APPROXIMATIONS IN ERGODIC THEORY

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Contents

Introduction

1. The idea of approximating metric automorphisms by periodic transformations arose in the early 1940's in papers by Halmos. Among the specific results in this direction we note the theorems of weak and uniform approximation [22], [23] by means of which Halmos and Rokhlin proved the well-known theorems on categories [23], [15] (see also [20]).

In recent years a new point of view on approximating automorphisms by periodic transformations has arisen, which makes it possible to study individual automorphisms by approximation. A technique for studying metric automorphisms has been developed, which we call the "method of approximations". By this method a number of problems of ergodic theory were solved that had not yielded to other means.

It should be noted that the use of periodic approximations in the study of an individual automorphism was proposed by Berezin at a seminar on functional analysis at the Moscow State University in 1964.

Some of the results of §§8 and 9 were communicated in one of the meetings at the school on ergodic theory in Khumsan and formed a topic of fruitful discussion among Bernshtein, Kushnirenko, Margulis, Oseledets, Rokhlin and Sinai.

2. We assume the reader to be acquainted with the fundamental concepts of general measure theory, the spectral theory of unitary operators and ergodic theory. A large part of the necessary material is contained in §§1 - 3 of Rokhlin's article "Lectures on the entropy theory of measure-preserving transformations". Definitions of or references to concepts not contained therein are given in the course of this paper.

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PART I
THE METHOD OF APPROXIMATIONS
§I. Definitions and examples

1. Let T be an automorphism of the Lebesgue space (M, μ) and $f(n)$ a monotonic sequence of positive numbers such that $f(n) \to 0$ as $h \to \infty$.

DEFINITION 1.1. We say that the automorphism T *admits an approximation of the first kind by periodic transformations (a. p. t. I)* with speed $f(n)$ if we can find decompositions ξ_n of M into a finite number q_n of measurable sets $C_{n,i} \subset M$ $(i = 1, 2, \ldots, q_n)$ that preserve the measure of the transformation T_n and for which:

A.1. $\xi_n \to \varepsilon$ as[1] $n \to \infty$,

A.2. $T^n \xi_n = \xi_n$,

A.3. $\sum_{i=1}^{q_n} \mu \, (TC_{n,i} \Delta T_n C_{n,i}) < f(q_n)$.

If, however, in addition to A.1 and A.2 we have

A.3'. $\sum_{i=1}^{q_n} \mu \, (TC_{n,i} \Delta T_n C_{n,i}) < f(p_n)$ and[2] $T_n \Rightarrow T$,

where p_n is the order of T_n on the factor-space M/ξ_n, we say that T *admits an approximation of the second kind by periodic transformations (a.p.t.II)* with speed $f(n)$.

If, in addition to A.1 - A.3, we have

A.4. T_n transposes the elements of the decomposition ξ_n cyclically, we say that T *admits a cyclic a.p.t. with speed* $f(n)$. When T admits an a.p.t.I (or a.p.t.II or cyclic a.p.t.) with speed $g(n) = o(f(n))$, it is convenient to say that T admits an a.p.t.I (or a.p.t.II or cyclic a.p.t., respectively) with speed $o(f(n))$.

A cyclic a.p.t. was defined in [8]; cyclic approximations of a special form were considered by S.A. Yuzvinskii [27]. The general definition of an a.p.t.I is given in [6] and of an a.p.t.II in [18].

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[1] The notation $\xi_n \to \varepsilon$ denotes that for any measurable set $A \subset M$ a sequence of ξ_n-sets A_n can be found such that $\mu(A_n \Delta A) \to 0$.

[2] $T_n \Rightarrow T$, by definition means that strong convergence $U_{T_n} \Rightarrow U_T$ holds.

The property of an automorphism T to admit an a.p.t. with a given speed $f(n)$ is the metric invariant of it. If $f(n)$ is chosen from certain standard one-parameter sets of "speeds", numerical invariants of an automorphism can be obtained. If, for example, $f_\lambda(n) = n^{-\lambda}$, we put $d(T)$ equal to the least upper bound of the numbers λ such that T admits a cyclic a.p.t. with speed $n^{-\lambda}$.

2. Let us illustrate the concept of approximation by examples.

EXAMPLE 1.1. Let $T^{(\alpha)}$ be an automorphism of the interval $[0, 1]$ defined by $T^{(\alpha)}x = \{x + \alpha\}$, where $\{x\}$ denotes the fractional part of the number x. We fix the sequence $f(n)$ and assume that α is irrational and

that there exists a sequence of irreducible fractions $\dfrac{p_n}{q_n}$ for which

$$\left| \frac{p_n}{q_n} - \alpha \right| < f(q_n) \quad (n = 1, 2, \ldots).$$

Let ζ_n be a decomposition of $[0, 1]$ into intervals

$$C_{n,i} = \left[\frac{i-1}{q_n} \; ; \; \frac{i}{q_n}\right] \quad (i = 1, \ldots, q_n), \text{ and let } T_n \text{ be defined by}$$

$$T_n x = \left\{x + \frac{p_n}{q_n}\right\}.$$

It is easily verified that

$$\sum_{i=1}^{q_n} \mu\left(T^{(\alpha)}C_{n,i} \Delta T_n C_{n,i}\right) < 2q_n f(q_n)$$

and so the automorphism $T^{(\alpha)}$ admits a cyclic a.p.t. with speed $2nf(n)$.

Using continued fraction theory [25] we obtain:

1) *For any irrational α the automorphism $T^{(\alpha)}$ admits a cyclic a.p.t. with speed* $\dfrac{2}{\sqrt{5n}}$.

2) *For almost all α the automorphism $T^{(\alpha)}$ admits a cyclic a.p.t. with speed* $o\left(\dfrac{1}{n}\right)$.

3) *For each speed $f(n)$ there exist numbers α for which the automorphism $T^{(\alpha)}$ admits a cyclic a.p.t. with speed $f(n)$.*

EXAMPLE 1.2. Let T be an ergodic automorphism with a discrete spectrum, the eigenvalues of the operator U_T being roots of unity. We denote by Σ_n the set of eigenvalues that are roots of unity of degree not greater than n, and by $\mathfrak{A}_n$ the subalgebra in the unital ring $L_2(M)$ generated by the eigenfunctions belonging to the eigenvalues from Σ_n. We know [14] that $\mathfrak{A}_n$ is the subalgebra of all functions that are constant on elements of a finite decomposition ζ_n of M. Let us put $T_n = T$. We note that $\mathfrak{A}_n$ is generated by a single eigenfunction and that $\mathfrak{A}_n \nearrow L_2(M)$. Hence for any $f(n)$ the automorphism T admits a cyclic a.p.t. with speed $f(n)$. If, in particular, the group of eigenvalues of T coincides with the group of roots of unity of degree 2^n ($n = 0, 1, \ldots$), then T can be realized as an automorphism K of the interval $[0, 1]$ such that ζ_{2^n} is a decomposition into intervals $\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]$ ($i = 1, \ldots, 2^n$).

REMARK 1.1. The assumption that T is ergodic is essential only for the cyclicity of the constructed a.p.t.

EXAMPLE 1.3. Let $V^{(\alpha)}$ be a transformation of the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ given by $V^{(\alpha)}(x, y) = (\{x + \alpha\}, \{x + y\})$. As in

Example 1.1 we suppose that α is irrational and that $\frac{p_n}{q_n}$ are irreducible

fractions such that $|\frac{p_n}{q_n} - \alpha| < f(q_n)$. We fix a sequence of even numbers r_n,

where $r_n \to \infty$. Let ζ_n be a decomposition of the square into rectangular elements

$0 \leqslant x \leqslant \frac{1}{q_n}$, $\frac{k}{r_n} \leqslant y \leqslant \frac{k+1}{r_n}$ ($k = 0, 1, \ldots, r_n - 1$) and $\frac{1}{q_n} \leqslant x < 1$. We

put $T_n(x, y) = \left(\left\{ x + \frac{p_n}{q_n} \right\}, \left\{ y + \frac{[q_n x]}{q_n} \right\} \right)$ and

$\zeta_n = \zeta_n \cdot T_n \zeta_n \cdot \ldots \cdot T_n^{q_n - 1} \zeta_n$. The number of elements of ζ_n is $Q_n = r_n q_n$, and if we denote these elements by $C_{n,i}$ ($i = 1, \ldots, Q_n$), we obtain

$$\sum_{i=1}^{Q_n} \mu \left(V^{(\alpha)} C_{n,i} \Delta T_n C_{n,i} \right) < 2 q_n f (q_n) + \frac{r_n}{q_n} .$$

Since for any α we can put $f(n) = \frac{1}{\sqrt{5} n^2}$, we see that the automorphism

$V^{(\alpha)}$ admits an a.p.t. I with speed $\frac{a_n}{n}$, where a_n is an arbitrary

preassigned sequence and $a_n \to \infty$.

3. We recall the definition of the weak topology in the group of all automorphisms of a Lebesgue space determined mod 0. A neighbourhood of the automorphism T is the intersection of a finite number of sets of the form

$$\{S: \mu (SE\Delta TE) < \varepsilon\},$$

where E is a measurable set and $\varepsilon > 0$. We denote by $\mathfrak{U}$ the topological group thereby obtained.

We now state a theorem which we shall require repeatedly. It is proved in a somewhat different form by Yuzvinskii in [27].

THEOREM 1.1. *The set of automorphisms admitting a cyclic a.p.t. with a fixed speed* $f(n)$ *contains a set of the type* G_δ *that is everywhere dense in* U.

PROOF. We assume that M is the interval $[0, 1]$ with Lebesgue measure. Let ζ_n be a decomposition of $[0, 1]$ into intervals

$C_{n,i} = \left[\frac{i-1}{2^n}, \frac{i}{2^n} \right]$ ($i = 1, \ldots, 2^n$), and $\mathfrak{U}_n$ a set of automorphisms from

U cyclically transposing the $C_{n,i}$ and linear on each of them.

We put

$$G_n = \bigcup_{T_n \in \mathfrak{U}_n} \left\{ T \in \mathfrak{U}: \sum_{i=1}^{2^n} \mu \left(T C_{n,i} \Delta T_n C_{n,i} \right) < f(2^n) \right\}.$$

G_n is clearly an open set; therefore $\quad G = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} G_n \quad$ is a set of type

G_δ. All automorphisms in G obviously admit a cyclic a.p.t. with speed $f(n)$. We prove that G is everywhere dense in $\mathfrak{U}$. The automorphism K from §1.2 above belongs to any of the sets G_n and therefore to G. Let $\mathfrak{W}$ be a

set of automorphisms linear on all the elements of any of the ζ_n. We put

$$\Omega = \{T \in \mathfrak{U}: \ T = SKS^{-1}, \ S \in \mathfrak{W}\}.$$

Note that $\Omega \subset \bigcap_n G_n$. It is therefore sufficient to prove that Ω is every-where dense. This follows from the following assertions.

1. The set of automorphisms conjugate to a given aperiodic automorphism is everywhere dense in $\mathfrak{U}$ ([20], 108).

2. The set $\mathfrak{W}$ is everywhere dense in $\mathfrak{U}$ ([20], 92).

3. The mapping $F_K: \mathfrak{U} \longrightarrow \mathfrak{U}$, defined by $F_K(T) = T^{-1}KT$ is continuous. This follows from the fact that $\mathfrak{U}$ is a topological group ([20], 89).

In fact, by 1, the set $F_K(\mathfrak{U})$ is everywhere dense in $\mathfrak{A}$ and, by 2 and 3, the set $\Omega = F_K(\mathfrak{W})$ is everywhere dense in $F_K(\mathfrak{U})$ and therefore also in $\mathfrak{U}$. The theorem is proved.

By means of this theorem and of theorems relating approximations to various properties of automorphisms (ergodicity, mixing, multiplicity of the spectrum and so on) we shall obtain in §§2 – 4 a number of results, some well known and some new, on the topological structure of different sets of automorphisms in the group $\mathfrak{U}$.

§2. Approximations, ergodicity and mixing

The fact that the automorphism T admits a.p.t.'s with definite properties (for example, cyclicity) and with "sufficiently high" speed enables us to draw conclusions relating to the properties of T such as ergodicity, mixing and multiplicity of the spectrum.

The following theorem clarifies the question of what speed of approximation is sufficient for T to be ergodic.

In the proof of this and subsequent theorems, ζ_n and T_n denote the decompositions and automorphisms in the definition of an a.p.t., and $C_{n,i}$ ($i = 1, \ldots, q_n$) denotes the elements of the decomposition ζ_n.

I. THEOREM 2.1. *If the automorphism T admits a cyclic a.p.t. with*

speed $\dfrac{\theta}{n} (\theta \geqslant 2)$, *then the number of ergodic components of T does not*

exceed $\dfrac{\theta}{2}$.

PROOF. Suppose that there exist m disjoint invariant sets of positive measure: $F_1, \ldots, F_m$. Let $f = \min_j \mu(F_j)$ ($j = 1, \ldots, m$) and let δ be fixed > 0. By A.1 for $n > n_0(\delta)$ there exist sets $F_1^{(n)}, \ldots, F_m^{(n)}$ composed of elements of ζ_n and such that $\sum_{j=1}^m \mu \, (F_j \triangle F_j^{(n)}) < \delta \cdot f$. It follows that an increasing sequence of numbers $0 = k_1 < k_2 < \ldots < k_m < k_{m+1} = q_n$ and an element $C \in \zeta_n$ can be found such that

$\mu \, (T_n^{k_j} C \cap F_{r_j}) > \dfrac{1-\delta}{q_n}$, where $r_1, \ldots, r_j, \ldots, r_m$ is a permutation of 1, 2, ..., m.

Since the sets F_j are invariant, for any integer r we have

$$\mu\,(T^r T_n^{k j}\,C\,\cap\,F_{r_j}) = \mu\,(T_n^{k j} C \cap F_{r_j}) > \frac{1-\delta}{q_n}\,.$$

Also $\mu\,(T_n^{k j+1} C \cap F_{r_{j+1}}) > \dfrac{1-\delta}{q_n}$ **and hence** $\mu\,(T_n^{k j+1} C \cap F_{r_j}) < \dfrac{\delta}{q_n}\,.$
Therefore for $1 \leqslant j < m$

$$\mu\,(T^{k j+1 - k j}\,(T_n^{k j} C)\,\Delta T_n^{k j+1 - k j}\,(T_n^{k j} C)) > \frac{2-2\delta}{q_n}\,. \tag{2.1}$$

This inequality also holds for $j = m$, **because**

$$T_n^{k m+1} C = T_n^{q_n} C = C.$$

For any element $\overline{C} \in \xi_n$ **we have**

$$\mu\,(T^k \overline{C} \Delta T_n^k \overline{C}) \leqslant \sum_{i=0}^{h-1} \mu\,(T\,(T_n^i \overline{C})\,\Delta T_n\,(T_n^i \overline{C}))\,. \tag{2.2}$$

By (2.1) and (2.2) we obtain

$$\frac{m\,(2-2\delta)}{q_n} < \sum_{j=1}^{m} \mu\,(T^{k j+1 - k j}\,(T_n^{k j}\,C))\,\Delta\,(T_n^{k j+1 - k j}\,(T_n^{k j} C)) \leqslant$$

$$\leqslant \sum_{i=0}^{q_n-1} (T\,(T_n^i C)\,\Delta T_n\,(T_n^i C)) < \frac{\theta}{q_n}\,, \tag{2.3}$$

or $m < \theta(2 - 2\delta)^{-1}$. **Since** δ **is arbitrary, the theorem is proved.**
     COROLLARY 2.1.   *The automorphism T admitting a cyclic a.p.t.*
with speed $\dfrac{\theta}{n}$ *for* $\theta < 4$ *is ergodic.*

     The conditions of Theorem 2.1 cannot be relaxed. More precisely:
     a) *for any integer $m > 1$ there exists an automorphism K_m containing*

m ergodic components and admitting a cyclic a.p.t. with speed $\dfrac{2m+\delta}{n}$,

where δ is arbitrary and > 0:
     b) *there exists an automorphism K_{con} with an infinite (and even
continuous) decomposition into ergodic components and admitting a cyclic*

a.p.t. with arbitrary speed of the form $\dfrac{a_n}{n}$ *, where $a_n \to \infty$.*

     Let $M = [0,\,1] \times Z_m$, where Z_m is the space of m points

$\left\{ e^{\frac{2\pi i l}{m}} \right\} (l = 1, \ldots, m)$ with equal measures. We introduce the automorphism
$K_m = K \times I_m$, where K is the automorphism referred to in Example 1.2 and
I_m is the identity transformation of Z_m. We put $\xi_n = \eta_n \times \varepsilon_m$, where η_n
is the decomposition of $[0,\,1]$ into intervals

$\left[\dfrac{i-1}{2^n},\, \dfrac{i}{2^n} \right] (i = 1,\, \ldots,\, 2^n)$ and ε_m is the decomposition of Z_m into separate
points, and we define

$$T_n\ (x,\ \zeta) = \begin{cases} (Kx,\ \zeta) & \text{for}\quad 0 \leqslant x \leqslant \frac{2^n-1}{2^n}, \\ \left(Kx,\ e^{\frac{2\pi i}{m}}\zeta\right) & \text{for}\quad \frac{2^n-1}{2^n} < x \leqslant 1. \end{cases}$$

We leave it to the reader to verify the properties A.1 - A.4.

$K\infty$ can be taken to be $K \times I$, where I is the identity automorphism of $[0,\ 1]$.

2. **THEOREM 2.2.** *If the automorphism T admits an a.p.t. II with*

speed $\frac{\theta}{n}$, *where $\theta < 2$, then T has no mixing.*

PROOF. We assume that $p_n \to \infty$, since otherwise it follows from A.3' that T is periodic. Let $F_j(j=1,\ 2,\ \ldots,\ k)$ be disjoint sets of measure $\frac{1}{k}$. We fix $\delta > 0$. For $n \geqslant n_0(\delta)$ sets $F_j^{(n)}$ consisting of elements of ζ_n can be found such that $\mu(F_j \Delta F_j^{(n)}) < \frac{\delta}{k}$. For any integer l we have

$$\sum_{i=1}^{q_n} \mu\ (T^l C_{n,\ i} \Delta T_n^l C_{n,\ i}) \leqslant |l| \cdot \sum_{i=1}^{q_n} \mu\ (T C_{n,\ i} \Delta T_n C_{n,\ i}), \tag{2.4}$$

where, as always, the $C_{n,\ i}$ $(i = 1,\ 2,\ \ldots,\ q_n)$ are elements of ζ_n. Furthermore,

$$\sum_{j=1}^{k} \mu\ (T^{p_n} F_j^{(n)} \Delta F_j^{(n)}) \leqslant \sum_{i=1}^{q_n} \mu\ (T^{p_n} C_{n,\ i} \Delta C_{n,\ i}). \tag{2.5}$$

Therefore

$$\sum_{j=1}^{k} \mu\ (T^{p_n} F_j \Delta F_j) \leqslant \sum_{j=1}^{k} \mu\ (T^{p_n} F_j \Delta T^{p_n} F_j^{(n)}) +$$

$$+ \sum_{j=1}^{k} \mu\ (T^{p_n} F_j^{(n)} \Delta F_j^{(n)}) + \sum_{j=1}^{k} \mu\ (F_j^{(n)} \Delta F_j) < 2\delta + \theta.$$

Hence

$$\sum_{j=1}^{k} \mu\ (T^{p_n} F_j \cap F_j) = \sum_{j=1}^{k} \mu\ (F_j) - \frac{1}{2} \sum_{j=1}^{k} \mu\ (T^{p_n} F_j \Delta F_j) \geqslant 1 - \delta - \frac{\theta}{2}. \tag{2.6}$$

We assume that T is mixing and in (2.6) we pass to the limit as $p_n \to \infty$: $\frac{1}{k} > 1 - \delta - \frac{\theta}{2}$. Since δ can be taken arbitrarily small and k arbitrarily large, we have $\theta \geqslant 2$, and the theorem is proved.

The following general assertion also follows from the proof of Theorem 2.2.

THEOREM 2.3. *The automorphism T has no mixing if there exists a*

sequence of decompositions $\zeta_n \to \varepsilon$ for which $\sum_{C \in \zeta_n} \mu\ (T^{p_n} C \Delta C) < \theta < 2$,

where $p_n \to \infty$ as $n \to \infty$ and θ does not depend on n.

REMARK 2.1. A consequence of Theorems 1.1 and 2.2 is Rokhlin's theorem [15]: *mixing automorphisms form a set of the first category in the*

group $\mathfrak{U}$.

§3. Approximations and the spectrum

I. We recall that U_T denotes the unitary operator in $L_2(M)$ adjoint to the automorphism T: $U_T f(x) = f(Tx)$. We shall clarify the relation between the approximability of T and the multiplicity of the spectrum of U_T.

THEOREM 3.1. *If the automorphism T admits a cyclic a.p.t. with speed $\dfrac{\theta}{n}$, where $\theta < \dfrac{1}{2}$, then the spectrum of U_T is simple.*

In the proof we use the following result from the spectral theory of operators.

LEMMA 3.1. *Let A be a bounded normal operator with a non-simple spectrum in a Hilbert space H. Then two orthogonal normalized vectors h_1, $h_2 \in H$ can be found such that, for any invariant subspace H' that is cyclic with respect to A, we have*

$$\rho\,(h_1,\,H') + \rho\,(h_2,\,H') \geqslant 1.$$

PROOF OF THEOREM 3.1. Let T admit a cyclic a.p.t. with speed $\dfrac{\theta}{n}$ and suppose that the spectrum of U_T is non-simple. Let $C \in \xi_n$. We introduce the set $A = \bigcap\limits_{i=0}^{q_n-1} T^{-i}(S_n^i C)$. It is obvious that $T^k A \subset S_n^k C$ for $k = 0,\ 1,\ \ldots,\ q_n - 1$. Furthermore,

$$\mu\,(A) \geqslant \mu\,(C) - \frac{1}{2} \sum_{k=0}^{q_n-1} (T\,(S_n^k C)\,\Delta S_n\,(S_n^k C)) > \frac{1 - \dfrac{\theta}{2}}{q_n}\,.$$

Let χ_C and χ_A denote the characteristic functions of the sets C and A, respectively, H_n the subspace of $L_z(M,\,\mu)$ consisting of the constant functions on the elements of ξ_n, and H_n' the cyclic subspace generated by the functions χ_A, that is, the closure of the linear span of the functions $U_T^k \chi_A$, $k = 0,\ \pm 1,\ \pm 2,\ \ldots$

Let $h \in L_2(M,\,\mu)$, $\|\,h\,\| = 1$. The existence of a sequence of functions $h_n \to h$, $h_n \in H_n$, $\|\,h_n\,\| = 1$, follows from A.1. Suppose that $\|\,h - h_n\,\| < \delta$ for $n > n_0(\delta)$. From A.4 it follows that

$$h_n\,(x) = \sum_{k=0}^{q_n-1} b_n \chi_C\,(S_n^{-k} x).$$

We put $h_n'\,(x) = \sum\limits_{k=0}^{q_n-1} b_k \chi_A\,(T^{-k} x)$ and seek a bound for $\|\,h - h_n'\,\|$.

Obviously $\|\,h - h_n'\,\| < \delta + \|\,h_n - h_n'\,\|$. Further,

$$\| h_n - h_n' \| = \left\| \sum_{k=0}^{q_n-1} b_k \left(\chi_C \left(S_n^{-k} x \right) - \chi_A \left(T^{-k} x \right) \right) \right\| =$$

$$= \left(\sum_{k=0}^{q_n-1} | b_k |^2 \| \chi_C (S_n^{-k} x) - \chi_A (T^{-k} x) \| \right)^{\frac{1}{2}} = \left(\sum_{k=0}^{q_n-1} | b_k |^2 \mu \left(S_n^k C \setminus T^k A \right) \right)^{\frac{1}{2}} \leqslant$$

$$\leqslant \sqrt{\frac{\theta}{2}} \left(\sum_{k=0}^{q_n-1} | b_k |^2 \cdot q_n^{-1} \right)^{\frac{1}{2}} = \sqrt{\frac{\theta}{2}} \, .$$

We now choose vectors h_1 and h_2 as in Lemma 3.1. If n is sufficiently

large then $\rho(h_i, H_n') \leqslant \sqrt{\frac{\theta}{2}} + \delta$, $i = 1$, 2. Therefore $2 \left(\sqrt{\frac{\theta}{2}} + \delta \right) \geqslant 1$.

Since δ is arbitrary, we have $\theta \geqslant \frac{1}{2}$, and the theorem is proved.

The simplicity of the spectrum of an automorphism admitting a cyclic

a.p.t. with speed $o \left(\frac{1}{n^2} \right)$ was first proved by Oseledets.

Yuzvinskii [27] has proved the following interesting theorem.

THEOREM 3.2. *The complement of a set of automorphisms with a simple continuous spectrum is a set of the first category in the group* $\mathfrak{U}$.

PROOF. It follows from Theorems 1.1 and 3.1 that the set of automorphisms with a simple spectrum is a set of the first category in $\mathfrak{U}$. But, as was shown by Halmos [23], automorphisms with a discrete component in the spectrum (that is, automorphisms not possessing weak mixing) also form a set of the first category in $\mathfrak{U}$. So the theorem is proved.

2. Another group of results on the connection between approximability and the spectrum are theorems that enable us to draw conclusions on the structure of an operator U_T of maximal spectral type from the existence of an approximation. In these theorems it is sufficient to assume an a.p.t. II instead of a cyclic a.p.t.

THEOREM 3.3. *If the automorphism T admits an a.p.t. II with speed* $o \left(\frac{1}{n} \right)$, *then there is a strong convergence* $U_T^{p_n} \Rightarrow E$ *(where E is the identity operator) in $L_2(M)$.*

PROOF. As in Theorem 2.2, we confine ourselves to the case $p_n \to \infty$. It is sufficient to prove the convergence $U_T^{-p_n} h \to h$ for bounded functions $h \in L_2(M)$. Let H_n be defined as in the proof of Theorem 3.1 and let h_n be the projection of the vector h on H_n. We assume that $| h(x) | < C$, and then also $| h_n(x) | < C$.

Since

$$\| U_T^{-p_n} h - h \| \leqslant \| U_T^{-p_n} h - U_T^{-p_n} h_n \| + \| U_T^{-p_n} h_n - h_n \| + \| h_n - h \|$$

and $\| U_T^{-p_n} h - U_T^{-p_n} h_n \| = \| h_n - h \| \to 0$ as $n \to \infty$ by A.1, it is sufficient

to prove that $\| U_T^{-p_n} h_n - h_n \| \to 0$. Note that $h_n(x) = h_n(T^{-p_n} x)$ outside

the set $E_n = \bigcup_{i=1}^{q_n} (T^{p_n} C_{n,i} \triangle C_{n,i})$. From (2.4) we have $\mu(E_n) \to 0$ as $n \to \infty$.

Since $\| U_T^{-p_n} h_n - h_n \| < 2C \sqrt{\mu(E_n)}$, the theorem is proved.

COROLLARY 3.1. *If the automorphism T admits an a.p.t. II with*

speed $o \left(\dfrac{1}{n} \right)$, *then the maximum spectral type of the operator* U_T *is*

singular.

This assertion is also valid under the assumption that T admits an

a.p.t. II with speed $\dfrac{\theta}{n}$ for $\theta < \dfrac{1}{2}$.

COROLLARY 3.2. *The set of automorphisms whose maximum spectral type subordinates an absolutely continuous type is a set of the first category in* $\mathfrak{U}$.

3. Let h be a vector of the maximum spectral type of the operator U_T and σ be the spectral measure of h. Under the assumptions of Theorem 3.3

we have $\displaystyle\int_{|\zeta|=1} \zeta^{p_n} d\sigma \longrightarrow 1$ as $n \to \infty$.

By an investigation of the approach to unity of the Fourier coefficients of the measure σ Corollary 3.1 can be made more precise.

THEOREM 3.4. *Let the automorphism T admit an a.p.t. II with speed*

$f(n) = o \left(\dfrac{1}{n} \right)$ *and let* a_n *be a fixed sequence such that* $a_n \to \infty$, $a_n > 0$.

We put

$$G_n = \{ \zeta : \ |\zeta| = 1, \ |\zeta^{p_n} - 1| < a_n \sqrt[4]{p_n f(p_n)} \}.$$

Then a subsequence n_k *can be found for which the set* $G = \displaystyle\bigcap_{l=1}^{\infty} \bigcup_{k \geq l} G_{n_k}$ *has*

full measure with respect to the maximum spectral type of the operator U_T.

We mentioned in §1.2 that automorphisms with a discrete spectrum whose eigenvalues are all roots of unity admit an a.p.t. with any preassigned speed.

The preceding theorem combined with a technical proposition of measure theory (see Lemma 1 of [18]) enables us to invert this proposition.

THEOREM 3.5. *If the automorphism T admits an a.p.t. II with any speed, then T has a discrete spectrum and all its eigenvalues are roots of unity.*

REMARK 3.1. Since the order p of any permutation of q elements satisfies the inequality $p < a^q$, where a does not depend on q, Theorem 3.5 remains valid if T admits an a.p.t. I with any speed.

REMARK 3.2. From Theorems 3.4 and 3.5 the following well-known result [16] is easily derived: *automorphisms whose maximum spectral types are subordinate to a fixed type form a set of the first category in* $\mathfrak{U}$.

§4. Approximations and entropy

We have not yet clarified whether an arbitrary automorphism T admits an a.p.t. with some speed $f(n) = o$ (1) or other. This question can be answered in the affirmative. Theorem 4.1, which is proved below, asserts

that any automorphism admits an a.p.t. I with speed $\dfrac{a_n}{\lg n}$, where a_n is an

arbitrary monotonic sequence of positive numbers tending to ∞ and the logarithm is binary. This theorem suggests that it is convenient to take $\frac{\theta}{\lg n}$ as one of the "standard speed units" for an a.p.t. It turns out that the existence of an a.p.t. I with such speeds is very closely related to the entropy of the automorphism (see Theorems 4.2 - 4.4).[1] As distinct from the properties considered in §§2 - 3, entropy and approximations (at least for ergodic automorphisms) completely determine one another.

THEOREM 4.1. *Every automorphism admits an a.p.t. I with speed* $\frac{a_n}{\lg n}$, *where* a_n *is an arbitrary monotonic sequence of positive numbers tending to* ∞ .

PROOF. We confine ourselves to the fundamental case of an aperiodic automorphism T. The proof is based on the following simple lemma (see [20], 99).

LEMMA 4.1. *For any aperiodic automorphism T, positive integer n and* $\varepsilon > 0$, *there exists a measurable set* $A_{n,\varepsilon}$ *such that the sets*

$$T^k A_{n,\varepsilon} \quad (k = 0, 1, \ldots, n-1) \quad \text{are disjoint and} \quad \mu\left(\bigcup_{h=0}^{n-1} T^k A_{n,\varepsilon}\right) > 1 - \varepsilon.$$

We choose a sequence of decompositions $\eta_n \to \varepsilon$ such that the number k_n of elements of η_n beginning with a certain n_0 satisfies $k_n < \min (a_n, n)$ and $8\lg k_n < k_n$. The meaning of these inequalities becomes clear in the proof.

For each $n > n_0$ we construct the set $A_{n,n-2}$ of Lemma 4.1. Let ζ_n be the decomposition whose elements are the intersections of the elements of $\eta_n \cdot T^{-1}\eta_n \cdot \ldots \cdot T^{-n+1}\eta_n$ with $A_{n,n-2}$ and the complement of $A_{n,n-2}$. We put $\zeta_n = \eta_n \cdot \zeta_n \cdot T\zeta_n \cdot \ldots \cdot T^{n-1}\zeta_n$. The elements of ζ_n belonging to $T^k A_{n,n-2}$ are the intersections of this set with the elements of $T^k \eta_n \ldots T^{k-n+1}\eta_n$. Obviously $\zeta_n \to \varepsilon$. Denoting the number of elements of ζ_n by q_n, we have $q_n \leqslant n(k_n)^n + k_n$. We now introduce the periodic automorphism T_n :

$$T_n(x) = \begin{cases} Tx, & \text{if } x \in T^k A_{n,\,n-2}\,(k = 0, 1, \ldots, n-2), \\ T^{-n+1}x, & \text{if } x \in T^{n-1}A_{n,\,n-2}, \\ x, & \text{if } x \in \bigcup_{h=0}^{n-1} T^k A_{n,\,n-2}. \end{cases}$$

It is not difficult to see that $T_n \zeta_n = \zeta_n$. Denoting the elements of ζ_n, as always, by $C_{n,i}$ ($i = 1, \ldots, q_n$), we obviously have

$$\sum_{i=1}^{q_n} \mu\,(TC_{n,\,i}\Delta T_n C_{n,\,i}) < 2\mu\,(M \setminus \bigcup_{h=0}^{n-2} T^k A_{n,\,n-2}) < \frac{2}{n} + \frac{2}{n^2}.$$

On the other hand,

---

[1]   The entropy of an automorphism is defined, for example, in [16] and in §9 of Rokhlin's article "Lectures on the entropy theory of measure-preserving transformations" in this issue.

$$q_n < (k_n)^{2n} \text{ and } \frac{4}{n} \leqslant \frac{8 \lg k_n}{\lg q_n} < \frac{k_n}{\lg q_n} < \frac{u_n}{\lg q_n} .$$

REMARK 4.1. It follows incidentally from the proof of Theorem 4.1 that every automorphism admits an a.p.t. II with speed $\dfrac{2 + \delta}{n}$ for any constant $\delta > 0$.

THEOREM 4.2. *An automorphism T with finite entropy $h(T)$ admits an a.p.t. I with speed* $\dfrac{2h(T) + \delta}{\lg n}$, *where δ is an arbitrary positive number.*

We give an outline of the proof of this theorem and, to avoid technical complications, we confine ourselves to the case of ergodic T.

Lemma 4.1 can be strengthened if we require in addition that

$$\mu (A_{n, \varepsilon} \cap B) < \frac{\mu (B)}{n} \text{ for a given measurable set } B \subset M.$$

We fix the sequence $\delta_n > 0$, $\delta_n \to 0$ and suppose, as above, that η_n is a sequence of finite measurable decompositions and that $\eta_n \to \varepsilon$ as $n \to \infty$. Let

$$B_k^{(n)} = \left\{ x \colon x \in C, \ C \in \eta_n \cdot T^{-1}\eta_n \cdot \ \ldots \ \cdot T^{-h}\eta_n, \ \mu (C) < 2^{-k(h(T, \eta_n)+\frac{\delta}{10})} \right\} .$$

According to Macmillan's theorem [24] $\mu(B_k^{(n)}) \to 0$ as $k \to \infty$. Let k be chosen sufficiently large so that $\mu(B_k^{(n)}) < \delta_n$. We construct the set $A_{k, k-2}$ satisfying the additional requirement that $\mu (A_{k, k-2} \cap B_k^{(n)}) < \dfrac{\mu (B_k^{(n)})}{k}$.

We introduce the decomposition ζ_n whose elements are the sets $M \backslash A_{k, k-2}$, $B_k^{(n)} \cap A_{k, k-2}$ and the intersections with $A_{k, k-2}$ of the elements of $\eta_n \cdot T^{-1}\eta_n \cdot \ \ldots \ \cdot T^{-k}\eta_n$ that are not contained in $B_k^{(n)}$. Also we put $\tilde{\zeta}_n = \zeta_n \cdot T\zeta_n \cdot \ \ldots \ \cdot T^{k-1}\zeta_n$ and

$$T_n x = \begin{cases} Tx, & \text{if } x \in \bigcup_{l=0}^{k-2} T^l A_{k, k-2}, \\ T^{-k+1}x, & \text{if } x \in T^{k-1} A_{k, k-2}, \\ x, & \text{if } x \in \bigcup_{l=0}^{k-1} T^l A_{k, k-2}. \end{cases}$$

Simple calculations show that the $\tilde{\zeta}_n$ and the T_n satisfy A.1 - A.3.

For ergodic automorphisms Theorem 4.2 can be inverted.

THEOREM 4.3. *If an ergodic automorphism T admits an a.p.t. I with speed* $\dfrac{\theta}{\lg n}$, *then* $h(T) \leqslant \dfrac{\theta}{2}$.

A somewhat weaker result holds in the general case:

THEOREM 4.4. *If an automorphism T admits an a.p.t. I with speed* $\dfrac{\theta}{\lg n}$, *then* $h(T) \leqslant \theta$.

These theorems are proved by a more refined argument than in the case of Theorems 4.1 and 4.2. Theorem 4.3 is proved in [6].

COROLLARY 4.1. *A necessary and sufficient condition for an*

automorphism T *to have zero entropy is that* T *admits an a.p.t.I with speed* $o\left(\frac{1}{\lg n}\right)$.

COROLLARY 4.2. *If* $h(T) = \infty$, *then a necessary and sufficient condition for an automorphism* T *to admit an a.p.t.I with speed* $f(n)$ *is that the sequence* $f(n)\lg n$ *is unbounded.*

COROLLARY 4.3. *If* T *is an ergodic automorphism, then a new definition of the entropy of* T *can be stated as follows:* $h(T) = \dfrac{c(T)}{2}$, *where* $c(T)$ *is the greatest lower bound of numbers* θ *such that* T *admits an a.p.t.I with speed* $\dfrac{\theta}{\lg n}$.

COROLLARY 4.4. *In the general case we have* $h(T) \leq c(T) \leq 2h(T)$.

It is highly probable, though not yet proved, that for any automorphism $c(T) = 2h(T)$.

§5. Fibre bundles

I. In this section we apply the results of §§2 – 3 to a special class of automorphisms, which are a particular case of fibre bundles introduced by Anzai [1].

DEFINITION 5.1. Let T be an automorphism of the space (M, μ); (M', μ') be the direct product of (M, μ) and the two-point space $Z_2 = \{+1, -1\}$ with measures $\left(\frac{1}{2}, \frac{1}{2}\right)$; and w be a measurable function on M with values ± 1. We call the automorphism T' of the space (M', μ') defined by

$$T'(x, j) = (Tx, w(x)j), \text{ where } x \in M, j \in Z_2,$$

a *fibre bundle with base* T *and function* w. Let $g(n)$ be a sequence of positive numbers.

DEFINITION 5.2. *We say that the set* $F \subset M$ *is oddly approximated with respect to a sequence of decompositions* η_n *with speed* $o\,[g(n)]$ *if, for a certain subsequence* n_k, *there exist sets* F_k *consisting of an odd number of elements of the decomposition* η_{n_k} *such that* $\mu(F \triangle F_k) = o\,[g(q_{n_k})]$, *where* q_n *is the number of elements of* η_n.

In Theorems 5.1 – 5.4 below ζ_n denotes, as before, the sequence of decompositions in the definition of an a.p.t.

THEOREM 5.1. *If an automorphism* T *admits a cyclic a.p.t. with speed* $o\left(\frac{1}{n}\right)$ *and the function* $w(x)$ *is such that the set* $w^{-1}(-1)$ *is oddly approximated with respect to* ζ_n *with speed* $o\left(\frac{1}{n}\right)$, *then the spectrum of the operator* $U_{T'}$ *is simple (* T' *is a fibre bundle with base* T *and function* w).

The proof consists in constructing with respect to an assigned a.p.t. of an automorphism T a cyclic a.p.t. with speed $o\left(\frac{1}{n}\right)$ for the automorphism T' and applying Theorem 3.1.

Let H_1 be a subspace of those functions from $L_2(M')$ for which $f(x, -j) = f(x, j)$, and let H_{-1} be the orthogonal complement to H_1 in

$L_2(M')$.

THEOREM 5.2. *If the propositions of Theorem 5.1 hold, then the strong convergence*

$$U_{T'}^{q_n} \Rightarrow -E,$$

where q_n is the number of elements of the decomposition ξ_n, exists in the subspace $H -_1$.

The proof is analogus to that of Theorem 3.3.

With suitable restrictions on w and the base automorphism T conclusions can also be drawn concerning the continuity of the spectrum of $U_{T'}$.

THEOREM 5.3. *If an automorphism T admits a cyclic a.p.t. with speed $o\left(\dfrac{1}{n}\right)$, the operator U_T has a continuous spectrum and the function $w(x)$ is such that the sets $w^{-1}(-1)$ and $w^{-1}(1)$ are oddly approximated with respect to ξ_n with speed $o\left(\dfrac{1}{n}\right)$, then the spectrum of the operator $U_{T'}$ is continuous.*

PROOF. Let $f(x, j) = f_1(x) + jf_{-1}(x)$ and $U_{T'}f = \zeta f$. Then $f_1(Tx) = \zeta f_1(x)$ and $w(x) f_{-1}(Tx) = \zeta f_{-1}(x)$. Squaring the second relation we have $f_{-1}^2(Tx) = \zeta^2 f_{-1}^2(x)$; but U_T has a continuous spectrum and therefore $\zeta^2 = 1$. In the case $\zeta = -1$ we have $-w(x) f_{-1}(Tx) = f_{-1}(x)$, which contradicts the ergodicity of the fibre bundle with base T and function $-w(x)$. So the theorem is proved.

Theorems 5.1 and 5.3 are closely related to the question of the existence of a measurable solution f of the functional equation

$$f(Tx) = \omega(x) f(x), \tag{5.1}$$

where $\omega(x)$ is a given function.

The method of approximations shows that in a number of cases (5.1) cannot be solved. One of these results is obtained in §8.

§6. Flows

I. Let $\{S_t\}$ be a measurable flow on the Lebesgue space (M, μ). A measurable flow is defined in [16].

DEFINITION 6.1. *We say that the flow $\{S_t\}$ admits an a.p.t.I with speed $f(n)$ if a sequence of moments of time t_n, decompositions ξ_n of the space M into a finite number q_n of measurable sets*

$$C_{n,i} \subset M \qquad (i = 1, 2, \ldots, q_n)$$

and measure-preserving transformations T_n can be found such that

A.F.1. $\xi_n \longrightarrow \varepsilon$ as $n \longrightarrow \infty$,

A.F.2. $T_n \xi_n = \xi_n$,

A.F.3. $\displaystyle\sum_{i=1}^{q_n} \mu\,(S_{t_n} C_n,\, {}_i \Delta T_n C_n,\, {}_i) < f\,(q_n)$,

A.F.4. $t_n p_n \longrightarrow \infty$ as $n \longrightarrow \infty$,

where as in §1 the order of the automorphism T_n on the factor space M/ζ_n is denoted by p_n.

Definitions of an a.p.t.II and a cyclic a.p.t. can be similarly adapted to the case of flows.

The basic theorems of §§2 - 3 carry over without difficulty to the case of flows and the results may be stated as follows:

THEOREM 6.1. *If the flow* $\{S_t\}$ *admits a cyclic a.p.t. with speed* $\dfrac{\theta}{n}$ *, where* $\theta \geqslant 2$, *the number of ergodic components of the flow does not exceed* $\dfrac{\theta}{2}$.

COROLLARY 6.1. *The flow* $\{S_t\}$ *admitting a cyclic a.p.t. with speed* $\dfrac{\theta}{n}$ *for* $\theta < 4$ *is ergodic.*

THEOREM 6.2. *If the flow* $\{S_t\}$ *admits an a.p.t.II with speed* $\dfrac{\theta}{n}$ *, where* $\theta < 2$, *then* $\{S_t\}$ *possesses no mixing.*

Let $\{U_t\}$ be a one-parameter group of unitary operators adjoint to the flow $\{S_t\}$.

THEOREM 6.3. *If the flow* $\{S_t\}$ *admits a cyclic a.p.t. with speed* $\dfrac{\theta}{n}$ *, where* $\theta < \dfrac{1}{2}$ *, the spectrum of the group* $\{U_t\}$ *is simple.*

THEOREM 6.4. *If the flow* $\{S_t\}$ *admits an a.p.t.II with speed* $o\left(\dfrac{1}{n}\right)$ *, then there is a strong convergence*

$$U_{p_n t_n} \Longrightarrow E$$

in $L_2(M)$.

THEOREM 6.5. *If the flow* $\{S_t\}$ *admits an a.p.t.II with speed* $\dfrac{\theta}{n}$ *, where* $\theta < \dfrac{1}{2}$ *, then the spectrum of the group* $\{U_t\}$ *is singular.*

Theorems analogous to those of §4 hold if in the definition of an a.p.t. the condition A.F.3 is replaced by

A.F.3a. $\displaystyle\sum_{i=1}^{q_n} \mu\,(S_{t_n}\, C_{n,i}\, \Delta T_n C_{n,i}) < t_n f(q_n),\quad t_n \longrightarrow 0.$

§7. Some unsolved problems

I. It is known that there exist mixing automorphisms with a singular spectrum. In connection with Theorem 2.2. it is of interest to know how these automorphisms are approximated. In particular, can they admit an

a.p.t.I with speed $\dfrac{1}{\sqrt{n}}$?

2. What can be said about the approximability of the product of two commuting automorphisms T and S if it is known that T admits an a.p.t.I with speed $f_1(n)$ and S admits an a.p.t.I with speed $f_2(n)$? The theorems of §4 show that this question is related, in particular, to the problem of estimating the entropy of the product of commuting automorphisms T and S in terms of $h(T)$ and $h(S)$.

3. Suppose that T admits an a.p.t. with speed $f(n)$ and T_1 is a factor-automorphism of the automorphism T. Is it true that T_1 also admits an a.p.t. with speed $f(n)$? Is it possible at least to estimate the speed of approximation of T_1 in terms of $f(n)$?

4. Suppose that we are given a sequence $\varphi(n)$, where $\varphi(n) > 0$ and $\varphi(n) \to 0$, and that the maximum spectral type σ of the operator U_T has support G of the form

$$G = \bigcap_{l=1}^{\infty} \bigcup_{k \geqslant l} G_k, \quad \text{where } G_k = \{\zeta : |\zeta| = 1, \ |\zeta^{r_k} - 1| < \varphi(r_k)\},$$

and r_k is a monotonic sequence of natural numbers. What can be said in this case about the approximability of the automorphism T ?

5. Is the invariant $d(T)$ of §1 expressible in terms of known metric invariants of the automorphism? A negative answer to this question is highly probable.

PART II

APPLICATIONS

§8. Shifting of intervals

1. All spectra of automorphisms not of a probability origin investigated until recently have turned out to be discrete or countably-multiple Lebesgue or a combination of these types. Continuous or mixed non-Lebesgue spectra and, in particular, a simple continuous spectrum have been found only for automorphisms of a probability origin [19], [5], [3], [4]. The results of §§2, 3 provide a systematic approach to the study of spectral properties of automorphisms with singular spectra. In this section we apply these results to the so-called automorphisms of "shifting three intervals". The problem of studying such automorphisms was raised by Rokhlin and Arnol'd [2].

2 Let $0 < \alpha < \beta < 1$. We consider the automorphisms $P_{\alpha, \beta}$ of the interval $[0, 1]$ defined by

$$P_{\alpha, \beta}(x) = \begin{cases} x + 1 - \alpha & \text{for } 0 \leqslant x < \alpha, \\ x + 1 - \alpha - \beta & \text{for } \alpha \leqslant x < \beta, \\ x - \beta & \text{for } \beta \leqslant x \leqslant 1. \end{cases}$$

We put $A = \dfrac{1-\alpha}{1+\beta-\alpha}$, $B = \dfrac{\beta-\alpha}{1+\beta-\alpha}$. Then $P_{\alpha,\beta}$ is isomorphic to the derived automorphism [16] induced by the automorphism $T^{(A)}$ in Example 1.1 on the interval $[0, 1 - B]$. We investigate the existence of a discrete component in the spectrum of the operator $V_{\alpha,\beta}$ adjoint to the automorphism $P_{\alpha,\beta}$. Let $V_{\alpha,\beta}f = \zeta f$. Then the function

$$f^*(x) = \begin{cases} f\left(\dfrac{x}{1+\beta-\alpha}\right) & \text{for } 0 \leqslant x < 1-B \\ f\left(\dfrac{x}{1-\beta-\alpha} + A\right) & \text{for } 1-B < x \leqslant 1 \end{cases}$$

satisfies the equation

where $\qquad\qquad g(x)f^*(x) = f^*(\{x + A\}),$ \hfill (8.1)

$$g(x) = \begin{cases} \zeta & \text{for } 0 \leqslant x \leqslant 1-B, \\ 1 & \text{for } 1-B < x \leqslant 1. \end{cases}$$

The insolubility of (8.1), and hence the continuity of the spectrum of $V_{\alpha,\beta}$ under certain arithmetic limitations on α and β, guarantees Lemma 8.1 to be derived below, which we also use in §9.

We say that the ordered pair of numbers (A, B) in $[0, 1]$ *satisfies condition* C if A is irrational and there exists a sequence of irreducible fractions $\dfrac{p_n}{q_n}$ such that

C.1. $\qquad \left| \dfrac{p_n}{q_n} - A \right| = o\left(\dfrac{1}{q_n^2}\right).$

C.2. There exists $c > 0$ such that for all integers r

$$\left| \frac{r}{q_n} - B \right| > \frac{c}{q_n} .$$

Let $|\zeta_1| = |\zeta_2| = 1$, $\zeta_1 \neq \zeta_2$. We put

$$\omega(x) = \begin{cases} \zeta_1 & \text{for } 0 \leqslant x < 1-B, \\ \zeta_2 & \text{for } 1-B \leqslant x < 1. \end{cases}$$

LEMMA 8.1. *If the pair of numbers* (A, B) *satisfies condition* C, *then the equation*

$$\omega(x)h(x) = h(\{x + A\}) \hfill (8.2)$$

has no measurable solution $h(x)$, *where* $|h(x)| = 1$.

PROOF. We put $\omega_n(x) = \prod\limits_{k=0}^{q_n-1} \omega(\{x + kA\})$. If $h(x)$ is a solution of

(8.2), then $\omega_n(x) = \dfrac{h(\{x + q_nA\})}{h(x)}$. It follows from the integral

continuity of $h(x)$ that $\|\omega_n(x) - 1\| \to 0$ as $n \to \infty$ in $L_2(0, 1)$. We show,
on the other hand, that if condition C is satisfied, then for sufficiently
large n we have $\|\omega_n(x). - 1\| > D > 0$, where D does not depend on n. We put

$\omega_n^*(x) = \prod\limits_{k=0}^{q_n-1} \omega\left(\left\{x + \dfrac{kp_n}{q_n}\right\}\right)$. Obviously,

$$\|\omega_n - 1\| \geqslant \|\omega_n^* - 1\| - \|\omega_n^* - \omega_n\|. \tag{8.3}$$

Let $B = \dfrac{r_n}{q_n} + \dfrac{\theta_n}{q_n}$, where the r_n are integers and $0 \leqslant \theta_n < 1$. It follows
from C.2 that $c < \theta < 1 - c$. It is easy to verify that

$$\omega_n^*(x) = \begin{cases} \zeta_2^{r_n}\zeta_1^{q_n - r_n} & \text{for } \{q_nx\} \leqslant \theta_n, \\ \zeta_2^{r_n+1}\zeta_1^{q_n - r_n - 1} & \text{for } \{q_nx\} > \theta_n. \end{cases}$$

Therefore

$$\|\omega_n^* - 1\| > \frac{1}{2}\left|\sqrt{\frac{\zeta_2}{\zeta_1}} - 1\right|\sqrt{c}. \tag{8.4}$$

Let $M_n = \{x: \omega_n^*(x) \neq \omega_n(x)\}$. By C.1. we have $\mu(M_n) \to 0$ as $n \to \infty$, since

$$\|\omega_n^*(x) - \omega_n(x)\| \leqslant 4\sqrt{\mu(M_n)}, \tag{8.5}$$

and the lemma now follows from (8.3), (8.4) and (8.5).

If α and β are subject to other arithmetic limitations, then a.p.t.'s
of the automorphism $P_{\alpha,\beta}$ can be constructed satisfying the conditions of
Theorems 3.1 and 3.3. Thus:

I. *Suppose that for the pair of numbers* $A = \dfrac{1-\alpha}{1+\beta-\alpha}$ *and* $B = \dfrac{\beta-\alpha}{1+\beta-\alpha}$

sequences of rational numbers $\dfrac{p_n}{q_n}$ *and* $\dfrac{r_n}{q_n}$ *exist satisfying the*

conditions:

8a. p_n *and* q_n *are relatively prime and* $q_n \to \infty$.

8b. $\left|\dfrac{p_n}{q_n} - A\right| = o\left(\dfrac{1}{q_n^2}\right)$.

8c. $\left|\dfrac{r_n}{q_n} - B\right| = o\left(\dfrac{1}{q_n}\right)$.

Then the automorphism $P_{\alpha,\beta}$ *admits a cyclic a.p.t. with speed* $o(n^{-1})$.

Conditions C and 8.a — 8.c are satisfied for almost all pairs (A, B)
in the sense of Lebesgue measure on the square $0 \leqslant A \leqslant 1$, $0 \leqslant B \leqslant 1$.
It follows that *for almost all* (α, β) *the operator* $V_{\alpha,\beta}$ has a simple
continuous singular spectrum and has no mixing. Therefore in addition to
Theorem 3.2 we obtain very simple geometrical examples of automorphisms
with a simple continuous spectrum.

REMARK 8.1. Using Theorem 2.3 it can be shown that the automorphism

$P_{\alpha,\beta}$ has no mixing for any α and β.

§9. The group property of the spectrum

Long ago Kolmogorov conjectured (see, for example, [17]) that the maximum spectral type of an ergodic automorphism always subordinates its convolution. This property is a natural continual analogue of the group property of the spectrum of an ergodic automorphism with a discrete spectrum and was proved by Sinai for a special class of automorphisms satisfying condition A (see [17]). However, the conjecture is not true in general. In this section we construct ergodic automorphisms whose maximum spectral types do not have the "group property" in the above sense.

Let (M, μ) be the direct product of the interval $[0, 1]$ with Lebesgue measure and the two-point space $Z_2 = \{ +1, -1 \}$ with measures $\left(\frac{1}{2}, \frac{1}{2} \right)$

We consider the automorphism $T^{(\alpha, \beta)}$ of M that is a fibre bundle in the sense of §5 with base $T^{(\alpha)}$ and function

$$w_\beta(x) = \begin{cases} -1 & \text{for} \quad 0 < x < \beta, \\ 1 & \text{for} \quad \beta < x < 1. \end{cases}$$

We denote by $U_{\alpha,\beta}$ the unitary operator adjoint to $T^{(\alpha,\beta)}$. Let $L_2(M) = H_1 \oplus H_{-1}$ be the direct sum described in §5. The operator $U_{\alpha,\beta}$ has a discrete spectrum in the subspace H_1. Let $U_{\alpha,\beta}$ have an eigevector in the subspace H_{-1} with eigenvalue ζ. Then, as was shown in §5, there exists a function $f(x)$ satisfying the equation $w_\beta(x) \cdot f(\{ x + \alpha \}) = \zeta f(x)$. If the pair of numbers (α, β) satisfies condition C of §8, it follows from Lemma 8.1 that $U_{\alpha,\beta}$ has a continuous spectrum in the invariant subspace H_{-1}. It is clear that in this case $T^{(\alpha,\beta)}$ is ergodic.

Let us suppose further that there exist sequences of fractions $\frac{k_n}{m_n}$ and $\frac{l_n}{m_n}$ such that :

9a. l_n is odd for all n,

9b. $\left| \alpha - \frac{k_n}{m_n} \right| = o(m_n^{-2})$.

9c. $\left| \beta - \frac{l_n}{m_n} \right| = o(m_n^{-1})$.

If these conditions hold, then w_β and the approximations of $T^{(\alpha)}$ described in §1.2 satisfy the conditions of Theorem 5.2. Hence there is strong convergence $U_{\alpha,\beta}^{m_n} \to -E$ in H_{-1}. Let $h \in H_{-1}$ be a normalized vector of maximum spectral type with respect to $U_{\alpha,\beta}$ in H_{-1}, and σ_{-1} the spectral measure corresponding to h. Then $\int_{|\zeta|=1} \zeta^{m_n} d\sigma_{-1} \to -1$, as $n \to \infty$, since $U_{\alpha,\beta}^{m_n} h \to -h$ as $n \to \infty$.

LEMMA 9.1. *If the normalized measure σ on the unit circle is such that for a certain sequence r_n*

$$\int_{|\zeta|=1} \zeta^{rn} d\sigma \longrightarrow \zeta_0 \quad \text{as} \quad n \longrightarrow \infty$$

and $|\zeta_0| = |$, $\zeta_0 \neq 1$, *then the measures σ and $\sigma * \sigma$ are mutually singular.*

We omit the proof of this lemma. Since $U_{\alpha,\beta}$ has a discrete spectrum in the invariant subspace H_1, the measure σ of the maximum spectral type of $U_{\alpha,\beta}$ is of the form $\sigma_1 + \sigma_{-1}$, where σ_1 is a discrete measure and σ_{-1} a continuous measure mutually singular with its convolution. It follows that σ does not subordinate its convolution.

REMARK 9.1. A similar construction provides an example of an ergodic flow whose maximum spectral type does not subordinate its convolution (see [8]).

The automorphism $T^{(\alpha, \frac{1}{2})}$ suggested by von Neumann was the first to which arguments of periodic approximations were applied. Oseledets proved [12] that the spectrum of the operator $U_{\alpha, \frac{1}{2}}$ is continuous and raised

problems concerning its singularity and simplicity which were actively discussed by participants of the Khumsan school on ergodic theory. Its singularity was proved by Bernshtein, and by using concepts similar to approximations Kushnirenko established that numbers α exist for which $U_{\alpha, \frac{1}{2}}$ has a simple spectrum.

§10. Square roots of automorphisms

1. An automorphism S is called a square root of an automorphism T if $S^2 = T$. A square root of T is denoted by $\sqrt{T}$. The problem of describing the square roots of a given automorphism T is completely solvable only when T has a discrete [20], [21] or quasi-discrete spectrum [11]. Until recently it was not even known whether every automorphism with a continuous spectrum had a square root. In this section we make a construction, using essentially the results of §§3, 5, showing that this is not the case.

We say that an automorphism T of the Lebesgue space (M, μ) admits a Z_2-fibering if there exists an involutory automorphism J of M commuting with T and having no fixed points (mod 0). J is called a Z_2-fibering *of* T.

LEMMA 10.1. *An ergodic automorphism T admits a Z_2-fibering if and only if there exists in the unitary ring $L_2(M)$ a multiplicative involutory unitary operator $U \neq E$ commuting with U_T.*

LEMMA 10.2. *If an automorphism T with a simple spectrum admits a Z_2-fibering J, then $\sqrt{T}$ (if it exists) also admits J.*

PROOF. Both U_J and $U_{\sqrt{T}}$ commute with U_T. Since U_T has a simple spectrum, both U_J and $U_{\sqrt{T}}$ are functions of U_T and therefore U_J commutes with $U_{\sqrt{T}}$. We then use Lemma 10.1.

The following statement of Lemma 10.2 is used subsequently: *If a fibre bundle T' with base T has a simple spectrum, then the square root of T' is also a fibre bundle with base $\sqrt{T}$.*

2. We now pass to the construction of an automorphism with a continuous spectrum but having no square root.

Let R be an automorphism of the Lebesgue space (M, μ) having a continuous spectrum. We assume that R admits a cyclic a.p.t. with speed $o(n^{-1})$, the corresponding decompositions ζ_n consisting of an odd number of elements. The existence of these automorphisms can be established by modifying the proof of Theorem 1.1; they can also be found among the automorphisms $P_{\alpha,\beta}$ of §8.

Let S be a fibre bundle with base R and function w operating in the space $M' = M \times Z_2$. Let us suppose that the sets $w^{-1}(1)$ and $w^{-1}(-1)$ are oddly approximated with respect to the decompositions ζ_n with speed $o(n^{-1})$. Then in accordance with Theorems 5.1 and 5.3 the automorphism S has a simple continuous spectrum.

We now consider the fibre bundle T with base S and function $\omega(y) = \omega(x, j) = j$, where $y = (x, j) \in M'$, $x \in M$, $j = \pm 1$. It is clear that $\omega^{-1}(1)$ and $\omega^{-1}(-1)$ are oddly approximated with respect to the sequence of decompositions $\zeta_n' = \zeta_n \times \varepsilon_2$ of the space M' with speed $o(n^{-1})$ (ε_2 is the decomposition of Z_2 onto the points $+1$ and -1). Therefore the automorphism T has a simple continuous spectrum. By Lemma 10.2 the square root of T is a fibre bundle with base $\sqrt{S}$ and function $\Delta(y)$, where $\Delta(y)$ must satisfy

$$\Delta\left(\sqrt{S}y\right) \Delta\left(y\right) = \omega\left(y\right). \tag{10.1}$$

We put $J(x, j) = (x, -j)$. Then $\omega(Jy) = -\omega(y)$, and by (10.1),

$$\Delta\left(\sqrt{S}Jy\right) \Delta\left(Jy\right) = -\omega\left(y\right). \tag{10.2}$$

Multiplying (10.1) and (10.2) we obtain

$$\Delta\left(y\right) \Delta\left(Jy\right) \Delta\left(\sqrt{S}y\right) \Delta\left(\sqrt{S}Jy\right) = -1. \tag{10.3}$$

We put $\Delta(y)\Delta(Jy) = \Theta(y)$. Since J is a Z_2-fibering of S and S has a simple spectrum, by Lemma 10.2 the automorphisms J and $\sqrt{S}$ commute and therefore $\Delta\left(\sqrt{S}y\right) \Delta\left(\sqrt{S}Jy\right) = \Theta\left(\sqrt{S}y\right)$. Then (10.3) takes the form

$$\Theta\left(y\right) \Theta\left(\sqrt{S}y\right) = -1. \tag{10.4}$$

Since $\Theta(y)$ takes the values $+1$ and -1, it follows from (10.4) that $\Theta(y) \neq$ const. On the other hand, $\Theta(Sy) = -\Theta(\sqrt{S}\,y) = \Theta(y)$, which contradicts the ergodicity of S.

3. In the above construction we required w to be a function with values ± 1 such that $w^{-1}(1)$ and $w^{-1}(-1)$ were oddly approximated with speed $o(n^{-1})$ with respect to a sequence of decompositions ζ_n. We now state a proposition showing that there are "sufficiently many" such functions. Let (M, μ) be a Lebesgue space and n and m measurable functions on M taking the values ± 1. We put $\rho(m, n) = \mu(m^{-1}(1)\Delta n^{-1}(1))$. The space of functions on M with values ± 1 and metric ρ is denoted by B.

LEMMA 10.3. *Let ζ_n be a sequence of decompositions of M tending to*

ε. *The set of functions* $w \in B$ *such that* $w^{-1}(1)$ *and* $w^{-1}(-1)$ *are oddly approximated with respect to the sequence* ξ_n *with given speed* $f(n)$ *contains a set of the type* G_δ *everywhere dense in* B.

R E M A R K 10.1. If $P_{\alpha,\beta}$ in §8 is taken to be the automorphism R in the above construction it is obviously not difficult to construct a function with the required properties.

§11. Flows on a two-dimensional torus

In the final two sections we apply methods developed in Part I to classical dynamical systems. In this section we investigate the spectral properties of flows on a two-dimensional torus by means of the theorems in §6 on flows admitting "good" approximations. In §12 we apply the theorems of §4 to find bounds for the entropy of classical dynamical systems of the general form.

Let T^2 be a two-dimensional torus and (x, y) coordinates on T^2, where x, y are real numbers $m \cdot \cdot$. The flows of interest to us are one-parameter groups of shifts $\{S_t\}$ ug trajectories of the system of differential equations

$$\frac{dx}{dt} = A(x, y), \qquad \frac{dy}{dt} = B(x, y) \tag{11.1}$$

with invariant measu of the form $d\mu = F(x, y)\, dx\, dy$. We assume that the vector field (11.1) has no singular points, that is, $A^2 + B^2 > 0$, and that the functions A, B and F are sufficiently smooth. The topological structure of the trajectories of the flow $\{S_t\}$ depends on the number

$$\alpha = \frac{\lambda_1}{\lambda_2} = \frac{\int AF\, dx\, dy}{\int BF\, dx\, dy}.$$

The case of rational α is of little interest, because we then have closed trajectories. Kolmogorov [9] has shown that if α is irrational, the metric properties of $\{S_t\}$ depend on the speed of approximation of α by rational numbers. If α is "not too well" approximated by rational numbers, then $\{S_t\}$ has a discrete spectrum. If, on the other hand, α is "very well" approximated by rational numbers, then $\{S_t\}$ can have a continuous spectrum [9], [26]. In both these cases, however, the spectrum of $\{S_t\}$ has properties characteristic of "well approximated" flows.

Let $\{U_t\}$ be a one-parameter group of unitary operators in $L_2(T^2)$ adjoint to the flow $\{S_t\}$ defined by (11.1).

THEOREM 11.1. *If the right-hand sides A and B of the system of differential equations* (11.1) *and the density of the invariant measure F have continuous partial derivatives up to the fifth order and the number α is irrational, then*

1. *the spectrum of the group* $\{U_t\}$ *is simple*,
2. *the maximum spectral type of the group* $\{U_t\}$ *is singular*,
3. *the flow* $\{S_t\}$ *has no mixing*.

Theorem 11.1 follows from two propositions.

THEOREM 11.2. *If the conditions of Theorem* 11.1 *are satisfied and the number α is such that, for any integers p and q, for some $c > 0$*

$$\left| \alpha - \frac{p}{q} \right| > \frac{c}{q^4} , \tag{11.2}$$

then the flow $\{ S_t \}$ *is ergodic and has a discrete spectrum.*

THEOREM 11.3. *If the conditions of Theorem* 11.1 *are satisfied and there exists a sequence of irreducible fractions* $\frac{p_n}{q_n}$ *such that*

$$\left| \alpha - \frac{p_n}{q_n} \right| = o \left(\frac{1}{q_n^4} \right) , \tag{11.3}$$

then the flow $\{ S_t \}$ *admits a cyclic a.p.t. with speed* $o(n^{-2})$.

For if α satisfies the conditions of Theorem 11.2, it is sufficient to use the fact that the assertions of Theorem 11.1 are satisfied for ergodic flows with a discrete spectrum.

If, on the other hand, α satisfies the conditions of Theorem 11.3, then Theorem 11.1 follows from Theorems 6.3, 6.5 and 6.2.

It is not difficult to show that $\{ S_t \}$ is isomorphic to a special flow constructed with respect to an automorphism $T^{(\beta)}$ and function $f(x)$ having five continuous derivatives (a special flow is defined in the survey paper [16]). We have $\beta = \frac{m\alpha + n}{p\alpha + q}$, where m, n, p and q are integers and therefore α and β both satisfy or do not satisfy (11.2).

Theorem 11.2 for a special flow is proved similarly to Theorem 2 in Kolmogorov's paper [9].

Let us describe in brief the construction of a cyclic a.p.t. with speed $o(n^{-2})$ for a special flow $\{ R_t \}$ constructed with respect to the automorphism $T^{(\beta)}$, where β satisfies (11.3), and with $f(x)$ having five continuous derivatives. The flow $\{ R_t \}$ operates on the domain

$$K = \{ x, y \colon\ 0 \leqslant x < 1,\ 0 \leqslant y < f(x) \}.$$

Let

$$\gamma = \int_0^1 f(x)\, dx \ \text{ and } \ \left| \beta - \frac{p_n}{q_n} \right| = \frac{r_n}{q_n^4} .$$

We choose numbers $\delta_n \to 0$ such that the numbers $q_n \gamma \delta_n^{-1} = Q_n$ are integral. In addition, we require that $\delta_n > \max \ (r_n^{\frac{1}{4}},\ q_n^{-\frac{1}{4}})$ and $\delta_n < \frac{1}{4} \min f(x)$. We fix n and take for definiteness $\beta > \frac{p_n}{q_n}$. We denote the rectangle

$\{(x, y) \colon 0 \leqslant x \leqslant \frac{1}{q_n} - \{ q_n \beta \},\ 0 \leqslant y \leqslant \delta_n \}$ by A_n. It turns out that, for sufficiently large n, the sets $R_{k \delta_n} A_n$, where $k = 0, 1, \ldots, Q_n - 2$, are mutually disjoint and the measures $\mu(R_{(Q_n-1)\delta_n A_n} \cap A_n)$ and $\mu(R_{Q_n \delta_n} A_n \triangle A_n)$ are sufficiently small. The decomposition ξ_n can be constructed as follows: we take the sets $R_{k \delta_n} A_n$ for $k = 0, 1, \ldots, Q_n - 2$ and $R_{(Q_n-1)\delta_n A_n} \backslash A_n$ and add to each of these sets part of the set

$K \backslash \bigcup_{k=0}^{Q_n - 1} R_{k \delta_n} A_n$ in order to obtain on Q_n a decomposition of sets of equal

measure $\frac{1}{Q_n}$. We can take for T_n any automorphism that transforms an element containing A_n into an element containing $R\delta_n A_n$, this in turn being transformed into an element containing $R_2\delta_n A_n$ and so on, until finally an element containing $R_{(Q_n-1)}\delta_n A_n \backslash A_n$ is transformed into an element containing A_n.

The condition $\zeta_n \to \varepsilon$ follows from the fact that $\underset{0\leqslant k\leqslant Q_n-2}{\max \ \text{diam}} (R_k\delta_n A_n) \to 0$

as $n \to \infty$. A complete proof of Theorem 11.1 is given in [7].

§12. Entropy of classical dynamical systems

An upper bound of the entropy of a classical dynamical system was first obtained by Kushnirenko [10] who proved, in particular, that this entropy is always finite.

Very recently, by means of the method of approximations, Kushnirenko's estimate has been improved upon. The estimates below, in contrast to Kushnirenko's estimate, depend on the properties of a dynamical system on a set of full measure and not at all points.

Let M^m be an m-dimensional Riemannian manifold and T be a twice continuously differentiable homeomorphism of M^m onto itself. We assume that T preserves the normalized measure μ which is absolutely continuous with respect to the measure σ induced by the Riemannian metric. Let $w \in M^m$, $\omega \in R_w^m$ be the tangent vector to M^m at w, and $(dT)_w$ the differential of the mapping T at w.

THEOREM 12.1. *If the automorphism T satisfies the above assumptions and is ergodic, the entropy of T is given by*

$$h(T) \leqslant H = \max \left(\frac{m}{2} \int_{M^m} \lg \| (dT)_w \| \, d\mu, \ \frac{m}{2} \int_{M^m} \lg \| (dT^{-1})_w \| \, d\mu \right).$$

The proof of Theorem 12.1 is based on the construction of an a.p.t. of the automorphism $T \times T^{(\alpha)}$ with a specially chosen α with speed $\frac{2H}{\lg n} + o\left(\frac{1}{\lg n}\right)$ and on an application of Theorem 4.3.

In the general case we assume a decomposition η of M^m into ergodic components of the automorphism T, $C_\eta \in \eta$.

THEOREM 12.2. *The following inequalities hold:*

$$h(T) \leqslant \max \left(m \cdot \operatorname{ess\,sup}_\eta \int_{C_\eta} \lg \| (dT)_w \| \, d(\mu/C_\eta) \right),$$

$$m \cdot \operatorname{ess\,sup}_\eta \left(\int_{C_\eta} \lg \| (dT^{-1})_w \| \, d(\mu/C_\eta) \right).$$

COROLLARY 12.1. *Let $\lambda_1, \ldots, \lambda_m$ be characteristic exponents* [13] *of the automorphism T. If T is ergodic, then $h(T) \leqslant \frac{m}{2} \max | \lg \lambda_i |$.*

COROLLARY 12.2. *If for almost all $w \in M^m$ we have*

$\| (dT^n)_w \| = 2^{|n|a_n(w)}$, *where $a_n(w) \to 0$ as $|n| \to \infty$, then $h(T) = 0$.*

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