DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume 2, Number 3, July 1996 pp. 397–411

HYPERBOLIC MEASURES AND COMMUTING MAPS IN LOW DIMENSION

ANATOLE KATOK

Department of Mathematics
The Pennsylvania State University
University Park, PA 16802, USA

Abstract. We study invariant measures with non–vanishing Lyapunov characterictic exponents for commuting diffeomorphisms of compact manifolds. In particular we show that for $k = 2, 3$ no faithful $\mathbb{Z}^k$ real–analytic action on a k–dimensional manifold preserves a hyperbolic measure. In the smooth case similar statements hold for actions faithful on the support of the measure. Generalizations to higher dimension are proved under certain non–degeneracy conditions for the Lyapunov exponents.

0.Introduction. The purpose of this paper is to provide a partial rigorous justification to the observation that there is not too much room for many commuting diffeomorphisms to act on a compact differentiable manifold. To be slightly more specific, one expects that on such a manifold a "sufficiently large" action of the group $\mathbb{Z}^k$ may exist only if the rank k of the group is not too big compared to the dimension m of the manifold. Naturally, one needs to define a proper notion (or, rather several different notions) of "sufficiently large". Any of the natural definitions which we discuss in this paper would show that the best inequality to be expected is

$$k \leq m - 1. \tag{0.1}$$

(See Corollary 1.4 below). Our ideas of "sufficiently large" revolve around various versions of the notion of hyperbolicity. The strongest notion of hyperbolicity is that of *Anosov actions* [PS], [KS1]. Anosov actions of both $\mathbb{Z}^k$ and $\mathbb{R}^k$ display a remarkable array of rigidity properties [KL1], [KS1], [KS2], [KS3]. In the present paper we consider a much weaker and more flexible notions of hyperbolicity in a non–uniform and asymptotic sense which involves consideration of invariant measures of the action. It is sufficient to consider only ergodic invariant measures. Such a measure is called *hyperbolic* if some element of the action has non–zero Lyapunov characteristic exponents and *strongly hyperbolic* if in addition no non–zero element has all Lyapunov characteristic exponents equal to zero. Our first observation which involves a rather simple application of Pesin theory is that the latter notion is strong enough to guarantee inequality (0.1) for any non–atomic measure (Corollary 1.4). Inequality (0.1) is sharp as can be seen from standard examples of actions by hyperbolic automorphisms of tori and their modifications.

The main point of the present paper is to show that in dimension two and three the same is true for any hyperbolic measure unless the action is essentially non–faithful, i.e. contains an element which fixes every point in the support of the measure. (Corollaries 3.1, 3.2, Theorem 4.1 and Corollary 4.3). In dimension two there are further restrictions which are particularly strong for analytic actions.

Partially supported by NSF grant DMS-94 04061

Namely, any faithful action consists of maps with zero topological entropy (Corollary 3.3) and in the area–preserving case existence of a single hyperbolic periodic point for a faithful action implies that essentially the action is completely intergable or embeds into a smooth flow, i.e. an action of $\mathbb{R}$ (Theorem 3.4). In the higher dimension (0.1) still holds for invariant measures which are sufficiently close to strongly hyperbolic (Theorem 4.4).

Two principal ingredients used in the proofs are the structural theory of non–uniformly hyperbolic systems (Pesin theory) [P], [KM] and the classical theory of local normal forms [S], [C], [B], [BK]. When possible we use [KH] and its supplement [KM] as standard references rather than quoting original sources.

1.Preliminaries.

1.1. Local $\mathbb{Z}^k$ actions and locally maximal sets. Let M be a differentiable manifold, $U \subset M$ an open set and $\Lambda \subset U$ a compact subset. Let $f_1, \ldots, f_k : U \to M$ be commuting diffeomorphic embeddings of class $C^{1+\epsilon}$ for some $\epsilon > 0$ preserving the set Λ. Obviously, the maps $f_1, \ldots, f_k$ restricted to the set Λ generate an action of the group $\mathbb{Z}^k$ on Λ. Outside Λ the action of the whole group $\mathbb{Z}^k$ may not be defined. We will call this situation a *local* $\mathbb{Z}^k$ *action near* Λ. Our standard notation for a local action will be F so that for $n = (n_1, \ldots, n_k) \in \mathbb{Z}^k$ one has $F(n_1, \ldots, n_k) = f_1^{n_1} \circ \cdots \circ f_k^{n_k}$. The set Λ is called a *locally maximal set* for the local action F if for some open set $V \supset \Lambda$ the set Λ is the biggest invariant set contained in V, i.e.

$$\Lambda = \bigcap_{n \in \mathbb{Z}^k} F(n)V.$$

Any such set V will be called a *separating neighborhood* for Λ. The $C^{1+\epsilon}$ differentiability assumption is needed for applicability of Pesin theory (see proof of Proposition 1.3 and Section 1.5 below).

1.2. Lyapunov exponents and hyperbolic measures. Let F be a local $\mathbb{Z}^k$ action near a compact set Λ and μ be a Borel probability F-invariant ergodic measure such that $\operatorname{supp} \mu \subset \Lambda$. The *Multiplicative Ergodic Theorem* for $\mathbb{Z}^k$ actions [H], [K] allows us to define the *Lyapunov characteristic exponents* (of F with respect to μ) as linear functionals $\chi_1, \ldots, \chi_l$ on $\mathbb{R}^k$ and for μ-a.e. point $p \in \Lambda$ the *(fine) Lyapunov decomposition* of the tangent space $T_pM = E_{\chi_1}(p) \oplus \cdots \oplus E_{\chi_l}(p)$ such that for $i = 1, \ldots, l$ and for any $v \in E_{\chi_i}(p)$ one has

$$\lim_{n \to \infty} \frac{\log \|DF(n)_p(v)\| - \chi_i(n)}{\|n\|} = 0.$$

Dimension of the space $E_{\chi_i}(p)$ is called the *multiplicity* of the exponent χ_i.

Remarks. 1.For any $n \in \mathbb{Z}^k$ the values $\chi_i(n), i = 1, \ldots, l$ are equal to the Lyapunov characteristic exponents of the map $F(n)$ in the usual sense [KM, Section S2]. Naturally, for a particular n the values of $\chi_i(n)$ may coincide for some i so the map $F(n)$ may have fewer than l distinct exponents with higher multiplicities. However, for "most" values of n this does not happen.

2.One should think of the group $\mathbb{Z}^k$ as embedded into $\mathbb{R}^k$ as the standard integer lattice. The values $\chi_i(t)$ for $t \in \mathbb{R}^k \setminus \mathbb{Z}^k$ do not make sense in the context of the action F. However they are easily interpreted as the Lyapunov characteristic exponents of the elements of the *suspension* action. [KS, Section 2.2].

The hyperplane $\ker\chi \subset \mathbb{R}^k$, where χ is a non-zero Lyapunov exponent, is called a *Lyapunov hyperplane.* The subspace $\chi^{-1}(-\infty,0)$ (corr $\chi^{-1}(0,\infty)$) is called a *negative* (corr. *positive) Lyapunov half-space.* A Lyapunov hyperplane L is called *rational* if $L\cap\mathbb{Z}^k$ is a lattice in L, *totally irrational* if $L\cap\mathbb{Z}^k=\{0\}$, and *partially irrational* otherwise. For k=2 we will call Lyapunov hyperplanes *Lyapunov lines.* Naturally , every Lyapunov line is either rational or totally irrational; in the latter case we will call it simply irrational.

Definition 1.1. An invariant ergodic measure μ for a local $\mathbb{Z}^k$ action F is called *partially hyperbolic* if there is at least one non-zero Lyapunov exponent and *hyperbolic* if all Lyapunov exponents are different from zero.

An element $n\in\mathbb{Z}^k$ is called *regular* is it does not belong to any of the Lyapuinov hyperlpanes. A regular element with respect to a hyperbolic measure is called *hyperbolic.* A *Weyl chamber* is a connected component of the complement to the union of all Lyapunov hyperplanes.

Each Weyl chamber is an open convex polyheadral cone in $\mathbb{R}^k$. Inside a Weyl chamber each non-zero Lyapunov exponent has a constant sign. Conversely, the locus of points in $\mathbb{R}^k$ for which each non-zero Lyapunov exponent has a particular sign is either empty or is a Weyl chamber. Thus any Weyl chamber can be characterized as a minimal non-empty intersection of positive and negative Lyapunov half-spaces.

1.3. Strongly hyperbolic measurs. In the case of multiple exponents it is convenient to count each exponent the number of times equal to its multiplicity so that there are always $m=\dim M$ exponents. Since this will not cause any confusion from now on we will denote exponents listed this way by $\chi_1,\dots,\chi_m$. This allows us to define the *Lyapunov map* $\Psi_\mu:\mathbb{R}^k\to\mathbb{R}^m$ by $\Psi=(\chi_1,\dots,\chi_m)$. The Lyapunov map is defined up to a permutation of coordinates.

Thus the measure μ is hyperbolic if and only if $\operatorname{Im}\Psi_\mu$ does not lie in any coordinate hyperplane. We will call $\dim\operatorname{Ker}\Psi_\mu$ the *defect* of μ and denote it by $d(\mu)$. Equivalently, $d(\mu)$ is equal to the dimension of the intersection of all Lyapunov hyperplanes. Furthermore, $\dim\operatorname{Im}\Psi_\mu$ is called the *rank* of μ and is denoted by $r(\mu)$. Equivalently, $r(\mu)$ is equal to the maximal number of linearly independent Lyapunov exponents.

Definition 1.2. A hyperbolic measure is called *strongly hyperbolic* if the intersection of all Lyapunov hyperplanes consists of the origin.

Obviously, $d(\mu)+r(\mu)=k$.
All our discussion from the beginning of Section 1.2 and up to this point was not specific to the differentiable case. In fact, it makes sense in the more general context of linear extensions of actions of $\mathbb{Z}^k$ by measure–preserving transformations [K]. Now we are coming back to the smooth situation. Since $r(\mu)\le\dim M$, we have

$$k\le\dim M+d(\mu).$$

In particular, for any strongly hyperbolic measure for a local $\mathbb{Z}^k$ action on M

$$k\le\dim M. \tag{1.1}$$

Proposition 1.3. *If* $\operatorname{Im}\Psi_\mu$ *intersects the positive octant* $\mathbb{R}^m_+$ *(and hence also the negative octant* $-\mathbb{R}^m_+$*) then the measure* μ *is atomic.*

Proof. The pre–image $(\Psi_\mu)^{-1}(-\mathbb{R}^m_+)$ is the *negative* Weyl chamber $\mathcal{N}$ where all Lyapunov exponents take negative values. Take $n \in \mathcal{N} \cap \mathbb{Z}^k$ and let $f = F(n)$. There is a set of full measure of points (we will call them *regular*) which satisfy the assertions of Theorem S.3.1 of [KM] with respect to the map f .This set contains a subset of large measure which consist of points which have stable and unstable local manifolds of fixed size with respect to f. (See subsection 1.5 below for a more detailed discussion.) Denote this set by $\mathcal{R}$. Pick a point $x \in \mathcal{R}$ such that the f–orbit of x returns to the intersection of $\mathcal{R}$ with an arbitrary small ball around x. We can also assure that the Lyapunov characteristic exponents at x coincide with those for the measure. By the Closing Lemma ([KM, Theorem S.4.13]) one can find regular f–periodic points for f arbitrary close to x. Let p be such a point, of period, say, N. The Lyapunov characteristic exponents with respect to f at p are close to those at x, hence negative. Hence the local stable manifold $W^s_f(p)$ is open , and, since its size is uniformly bounded from below, if p is chosen close enough to x , we conclude that $x \in W^s_f(p)$. Consequently, $f^{kN}x \to p$ as $k \to \infty$. Since x is a recurrent point this implies that $x = p$. But this argument applies as well to *any* regular periodic point constructed near x. Thus x is an isolated point in a set of positive measure, so μ must be an atomic measure. □

Corollary 1.4. *If* μ *is a hyperbolic measure for a local* $\mathbb{Z}^k$ *action and* $k = \dim M + d(\mu)$*, then* μ *is atomic. In particular, any strongly hyperbolic ergodic invariant measure for a local* $\mathbb{Z}^k$ *action on a* k*–dimensional manifold is atomic.*

Proof. Obviously $r(\mu) = \dim M$ so $\operatorname{Im}\Psi_\mu = \mathbb{R}^{\dim M} \supset \mathbb{R}^{\dim M}_+$. □

Corollary 1.5. *If* μ *is a non-atomic hyperbolic measure for a local* $\mathbb{Z}^k$ *action* F *and* $r(\mu) = \dim M - 1$ *then there exists a regular element* $n \in \mathbb{Z}^k$ *such that* $F(n)$ *has exaclty one positive Lyapunov exponent and this exponent is simple.*

Proof. Let as before $\dim M = m$. Then the image of the map Ψ_μ is a hyperplane $S \subset \mathbb{R}^m$ and by Proposition 1.3 it does not contain points all of whose coordinates are negative. Thus it is given by a single linear equation all of whose non-zero coefficients have the same sign. Hyperbolicity implies that there are at least two non-zero coefficients. Hence one can find an open set of points in S with one positive coordinate and the rest negative. Among pre–images of such points there are regular elements of $\mathbb{Z}^k$. □

As was advertised in the introduction Corollary 1.4 immediately implies that if a local $\mathbb{Z}^k$ action on an n–dimensinal manifold possesses a strongly hyperbolic non–atomic invariant measure then

$$k \le m - 1.$$

This inequality cannot be improved. In fact, for any $m \ge 2$ there is a $\mathbb{Z}^{m-1}$ action on the m–dimensional torus $\mathbb{T}^m$ by hyperbolic automorphisms for which Lebesgue measure is invariant and strongly hyperbolic See e.g.[KL1]. A "surgery" described in [KL2] allows to produce actions of $\mathbb{Z}^{m-1}$ with strongly hyperbolic absolutely continuous invariant measures on certain m–dimensional manifolds ofther than torus. The rank of these measures is equal to $m - 1$.

The principal purpose of this paper is to show that in the low-dimensional cases, namely for globally defined actions on manifolds of dimension two and three, inequality (0.1) holds not only for strongly hyperbolic but for non-atomic hyperbolic invariant measures unless the action is essentially not faithful, i.e. some of its elements have fixed point sets of full measure. In dimension higher than three this assertion is true under the extra condition $d(\mu) = 1$. On the other hand, if dimension is high enough one can produce an effective hyperbolic (but not strongly hyperbolic) action of $\mathbb{Z}^k$ by automorphisms of $\mathbb{T}^m$ such that $k > m$. Appropriate examples which contain lots of unipotent elements were found recently first by A.Starkov and the author in dimension ≥ 19 and then were shown to exist by A.Starkov in dimension 16 (unpublished).

1.4. Periodic points for commuting maps. The following general lemma about periodic orbits which is undoubtedly well-known will be used on several occasions later.

Let $f : U \to M$ be a C^1 diffeomorphic embedding , $p \in U$ be a transverse periodic point for f , i.e $f^N p = p$ and the derivative Df_p^N does not have one as an eigenvalue. Let g be another diffeomorphic embedding defined in a neighborhood of the orbit of the point p and commuting with f.

Lemma 1.6. *The sequence $\{g^n p\}$, $n = 0, 1 \ldots$ has no accumulation points in the domain U of the map f.*

Remark. The lemma implies that one of the following possibiliteis takes place:

(i) p is a periodic point for g;
(ii) the closure of U is not compact;
(iii) only finitely many iterates of the map g are defined at p;
(iv) limit points of the sequence $g^n p$ lie on the boundary of U where the map f is not defined.

The second possibility is excluded by our standing assumption that the ambient manifold M is compact. Under a stronger assumption the last two possibilities can also be excluded.

Corollary 1.7. *If f, g are commuting C^1 diffeomorphisms of a compact manifold , than any transverse f–periodic point is also g–periodic.*

Proof of Lemma 1.6. Assume that all iterates of g are defined at the orbit of p and assume that $g^{n_k} p \to q \in U$ where all points in the sequence $g^{n_k} p$, $k = 1, \ldots$ are distinct. We have $f^N(g^k p) = g^k(f^N p) = g^k p$ so that the point $g^k p$ is f–periodic with the same period N as p. Furhtermore, taking the derivative of both sides of the equality $f^N = g^k \circ f^N \circ g^{-k}$ at the point $g^k p$ we obtain

$$(Df^N)_{g^k p} = (Dg^k)_p \circ (Df^N)_p \circ (Dg^{-k})_{g^k p} = (Dg^k)_p \circ (Df^N)_p \circ [(Dg^k)_p]^{-1}.$$

Thus, all linear operators $(Df^N)_{g^k p}, k = 0, 1 \ldots$ are conjugate and hence have the same eigenvalues. Since f is a C^1 map and $q \in U$, by the continuity of the derivative of f we conclude that $f^N q = q$ and $(Df^N)_q$ has the same eigenvalues as $(Df^m)_p$, so that q is a transverse and hence isolated periodic point of period N for f, a contradiction. □

1.5. Regular points. Suppose $f : U \to M$ is a $C^{1+\epsilon}$ diffeomorphic embedding and μ is an f–invariant measure with compact support, not necessarily ergodic but with non-zero characteristic exponents almost everywhere. This situation appears when we consider a regular element for a $\mathbb{Z}^k$ action with a hyperbolic invariant measure. Then one can find a nested family of compact sets R_θ; $0 < \theta < \theta_0$; $R_{\theta_1} \subset R_{\theta_2}$ for $\theta_2 < \theta_1$, where parameter θ signifies "guaranteed size" of local stable and unstable manifolds, with the following properties (see [KM]):

(1) the union of sets R_θ ; $0 < \theta < \theta_0$ has full measure;
(2) for each point $x \in R_\theta$ there are C^1 local stable and unstable manifolds $W^s_f(x) = \exp_x B^s_\theta$ and $W^u_f(x) = \exp_x B^u_\theta$ where B^s_θ and B^u_θ are the balls of radius θ around the origin in the stable and unstable subspaces $E^-(x)$ and $E^+(x)$ of the tangent space T_xM correspondingly;
(3) the manifolds $W^s_f(x)$ and $W^u_f(x)$ depend on $x \in R_\theta$ in a Hoelder way with the Hoelder exponent independent of θ but the constant dependent on it;
(4) each set R_θ contains a dense set of periodic points; if Lyapunov characteristic exponents of f with respect to μ are constant almost everywhere one can find such points with the Lyapunov characteristic exponents arbitrary close to those for the measure μ.

Unless an ambiguity may appear , we would usually refer to the sets $R_\theta \cap \operatorname{supp} \mu$ as *regular* sets. We will denote by $R^-(f, \mu)$ and $R^+(f, \mu)$ correspondingly, the closure of the union of global stable and unstable manifolds of all points from the set $R \cap \operatorname{supp} \mu$.

Remark. The sets R_θ may contain points outside of the support of the measure μ; for example, periodic points from (4) may not belong to $\operatorname{supp} \mu$; this would happen, in particular, if $\operatorname{supp} \mu$ is a minimal set for f.

2.The main theorem. In this section we establish a criterion for an action preserving a hyperbolic measure to contain elements with large (in fact, full measure) sets of fixed points. Later we will show that the conditions of that criterion are satisfied in a variety of natural situations.

Let p be a hyperbolic fixed point of a C^r map f where $1 \le r \le \infty$ and $W^s(p)$ be its local stable manifold. For $1 \le t \le r$ we denote by $\mathcal{S}^t(f, p)$ the group of local C^t diffeomorphisms of $W^s(p)$, fixing the point p and commuting with f. Given the eigenvalues of DF_p , one can find r_0 and t_0 such that if $r \ge r_0$ and $t \ge t_0$ then $\mathcal{S}^t(f, p)$ is a finite–dimensional Lie group whose dimension is uniformly bounded for all f with the given derivative at p and which for a given f does not depend on t. Thus we will denote this group by $\mathcal{S}(f, p)$. This can be deduced from the theory of local normal forms [B], [BK], [C]. We will call the maximal dimension of an abelian subgroup of $\mathcal{S}(f, p)$ *the stable rank of f at p* and denote it by $\mathcal{R}^s(f, p)$. For a periodic point p of period N we define $\mathcal{R}^s(f, p)$ to be equal to $\mathcal{R}^s(f^N, p)$.

The importance of these notions to our study of $\mathbb{Z}^k$ actions lies in the fact that the stationary subgroup of a periodic point p of a regular element acts on the stable manifold of that point. The stable rank determines whether the resulting homomorhism of the stationary subgroup (which is isomorphic to $\mathbb{Z}^k$ by Corollary 1.7) into $\mathcal{S}^t(f, p)$ may me a monomorphism with discrete image. In what follows we assume that $r \ge r_0$ and $t \ge t_0$ for any periodic point mentioned.

Theorem 2.1. *Let f be a C^r diffeomorphism of a compact manifold M and μ be an f-invariant ergodic measure with non-zero characteristic exponents, one positive and the rest negative. Suppose that the stable rank of f at any periodic point described in*

Section 1.5 (4), is less than k. Then any C^t action of the group $\mathbb{Z}^k$ which includes f also contains an element which fixes every point of the set $R^-(f,\mu) \cup R^+(f,\mu)$.

Proof. Fix $\theta > 0$ and a point $x \in R_\theta \cap \operatorname{supp} \mu$.

Step 1. First, let us explain that it is enough to find an element g of the action such that for a small enough neighborhood $V \ni x$ the local stable and unstable manifolds of all periodic points from the intersection $V \cap R_\theta$ are fixed points of the map g. The set of fixed points of any map commuting with f is closed and f–invariant. Thus, by the density of periodic points in R_θ and the continuity of the stable and unstable manifolds on R_θ the above statement will imply that g fixes every point of $V \cap R_\theta$ as well as of their local stable and unstable manifolds. Since $\mu(V \cap R_\theta) > 0$ and f is ergodic with respect to μ this implies that the fixed-point set of g contains $\operatorname{supp} \mu$ as well as open pieces of stable and unstable manifold of an almost every point in R. But every closed invariant set which contains such pieces has to contain $R^-(f,\mu) \cup R^+(f,\mu)$.

Step 2. Pick a periodic point $p \in R_\theta$ very close to x. Let N be the minimal positive period of p. Lyapunov characteristic exponents at p are close to those for the measure μ, hence p is a hyperbolic fixed point for f^N with one eigenvalue of absolute value greater than one and the rest less than one; in other words, the local unstable manifold $W^u(p)$ of p with respect to f is one–dimensional and the stable manifold has codimension one. By the continuity of the stable and unstable manifolds on the set R_θ the same is true for any nearby point $x \in R_\theta$. The local stable manifold $W^s(p)$ of the point p intersects transversally at a single point the local unstable manifold of any point $y \in V \cap R_\theta$; similarly, the local unstable manifold $W^u(p)$ of p intersects transversally at a single point the local stable manifold of any such point y. Let us denote these intersection points $[p;y]$ and $[y;p]$ correspondingly.

By Corollary 1.7 the point p is periodic for our $\mathbb{Z}^k$ action. Thus its stationary subgroup G_p has finite index in $\mathbb{Z}^k$ and hence is itself isomorphic to $\mathbb{Z}^k$. All elements of the stationary subgroup preserve the stable manifold $W^s(p)$ and thus we obtain a representation $\mathcal{R} : G_p \to \mathcal{S}^t(f,p)$ whose image must lie in an abelian subgroup of $\mathcal{S}^t(f,p)$. By the rank assumption one of the following possibilities hold:

(1) $\mathcal{R}$ is not discrete;
(2) the image of of $\mathcal{R}$ has a non-trivial kernel.

We will show by contradiction that the former possibility is not possible.

Assume (1), pick another periodic point $q \in R_\theta$ very close to p and consider the intersection of the stationary subgroups of p and q. This is still a finite index subgroup of $\mathbb{Z}^k$, hence its image is not discrete either. Pick $g \in G_p \cap G_q$ such that its image is C^1 close to identity. In particular, both points $g([p;q])$ and $g^{-1}([p;q])$ still belong to the local stable manilold $W^s(p)$. But the uniqueness of the intersection of local stable and unstable manifolds implies that none of these points belong to the local unstable manifold $W^u(q)$. This is a contradiction. For, consider the segment of $W^u(q)$ between q and $[p;q]$. Either the map g or its inverse map this segment into itself so that one of the two points must belong to the local unstable manifold $W^u(q)$.

Step 3. Thus we can assume (2). Let $g \in \operatorname{Ker} \mathcal{R}$. Now again pick a periodic point $q \in R_\theta$. In order to show that the map g fixes every point on the local stable manifold $W^s(q)$ we use the Inclination lemma [KH, Proposition 6.2.23]. It implies that $W^s(q)$ belongs to the closure of the f–orbit of $W^s(p)$ and hence to the set of fixed points of the map g. In particular, g fixes the point $[q;p]$. But by the Poicaré Linearization Theorem the map f which is an expansion on the one–dimensional manifold $W^u(p)$ can be linearized on $W^u(p)$ simultaneously with any C^1 map commuting with it (See e.g.[KH; Sections 2.1 and 6.6]) ; since g is such a map and it preserves two points p and $[q;p]$, it must be the identity on $W^u(p)$. Applying the Inclination lemma again we deduce that it also fixes every point on $W^u(q)$.

By the reduction explained in the Step 1 this finishes the proof of the theorem. □

Remark. The only place we used the fact that the action is defined globally rather than in a neighborhood of an invariant compact set was in our use of Corollary 1.7 to infer that f–periodic points are actually periodic for the whole action. This point looks rather technical and we believe that the statement of the theorem holds for local actions as well.

Since the set of fixed points of g contains a subset diffeomorphic to the product of a codimension one disc and an infinite set, if we assume that our action is real–analytic the conclusion is that g is the identity map.

Corollary 2.2. *Let f be a real–analytic diffeomorphism of a compact manifold M and μ be an f-invariant ergodic measure with non-zero characteristic exponents, one positive and the rest negative. Suppose that the stable rank of f at all periodic points described in Section 1.5 (4) is less than k. Then any real–analytic action of the group $\mathbb{Z}^k$ which includes f is not faithful.*

3. The two–dimensional case.

3.1. $\mathbb{Z}^2$ actions are not faithful. In dimension two both stable and unstable manifolds for any hyperbolic periodic point are one–dimensional unless the point is either contracting or expanding. The latter possibilities do not appear for periodic orbits accompanying non-atomic hyperbolic invariant measures due to Proposition 1.3. We already used Poincaré Linearization for an expanding (or contracting) map on the line near its fixed point. This classical fact, namely, existence and uniqeness of a C^1 linearization, immediately implies that the group of C^1 local diffeomorphisms of $W^s(p)$ commuting with f is one-dimemsional, i.e using the language of the previous section for $r \geq 1$ one can take $r_0 = 1$ and the stable rank of f at the point p is equal to one. Thus, Theorem 2.1 immedialety implies

Corollary 3.1. *Let f be a $C^{1+\epsilon}, (\epsilon > 0)$ diffeomorphism of a compact two-dimensional manifold M, μ an f-invariant ergodic measure with non-zero characteristic exponents and g be a C^1 diffeomorphism of M commuting with f. Then there exist non-zero integers k and l such that the map $f^k g^l$ fixes every point of the set $R^-(f,\mu) \cup R^+(f,\mu)$.*

Since the set R_μ for a non-atomic hyperbolic measure μ contains a set diffeomorphic to the product of an interval with an infinite set we immediately obtain

Corollary 3.2. *Any real-analytic action of $\mathbb{Z}^2$ on a compact two–dimensional manifold which preserves a hyperbolic measure is not faithful.*

By the Variational Principle [KH,Theorem 4.5.3] and Ruelle inequality [KM,Theorem S.2.13] the topological entropy of a diffeomorphism of a compact two–dimensional manifold is equal to the supremum of entropies with respect to ergodic invariant measures with non-zero characteristic exponents. Any non-regular element in our situation has to have zero exonents. Thus we obtain

Corollary 3.3. *Any element of a faithful real–analytic action of $\mathbb{Z}^2$ on a compact two–dimensional manifold has zero topological entropy.*

3.2. Area-preserving analytic actions. Combining Corollaries 3.2 and 3.3 we conclude that a faithful real–analytic action of $\mathbb{Z}^2$ on a compact two–dimensional manifold may have only two types of invariant measures: non-hyperbolic (one exponent is identically equal to zero) and atomic hyperbolic. We can develop this theme further by asking how such measures may coexist. Not much can be said for an arbitrary non-hyperbolic measure but if we assume that it is *smooth*, i.e is given by a nice density, a surprisingly strong conlusion can be drawn in the real–analytic case. So we assume that F is a real–analytic *area–preserving* action of $\mathbb{Z}^2$ on a compact surface which possesses a hyperbolic periodic point p. Passing to the stationary subgroup of the point p which is in itself isomorphic to $\mathbb{Z}^2$ we can assume that p is a fixed point. We start from the description of the local real–analytic centralizer of a real–analytic area–preserving diffeomorphism f in a neighborhood of a hyperbolic fixed point p. This can be summarized as follows: there exists a real–analytic local coordinate system (x_1, x_2) near the point p with p as the origin such that f as well as any area-preserving real–analytic diffeomorphism g fixing the point p and commuting with f have the following form:

$$g(x_1, x_2) = (x_1\omega(x_1x_2), x_2(\omega(x_1x_2))^{-1}),$$

where $\omega(t) = \sum_{n=0}^{\infty} \omega_{2n+1}t^n$ is a converging power series [M]. This implies in particular that g preseves each hyperbola H_c defined by the equation $x_1x_2 = c$. Furthermore, if $\omega_0 > 0$ than each branch of the hyperbola H_c is invariant and the map g allows the following convenient geometric description. From now on we will slightly abuse our notation and will use the same word "hyperbola" and the same notation for a branch. Introduce hyperbolic coordinates (ρ, θ) where $\frac{x_1+x_2}{\sqrt{2}} = \rho\cosh\theta$, $\frac{x_1-x_2}{\sqrt{2}} = \rho\sinh\theta$. In these coordinates

$$g(\rho, \theta) = (\rho, \theta + \tau(\rho)) \tag{3.1}$$

where $\tau(t)$ ia another converging power series with the constant term $\log\omega_0$. Thus,the map g resrtricted to a branch of the hyperbola H_ρ is a *hyperbolic rotation* by the angle $\tau(\rho)$. If $\tau(\rho) = \text{const}$, the map g is linear and is itself called a hyperbolic rotation; otherwise we will call g a *hyperbolic twist*. The function τ will be called *the twist function* (even in the case when it is a constant).

Now consider our $\mathbb{Z}^2$ action F in a neighborhood of the fixed point p. By passing if necessary to a subgroup of index two we can assume that the element $F(m, n)$ has the form (3.1) with the twist function $\tau_{m,n}$. The group property implies additivity of the twist functions:

$$\tau_{m_1+m_2,n_1+n_2} = \tau_{m_1,n_1} + \tau_{m_2,n_2}$$

Thus, on each hyperbola H_ρ our action induces a local group of hyperbolic rotations. For a given ρ here are two possibilities:

(i) $\tau_{m,n}(\rho) = 0$ for some $(m, n) \neq (0, 0)$; in this case we will call H_ρ *a rational hyperbola*;

(ii) F acts on H_ρ with dense orbits; then H_ρ is called *an irrational hyperbola*;

If the twist functions for different $(m, n) \in \mathbb{Z}^2$ are all proportional then all hyperboli H_ρ are either simultaneously rational or simultaneously irrational. We will call the two cases the *rational resonance* and the *irrational resonance* correspondingly. Otherwise the ratio $\frac{\tau_{1,0}(\rho)}{\tau_{0,1}(\rho)}$ is not constant and hence it takes rational values for a dense set of ρ's. We will call this situation the *non-resonance* case.

Now we can formulate our structural result:

Theorem 3.4. *Let F be a real-analytic area-preserving action of $\mathbb{Z}^2$ on a compact two-dimensional manifold and p be a hyperbolic fixed point for F such that the eigenvalues of $DF(m,n)_p$ are positive. Then one of the following possibilities hold:*

(1) *the action is not faithful (the rational resonance case);*
(2) *there exists an open F-invariant set U which contains p in its closure such that F has an analytic first intergral on U i.e. U splits into an analytic family of F-invariant analytic closed curves (the non-resonance case);*
(3) *there exists an open F-invariant set U which contains p and an analytic non-vanishing complete vector-field v defined on the set U such that the restriction of F to U embeds to the one-paremeter flow generated by the vector-field v (the irrational resonance case).*

Proof. This theorem follows from several simple observations on how the local picture described above may be fitted into a global action on a compact manifold. At this point it is important to distinguish between pieces of hyperboli in the local coordinates near the point p and the F-invariant immersed images of $\mathbb{R}$ which are obtained by applying the elements of F to the local hyperboli and extending the hyperbolic angle parameter along these images. To fix our notation we will continue to denote the local curves by H_ρ and will denote the global objects by Γ_ρ. Naturally, Γ_ρ may have points of self-intersection; since the set of such points on each Γ_ρ is F-invariant if such points appear at all they would appear at the local hyperbola H_ρ. We may define the *first positive return map* of the local hyperbola H_ρ to a fixed small neighborhood V of the point p. It is defined by extending the hyperbolic angle parameter in the positive direction by applying the elements of F which effect hyperbolic rotations by positive angles until a parameter value is reached which maps a point from H_ρ back to the neighborhood V. If a point $x \in H_\rho$ returns to V then the whole interval around x also returns. Thus the set of points for which the first positive return is defined is open and by Poincaré recurrence theorem it has full measure. Hence it is open and dense.

Now we consider how the first positive return may appear. There are two "good" possibilities: namely, when the image of a piece of H_ρ under the first positive return coincides with a piece of a local hyperbola, either H_ρ itself, or another one. We will examine these possibilities later and now assume that the image does not lie on a single local hyperbola. First, a point $x \in V$ has infinite stationary subgroup if and only if it belongs to a rational hyperbola; hence the images of a rational hyperbola may intersect only rational hyperboli. But this is possible only in the rational resonance case. In that case an open set of points has the same infinite stationary subgroup; hence by analyticity the action is not fathful. This is the case (1).

Now consider the first return for an irrational hyperbola. In the non-resonance case this image must intersect rational hyperboli, which is impossible, since as we have pointed out a point in V has infinite stationary subgroup if and only if it belongs to a rational hyperbola. The only remaining possibility for a "bad" return then would be in the irratinal resonance case. This is impossible by the following reason: the action F restricted to an irrational hyperbola includes elements effecting arbitrary small hyperbolic rotations. The same is true for a piece which appears as the image of a piece of an irrational hyperbola under the first positive return. So if this piece intersects different hyperboli, the small shift would not keep each point at the same local hyperbola, a contradiction.

Thus, we found that the image of a piece of a hyperbola under the first positive return map is a piece of a single hyperbola and, moreover, rational hyperboli return only on rational ones and irrational on irrational ones. Fix ρ for which H_ρ is an irrational hyperbola and assume that it returns to the hyperbola $H_{\rho'}$. Consider those values $(m,n) \in \mathbb{Z}^2$ for which $\tau_{m,n}(\rho)$ is not too big. Using Euclid algorithm one immediately sees that any transltation by $\tau_{m,n}$ can be effected as the composition of small translations. This implies that the functions $(m,n) \to \tau_{m,n}(\rho)$ are identical for the hyperboli H_ρ and $H_{\rho'}$. Since the twist functions are real–analytic, this means that if the neighborhood V is chosen small enough then in the non-resonance case $\rho = \rho'$. Thus, each irrational hyperbola returns to itself and by continuity this is true for rational hyperboli too. This proves (2). It remains to consider the irrational resonance case. The vector–field v is defined locally in the hyperbolic coordinates as $\tau_{1,0}(\rho)\frac{\partial}{\partial\theta}$ and it is invariant under the return map. An obvious local calculation shows that v is analytic at the point p. The set U is the union of images of V under all elements of the action F. $\square$

It is worth pointing out that for real–analytic actions integrable behavior may change into a non–integrable one. For example, one can construct a real–analytic area–preserving $\mathbb{Z}^2$ action on the torus $\mathbb{T}^2$ which has analytic first integral on an open invariant set U, so that U splits into a family of closed invariant curves, but such that the boundary of U is non–differentiable and outside of U the action is ergodic on an open set.

4. Dimension three and higher.

4.1. The three–dimensional case. The following result is a counterpart of Corollary 3.1. Notice, however, stronger smoothness asumptions.

Theorem 4.1. *Let f be a C^∞ diffeomorphism of a compact three-dimensional manifold M, μ be an f-invariant ergodic measure with non-zero characteristic exponents, g, h two commuting C^∞ diffeomorphisms of M commuting with f. Then there exists $(k_1,k_2,k_3) \in \mathbb{Z}^3 \setminus (0,0,0)$ such that the map $f^{k_1}g^{k_2}h^{k_3}$ fixes every point of the set $R^-(f,\mu) \cup R^+(f,\mu)$.*

Proof. In dimension three any hyperbolic periodic point which is neither attracting nor expanding has either one–dimensional stable manifold or one–dimensional unstable manifold. By replacing f with f^{-1} if necessary we may always assume that for periodic points from R_θ the latter is the case. Thus in order to apply Theorem 2.1 we need to show that for any such point p the stable rank of f at p does not exceed two. This follows from a rather elementary normal form analysis which we will present here in detail.

Let N be the smallest positive period of the point p. The derivative of f^N at the point p has three eigenvalues: $\alpha_1, \alpha_2, \alpha_3$ such that $|\alpha_1| \le |\alpha_2| < 1 < |\alpha_3|$ (the first two may coincide). By taking a power if necessary we may assume that none of the three eigenvalues is a negative real number.

Taking derivatives of the smallest positive powers of g and h which fix the point p we obtain the eigenvaues $\beta_1, \beta_2, \beta_3$ and $\gamma_1, \gamma_2, \gamma_3$ correspondingly. There are two possible situations:

(i) the derivatives of all elements of the stationary subgroup of the point p are simultaneously diagonalizable. In this case we define three linear functionals $l_1(x,y,z) = \log|\alpha_1|x + \log|\beta_1|y + \log|\gamma_1|z$, $l_2(x,y,z) = \log|\alpha_2|x + \log|\beta_2|y + \log|\gamma_2|z$, and $l_3(x,y,z) = \log|\alpha_3|x + \log|\beta_3|y + \log|\gamma_3|z$ and conclude that they must be linearly dependent to avoid having lattice points where all three functionals have negative values which is prohibited due to Proposition 1.3.

(ii) $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$. In this case all three functionals l_1, l_2, l_3 are proportional.

Now we consider the normal form and the centralizer of the map f^N in the two–dimensional local stable manifold $W^s(p)$. It is exactly at this point that we will need our high differentiability assumption.

Lemma 4.2. *Any maximal abelian subgroup in the group $\mathcal{S}(f,p)$ of C^∞ local diffeomophisms of $W^s(p)$ fixing the point p and commuting with f is a two–dimensional Lie group, i.e the stable rank of the map f at the point p is equal to two.*

Proof of the lemma. There are the following possibilities:

Non-resonance case. Both eigenvalues α_1, α_2 are real and $\frac{\log \alpha_1}{\log \alpha_2}$ is not an integer. In this case there is a unique invariant affine structure in $W^s(p)$ and any C^∞ diffeomorphism commuting with f and fixing p is linear and diagonalizable simultaneously with f [S].

Resonance case. Both eigenvalues α_1, α_2 are real and different and $\frac{\log \alpha_1}{\log \alpha_2}$ is an integer, say, $\frac{\log \alpha_1}{\log \alpha_2} = k$, i.e. $\alpha_1 = (\alpha_2)^k$. In this case there is a C^∞ coordinate system (x_1, x_2) in $W^s(p)$ with p as the origin such that f has the form

$$f(x_1, x_2) = (\alpha_1 x_1 + c x_2^k, \alpha_2 x_2,)$$

for some constant c. If $c \neq 0$ the C^∞ centralizer of this map consists of maps of the form

$$\mathcal{G}(x_1, x_2) = (\lambda^k x_1 + d x_k^2, \lambda x_2)$$

with λ, $d \in \mathbb{R}$. If $c = 0$, the centralizer is three-dimensional. Namely, it consists of maps

$$\mathcal{G}(x_1, x_2) = (\eta x_1 + d x_k^2, \lambda x_2)$$

with λ, η, $d \in \mathbb{R}$, but any maximal abelian subgroup in it is still two–dimensional.

Eigevalues with equal absolute values. This means either complex eigenvalues or $\alpha_1 = \alpha_2$. In both cases the map f on $W^s(p)$ is C^∞ linearizable and and its centralizer consists of linear maps. In the forner case the centralizer is abelian and two–dimwnsional and in the latter it consists of all linear maps, but any maximal abelian subgroup of the centralizer is still two–dimensional. This finishes the proof of the lemma. □

Now Therem 2.1 directly applies. □

Corollary 4.3. *Any real–analytic action of $\mathbb{Z}^3$ on a compact three–dimensional manifold which preserves a hyperbolic measure is not faithful.*

4.2. Defect and rank. Attempts to extend Corollary 3.1 and Theorem 4.1 to higher dimension face two problems:

(1) beginning from dimension four neither stable nor unstable manifold of a hyperbolic periodic point has to be one–dimensional;
(2) the dimension of an abelian subgroup in the local centralizer of a smooth contraction near a fixed point on an m–dimensional manifold for $m \geq 3$ may be greater than $m-1$ in certain resonance cases; for example, for $m = 3$ it is equal to three if the eigenvalues have the form $\lambda, \lambda^2, \lambda^4$ for some $0 < \lambda < 1$. Hence one can find a map in dimension four with a hyperbolic fixed point which has one–dimensional unstable manifold and the stable rank at the point equals to three.

Both problems may be avoided by putting some extra conditions on the Lyapunov exponents of the invariant measure. For example, if one assumes that no two different Lyapunov exponents are proportional with a positive coefficient (multiple exponents are allowed) then one can find a regular element with non-resonance exponents and hence periodic points for which the local centralizer in the stable manifold consists of linear maps. If one excludes multiple exponents this solves the problem with the dimension of the centralizer. A stronger condition which first appeared in Corollary 1.5 allows also to guarantee existence of elements with one-dimensional unstable manifolds. Namely we will consider measures which are as close as possible to strongly hyperbolic.

Theorem 4.4. *Let F be a C^∞ $\mathbb{Z}^k$ action on a k–dimensional compact manifold with an invariant hyperbolic non-atomic measure μ such that the defect $d(\mu) = 1$. Then there exist a regular element $f = F(n_0)$ and a non-zero element $n \in \mathbb{Z}^k$ such that $F(n)$ fixes every point of the set $R^-(f,\mu) \cup R^+(f,\mu)$.*

Proof. Since $d(\mu) + r(\mu) = k$ we have $r(\mu) = k - 1$ and by Corollary 1.5 there are regular elements with one dimensional unstable manifold. In particular, fixing an appropriate octant $\mathcal{O} \in \mathbb{R}^k$ corresponding to one positive coordinate and the rest negative and taking the pre–image of $\mathcal{O}$ under the Lyapunov map Ψ we find a Weyl chamber $\mathcal{W}$ such that any map $F(n)$ where $n \in \mathbb{Z}^k \cap \mathcal{W}$ has one positive Lyapunov exponent and $k-1$ (counting with multiplicities) negative ones. Hence any $F(n)$ regular points (and consequently the periodic points described in section 1.5.(4) have one–dimensional unstable manifolds and stable manifolds of codimension one. Let p be such a periodic point and $\lambda_1, \ldots, \lambda_{k-1}$ be its eigenvalues of absolute value less than one. If N is a period of p then the numbers $\frac{\log|\lambda_1|}{N}, \ldots, \frac{\log|\lambda_{k-1}|}{N}$ are close to the negative Lyapunov exponents of $F(n)$. Let $\alpha_i = \log|\lambda_i|$, $i = 1, \ldots, k-1$. The following *non-resonance* condition is sufficient for the stable rank of $F(n)$ at the pont p to be equal to $k-1$.

(NR) If $m_1, \ldots m_{k-1}$ are non–negative integers and

$$\alpha_i = \sum_{j=1}^{k-1} m_j \alpha_j, \text{ then } m_1 = \cdots = m_{i-1} = m_{i+1} = \ldots m_{k-1} = 0.$$

For, if the non-resonance condition is satisfied the restriction of $F(n)$ to the stable manifold $W^s(p)$ can be C^∞ linearized and brought to a diagonal form. But any map C^∞ map commuting with a linear diagonal contracting non-resonance map is again linear and diagonal so the stable rank of the map $F(n)$ at such a point p is equal to $k-1$.

Now we will show that if we pick $n \in \mathbb{Z}^k \cap \mathcal{W}$ such that the Lyapunov characteristic exponents $x_i = \chi_i(n)$, $i = 1, \dots, k-1$ satisfy the non-resonance condition (NR) then the periodic points from 1.5, (4) can be chosen in such a way that the non-resonance condition is satisfied for the α's. Let us consider an equation $x_i = \sum_{j=1}^{k-1} m_j x_j$ with non-negative integer coefficients $m_1, \dots m_{k-1}$. The $k-1$ Lyapunov exponents of F which are negative in the Weyl chamber $\mathcal{W}$ can not satisfy such a relation identically. For, by our rank assumption all k Lyapunov exponents satisfy only one linearly independent relation and that relation can be expressed by an equation with non-negative coefficients. The only relations of the above form

$$(m_i - 1)x_i + \sum_{j \neq i} m_j x_j = 0 \tag{4.1}$$

which possesses this property would have to have positive coefficient m_i. But no such relation can hold at any point of $\mathcal{W}$ where all variables $x_1, \dots, x_{k-1}$ are negative. Thus the only possible relations have $m_i = 0$. Now consider the hyperplane $L = \operatorname{Im} \Psi_\mu \subset \mathbb{R}^k$ and its intersection with the octant $\mathcal{O}$. Any relation of the form (4.1) with $m_i = 0$ determines a hyperlane in $\mathbb{R}^k$. If at least one of the coefficients m_j, $j \neq i$ is large enough then the intersection of this hyperplane with $\mathcal{O}$ lies in a small neighborhood (in the projective sense) of the hyperplane $x_i = 0$. Since for any $i \in \{1, \dots k-1\}$ there are only finitely many relations of the form

$$x_i = \sum_{j \neq i} m_j x_j$$

with bounded coefficients we conclude that the complement to the intersection of $\Psi_\mu(\mathcal{W}) = L \cap \mathcal{O}$ to the union of all planes of the form (4.1) is open and dense in $\operatorname{Im} \mathcal{W}$; hence its pre-image which is open and dense in $\Psi_\mu(\mathcal{W})$ contains points of the lattice $\mathbb{Z}^k$. Picking any such lattice point n we can guarantee that the periodic points from Section 1.5.(4) can be chosen to satisfy the non-resonance condition. □

5. Remarks on continuous time actions. All results of this paper have natural counterparts for continuous time systems, i.e. for actions of $\mathbb{R}^k$. Naturally the dimension of the phase space has to be increased by the dimension of the group, i.e. by k. It is natural to consider measures supported on the set where the action is locally free. Accordingly, in the definitions of hyperbolic, strongly hyperbolic etc measure one should allow for k zero exponents corresponding to the orbit direction. The role of atomic measures is played by measures concentrated on compact orbits.

Thus, the following facts hold: any strongly hyperbolic measure for an $\mathbb{R}^k$ action on a $2k$-dimensional manifold is atomic; the group $\mathbb{R}^2$ acting on a three-dimensional manifold cannot have any hyperbolic measures, not concentrated on compact orbits; any $C^{1+\epsilon}$ action of $\mathbb{R}^2$ on a four-dimensional manifold or a C^∞ action of $\mathbb{R}^3$ on a six-dimensional manifold with a non-atomic hyperbolic measure contains a one-parameter subgroup which fixes every point in the support of the measure as well as on the stable and unstable manifolds of the regular points from the support.

The proofs of these facts are obtained by translating essentially verbatim our proofs in the discrete time case using proper versions of results from Pesin theory and from local normal forms theory. All examples and counterexamples are extended to the continuous time situation via suspension construction.

References

[B] G.R. Belitskii, *Normal forms, invariants and local mappings*, Naukova dumka, 1979.

[BK] I. U. Bronstein and A. Ya. Kopanskii, *Smooth invariant manifolds and normal forms*, World Scientific, 1994.

[C] K.T.Chen, *Equivalence and decomposition of vector fields about an elementary critical point*, Amer. Journal Math. **85** (1963), 693–733.

[H] Huyi Hu, *Some ergodic properties of commuting diffeomorphisms*, Erg. Theory and Dynam. Systems **13** (1993), 73–100.

[K] A.Katok, *Hyperbolic measures for actions of higher–rank abelian groups*, preprint.

[KH] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.

[KL1] A. Katok and J. Lewis, *Local Rigidity of certain groups of toral automorphisms*, Israel J. Math **75** (1991), 203-241.

[KL2] A. Katok and J. Lewis, *Global rigidity results for lattice actions on tori and new examples of volume–preserving actions*, Israel J. Math. **93** (1996), 253–280.

[KM] A. Katok and L. Mendoza, *Dynamical systems with nonumiformly hyperbolic behavior*, Supplement to "Introduction to the modern theory of dynamical systems" (A.Katok and B.Hasselblatt), 1995, pp. 659–700.

[KS1] A. Katok and R. Spatzier, *First cohomology of Anosov actions of higher rank Abelian groups and applications to rigidity*, Publ. Math. IHES **79** (1994), 131–156.

[KS2] A. Katok and R. Spatzier, *Invariant measures for higher rank hyperbolic Abelian actions*,, Erg. Theory and Dynam. Systems, **16** (1996).

[KS3] A. Katok and R. Spatzier, *Differential Rigidity of Anosov Actions of Higher Rank Abelian Groups and Projective Lattice Actions*, preprint.

[M] J.Moser, *The analytic invariants of an area preserving mapping near a hyperbolic fixed point*, Comm. Pure Appl. Math. **11** (1958), 673–692.

[P] Ya. B. Pesin, *Characteristic Lyapunov exponents and smooth ergodic theory*, Russian Math. Surveys **32** (1977), 55–114.

[PS] C. Pugh and M. Shub, *Ergodicity of Anosov actions*, Inventiones Math. **15** (1972), 1–23.

[S] S. Sternberg, *Local contractions and a theorem of Poincaré*, Amer. Journal Math. **79** (1957), 809–824.

E-mail address: katok_a@math.psu.edu

Received for publication May 1996.