partition. Observe also that ξ is also a full-entropy partition for the fiber-entropy $\sup_\gamma h(\gamma\vee\zeta,T)$. □

6.2. **Matching of Lyapunov half-spaces.** Here we assume α and α_0 are $\mathbb{Z}^k$-actions on an infranilmanifold M as in Theorem 2.5.

The unstable foliation $W^u_{\alpha_0(\mathbf{m})}$ for an element of the algebraic action α_0 is right homogeneous. Lyapunov foliations W^i that are one-dimensional under our assumptions are intersections of unstable foliations of different elements and are projections of cosets of one-parameter subgroups in N.

Let $p\colon M\to M$ be the semiconjugacy between these actions and let μ be an ergodic large measure invariant under α. We want to prove that Weyl chambers for both actions match. We do this in two steps. First we prove a general result which requires no assumption on the linear action:

LEMMA 6.3. *If L is a Lyapunov hyperplane for α_0 then L is also a Lyapunov hyperplane for α and the Lyapunov half-spaces match.*

REMARK 6. Note that this lemma implies that the number of nonproportional (and hence nonzero) Lyapunov exponents for the nonlinear action is greater than or equal to the number of coarse Lyapunov distributions for the linear action. Then later we prove that under our assumptions the Lyapunov hyperplanes for the nonlinear action also correspond to Lyapunov hyperplanes for the linear action.

Proof of Lemma 6.3. Assume by contradiction that there are two elements $\mathbf{n},\mathbf{m}\in\mathbb{Z}^k$ on different sides of L, but in the same Weyl chamber for α. Then we will have $\mathscr{W}^u_{\alpha(\mathbf{n})}(x)=\mathscr{W}^u_{\alpha(\mathbf{m})}(x)$ but $W^u_{\alpha_0(\mathbf{n})}\neq W^u_{\alpha_0(\mathbf{m})}$. Since the semiconjugacy maps unstable manifolds for α into unstable manifolds for α_0, either for $\mathbf{n}$ or $\mathbf{m}$ the following is true (we assume it is for $\mathbf{n}$):

$$p\big(\mathscr{W}^u_{\alpha(\mathbf{n})}(x)\big)\subset p(x)W^u_{\alpha_0(\mathbf{n})}\cap E^u_{\alpha_0(\mathbf{m})}\subsetneq p(x)W^u_{\alpha_0(\mathbf{n})}.$$

Now, we can take an increasing full-entropy partition $\tilde{\xi}$ for $\alpha(\mathbf{n})$ subordinate to $\mathscr{W}^u_{\alpha(\mathbf{n})}$ like the one built in [17] and an increasing partition η subordinate to

$yW^u_{\alpha_0(\mathbf{n})} \cap W^u_{\alpha_0(\mathbf{m})}$, again like in [17], and we can build them in such a way that $p^{-1}\eta < \tilde{\xi}$.

Since the negative iterates of $\alpha_0(\mathbf{n})$ contract $W^u_{\alpha_0(\mathbf{n})} \cap W^u_{\alpha_0(\mathbf{m})}$ we have that $\eta_{\alpha_0(\mathbf{n})} = \varepsilon$ and hence $(p^{-1}\eta)_{\alpha(\mathbf{n})} = \zeta$. So we have that, using Proposition 6.2, η is a full-entropy partition for $\alpha_0(\mathbf{n})$.

But since we are considering Lebesgue measure for α_0, $W^u_{\alpha_0(\mathbf{n})} \cap W^u_{\alpha_0(\mathbf{m})} \subsetneq E^u_{\alpha_0(\mathbf{n})}$ and η is subordinate to $yW^u_{\alpha_0(\mathbf{n})} \cap W^u_{\alpha_0(\mathbf{m})}$ we have that

$$h(\eta, \alpha_0(\mathbf{n})) < h(\alpha_0(\mathbf{n})),$$

which gives a contradiction. □

Now, we consider the case at hand: there are no proportional Lyapunov exponents for the linear action, *i.e.,* there are different exponents $\chi_1, \dots, \chi_n$ and none of them are proportional. Thus in this case α also has n different nonproportional Lyapunov exponents and that the Weyl chambers coincide. In particular there are positive numbers c_i such that $\tilde{\chi}_i = c_i \chi_i$ for $i = 1, \dots, n$. So, we have already excluded zero exponents.

COROLLARY 6.4. *For every $i = 1, \dots, n$, there is a Lyapunov foliation $\mathscr{W}^i$ associated to $\tilde{\chi}_i$, such that the leaf $\mathscr{W}^i(x)$ is mapped by the semiconjugacy p into the corresponding coset $W^i(p(x))$.*

6.3. **Conclusion of the proof.**

PROPOSITION 6.5. *The conditional measures along $\mathscr{W}^i$ are equivalent to Lebesgue measure a.e.*

Proof. We shall argue as in the proof of preservation of Weyl chambers. Using Theorem 2.9 and arguing by contradiction we can assume that the conditional measures along $\mathscr{W}^i$ are atomic a.e. Then, take $\mathbf{t}$ and $\mathbf{s}$ as in Theorem 2.10 for the suspended action and we can take $\mathbf{s} \in \mathbb{Z}^k$. We will use the same notation α and $\tilde{\alpha}$ for the suspended actions of $\mathbb{R}^k$. Take now an $\alpha(-\mathbf{s})$ increasing partition ξ subordinate to $\mathscr{W}^s_{\alpha(\mathbf{t})}(x) \subsetneq \mathscr{W}^s_{\alpha(\mathbf{s})}(x)$. Then, since $p(\mathscr{W}^s_{\alpha(\mathbf{t})}(x)) \subset p(x)W^s_{\tilde{\alpha}(\mathbf{t})} \subsetneq p(x)W^s_{\tilde{\alpha}(\mathbf{s})}$ we can build another partition η subordinate to $p(x)W^s_{\tilde{\alpha}(\mathbf{t})}$ such that $p^{-1}\eta < \xi$. Since by Theorem 2.10 the conditional measure along $\mathscr{W}^s_{\alpha(\mathbf{s})}(x)$ is in fact supported in $\mathscr{W}^s_{\alpha(\mathbf{t})}(x)$ we have that ξ is a full-entropy partition for $\alpha(-\mathbf{s})$ and then by Proposition 6.2 η should be also a full-entropy partition for $\tilde{\alpha}(-\mathbf{s})$, but this is impossible since $W^s_{\tilde{\alpha}(\mathbf{t})} \subsetneq W^s_{\tilde{\alpha}(\mathbf{s})}$. □

Now we can use Theorem 5.2 and conclude that μ is an absolutely continuous measure. This concludes the proof of Theorem 2.5(a).

Theorem 2.5(b) follows exactly as in [8]. Or, more precisely, it is proven there using information that we already possess. Namely, [8, Lemma 4.4] asserts that the semiconjugacy restricted to a.e. leaf of a Lyapunov foliation is a diffeomorphism. Hence it matches asymptotic rates of expansion/contraction along the foliations and thus Lyapunov exponents. □

7. PROOFS OF THEOREMS 2.6 AND 2.7

7.1. **Invariant affine structures and Whitney smoothness of conjugacy.** We will use the invariant affine structures on stable manifolds of the action α both for the proof of Theorem 2.7 and for the proof of uniqueness in Theorem 2.6. We shall prove that the affine structures for unstable manifolds of the nonlinear action α are intertwined by the semiconjugacy with the standard affine structure of unstable spaces for the linear action α_0. Note that due to Theorem 2.5 no resonance condition holds for α. Existence of these affine structures is guaranteed by the nonresonance condition, see [7, Section 6.2].

PROPOSITION 7.1. *For every $\mathbf{t} \in \mathbb{Z}^k$ and on each leaf of $\mathcal{W}^s_{\alpha(\mathbf{t})}$ there is a unique smooth α-invariant affine structure together with a frame such that for any regular point x and any j such that $\chi_j(\mathbf{t}) < 0$*

- *the one-dimensional leaf $\mathcal{W}^j(x)$ is a coordinate line in $\mathcal{W}^s_{\alpha(\mathbf{t})}(x)$ and*
- *for any regular point $z \in \mathcal{W}^s_{\alpha(\mathbf{t})}(x)$ the affine structure on $\mathcal{W}^j(z)$ coincides with the restriction of the affine structure on $\mathcal{W}^s_{\alpha(\mathbf{t})}(x)$.*

We will use additive notation for the various invariant foliations associated with the action α_0.

PROPOSITION 7.2. *For almost every regular point z the restriction of the semiconjugacy h to the leaf $\mathcal{W}^s_{\alpha(\mathbf{t})}(z)$ is an affine bijection between $\mathcal{W}^s_{\alpha(\mathbf{t})}(z)$ and the hyperplane $p(z) + E^s_{\tilde{\alpha}(\mathbf{t})}$.*

Proof. Take z for which almost every point of the leaf $\mathcal{W}^s_{\alpha(\mathbf{t})}(z)$ with respect to the s-dimensional volume is regular. Since conditional measures are equivalent to Lebesgue measure, the set of such points is of full μ measure. Thus there is a dense subset of $\mathcal{W}^i(z)$ where leaves of $\mathcal{W}^j$ for all $j \neq i$ are defined. By Proposition 7.1 any such manifold is a part of a corresponding line and its affine parameterization agrees with the one coming from the affine structure in $\mathcal{W}^i(z)$. But we already know that the semiconjugacy on any leaf of $\mathcal{W}^j$ is affine. Furthermore for each regular $y \in \mathcal{W}^i(z)$ the manifold $\mathcal{W}^j(y)$ cannot be just an interval but must be the whole line in the affine structure. Thus we know that h restricted to $\mathcal{W}^i(z)$ is affine on a dense set of lines parallel to each coordinate direction. Hence it is affine. □

Now, Theorem 2.7 follows as an immediate consequence of Proposition 7.2 and the following result by R. de la Llave [4]. De la Llave proved a version of Journé's Theorem for functions defined on sets of positive Lebesgue measure, see [4, Theorem 5.7 and Proposition 5.13]. Let us summarize these results.

THEOREM 7.3. *Let M be a manifold and let P be a positive-measure set such that along points of P there are two continuous (uniformly) transverse families of C^r manifolds $\mathcal{W}^u$ and $\mathcal{W}^s$. Let $\phi\colon M \to \mathbb{R}$ be a measurable map which is defined and uniformly C^r along the manifolds $\mathcal{W}^u$ and $\mathcal{W}^s$. Then there are sets of arbitrarily large measure $\Omega_\varepsilon \subset P$ such that the restrictions $\phi|\Omega_\varepsilon$ are $C^{r-\varepsilon}$ in the Whitney sense.*

REMARK 7. Let us point out that in the C^∞ category, the extension exists in a weaker sense: for every $\varepsilon > 0$ there is a subset $\Omega_\varepsilon \subset P$ such that $\mathrm{Leb}(P \smallsetminus \Omega_\varepsilon) \leq \varepsilon$ and $\phi|\Omega_\varepsilon$ has a C^r extension for every $r < \infty$.

It is possible that those extensions can be "synchronized" to obtain a single C^∞ extension but the proof of Theorem 7.3 does not provide for this.

7.2. **Uniqueness.** Now we turn to the proof of uniqueness of large invariant measures.

LEMMA 7.4. *For any point z satisfying the assertion of Proposition 7.2, the manifold $\mathcal{W}^i(z)$ is a complete manifold properly embedded into $\mathbb{R}^n$ and at a bounded distance from E^i. Indeed, if we denote the semiconjugacy restricted to $\mathcal{W}^i(z)$ by $h_i^z\colon \mathcal{W}^i(z) \to h(z)+E^i$, then its inverse, $p_i^z\colon h(z)+E^i \to \mathcal{W}^i(z)$, is a proper diffeomorphism at a bounded distance from the inclusion.*

Proof. Proposition 7.2 implies this statement for any compact part of $\mathcal{W}^{i,+}(z)$. But since h is a bounded distance away from the identity for any sequence of points on $h(z)+E^i$ that goes to infinity, the preimages go to infinity, too. The assertion about the inverse of the semiconjugacy follows from the fact that the semiconjugacy is at a bounded distance from identity. □

Now we shall show how the Hopf argument applies in this case to get uniqueness similar to what is done for instance in [19]. To this end we will need that for any two given regular points x_1 and x_2 (possibly regular with respect to different large measures), the stable manifold of one intersects the unstable manifold of the other. This is done through an index argument.

LEMMA 7.5. *Let $E^i \subset \mathbb{R}^n$, $i = 1,2$, be two subspaces such that $E^1 \oplus E^2 = \mathbb{R}^n$. Let $p_i\colon E^i \to \mathbb{R}^n$, $i = 1,2$, be two proper embeddings at a bounded distance from the inclusion. Define $W_i = p_i(E^i)$, $i = 1,2$. Then $W_1 \cap W_2 \neq \emptyset$.*

Proof. Let us assume by contradiction that $W_1 \cap W_2 = \emptyset$. Let D_i be closed unit disks in E_i and define for $0 < t \leq 1$,

$$X_t\colon D_1 \times D_2 \to S^{n-1} \subset E^1 \oplus E^2$$

by

$$X_t(v_1, v_2) = \frac{p_1(v_1/t) - p_2(v_2/t)}{\|p_1(v_1/t) - p_2(v_2/t)\|}.$$

Observe that X_t is well defined since the denominator is never 0. If we write $p_i(z) = z + \psi_i(z)$, then there is $C > 0$ such that $\|\psi_i(z)\| \leq C$ for every $z \in E^i$.

Let us see that as $t \to 0$ we have that X_t restricted to $\partial(D_1 \times D_2)$ converges uniformly to

$$X_0(v_1, v_2) = \frac{v_1 - v_2}{\|v_1 - v_2\|}.$$

Indeed,

$$p_1(v_1/t) - p_2(v_2/t) = \frac{v_1 - v_2}{t} + \psi_1(v_1/t) - \psi_2(v_2/t).$$

Hence,

$$X_t(v_1, v_2) = \frac{p_1(v_1/t) - p_2(v_2/t)}{\|p_1(v_1/t) - p_2(v_2/t)\|} = \frac{v_1 - v_2 + t\big(\psi_1(v_1/t) - \psi_2(v_2/t)\big)}{\big\|v_1 - v_2 + t\big(\psi_1(v_1/t) - \psi_2(v_2/t)\big)\big\|}.$$

Since the ψ_i are uniformly bounded and the denominator is bounded away from zero when $(v_1, v_2) \in \partial(D_1 \times D_2)$ and t is small, we get $X_t \to X_0$ uniformly. But then it is known that X_0 is a map of nonzero degree (it is a homeomorphism), while X_1 restricted to $\partial(D_1 \times D_2)$ should have zero degree since it extends to $D_1 \times D_2$. □

Now, take μ_1 and μ_2 two ergodic large measures. Fix an element of the action $f := \alpha(\mathbf{n})$ with all exponents nonzero. We shall prove uniqueness using f. Let us call G the set of points satisfying the conclusion of Proposition 7.2, we have that G has full measure for every large measure.

Take a continuous function ϕ, we will prove that $\int \phi\, d\mu_1 = \int \phi\, d\mu_2$. Let us take a set $B_1 \subset G$ of full μ_1 measure such that for x in B_1, $\phi^+(x) = \phi^-(x) = \int \phi\, d\mu_1$; here ϕ^+ and ϕ^- denote forward and backward Birkhoff averages (with respect to f) respectively. Similarly take a set $B_2 \subset G$ of full μ_2 measure where $\phi^+(x) = \phi^-(x) = \int \phi\, d\mu_2$ for $x \in B_2$. Now take sets $A_i \subset B_i$ of full μ_i measure such that if a point x is in A_i then $\mathrm{Leb}_{W^u(x)}$ almost every point y in $W^u(x)$ is in B_i. We have that the A_i have full measure by the absolute continuity of the stable and unstable foliations.

We know that ϕ^+ is constant on stable manifolds and ϕ^- is constant on unstable manifolds.

We now lift all the objects to the universal covering in order to define holonomy maps in a more clear manner; we denote points in the universal covering and in the manifold in the same manner and this should not cause any confusion. Take now two points $x_1 \in A_1$ and $x_2 \in A_2$. By Lemmas 7.4 and 7.5 we have that $W^s(x_1) \cap W^u(x_2) \neq \emptyset$, but we do not know *a priori* what this intersection looks like. Now, the semiconjugacy must send this intersection into the intersection of $h(x_1) + E^s$ and $h(x_2) + E^u$, which is a point. By Proposition 7.2, we know that the semiconjugacy restricted to $W^u(x_2)$ is one-to-one, hence this intersection is a point. Still we do not know if this intersection is transverse, so we can not follow the usual Hopf argument. In any case, since almost every point in $W^u(x_1)$ is in G we can define the holonomy map $\pi\colon W^u(x_1) \to W^u(x_2)$ by $\pi(z) = W^s(z) \cap W^u(x_2)$. Observe that π is *a priori* only defined on a set of full Lebesgue measure in $W^u(x_1)$. We want to prove that π is absolutely continuous but since the intersection defining π is not transverse *a priori* we can not follow the usual absolute continuity proof. What we have is that the semiconjugacy restricted to $W^u(x_i)$ is smooth, in fact it is affine with respect to the affine structure and the semiconjugacy also conjugates the holonomies, that is: if we define $\mathrm{Hol}\colon h(x_1) + E^u \to h(x_2) + E^u$ as we did with π we have that $h \circ \pi = \mathrm{Hol} \circ h$ for every point in $W^u(x_1)$ where π is defined. But Hol is a smooth map since Hol is simply a translation. Also, h restricted to $W^u(x_i)$ is smooth, so $\pi = h^{-1} \circ \mathrm{Hol} \circ h$ coincides a.e. with a smooth map and hence it is absolutely continuous. Now $B_1 \cap W^u(x_1)$ has full Lebesgue measure in $W^u(x_1)$ and hence its image under π has also full Lebesgue measure in $W^u(x_2)$ and hence this image intersects B_2, that is, we can take a point $a \in B_1$ whose stable manifold contains a point $b \in B_2$ hence we have that $\int \phi \, d\mu_1 = \phi^+(x_1) = \phi^+(x_2) = \int \phi \, d\mu_2$ and we are done.

7.3. **Semiconjugacy and measurable isomorphism of α and α_0.** Let us see that the semiconjugacy is one-to-one over a set of full measure. Let R be the set of regular points satisfying the conclusion of Lemma 7.4. We shall see that the restriction of h to R is one-to-one. Let us fix a nonsingular element of the action. We already know that the restriction of h to stable and unstable manifolds of regular points is a diffeomorphism. Take $x, y \in R$ and assume by contradiction that $h(x) = h(y) = a$. By Lemmas 7.4 and 7.5, we know that $\mathscr{W}^s(x) \cap \mathscr{W}^u(y) \neq \emptyset$. Take z in this intersection. Then $h(z) \in (h(x) + E^s) \cap (h(y) + E^u)$ but since $h(x) = h(y) = a$ this last intersection is a and hence $h(z) = a$. Now, injectivity along stable manifolds gives a contradiction since $z \in \mathscr{W}^s(x)$ and $h(z) = a = h(x)$. Since the image of R has full measure, the restriction of h to R gives a measurable isomorphism between α and α_0 and thus we finish the proof of Theorem 2.6.

8. PROOF OF THEOREM 2.8

We first need to properly describe the classes of cocycles considered in Theorem 2.8. Let us fix a small positive number ε and consider Pesin sets $\mathfrak{Re}_\varepsilon^l$ as defined in (3.1).

Let us consider a Lyapunov Riemannian metric on each Lyapunov distribution defined on the full- measure set $\mathfrak{Re}_\varepsilon = \bigcup_l \mathfrak{Re}_\varepsilon^l$. It is defined similarly to (3.2) with summation over $\mathbb{Z}^k$ instead of integration. By [10, Proposition 5.3] this metric is Hölder-continuous on each $\mathfrak{Re}_\varepsilon^l$. Now consider a system of neighborhoods $P_\varepsilon(x)$, sometimes called *Pesin boxes*, of points in $\mathfrak{Re}_\varepsilon$ whose size depends on l and slowly oscillates with the action, similarly to the function K_ε from Proposition 3.1. Using a local coordinate system from a fixed finite atlas project the Lyapunov metric from T_x to the Pesin box around x with constant coefficients. Thus we obtain a system of locally defined metrics.

DEFINITION 8.1. A cocycle β defined on $\mathfrak{Re}_\varepsilon$ is called *Lyapunov Hölder* if for any $l, x \in \mathfrak{Re}_\varepsilon^l$, β is Hölder-continuous on $\mathfrak{Re}_\varepsilon^l \cap P_\varepsilon(x)$ with Hölder exponent and constant independent of x and l.

Similarly, define Lyapunov smooth cocycles by requiring smoothness along local stable manifolds of points in $\mathfrak{Re}_\varepsilon^l$ with uniform bounds on the derivative with respect to a Lyapunov metric within Pesin boxes.

Note that by Proposition 7.1 the semiconjugacy h between α and the linear action α_0 is bijective on an increasing sequence of compact Pesin sets as well as on stable and unstable manifolds of points from those sets with respect to all elements of the action α. The strategy of the proof is to use these bijections to construct cocycles over α_0 and then use the method of [11].

Take the image of a Pesin set $\mathscr{P}$ under the semiconjugacy. If a solution of the coboundary equation exists then along the stable manifold $\mathscr{W}$ of any element of the action it is given by the familiar telescoping sum, see [11, proof of Theorem 3.1]. This implies in particular that the solution (transfer function) is Lyapunov Hölder or Lyapunov smooth if the cocycle has one of those properties.

By absolute continuity $\mathscr{W} \cap \mathscr{P}$ has large conditional measure in $\mathscr{W}$ and the union of our Pesin sets has full conditional measure. Now one considers periodic cycles anchored at points of the Pesin sets. Any two successive points in such a cycle lie on a one-dimensional Lyapunov line and any three successive points lie in a stable manifold of some element. The last statement follows from the TNS condition that is weaker than our strongly simple property. One can simply consider the situation after the semiconjugacy, as a cocycle over the linear action. Arguing as in [11] we deduce that the solution can be constructed consistently from a single typical point to the union of Pesin sets which has full measure. Since the semiconjugacy is bijective on a full measure set and is smooth along almost every stable manifold the solution can be brought back, and, as we pointed out, is then Lyapunov Hölder or Lyapunov smooth. □

REMARK 8. In the absence of a semiconjugacy but assuming strongly simple and no resonance conditions one can still extend the solution along Lyapunov lines but due to the "holes" in the union of Pesin sets the argument works only locally. This leads to the following statement.

Let μ be a measure as in Theorem 2.4. The spaces of classes of Lyapunov Hölder (resp. Lyapunov smooth) cocycles modulo cohomology with Lyapunov Hölder (resp. Lyapunov smooth) transfer functions are finite-dimensional.

Even in the absence of such holes the solution can be constructed on the universal cover but cannot in general be projected to the original manifold since the possibility of the action preserving a nontrivial homology class cannot be excluded.

9. BEYOND THE STRONGLY SIMPLE CASE

9.1. **Summary.** We can tentatively claim partial generalizations of some results of this paper in the presence of multiple or positively proportional (but not negatively proportional) exponents. We can also outline the limits of applicability for our methods and formulate plausible conjectures.

One should consider separately the general case of hyperbolic measures for smooth actions as in Section 2.1 and large measures for actions on tori and nilmanifolds with hyperbolic homotopy data as in Section 2.2. At the level of linear algebra three effects may appear separately or in combinations:

1. *Negatively proportional exponents.* Our methods, which are essentially geometric are not suitable for this situation. The main problem with using the linear algebra of Lyapunov exponents is that in the representative symplectic case the picture of Lyapunov hyperplanes and Weyl chambers is the same as for the product of rank-one actions where rigidity does not

take place. Thus there is not much hope for developing a general theory along the lines of [10].

Even for algebraic actions on a torus, measure rigidity is established by different methods that take into account global Diophantine properties of stable foliations [5]. Another approach can be developed based on an unpublished preprint of J. Feldman and M. Smorodinsky from the early 1990s. Finding a nonuniform version of these arguments is a serious, albeit not hopeless, challenge.

Hyperbolic measures without negatively proportional Lyapunov exponents are said to be *totally nonsymplectic* (TNS).

2. *Multiple exponents.* The first central step of our approach is "freezing" the action in question along the walls of Weyl chambers. Note that for linear (and hence algebraic) actions this is possible in the semisimple case, *i.e.,* in the absence of Jordan blocks. For actions on tori and nilmanifolds assuming that the algebraic model (the homotopy data) is semisimple helps. In the presence of Jordan blocks the situation is less hopeful.
3. *Simple positively proportional exponents.* This case (assuming that no other effects appear) is the most hopeful and is discussed in more detail below. The key issue here is understanding resonances and invariant geometric structures that appear on coarse Lyapunov foliations.

9.2. **Hyperbolic measures for actions on manifolds.** The main difficulty here is that vanishing of a Lyapunov exponent does not guarantee that along Lyapunov foliations (even if the exponent is simple and if those exist) on a set of large measure the distances remain bounded. The technical device that allows us to overcome this problem in the strongly simple case is the synchronizing time change described in Section 3.3. This is easily modified to obtain bounded growth estimates like in Proposition 3.4 for any one Lyapunov direction, *e.g.,* the fastest for which Lyapunov distribution is integrable. However, in general time changes would be different so no simultaneous "freezing" is possible. This problem looks fundamental and probably cannot be overcome within the usual rank ≥ 2 assumption.

But in fact in our argument synchronization of one exponent is achieved along the whole Lyapunov hyperplane. If the rank is ≥ 3 than *simultaneous* synchronization of two exponents can be achieved along a codimension two subspace, and so on. This of course is under the assumption that exponents are simple. Thus the following statement holds:

STATEMENT B. *Simultaneous synchronization of all proportional exponents is possible if their number is less than the rank of the action.*

Now one considers an invariant geometric structure on the coarse Lyapunov foliation. In the absence of double resonances this structure is flat affine and one can show that Lyapunov distributions integrate to foliations into lines with respect to this structure. The critical π-partition argument holds in this case and allows us to show that conditional measures on the coarse Lyapunov foliation are supported on affine subspaces and invariant under transitive groups of affine transformations on those subspaces. Hence those conditional measures are either atomic or absolutely continuous on smooth submanifolds of the leaves of the coarse Lyapunov foliations.

The arguments outlined above lead to the proof of a generalization of Theorem 2.2 for TNS actions with simple positively proportional exponents, no double resonances if the number of exponents proportional to a given one does not exceed the rank of the action minus one. Detailed proofs will appear in a subsequent paper.

The case with double resonances is somewhat more complicated because for the slow directions there are no unique curves tangent to the slow directions similar to lines in the affine case. Instead there are some parametric families of such curves like parabolas in the case of 2 : 1 resonance. If one can prove that the Lyapunov distributions integrate to certain families of such curves, the rest of the argument should be similar to the non-resonance case.

An extension of Theorem 2.4 looks more problematic. The problem is that the full-entropy assumption does not catch contributions coming from different positively proportional exponents. One should look for appropriate "high entropy" assumptions that would lead to the assertion that conditional measures along the leaves of the coarse Lyapunov foliation are absolutely continuous. After that absolute continuity of the measure can be established, similarly to the proof of Theorem 2.4.

9.3. **Actions on tori and nilmanifolds.** As was mentioned above, our methods are restricted to the TNS case so we make this assumption for the algebraic action α_0. To be able to carry out the "freezing" argument we also need to avoid Jordan blocks for the action α_0, *i.e.,* to assume that its linear part is semisimple (diagonalizable over $\mathbb{C}$).

Then, the action along the Lyapunov hyperplanes is an isometry. The main issue is to prove that there are no new Lyapunov hyperplanes for the action α. Thus far, we can prove this in certain special cases. If new Lyapunov exponents for α not proportional to those of α_0 (and hence new Lyapunov hyperplanes) appear, corresponding Lyapunov foliations must collapse under the semiconjugacy.

Entropy considerations like in Section 6.1 provide for collapsing of certain directions, leading to entropy deficit, although the arguments become more involved. After that, one can follow the general line of the arguments in Sections 6 and 7 to obtain an extension of Theorem 2.6 to the TNS non-resonance case. A particular case where double exponents are allowed due to the existence of complex eigenvalues for α_0 is announced in [9]. Detailed proofs will appear in a subsequent paper.

Resonances both for α_0 and for α represent an additional difficulty but basically one should prove intertwining of geometric structures and hence smoothness of the semiconjugacy along the coarse Lyapunov foliations. Thus one can formulate the desired outcome as follows.

CONJECTURE. *Let α_0 be a totally nonsymplectic $\mathbb{Z}^k$ action by automorphisms of an infranilmanifold and α be an action with homotopy data α_0. Then every large invariant measure for α is absolutely continuous and has the same Lyapunov characteristic exponents as α_0.*

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